CHAPTER II

THE STANDARD MODEL

In this chapter a brief review of the Standard Model of particle physics is given. We begin with the gauge groups of the Standard Model in the first section and then provide, in Section 2.2, the construction of the Lagrangian whose form is dictated by gauge invariance. We next discuss the mechanism of electroweak symmetry breaking and the particle spectrum in Sections 2.3 and 2.4 respectively. The properties of fields under discrete symmetries (parity P, charge conjugation C, and the combined CP) are then investigated in Section 2.5. Finally, the electric dipole moment of the electron in the context of the Standard Model will be reviewed in the last section. The notations used in this chapter follow the book by Peskin and Schroeder [12].

2.1 The Standard Model gauge groups

The Standard Model (SM) of particle physics is the gauge field theory which successfully describes the nature of electromagnetic, weak and strong interactions among elementary particles. It combines Quantum Chromodynamics (QCD) with the Glashow-Weinberg-Salam model of electroweak interactions, and is based on the gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$. The $SU(3)_C$ group, the symmetry group of QCD which describes the strong interactions among colored quarks, is believed to be an exact symmetry of the SM. On the other hand, $SU(2)_L \times U(1)_Y$, the symmetry group of the standard electroweak theory which describes the electromagnetic and weak interactions among quarks and leptons, must be broken spontaneously via the so-called Higgs mechanism to the $U(1)_{EM}$ group of electromagnetism.

In the strong interaction sector, the generators of the $SU(3)_C$ group

$t^a (a = 1, \ldots, 8)$ are related to the 3×3 Gell-Mann matrices λ^a by $t^a = \lambda^a/2$, and obey the commutation relations

$$[t^a, t^b] = i f^{abc} t^c \tag{2.1}$$

where f^{abc} are the antisymmetric structure constants of $SU(3)_C$. The non-vanishing f^{abc} are given by the permutations of $f^{123} = 1$, $f^{147} = f^{246} = f^{257} = f^{345} = f^{516} = f^{637} = 1/2$, $f^{458} = f^{678} = \sqrt{3}/2$. Here are the explicit forms of the eight traceless hermitian Gell-Mann matrices:

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{2.2}$$

For each generator, there is a corresponding gauge boson, called a gluon and denoted by G^a_μ, which mediates the strong interactions. With the $SU(3)_C$ coupling constant g_3, the field strength tensor of the gluon field is defined by

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_3 f^{abc} G^b_\mu G^c_\nu. \tag{2.3}$$

In the electroweak sector, before the symmetry is broken, there are four gauge bosons, $W^a_\mu (a = 1, 2, 3)$ and B_μ, associated with the three $SU(2)_L$ weak isospin generators, τ^a, and the $U(1)_Y$ weak hypercharge generator, Y, respectively. The $SU(2)_L$ generators are related to the 2×2 Pauli spin matrices by $\tau^a = \sigma^a/2$ and satisfy the commutation relations

$$[\tau^a, \tau^b] = i\epsilon^{abc}\tau^c \tag{2.4}$$

where ϵ^{abc} is the totally antisymmetric structure constant with $\epsilon^{123} \equiv +1$. We will see later that the hypercharge generator is related to the third component of the weak isospin τ^3 and the electric charge Q in units of the positron charge $+e$ by

$$Y = Q - \tau^3.$$

The field strength tensors for W and B bosons are defined by

$$\begin{aligned} W^a_{\mu\nu} &= \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\epsilon^{abc}W^b_\mu W^c_\nu, \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, \end{aligned} \tag{2.5}$$

with g being the $SU(2)_L$ gauge coupling constant.

2.2 Lagrangian of the Standard Model

Before proceeding to consider the Lagrangian of the Standard Model, we first look at the particle content of the model. The field content of the SM consists of three generations of quarks and leptons, together with the gauge and Higgs bosons as listed in Table 2.1.

Each fermion field is conventionally decomposed into its left-handed and right-handed components as

$$\psi_L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\psi, \qquad \psi_R = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\psi. \tag{2.6}$$

The reason for this, besides their difference in the transformation properties under the Lorentz group, is that they couple differently to gauge bosons. As seen in the table, the left-handed fermions and the complex Higgs scalars form weak isodoublets while the right-handed fermions are weak isosinglets. Note that the neutrinos are assumed to be massless and exist only with left-handed components. Moreover, each quark flavor is a color triplet under the $SU(3)_C$ gauge group, while all other particles are color singlets and do not experience the strong interactions.

The Lagrangian density of the Standard Model can be decomposed as [13]

$$\mathcal{L}_{\text{SM}} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 \tag{2.7}$$

where

(1)

$$\mathcal{L}_1 = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} - \frac{1}{4}W^a_{\mu\nu}W^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \tag{2.8}$$

describes the pure gauge interactions, and contains the kinetic terms and self interactions of the vector gauge fields;

(2)

$$\begin{aligned}\mathcal{L}_2 &= \bar{L}^i_L(i\gamma^\mu D_\mu)L^i_L + \bar{e}^i_R(i\gamma^\mu D_\mu)e^i_R + \bar{Q}^i_L(i\gamma^\mu D_\mu)Q^i_L \\ &\quad + \bar{u}^i_R(i\gamma^\mu D_\mu)u^i_R + \bar{d}^i_R(i\gamma^\mu D_\mu)d^i_R\end{aligned} \tag{2.9}$$

is the matter Lagrangian which consists of the kinetic terms and gauge interactions of the fermion fields. The index i is summed over the three families of fermions. The covariant derivative of a quark field, for example, is defined by

$$D_\mu = (\partial_\mu - ig_3G^a_\mu t^a - igW^a_\mu \tau^a - ig'YB_\mu); \tag{2.10}$$

(3)

$$\mathcal{L}_3 = (D_\mu\phi)^\dagger(D^\mu\phi) - V(\phi) \tag{2.11}$$

is the Higgs boson Lagrangian which contains the kinetic term, gauge interactions, and self interactions of the Higgs boson. The Higgs potential is written as

$$V(\phi) = \mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2; \tag{2.12}$$

and (4)

$$\mathcal{L}_4 = -\lambda^{ij}_e\bar{L}^i_L\phi e^j_R - \lambda^{ij}_u\bar{Q}^i_L\tilde{\phi}u^j_R - \lambda^{ij}_d\bar{Q}^i_L\phi d^j_R + \text{h.c.} \tag{2.13}$$

is the generalization of the Yukawa interactions which couple the Higgs field to fermions. Here, $\tilde{\phi} = i\sigma_2\phi^*$ is the Higgs isodoublet with hypercharge $Y = -1$. The dimensionless couplings λ^{ij}_e, λ^{ij}_u and λ^{ij}_d are general 3×3 complex matrices which are not necessarily symmetric or hermitian.

Table 2.1: Particle content of the Standard Model.

Names	Notations	spin	$SU(3)_C, SU(2)_L, U(1)_Y$
quarks, Q (3 families)	$(u_L\ d_L)\ (c_L\ s_L)\ (t_L\ b_L)$ $u_R\ c_R\ t_R$ $d_R\ s_R\ b_R$	1/2 1/2 1/2	$(\mathbf{3},\ \mathbf{2},\ \frac{1}{6})$ $(\bar{\mathbf{3}},\ \mathbf{1},\ \frac{2}{3})$ $(\bar{\mathbf{3}},\ \mathbf{1},\ -\frac{1}{3})$
leptons, L (3 families)	$(\nu_e\ e_L)\ (\nu_\mu\ \mu_L)\ (\nu_\tau\ \tau_L)$ $e_R\ \mu_R\ \tau_R$	1/2 1/2	$(\mathbf{1},\ \mathbf{2},\ -\frac{1}{2})$ $(\bar{\mathbf{1}},\ \mathbf{1},\ -1)$
Higgs, ϕ	$(\phi_1\ \phi_2)$	0	$(\mathbf{1},\ \mathbf{2},\ 1)$
gluon	g	1	$(\mathbf{8},\ \mathbf{1},\ 0)$
W bosons	$W^1\ W^2\ W^3$	1	$(\mathbf{1},\ \mathbf{3},\ 0)$
B boson	B	1	$(\mathbf{1},\ \mathbf{1},\ 0)$

The Lagrangian (2.7) is invariant under the local $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge transformations under which the fields transform as follows:

1. For left-handed quarks Q_L:

$$Q_L \longrightarrow Q_L' = e^{i\gamma_a(x)t^a + i\alpha_a(x)\tau^a + i\beta(x)Y} Q_L.$$

2. For left-handed leptons L_L and the Higgs scalar ϕ:

$$L_L \longrightarrow L_L' = e^{i\alpha_a(x)\tau^a + i\beta(x)Y} L_L, \qquad \phi \longrightarrow \phi' = e^{i\alpha_a(x)\tau^a + i\beta(x)Y} \phi.$$

3. For right-handed quarks:

$$q_R \longrightarrow q_R' = e^{i\gamma_a(x)t^a + i\beta(x)Y} q_R.$$

4. For right-handed leptons:

$$l_R \to l'_R = e^{i\beta(x)Y} l_R.$$

5. For gauge bosons:

$$\begin{aligned}
G^a_\mu(x)t^a &\to e^{i\gamma_a(x)t^a}\left(G^a_\mu(x)t^a + \frac{i}{g_3}\partial_\mu\right)e^{-i\gamma_a(x)t^a}, \\
W^a_\mu(x)\tau^a &\to e^{i\alpha_a(x)\tau^a}\left(W^a_\mu(x)\tau^a + \frac{i}{g}\partial_\mu\right)e^{-i\alpha_a(x)\tau^a}, \\
B_\mu(x)Y &\to \left(B_\mu(x) + \frac{1}{g'}\partial_\mu\beta(x)\right)Y.
\end{aligned}$$

So far, all gauge fields and fermion fields in the theory are still massless. In order to generate masses for these fields, the $SU(2)_L \times U(1)_Y$ symmetry must be spontaneously broken and the Higgs mechanism, explored by Higgs, Kibble, Guralnik, Hagen, Brout, and Englert, will subsequently generate masses for the particles.

2.3 Spontaneous symmetry breaking

To discuss sponteneous electroweak symmetry breaking, we now consider the scalar potential of the Higgs field in Eq. (2.12). For positive λ and $\mu^2 < 0$, the minimum of the potential occurs at

$$\langle\phi^\dagger\phi\rangle_0 = \frac{v^2}{2}, \quad v = \sqrt{\frac{-\mu^2}{\lambda}}. \tag{2.14}$$

The ground state of the theory must be chosen such that only the neutral component of the Higgs doublet acquires a vacuum expectation value (VEV)

$$\langle\phi\rangle_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \tag{2.15}$$

if we assume that the physical vacuum is electrically neutral. The field configuration of this non-vanishing VEV lies along a certain direction in the $SU(2)_L \times U(1)_Y$ representation space, and is invariant only under the $U(1)$ rotation about that preferred direction. Therefore, the electroweak symmetry is spontaneously broken down to the $U(1)_{EM}$ symmetry.

In general, the Higgs doublet ϕ is conveniently expressed in the form

$$\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta_2(x) + i\zeta_1(x) \\ \sqrt{(v + h(x))^2 - |\vec{\zeta}|^2} - i\zeta_3(x) \end{pmatrix} = \frac{1}{\sqrt{2}} e^{i\eta_a(x)\tau^a/v} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} \tag{2.16}$$

where $\zeta_a(x)$ and $h(x)$ are real scalar fields with vanishing VEVs, and $\vec{\zeta}$ is related to $\vec{\eta}$ by $\vec{\zeta} = \sin(|\vec{\eta}|/2)(v + h)\vec{\eta}/|\vec{\eta}|$. This was done by first shifting the magnitude of ϕ by $h/\sqrt{2}$ and then rotating it to other direction in the $SU(2)$ space. Using the gauge invariance of the Lagrangian, it is possible to fix gauge to the unitary gauge in which only the physical Higgs field $h(x)$ is left, by performing a local $SU(2)$ rotation on ϕ,

$$\phi(x) \to e^{-i\eta_a(x)\tau^a} \phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} \tag{2.17}$$

so that the three remaining scalar fields ζ_a get absorbed into the $SU(2)$ gauge bosons through

$$W^a_\mu(x)\tau^a \to e^{i\eta_a(x)\tau^a} \left(W^a_\mu(x)\tau^a + \frac{i}{g}\partial_\mu \right) e^{-i\eta_a(x)\tau^a}. \tag{2.18}$$

2.4 The mass spectrum of the Standard Model

We will begin this section with the Higgs boson mass. From the scalar potential (2.12), substituting (2.17) into it yields

$$\begin{aligned} V &= \frac{\mu^2}{2} \begin{pmatrix} 0 & v + h \end{pmatrix} \begin{pmatrix} 0 \\ v + h \end{pmatrix} + \frac{\lambda}{4} \left| \begin{pmatrix} 0 & v + h \end{pmatrix} \begin{pmatrix} 0 \\ v + h \end{pmatrix} \right|^2 \\ &= \frac{\mu^2}{2}(v + h)^2 + \frac{\lambda}{4}(v + h)^4. \end{aligned}$$

Using the relation $\mu^2 = -\lambda v^2$, then

$$V = \frac{1}{2}\left(2\lambda v^2\right) h^2 + \lambda v h^3 + \frac{\lambda}{4} h^4 - \frac{1}{4}\lambda v^4. \tag{2.19}$$

From the potential above, the Higgs boson mass reads

$$m_h = \sqrt{2\lambda}v. \tag{2.20}$$

We see that the magnitude of the Higgs mass depends on its vacuum expectation value and the coupling constant λ.

Next, we consider the mass spectrum of gauge bosons. Since the Higgs particle interacts with gauge bosons only via the covariant derivative, $D_\mu\phi$, then by evaluating the kinetic terms of the Higgs field at its vacuum expectation value, the gauge boson mass terms can be obtained. The relevant terms are

$$\Delta\mathcal{L}_3 = \frac{1}{2}\begin{pmatrix} 0 & v \end{pmatrix}\left(gW_\mu^a\tau^a + g'YB_\mu\right)\left(gW^{b\mu}\tau^b + g'YB^\mu\right)\begin{pmatrix} 0 \\ v \end{pmatrix}. \tag{2.21}$$

Substitute $\tau^a = \sigma^a/2$ and $Y = 1/2$, then we get

$$\Delta\mathcal{L}_3 = \frac{v^2}{8}\left[g^2(W_\mu^1)^2 + g^2(W_\mu^2)^2 + (-gW_\mu^3 + g'B_\mu)^2\right]. \tag{2.22}$$

Define the mass eigenstate fields $W_\mu^\pm$, Z_μ and A_μ as follows

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2) \tag{2.23}$$

$$Z_\mu = \frac{1}{\sqrt{g^2+g'^2}}(gW_\mu^3 - g'B_\mu) \tag{2.24}$$

$$A_\mu = \frac{1}{\sqrt{g^2+g'^2}}(g'W_\mu^3 + gB_\mu). \tag{2.25}$$

In terms of these new fields, $\Delta\mathcal{L}_3$ becomes

$$\Delta\mathcal{L}_3 = \frac{g^2v^2}{4}(W_\mu^+W^{-\mu}) + (g^2+g'^2)\frac{v^2}{8}(Z_\mu)^2. \tag{2.26}$$

Since W_μ^+ and W_μ^- are hermitian conjugate to each other, then the first term in (2.26) can be rewritten as

$$\frac{1}{2}\frac{g^2v^2}{4}\left(|W_\mu^+|^2 + |W_\mu^-|^2\right). \tag{2.27}$$

Thus after electroweak symmetry breaking, three gauge vector bosons acquire masses via the Higgs mechanism; W_μ^+ and W_μ^- are mass degenerate states with mass

$$m_W = \frac{gv}{2}, \tag{2.28}$$

and Z_μ gets its mass equal to

$$m_Z = \sqrt{g^2 + g'^2}\frac{v}{2}. \tag{2.29}$$

The gauge boson A_μ, in contrast, remains massless and is identified with the unbroken $U(1)_{EM}$ gauge boson, the photon.

From the mixing of W^3_μ and B_μ in Eqs. (2.24) and (2.25), it is convenient to introduce the weak mixing angle θ_W, also known as the Weinberg angle, such that

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix}$$

where

$$\cos\theta_W = \frac{g}{\sqrt{g^2 + g'^2}}, \qquad \sin\theta_W = \frac{g'}{\sqrt{g^2 + g'^2}}. \tag{2.30}$$

Now, if we rewrite the electroweak covariant derivative in terms of mass eigenstates (2.23)-(2.25), it becomes

$$\begin{aligned} D_\mu &= \partial_\mu - i\frac{g}{\sqrt{2}}(W^+_\mu \tau^+ + W^-_\mu \tau^-) - i\frac{1}{\sqrt{g^2 + g'^2}} Z_\mu (g^2 \tau^3 - g'^2 Y) \\ &\quad - i\frac{gg'}{\sqrt{g^2 + g'^2}} A_\mu (\tau^3 + Y) \end{aligned} \tag{2.31}$$

where

$$\tau^{\pm} = (\tau^1 \pm i\tau^2) = \frac{1}{2}(\sigma^1 \pm i\sigma^2). \tag{2.32}$$

Since we have identified A_μ with photon, then we identify the electromagnetic coupling in the last term of Eq. (2.31) as the positron electric charge

$$e = \frac{gg'}{\sqrt{g^2 + g'^2}} = g\sin\theta_W \tag{2.33}$$

and the unbroken generator

$$Q = \tau^3 + Y \tag{2.34}$$

as the electric charge; this formula was mentioned earlier in Section 2.1.

We now turn our attention to the fermion mass spectrum. The fermion mass terms can be obtained by replacing the Higgs field ϕ in the Lagrangian (2.13) with its VEV. The result is

$$\Delta\mathcal{L}_4 = -\frac{1}{\sqrt{2}}\lambda_e^{ij} v \bar{e}_L^i e_R^j - \frac{1}{\sqrt{2}}\lambda_u^{ij} v \bar{u}_L^i u_R^j - \frac{1}{\sqrt{2}}\lambda_d^{ij} v \bar{d}_L^i d_R^j + \text{h.c.}. \tag{2.35}$$

This gives rise to the fermion mass and flavor mixing matrices

$$m_f^{ij} = \frac{1}{\sqrt{2}}\lambda_f^{ij} v \tag{2.36}$$

where f denotes e, u, d. Since these mass matrices need not be diagonal, then the mass eigenstates need not necessarily be identical to the gauge eigenstates but are linear combinations of them instead. In order to diagonalize the mass matrices, we perform biunitary transformations as follows. In the quark sector, define the unitary matrices U_u, U_d, W_u and W_d such that

$$\begin{aligned} U_u^\dagger \left(\lambda_u \lambda_u^\dagger\right) U_u &= W_u^\dagger \left(\lambda_u^\dagger \lambda_u\right) W_u = D_u^2, \\ U_d^\dagger \left(\lambda_d \lambda_d^\dagger\right) U_d &= W_d^\dagger \left(\lambda_d^\dagger \lambda_d\right) W_d = D_d^2, \end{aligned} \tag{2.37}$$

where D_u^2 and D_d^2 are diagonal matrices with positive eigenvalues. From the relations above, we have

$$D_u = U_u^\dagger \lambda_u W_u, \qquad D_d = U_d^\dagger \lambda_d W_d \tag{2.38}$$

where D_u and D_d are diagonal matrices whose diagonal elements are the positive roots of the eigenvalues of D_u^2 and D_d^2 respectively.[1] With these constructions,

[1]That such a diagonalization procedure is possible can be understood as follows. For any $n \times n$ matrix λ, the matrices $\lambda\lambda^\dagger$ and $\lambda^\dagger\lambda$ are unitary and therefore can be diagonalized by unitary transformations. Let $|\alpha\rangle$ be an eigenvector of $\lambda^\dagger\lambda$ with a non-zero eigenvalue α. Then it can be seen easily that $\lambda|\alpha\rangle$ is non-zero and is an eigenvector of $\lambda\lambda^\dagger$ with the same eigenvalue α. Thus $\lambda^\dagger\lambda$ and $\lambda\lambda^\dagger$ have the same set of eigenvalues, so they are equal after the diagonalization. Moreover, from $0 < \langle\alpha|\lambda^\dagger\lambda|\alpha\rangle = \alpha\langle\alpha|\alpha\rangle$, we see that all their eigenvalues are non-negative. Therefore, there exist the unitary matrices U and W such that $U^\dagger(\lambda\lambda^\dagger)U = W^\dagger(\lambda^\dagger\lambda)W = D^2$ with D^2 being a diagonal matrix with non-negative eigenvalues. Let $M \equiv U^\dagger\lambda W$, then $M^\dagger M = MM^\dagger = D^2$. As M and $M^\dagger$ commute, M itself must be a diagonal matrix. By adjusting the $U(1)$ factor of either U or W, we can make M real and positive, so M is a positive root of D^2.

the gauge eigenstates are related to the mass eigenstates (expressed with prime) by the transformations

$$u_L^i = U_u^{ij} u_L'^j, \qquad d_L^i = U_d^{ij} d_L'^j, \tag{2.39}$$

$$u_R^i = W_u^{ij} u_R'^j, \qquad d_R^i = W_d^{ij} d_R'^j. \tag{2.40}$$

Similarly, in the lepton sector we diagonalize the mass matrix λ_e by

$$D_e = U_e^\dagger \lambda_e W_e \tag{2.41}$$

and make the transformations on the lepton fields as follows:

$$e_L^i = U_e^{ij} e_L'^j, \qquad \nu_L^i = U_e^{ij} \nu_L'^j, \qquad e_R^i = W_e^{ij} e_R'^j. \tag{2.42}$$

Observe that ν_L transforms in the same way as e_L. This is possible because we have assumed that neutrinos are massless and have no right-handed components. Under the transformations (2.39), (2.40), and (2.42), the Lagrangian (2.35) becomes

$$\Delta\mathcal{L}_4 = -\frac{1}{\sqrt{2}} D_e^{ii} v \bar{e}_L'^i e_R'^i - \frac{1}{\sqrt{2}} D_u^{ii} v \bar{u}_L'^i u_R'^i - \frac{1}{\sqrt{2}} D_d^{ii} v \bar{d}_L'^i d_R'^i + \text{h.c.}.$$

This is the mass terms which are already diagonal in flavor and thus we have fermion masses for the ith generation

$$m_f^i = \frac{1}{\sqrt{2}} D_f^{ii} v. \tag{2.43}$$

Before proceeding further to the next section, we should ask whether transforming fermion fields into their mass eigenstates, which are the observable states, affects the form of the Lagrangian (2.9). The answer is yes; indeed, it affects only the charge-changing weak interactions, terms that fermions couple to charged W bosons. In terms of the fermion gauge eigenstates, the charged current interactions can be written as

$$\mathcal{L}_{CC} = g\left(W_\mu^+ J_W^{\mu+} + W_\mu^- J_W^{\mu-}\right) \tag{2.44}$$

where

$$J_W^{\mu+} = \frac{1}{\sqrt{2}}(\bar{\nu}_L\gamma^\mu e_L + \bar{u}_L\gamma^\mu d_L) \tag{2.45}$$

$$J_W^{\mu-} = \frac{1}{\sqrt{2}}(\bar{e}_L\gamma^\mu \nu_L + \bar{d}_L\gamma^\mu u_L). \tag{2.46}$$

It is clear that under the transformations (2.39) and (2.42), the second terms of both weak currents get extra factors

$$\frac{1}{\sqrt{2}}\bar{u}^i_L\gamma^\mu d^i_L = \frac{1}{\sqrt{2}}\bar{u}'^i_L\gamma^\mu (U_u^\dagger U_d)^{ij} d'^j_L, \tag{2.47}$$

$$\frac{1}{\sqrt{2}}\bar{d}^i_L\gamma^\mu u^i_L = \frac{1}{\sqrt{2}}\bar{d}'^i_L\gamma^\mu (U_d^\dagger U_u)^{ij} u'^j_L. \tag{2.48}$$

It is customary to define a unitary rotation matrix

$$V = U_u^\dagger U_d \tag{2.49}$$

operating on down-type quark mass eigenstates. Explicitly

$$\begin{pmatrix} d'' \\ s'' \\ b'' \end{pmatrix} = V \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}. \tag{2.50}$$

The marix V is well-known as the Cabibbo-Kobayashi-Maskawa (CKM) quark mixing matrix [14, 15]. Since it is a general 3×3 unitary matrix, then it can be parameterized by three real rotation angles (the Euler angles which parameterize the $SO(3)$ subgroup of $U(3)$) and six complex phases. However, not all of these phases are physical as they can be removed by performing phase rotations of the quark fields. At first, it seems one can remove all six phases by changing phases of all six quarks, but since V is invariant under the rotation of all quarks with the same phase, only 5 complex phases can be removed. So we are left with one physical phase which cannot be removed. There are several parameterizations of the CKM matrix. The standard one is parameterized by the angles $\theta_{12}, \theta_{23}, \theta_{13}$ and a phase δ as [16, 17, 18, 19]

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} e^{-i\delta/2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta/2} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix}$$

$$\times \begin{pmatrix} e^{i\delta/2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\delta/2} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \quad (2.51)$$

where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$. It was shown by Jarlskog [20] that for an arbitrary unitary matrix V, the combinations of the matrix elements $(V_{ij}V_{kl}V_{kj}^{*}V_{il}^{*})$ with $i \neq k$ and $j \neq l$ is invariant under the transformation

$$V_{ij} \longrightarrow e^{i(\alpha_i - \beta_j)}V_{ij}.$$

Applying this result to the CKM matrix, we see that if one of such combinations associated with the CKM matrix is complex then it cannot be made real by performing phase rotations of the quark fields that multiply V. With the above parameterization of the CKM matix, it can be checked that the nonvanishing imaginary parts of such combinations are proportional to the quantity

$$J \equiv c_{12}c_{13}^{2}c_{23}s_{12}s_{13}s_{23}\sin\delta \quad (2.52)$$

known as the Jarlskog parameter. As this parameter is invariant under the general redefinition of the quark phases

$$u_i \longrightarrow e^{i\beta_i}u_i, \qquad d_i \longrightarrow e^{i\alpha_i}d_i \quad (2.53)$$

and is non-zero only when the CKM matrix is complex, then it serves as an appropriate parameter for measuring "how much the CKM matrix is complex."

2.5 CP violation in the Standard Model

In addition to Lorentz and gauge symmetries, there is also the discrete CPT symmetry which is proposed to be the fundamental symmetry of Nature. This CPT symmetry is the combined operations of charge conjugation C, parity P, and time reversal T. The statement that a Lorentz-invariant quantum field theory is invariant under CPT operation is well-known as the CPT theorem which was discovered by Pauli in 1955 [21]. For the Standard Model, although the CPT theorem is still valid, the separate C, P or T symmetries can be violated. Actually, any chiral gauge theory will naturally violate charge conjugation and parity. Moreover, the CPT theorem implies, for example, that if time reversal is not a symmetry of the model, then neither is the combined CP. In this section, we will investigate the violation of CP symmetry in the SM. First, we begin with the actions of parity and charge conjugation on the Dirac particle.

The parity or space inversion operation, P, is the coordinate transformation which changes $(t, \mathbf{x})$ to $(t, -\mathbf{x})$. Under this operation the coordinate axes and hence the handedness of space are reversed. Consequently, the momentum of a particle is reversed while the particle's spin is kept unchanged. In the mathematical language, the unitary operator P implementing the parity transformation on a spinor is conventionally defined by

$$P\psi(t, \mathbf{x})P = \gamma^0 \psi(t, -\mathbf{x}). \tag{2.54}$$

Suppose we write $\psi(t, \mathbf{x}) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} (t, \mathbf{x})$. Then under parity,

$$\psi(t, \mathbf{x}) \to P\psi(t, \mathbf{x})P = \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix} (t, -\mathbf{x}) \equiv \psi^P(t, -\mathbf{x}).$$

We can see that the left-handed (right-handed) components in the coordinates $(t, \mathbf{x})$ become the right-handed (left-handed) components in the new coordinates $(t, -\mathbf{x})$. Similarly, the Dirac conjugate spinor transforms as

$$P\bar{\psi}(t, \mathbf{x})P = P\psi^\dagger(t, \mathbf{x})P\gamma^0 = (P\psi(t, \mathbf{x})P)^\dagger \gamma^0 = \bar{\psi}(t, -\mathbf{x})\gamma^0. \tag{2.55}$$

With the above transformations, the transformation rules under parity for the bilinear products of spinors, such as $\bar{\psi}\psi$ or $\bar{\psi}\gamma^\mu\psi$, can be easily obtained by doing some algebras on the gamma matrices.

Now we turn to consider the C operation. Charge conjugation is the operation which turns a fermion into its antiparticle with the same spin orientation. Hence, the C symmetry is also known as the particle-antiparticle symmetry. Mathematically, the unitary operator C implementing charge conjugation on a spinor gives the results

$$C\psi(x)C = -i\gamma^2\psi^*(x) = -i\gamma^2(\psi^\dagger)^T = -i(\bar{\psi}\gamma^0\gamma^2)^T \tag{2.56}$$

and

$$C\bar{\psi}(x)C = C\psi^\dagger C\gamma^0 = (-i\gamma^2\psi)^T\gamma^0 = (-i\gamma^0\gamma^2\psi)^T \tag{2.57}$$

which lead to the transformation rules under charge conjugation of the bilinear products of spinors. The transformation properties under C, P and CP of various useful quantities are listed as follows:

	$\bar{\psi}\psi$	$\bar{\psi}\gamma^\mu\psi$	$i\bar{\psi}\gamma^5\psi$	$\bar{\psi}\gamma^\mu\gamma^5\psi$	∂_μ	gauge fields
P	$+1$	$(-1)^\mu$	-1	$-(-1)^\mu$	$(-1)^\mu$	$(-1)^\mu$
C	$+1$	-1	$+1$	$+1$	$+1$	-1
CP	$+1$	$(-1)^\mu$	-1	$-(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu$

Here the shorthand $(-1)^\mu$ denotes

$$(-1)^\mu = \begin{cases} +1 & \text{for } \mu = 0 \\ -1 & \text{for } \mu = 1,2,3. \end{cases}$$

With the properties above, CP is a symmetry of every term in the Standard Model Lagrangian, except the terms that couple quarks to the W bosons in

Eq. (2.44). In terms of quark mass eigenstates, these are the terms that involve the CKM matrix

$$\mathcal{L}_{\mathrm{CC}} \supset \frac{g}{\sqrt{2}} \left(\bar{u}_L^{\prime i} \gamma^\mu V^{ij} d_L^{\prime j} W_\mu^+ + \bar{d}_L^{\prime i} \gamma^\mu V^{*ij} u_L^{\prime j} W_\mu^- \right). \tag{2.58}$$

Under CP, they transform to

$$\mathcal{L}_{\mathrm{CC}} \rightarrow \frac{g}{\sqrt{2}} \left(\bar{d}_L^{\prime i} \gamma^\mu V^{ij} u_L^{\prime j} W_\mu^- + \bar{u}_L^{\prime i} \gamma^\mu V^{*ij} d_L^{\prime j} W_\mu^+ \right). \tag{2.59}$$

From two equations above, we can see that the charge-changing weak interactions are invariant under CP if and only if V^{ij} is real for all i, j. Therefore, the Standard Model violates CP symmetry and the source of such violation is the complex phase δ of the CKM quark mixing matrix. As mentioned in the previous section, the Jarlskog parameter J is invariant under the phase transformations of quark fields and is non-zero only when the CKM matrix is complex. Thus any physical effect whose occurrence is due to CP violation must be an analytic function of J which has a zero at $J = 0$.

2.6 The EDM of the electron in the Standard Model

In quantum field theories, the general form of the electromagnetic form factor of a spin 1/2 particle f of mass m satisfying the Ward identity is [22]

$$\begin{aligned} \Gamma^\mu(q) &= F_1(q^2)\gamma^\mu + F_2(q^2) i\sigma^{\mu\nu} q_\nu/2m + F_A(q^2)(\gamma^\mu\gamma^5 q^2 - 2m\gamma^5 q^\mu) \\ &\quad + F_3(q^2)\sigma^{\mu\nu}\gamma^5 q_\nu/2m \end{aligned} \tag{2.60}$$

where p and p' are the 4-momenta of the initial- and final-state particles and $q = p' - p$ is the 4-momentum carried by the photon.

The EDM of a fermion f is defined as

$$d_f = -\frac{1}{2m} F_3(0). \tag{2.61}$$

The reason for this is that the last term in Eq. (2.60) with $F(q^2)$ evaluated at $q^2 = 0$ may be thought of as coming from a low energy effective Lagrangian

$$\mathcal{L}_{\mathrm{EDM}} = -\frac{i}{2} d_f \bar{\psi} \sigma^{\mu\nu} \gamma^5 \psi F_{\mu\nu} \tag{2.62}$$

which gives rise to the EDM interaction Hamiltonian of the form $\mathbf{H}_{EDM} = -\mathbf{d} \cdot \mathbf{E}$, with $\mathbf{d}$ and $\mathbf{E}$ being respectively the fermion electric dipole moment and an electric field. Here $\mathbf{d} = d_f \mathbf{S}$ where $\mathbf{S}$ is the fermion spin.

In a renormalizable theory, if the theory contains a source of CP violation, this interaction must be induced by loop diagrams because it is a dimension-5 operator, which is nonrenormalizable. In the Standard Model, although there is no CP violating phase in the lepton sector, the nonzero lepton EDM can be induced from the CKM mixing via quark loops. In 1990, the SM prediction for the EDM of the electron was obtained by Hoogeveen [23]. It was shown that the Feynman diagrams contributing to the electron EDM must have at least four charged vector bosons coupled to a quark loop. This is because if there are only two charged vector bosons in the quark loops, the diagram will depend only on the square of the absolute value $|V^{ij}|^2$ of the CKM matrix and therefore be independent of the CP-violating phase. However only a year later, Khriplovich and Pospelov [24] demonstrated that d_e is zero even at three-loop order and the first non-vanishing result appears below the level of

$$d_e \leq 10^{-38} e \text{ cm} \tag{2.63}$$

which is highly suppressed.

It is worth noting that the recent experimental discovery of neutrino masses makes possible for CP-violating phase to occur in the lepton sector similar to that in the quark sector. If we assume that neutrinos are Dirac particles, the lepton mixing matrix then contains three rotation angles and one CP-violating phase in complete analogy to the CKM matrix. On the other hand, if neutrinos are Majorana particles, two CP-violating phases will be introduced in the lepton mixing matrix [25]. In the latter case, it was shown that a non-vanishing contribution to the electron EDM is induced at two-loop level [26]. However, its value is still very tiny unless a fine-tuning of the neutrino masses is allowed [27].