ขอบเขตรายจุดของฟังก์ชันโฮโลมอร์ฟิกซึ่งกำลังสองอินทิเกรตได้เทียบกับเมเชอร์เกาส์และไม่แปรเปลี่ยนภายใต้การกระทำของกลุ่ม SO(d,C)

นางสาว อารีรักษ์ แก้วเทพ

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POINTWISE BOUND OF $SO(d,\mathbb{C})$-INVARIANT HOLOMORPHIC FUNCTIONS WHICH ARE SQUARE-INTEGRABLE WITH RESPECT TO A GAUSSIAN MEASURE

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ปริภูมิซีกัล-บาร์กแมน $\mathcal{H}L^2(\mathbb{C}^2,\mu_t)$ เป็นปริภูมิที่ได้รับการศึกษาอย่างกว้างขวาง ในที่นี้เราจะพิจารณา ปริภูมิ $\mathcal{H}L^2(\mathbb{C}^2,\mu_t)^{SO(d,\mathbb{C})}$ ซึ่งประกอบไปด้วยฟังก์ชัน F ใน $\mathcal{H}L^2(\mathbb{C}^2,\mu_t)$ ที่ไม่แปรเปลี่ยนภายใต้การกระทำของกลุ่ม $SO(d,\mathbb{C})$ ปริภูมิดังกล่าวนี้เป็นปริภูมิย่อยปิดของปริภูมิ $\mathcal{H}L^2(\mathbb{C}^2,\mu_t)$ ดังนั้นจึงเป็นปริภูมิฮิลเบิร์ต ในงานวิจัยนี้เราจะทำการสร้างขอบเขตรายจุดของฟังก์ชันในปริภูมินี้

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The Segal-Bargmann space $\mathcal{H}L^2(\mathbb{C}^d,\mu_t)$ has been extensively studied. Here, we consider the space $\mathcal{H}L^2(\mathbb{C}^d,\mu_t)^{SO(d,\mathbb{C})}$ consisting of all functions F in $\mathcal{H}L^2(\mathbb{C}^d,\mu_t)$ which are invariant under the action of the complex special orthogonal group $SO(d,\mathbb{C})$. It is a closed subspace of $\mathcal{H}L^2(\mathbb{C}^d,\mu_t)$, and hence is a Hilbert space. In this work, we establish a pointwise bound for a function in this space.

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Chapter 1

Statements of the results

The Segal-Bargmann space $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$ is the space of holomorphic functions on $\mathbb{C}^d$ that are square-integrable with respect to the Gaussian measure $\mu_t(z)\,dz = (\pi t)^{-d} e^{-|z|^2/t}\,dz$, where $|z|^2 = |z_1|^2 + \cdots + |z_d|^2$. Here t is a fixed positive real number. See [B], [F1], [GM], [H1], [H3], [H4], [HM] for details about the importance of this space.

Various generalizations of the Segal-Bargmann space have been considered. An important part of the study of such generalizations is to obtain sharp pointwise bounds on the functions. (See, for examples, [CL], [E], [H2], [H5], [HL].) Such bounds amount to estimates for the reproducing kernel on the diagonal.

In this work, we consider the subspace of the standard Segal-Bargmann space that is invariant under the special orthogonal group. The goal of the work is to compare two bounds for functions in this space, a simple bound obtained by minimizing the standard bounds in the full Segal-Bargmann space over the orbits of the group, and a sharp bound obtained by directly estimating the reproducing kernel for the subspace. We show that the sharp bounds are polynomially better than the simple bounds, with the difference between the two growing larger and larger as the dimension d goes to infinity.

This analysis is motivated in part by a comparison of [Dr] and [H2]. In [Dr], Driver obtains (among other things) bounds for a *generalized* Segal-Bargmann space by representing it as the subspace of a certain infinite-dimensional *standard* Segal-Bargmann space that is invariant under a certain group action. (See also [GM], [HS], [H6].) Meanwhile, in [H2], Hall obtains sharp bounds for the relevant generalized Segal-Bargmann space by directly estimating the reproducing kernel. The difference between the two bounds is significant; the sharper bounds of [H2] are essential, for example, in the analysis in [HL].

It is well-known that for any function $F \in \mathcal{H}L^2(\mathbb{C}^d, \mu_t)$, we have the pointwise bound

$$|F(z)|^2 \leq e^{|z|^2/t} \|F\|^2_{L^2(\mathbb{C}^d,\mu_t)} \qquad (z \in \mathbb{C}^d). \tag{1.1}$$

Now suppose that F is invariant under the action of $SO(d)$, and therefore, by analytic continuation, under the action of $SO(d, \mathbb{C})$. By minimizing (1.1) on each orbit, for any $SO(d)$-invariant function F in the Segal-Bargmann space, we obtain the preliminary estimate

$$|F(z)|^2 \leq e^{|(z,z)|/t} \|F\|^2_{L^2(\mathbb{C}^d,\mu_t)} \qquad (z \in \mathbb{C}^d), \tag{1.2}$$

where $(z, z) = z_1^2 + \cdots + z_d^2$. Since $|(z, z)| \leq |z|^2$, this is already an improvement over the pointwise bound in (1.1).

The $SO(d)$-invariance means that F is determined by its values on $\{(z, 0, ..., 0)\} \simeq \mathbb{C}^1$. (By holomorphicity, F is determined by its values on $\mathbb{R}^d$, then any point in $\mathbb{R}^d$ can be rotated into $\mathbb{R}^1$.) Conversely, any even holomorphic function on $\mathbb{C}^1$ has an extension to an $SO(d)$-invariant function on $\mathbb{C}^d$. Then the space of $SO(d)$-invariant functions in the Segal-Bargmann space over $\mathbb{C}^d$ can be expressed as an L^2-space of holomorphic functions on $\mathbb{C}^1$, with some non-Gaussian measure. By estimating the reproducing kernel for this space, we obtain a sharp bound for an $SO(d)$-invariant function F in $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$, which will be polynomially better than (1.2). This bound is described in the following theorem.

Theorem 1.1. *There exists a constant C, depending only on d and t, such that for each $SO(d)$-invariant function F in $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$, we have*

$$|F(z)|^2 \leq \frac{C\, e^{|(z,z)|/t}}{1 + |(z,z)|^{(d-1)/2}} \, \|F\|^2_{L(\mathbb{C}^d, \mu_t)} \qquad (z \in \mathbb{C}^d).$$

In the last chapter, we obtain the formula of the reproducing kernel for the space of $SO(d)$-invariant functions in the Segal-Bargmann space which is given by

$$K(z,z) = C \,\mathrm{BesselI}\left(\frac{d-2}{2}, \frac{|(z,z)|}{t}\right)\left(\frac{1}{|(z,z)|}\right)^{d/2-1}$$

for all $z \in \mathbb{C}^d$, where BesselI is the modified Bessel function of the first kind and C is a constant depending on d and t. The reproducing kernel for this space is asymptotically equal to the bound in Theorem 1.1 for a large argument. This implies that the bound in Theorem 1.1 is sharp.

Chapter 2

Orthogonal group

In this chapter, we consider some properties of the non-degenerate bilinear form on a finite-dimensional vector space and the special complex orthogonal group that will be used in later chapters.

2.1 Bilinear Form

Definition 2.1. Let V be a vector space over $\mathbb{C}$ and B a symmetric bilinear map on V.

1. For any subspace W of V, define $W^{\perp} = \{x \in V \mid B(x,y) = 0 \ \forall y \in W\}$.
2. A subset S of V is called an *orthonormal set* if $B(v,v) = 1$ for any $v \in S$ and $B(v,w) = 0$ for $v, w \in S$ and $v \neq w$.
3. We say that B is *non-degenerate* if for all $v \in V - \{0\}$ there is $w \in V$ such that $B(v,w) \neq 0$.

Remark. Let V be a vector space over $\mathbb{C}$ and B a non-degenerate symmetric bilinear form on V. For any subspace W of V, if B is non-degenerate on W then $W \cap W^{\perp} = \{0\}$.

Lemma 2.2. *Let V be a finite-dimensional vector space over $\mathbb{C}$ and B a non-degenerate symmetric bilinear form on V. If W is a subspace of V such that B is non-degenerate on W, then V/W and $W^\perp$ are linearly isomorphic.*

Proof. Consider a linear map $x \mapsto x + V$ from $W^\perp$ into V/W. It is injective, since $W \cap W^\perp = \{0\}$. Thus $\dim W^\perp \leq \dim V/W$. On the other hand, define a linear map $\phi\colon V/W \to (W^\perp)^*$ by

$$\phi(v + W)(x) = B(v, x)$$

for all $x \in W^\perp$. For any $v_1, v_2 \in V$, if $v_1 + W = v_2 + W$ then $v_1 - v_2 \in W$, so $B(v_1 - v_2, x) = 0$ for any $x \in W^\perp$. Thus $\phi(v_1)$ and $\phi(v_2)$ are equal. Hence ϕ is an injective map which implies that $\dim V/W \leq \dim(W^\perp)^* = \dim W^\perp$. So the lemma is proved. □

Theorem 2.3. *Let V be a finite-dimensional vector space over $\mathbb{C}$ and B a non-degenerate symmetric bilinear form on V. Then B is non-degenerate on a subspace W of V if and only if $V = W \oplus W^\perp$.*

Proof. Let W be a subspace of V. Assume that B is non-degenerate on W. The canonical map $\phi\colon V \to V/W$ is a linear map with kernel W. Then

$$\dim V = \dim W + \dim V/W = \dim W + \dim W^\perp = \dim(W + W^\perp) + \dim(W \cap W^\perp).$$

Using the fact that $W \cap W^\perp = \{0\}$, we have $V = W \oplus W^\perp$. We next assume that $V = W \oplus W^\perp$. Let $w \in W - \{0\}$. Then there is $v = x + y \in V$ such that $B(w, v) \neq 0$, where $x \in W$ and $y \in W^\perp$. Thus $B(w, x) = B(w, v) \neq 0$, so the theorem is proved. □

Theorem 2.4. *Let V be a finite-dimensional vector space over $\mathbb{C}$ and B a non-degenerate symmetric bilinear form on V. Let W be a subspace of V. If B is non-degenerate on W, then it is non-degenerate on $W^\perp$.*

Proof. Assume that the form B is non-degenerate on W. Then by Theorem 2.3 we have that $V = W \oplus W^{\perp}$. Let $w \in W^{\perp} - \{0\}$. Thus there exists $v = x + y \in V$ such that $B(w, v) \neq 0$, where $x \in W$ and $y \in W^{\perp}$. So $B(w, y) = B(w, v) \neq 0$. This implies that the form B is non-degenerate on $W^{\perp}$. □

Lemma 2.5. *Let V be a vector space over $\mathbb{C}$. If B is a symmetric non-degenerate symmetric bilinear form on V, then there is $v \in V$ such that $B(v, v) \neq 0$.*

Proof. Suppose that for all $v \in V$, $B(v, v) = 0$. Let $w \in V - \{0\}$. Since B is non-degenerate, there is $v \in V$ such that $B(w, v) \neq 0$. Then

$$\begin{aligned} 0 &= B(w + v, w + v) \\ &= B(w, w) + B(w, v) + B(v, w) + B(v, v) \\ &= B(w, v) + B(v, w). \end{aligned}$$

It follows that $2B(v, w) = 0$, a contradiction. Hence there is $v \in V$ such that $B(v, v) \neq 0$. □

Theorem 2.6. *If V is a finite-dimensional vector space over $\mathbb{C}$ and B a non-degenerate symmetric bilinear form on V, then V has an orthonormal basis.*

Proof. We will prove by induction on dimension of V. If $\dim V = 1$, then there exists $v \in V - \{0\}$ such that $B(v, v) \neq 0$ since B is a non-degenerate symmetric bilinear form. Hence $\{v/\sqrt{B(v, v)}\}$ is an orthonormal basis of V. Suppose $\dim V = n$. Assume the statement of this lemma is true for any vector space W with $\dim W \leq n-1$. Let $v \in V - \{0\}$ be such that $B(v, v) \neq 0$ and $V_1 = \text{span}\{v\}$. Then B is non-degenerate on V_1 and $V_1^{\perp}$. By the induction hypothesis, there is an orthonormal basis $\{v_2, \dots, v_n\}$ of $V_1^{\perp}$. By Theorem 2.3, $V = V_1 \oplus V_1^{\perp}$, so we have that $\{\frac{v}{\sqrt{B(v,v)}}, v_2, \dots, v_n\}$ is an orthonormal basis of V. □

2.2 Special Complex Orthogonal Group

Let $\mathbb{F}$ be the field $\mathbb{R}$ or $\mathbb{C}$. For each $d \in \mathbb{N}$, denote by $M_d(\mathbb{F})$ the set of all $d \times d$ matrices with entries in $\mathbb{F}$ and by $GL(d, \mathbb{F})$ the set of all invertible $d \times d$ matrices with entries in $\mathbb{F}$. We can regard $M_d(\mathbb{F})$ as the vector space $\mathbb{F}^{d^2}$, and hence it has an inherited topology from the usual topology on $\mathbb{F}^{d^2}$.

Consider the map $(\,\cdot\,,\,\cdot\,) : \mathbb{F}^d \times \mathbb{F}^d \to \mathbb{F}$ given by

$$(x, y) = x_1y_1 + \cdots + x_dy_d$$

for all $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d) \in \mathbb{F}^d$. It is easy to see that this map is a symmetric non-degenerate bilinear form. If we write $x \in \mathbb{F}^d$ as a column matrix, then

$$(x, y) = x^t y \quad \text{for all } x,\, y \in M_{d\times 1}(\mathbb{F}) \cong \mathbb{F}^d.$$

From this, we have

$$(Ax, y) = (Ax)^t y = x^t A^t y = (x, A^t y)$$

for all $x, y \in \mathbb{F}^d$ and all $A \in M_d(\mathbb{F})$.

Definition 2.7. An invertible $d \times d$ matrix A which preserves the bilinear form $(\,\cdot\,,\,\cdot\,)$, i.e.

$$(Ax, Ay) = (x, y) \qquad \text{for all } x, y \in \mathbb{F}^d,$$

is called an **orthogonal matrix**. Denote by $O(d, \mathbb{F})$ the set of all $d \times d$ orthogonal matrices and by $SO(d, \mathbb{F})$ the set of all A in $O(d, \mathbb{F})$ with $\det A = 1$.

Definition 2.8. The set of all $d \times d$ complex orthogonal matrices is called the **complex orthogonal group** $O(d, \mathbb{C})$ and the set of all $d \times d$ complex orthogonal matrices with determinant one is called the **special complex orthogonal group**

$SO(d,\mathbb{C})$. The set of $d \times d$ real orthogonal matrices is called the **(real) orthogonal group** $O(d)$. The set of $d \times d$ real orthogonal matrices with determinant one is called the **special (real) orthogonal group** $SO(d)$. Then $O(d)$ and $SO(d)$ are subgroups of $O(d,\mathbb{C})$, and hence of $GL(d,\mathbb{C})$. Moreover, they are closed subgroups of $GL(d,\mathbb{C})$. Geometrically, an element of $O(d)$ is either a rotation, or a combination of rotation and reflection. An element of $SO(d)$ is just a rotation.

For each $w \in \mathbb{C}$, we define

$$S_w = \{z \in \mathbb{C}^d \mid (z,z) = w^2\}.$$

In particular, S_1 is the complex unit sphere. Then $SO(d,\mathbb{C})$ acts on $\mathbb{C}^d$ by left multiplication. Under this action an orbit for each $z \in \mathbb{C}^d$ is a subset of $S_{\sqrt{(z,z)}}$. We next show that for each $d \geq 2$ if $(z,z) \neq 0$, then the orbit of z is equal to $S_{\sqrt{(z,z)}}$.

Definition 2.9. Let $\eta\colon G \times M \to M$ be an action of G on M on the left. The action is called **transitive** if whenever m and n belong to M there is g in G such that $\eta(g,m) = n$. For $m_0 \in M$, the set

$$G_{m_0} = \{g \in G \mid \eta(g,m_0) = m_0\}$$

is called the **isotropy group at** m_0. If G is a Lie group, then G_{m_0} is a closed subgroup of G.

Theorem 2.10. *For any $d \geq 2$, $SO(d,\mathbb{C})$ acts transitively on S_w for all $w \in \mathbb{C}^*$.*

Proof. Let $w \in \mathbb{C}^*$ and $z \in S_w$. Then a bilinear form $(\cdot,\cdot)$ is non-degenerate on $U = \operatorname{span}\{z\}^\perp$. By Lemma 2.6, we can find an orthonormal basis of $\mathbb{C}^d$ with a normalized z as its first element, denoted by $B_1 = \{\frac{z}{\sqrt{(z,z)}}, z_2, \ldots, z_d\}$. If $x \in S_w$

then choose an orthonormal basis $B_2 = \{\frac{x}{\sqrt{(x,x)}}, x_2, \dots, x_d\}$ of $\mathbb{C}^d$ and define a linear map $g\colon \mathbb{C}^d \to \mathbb{C}^d$ by

$$g\Big(\frac{z}{\sqrt{(z,z)}}\Big) = \frac{x}{\sqrt{(x,x)}} \qquad \text{and } g(z_i) = x_i$$

for any $i = 2, 3, \dots, d$. Then we have $[g]_{B_2,B_1} z = x$ and $\det[g]_{B_2,B_1} = \pm 1$. If $\det[g]_{B_2,B_1} = -1$, define a linear map $g'\colon \mathbb{C}^d \to \mathbb{C}^d$ by

$$g\Big(\frac{z}{\sqrt{(z,z)}}\Big) = \Big(\frac{x}{\sqrt{(x,x)}}\Big), \quad g(z_2) = -x_2 \text{ and } g(z_i) = x_i$$

for any $i = 3, \dots, d$. Then $[g]_{B_2,B_1} z = x$ and $\det[g]_{B_2,B_1} = 1$. $\square$

Next we will show that the smooth complex manifold S_1 is diffeomorphic to a homogeneous manifold $SO(d,\mathbb{C})/SO(d-1,\mathbb{C})$. Let us recall some theorem about homogeneous manifolds first.

If G is a Lie group and H is a closed subgroup of G, then we can define a differentiable structure on the quotient space G/H so that it is a smooth manifold, called a **homogeneous manifold**. Moreover, there is a natural transitive left-action of G on G/H. Conversely, if M is a smooth manifold and there is a transitive left-action by a Lie group G on M, then M can be identified with the quotient manifold G/G_{m_0}, where m_0 is a point in M. This is summarized in the following theorem.

Theorem 2.11 ([War] Theorem 3.62). *Let $\eta\colon G \times M \to M$ be a transitive left-action of the Lie group G on the manifold M. Let $m_0 \in M$, and let H be the isotropy group at m_0. Define a mapping $\beta\colon G/H \to M$ by $\beta(gH) = \eta(g, m_0)$. Then β is a diffeomorphism.*

Proposition 2.12. *The complex unit sphere S_1 is diffeomorphic to $SO(d,\mathbb{C})/SO(d-1,\mathbb{C})$.*

Proof. The Lie group $SO(d, \mathbb{C})$ acts transitively on S_1 with the isotropy group at $(1, 0, \ldots, 0)$ being the set of matrices in $SO(d, \mathbb{C})$ of the form

$$X = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix},$$

where the matrix A is a matrix in $SO(d-1, \mathbb{C})$. Hence, we identify the isotropy group at $(1, 0, \ldots, 0)$ with $SO(d-1, \mathbb{C})$. It follows from Theorem 2.11 that the homogeneous manifold $SO(d, \mathbb{C})/SO(d-1, \mathbb{C})$ is diffeomorphic to S_1. $\square$

Chapter 3

Holomorphic function spaces

In this chapter, we provide some results about a space of holomophic functions. First, we discuss a reproducing kernel, which exists on any Hilbert space such that each pointwise evaluation is bounded. After that is we consider the Segal-Bargmann space that is invariant under the action of $SO(d, \mathbb{C})$.

3.1 Basic properties of the holomorphic function spaces

In this section, we review some definitions and theorems about the holomorphic function space which are in [H3].

Definition 3.1. Let U be a non-empty open set in $\mathbb{C}^d$. A function $f : U \to \mathbb{C}$ is **holomorphic** if f is continuous and holomorphic in each variable with the other variables fixed. Let $\mathcal{H}(U)$ be the space of holomorphic functions on U.

Definition 3.2. Let α be a continuous strictly positive function on a non-empty open set U in $\mathbb{C}^d$. Denote by $\mathcal{H}L^2(U, \alpha)$ the space of L^2-holomorphic functions with respect to the weight α. In other words,

$$\mathcal{H}L^2(U,\alpha) = \left\{ F \in \mathcal{H}(U) \;\Big|\; \int_U |F(z)|^2 \,\alpha(z)\, dz \; < \infty \right\}.$$

Theorem 3.3. *1. For all $z \in U$, there exists a constant c_z such that*

$$|F(z)|^2 \leq c_z \|F\|^2_{L^2(U,\alpha)}$$

for all $F \in \mathcal{H}L^2(U,\alpha)$.

2. $\mathcal{H}L^2(U,\alpha)$ is a closed subspace of $L^2(U,\alpha)$, and therefore a Hilbert space.

Theorem 3.4. *There exists a complex-valued function K on $U \times U$, with the following properties*

1. *$K(z,u)$ is holomorphic in z and anti-holomorphic in u, and satisfies*

$$K(z,u) = \overline{K(u,z)} \quad (z,u \in U).$$

2. *For all $z,u \in U$,*

$$\int_U K(z,w)K(w,u)\alpha(w)dw = K(z,u).$$

3. *For each fixed $z \in U$ and for all $F \in \mathcal{H}L^2(U,\alpha)$*

$$F(z) = \int_U K(z,w)F(w)\alpha(w)dw.$$

4. *For all $z \in U$*

$$|F(z)|^2 \leq K(z,z)\|F\|^2,$$

and the constant $K(z,z)$ is optimal in the sense that for each $z \in U$ there is a non-zero $F_z \in \mathcal{H}L^2(U,\alpha)$ for which equality holds.

Theorem 3.5. *Let* $\{e_n\}$ *be an orthonormal basis for* $\mathcal{H}L^2(U,\alpha)$. *Then for all* $z, u \in U$

$$\sum_n |e_n(z)\overline{e_n(u)}| < \infty$$

and

$$K(z,u) = \sum_n e_n(z)\overline{e_n(u)}.$$

Definition 3.6. The **Segal-Bargmann space** is the holomorphic function space $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$, where

$$\mu_t(z) \; = \; (\pi t)^{-d}\, e^{-|z|^2/t}.$$

Here, $|z|^2 = |z_1|^2 + \cdots + |z_d|^2$ and t is a positive number.

Theorem 3.7. *The reproducing kernel of* $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$ *is given by*

$$K(z,u) \; = \; e^{\langle u,z\rangle/t} \quad (z, u \in \mathbb{C}^d).$$

where $\langle\cdot,\cdot\rangle$ *is an inner product on* $\mathbb{C}^d$. *Hence, we have the pointwise bound*

$$|F(z)|^2 \; \le \; e^{|z|^2/t}\|F\|^2 \quad (z, u \in \mathbb{C}^d), \tag{3.1}$$

for any $F \in \mathcal{H}L^2(\mathbb{C}^d, \mu_t)$.

3.2 $SO(d)$-invariant holomorphic function spaces

Definition 3.8. Let F be a holomorphic function on $\mathbb{C}^d$. We say that F is $SO(d)$-**invariant** if

$$F(Az) = F(z) \quad \text{ for all } A \in SO(d) \text{ and all } z \in \mathbb{C}^d.$$

In particular, if F is an $SO(d)$-invariant holomorphic function, then by analytic continuation it is $SO(d,\mathbb{C})$-invariant, i.e., $F(Az) = F(z)$ for all $A \in SO(d,\mathbb{C})$ and all $z \in \mathbb{C}^d$.

Notation. Denote by $\mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$ the set of all $SO(d,\mathbb{C})$-invariant holomorphic functions on $\mathbb{C}^d$, i.e.,

$$\mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})} \;=\; \{\, f \in \mathcal{H}(\mathbb{C}^d) \;|\; f \text{ is } SO(d,\mathbb{C})\text{-invariant}\}.$$

Then it is a linear subspace of $\mathcal{H}(\mathbb{C}^d)$.

Denote by $\mathcal{H}(\mathbb{C})^e$ the set of all holomorphic even functions on $\mathbb{C}$, i.e.,

$$\mathcal{H}(\mathbb{C})^e = \{g \in \mathcal{H}(\mathbb{C}) \mid g(w) = g(-w),\ \forall w \in \mathbb{C}\}.$$

Proposition 3.9. *For any $d \geq 2$, the map $\phi\colon \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})} \to \mathcal{H}(\mathbb{C})^e$ defined by*

$$\phi(f)(x) = f(x, 0, \ldots, 0)$$

for all $f \in \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$ and all $x \in \mathbb{C}$, is a linear isomorphism whose inverse is given by

$$\psi(g)(z) = g\left(\sqrt{(z,z)}\right)$$

for all $g \in \mathcal{H}(\mathbb{C})^e$ and all $z \in \mathbb{C}^d$.

Note that since g is even, the value of $\phi(g)(z)$ is independent of choice of square root of (z,z).

Proof. It is clear that ϕ is a linear map and $\phi(f)$ is a holomorphic function on $\mathbb{C}$ for any $f \in \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$. Moreover, $\phi(f)$ is even since

$$\begin{aligned}\phi(f)(w) &= f(w, 0, \ldots, 0)\\ &= f(A(w, 0, \ldots, 0))\\ &= f(w, 0, \ldots, 0)\end{aligned}$$

where $A = \operatorname{diag}(-1, -1, 1, 1, \ldots, 1)$. On the other hand, ψ is a linear map. Since g is even, $\psi(g)$ is given by a convergent power series in integer powers of (z,z), where

$(z,z) = z_1^2 + \cdots + z_d^2$, and therefore $\psi(g)$ is holomorphic on $\mathbb{C}^d$ for each $g \in \mathcal{H}(\mathbb{C})^e$. Moreover, $\psi(g)$ is $SO(d,\mathbb{C})$-invariant because the bilinear form is preserved under the action of the orthogonal group. It is easy to see that $\phi \circ \psi = \mathrm{id}_{\mathcal{H}(\mathbb{C})^e}$ and by analytic continuation we have $\psi \circ \phi = \mathrm{id}_{\mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}}$, so the theorem is proved. □

Corollary 3.10. *For each $f \in \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$ and each $w \in \mathbb{C}$, if x, $y \in S_w$, then $f(x) = f(y)$.*

Proof. Let $f \in \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$ and $w \in \mathbb{C}$. By Proposition 3.9, there exists a function $g \in \mathcal{H}(\mathbb{C})^{even}$ such that

$$f(z) = g\left(\sqrt{(z,z)}\right) \qquad \text{for all } z \in \mathbb{C}^d.$$

Hence, for all $x, y \in S_w$,

$$f(x) = g\left(\sqrt{(x,x)}\right) = g\left(\sqrt{(y,y)}\right) = f(y).$$

□

Chapter 4

Invariant measure on the complex sphere

In this chapter, we construct a rotation-invariant Borel measure α on the complex unit sphere. It will be shown that Lebesgue measure on $\mathbb{C}^d$ can be factored as a product of a measure on the open right-half plane of $\mathbb{C}$ and the rotation-invariant measure α on the complex unit sphere. We can give an explicit formula for α via the identification between the complex unit sphere and the cotangent bundle on the real unit sphere.

Notation. Let S be the set of all elements on $\mathbb{C}^d$ such that the argument of their bilinear forms lie on the negative real axis, including zero, i.e.,

$$S = \{z \in \mathbb{C}^d \mid (z, z) \in (-\infty, 0]\}.$$

Recall that $S_0 = \{z \in \mathbb{C}^d \mid (z, z) = 0\}$. Next we show that S is a subset of $\mathbb{C}^d$ with Lebesgue measure zero. Let $U \subset \mathbb{R}^n$ be an open set, and let $F\colon U \to \mathbb{R}^k$ be a smooth function. The *graph* of F is the subset of $\mathbb{R}^n \times \mathbb{R}^k$ defined by

$$\Gamma(F) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k \mid x \in U \text{ and } y = F(x)\}.$$

Theorem 4.1 (Implicit Function Theorem). *Let $U \subset \mathbb{R}^n \times \mathbb{R}^k$ be an open set, and let $(x, y) = (x^1, \ldots, x^n, y^1, \ldots, y^k)$ denote the standard coordinates on U. Suppose $\Phi\colon U \to \mathbb{R}^k$ is a smooth map, $(a, b) \in U$, and $c = \Phi(a, b)$. If the $k \times k$ matrix*

$$\Big(\frac{\partial \Phi^i}{\partial y^j}(a, b)\Big)$$

is nonsingular, then there exist neighborhoods V_0 of a and W_0 of b and a smooth map $F\colon V_0 \to W_0$ such that $\Phi^{-1}(c) \cap (V_0 \times W_0)$ is the graph of F, and hence it is an embedded n-dimensional submanifold of $\mathbb{R}^{n+k}$.

Proposition 4.2. *$S_0 - \{0\}$ and $S - S_0$ are submanifolds of $\mathbb{C}^d$ with dimension $d - 1$.*

Proof. Let $z \in \mathbb{C}^d - \{0\}$. Then $z_j \neq 0$ for some $1 \leq j \leq d$, so $\frac{\partial (z,z)}{\partial z_j} = 2z_j \neq 0$. By Implicit Function Theorem $S_0 - \{0\} = (\cdot, \cdot)^{-1}\{0\} - \{0\}$ is a smooth $(d-1)$-dimensional submanifold of $\mathbb{C}^d$. Since

$$\begin{aligned}(z, z) &= z_1^2 + z_2^2 + \cdots + z_d^2 \\ &= x_1^2 - y_1^2 + 2ix_1y_1 + \cdots + x_d^2 - y_d^2 + 2ix_dy_d\end{aligned}$$

where $z_j = x_j + iy_j$, we have that

$$\begin{aligned}\tan(\arg(z, z)) &= \frac{\mathrm{Im}(z, z)}{\mathrm{Re}(z, z)} \\ &= \frac{2x_1y_1 + \cdots + 2x_dy_d}{x_1^2 - y_1^2 + \cdots + x_d^2 - y_d^2}.\end{aligned}$$

Thus for any $z' \in S - S_0$, $\arg(z', z') = \pi$, $\mathrm{Im}(z', z') = 2x'_1y'_1 + \cdots + 2x'_dy'_d = 0$ and $\mathrm{Re}(z', z') = x'^2_1 - y'^2_1 + \cdots + x'^2_d - y'^2_d < 0$. Therefore $y'_j \neq 0$ for some $1 \leq j \leq d$

which implies

$$\begin{aligned}\frac{\partial \tan(\arg(z,z))}{\partial z_j}\Big|_{z=z'} &= \frac{1}{2}\frac{\partial\left(\frac{\mathrm{Im}(z,z)}{\mathrm{Re}(z,z)}\right)}{\partial x_j}\Big|_{z=z'} - \frac{i}{2}\frac{\partial\left(\frac{\mathrm{Im}(z,z)}{\mathrm{Re}(z,z)}\right)}{\partial y_j}\ |_{z'=z} \\ &= \frac{y_j'}{\mathrm{Re}(z',z')} - \frac{ix_j'}{\mathrm{Re}(z',z')} \\ &\neq 0.\end{aligned}$$

Applying the chain rule we have that $\frac{\partial \arg(z,z)}{\partial z_j}\Big|_{z=z'} \neq 0$. Hence we can conclude that $S - S_0 = (\arg(\cdot,\cdot))^{-1}\{\pi\}$ is a submanifold of $\mathbb{C}^d$ with dimension $d-1$. □

Corollary 4.3. *The set $S \subset \mathbb{C}^d$ has Lebesgue measure zero.*

Denote by $\mathbb{H}^+ = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$ the open right-half plane of $\mathbb{C}$. Define $\Psi\colon \mathbb{C}^d - S \to \mathbb{H}^+ \times S_1$ by

$$\Psi(z) = (w, z')$$

where $w = |(z,z)|^{1/2}e^{i\frac{\theta}{2}}$, θ is the principal value of $\arg(z,z)$, $\theta \in (-\pi,\pi)$, and $z' = \frac{z}{w}$. Then Ψ is a continuous bijection map whose inverse is $\Psi^{-1}(w,z') = wz'$. We can think of this map as a "complex polar form" of an element in $\mathbb{C}^d$ whose bilinear form is nonzero.

Let m be Lebesgue measure on $\mathbb{C}^d$ and m_* the Borel measure on $\mathbb{H}^+ \times S_1$ such that $m_*(E) = m(\Psi^{-1}(E))$. The next theorem shows that the push-forward measure m_* on $\mathbb{H}^+ \times S_1$ can be written as a product measure $m_* = \rho \times \alpha$, where ρ is a measure on $\mathbb{H}^+$ defined by

$$\rho(A) = \int_A |w|^{2d-2} dw$$

and α is an $SO(d,\mathbb{C})$-invariant Borel measure on S_1.

Theorem 4.4. *There is an $SO(d,\mathbb{C})$-invariant Borel measure α on S_1 such that $m_* = \rho \times \alpha$. If f is a Borel measurable function on $\mathbb{C}^d$ such that $f \geq 0$ or $f \in L^1(\mathbb{C}^d, m)$, then*

$$\int_{\mathbb{C}^d} f(z)\,dz = \int_{\mathbb{C}} \int_{S_1} f(wz')d\alpha(z')\,|w|^{2d-2}dw, \tag{4.1}$$

where dw denotes the two-dimensional Lebesgue measure on $\mathbb{C} = \mathbb{R}^2$.

Proof. Since S has Lebesgue measure zero, (4.1) is equivalent to

$$\int_{\mathbb{C}^d - S} f(z)\,dz = \int_{\mathbb{C}} \int_{S_1} f(wz')d\alpha(z')\,|w|^{2d-2}dw. \tag{4.2}$$

First, we need to construct α. Let E be a Borel set in S_1. For any $a > 0$, define $D_a = \{w \in \mathbb{C} \mid |w| < a\}$ and

$$E_a \;=\; \Psi^{-1}(D_a \cap \mathbb{H}^+ \times E) \;=\; \{wz' \mid\; w \in D_a \cap \mathbb{H}^+, z' \in E\}.$$

If (4.2) is to hold when $f = \chi_{E_1}$, we must have

$$m(E_1) = \frac{1}{2}\int_{D_1}\int_E d\alpha(z')\,|w|^{2d-2}dw = \frac{\pi}{2d}\,\alpha(E).$$

Hence, for any Borel set E in S_1, we define

$$\alpha(E) = \frac{2d}{\pi}\,m(E_1).$$

Since the map $E \mapsto E_1$ takes Borel sets to Borel sets and commutes with unions, intersections and complements, it is clear that α is a Borel measure on S_1. If E is a Borel set in S_1 and $A \in SO(d,\mathbb{C})$ then

$$\begin{aligned}(AE)_1 &= \{wz' \;\mid w \in \mathbb{H}^+,\; |w| < 1,\; z' \in AE\}\\ &= \{wAz' \;\mid w \in \mathbb{H}^+,\; |w| < 1,\; z' \in E\}\\ &= AE_1.\end{aligned}$$

Consequently,

$$\alpha(AE) = \frac{2d}{\pi} m((AE)_1) = \frac{2d}{\pi} m(A(E)_1) = \frac{2d}{\pi} \det(A) m(E_1) = \alpha(E),$$

where $\det(A)$ is the determinant of A over $\mathbb{R}$ which is 1. Hence α is $SO(d,\mathbb{C})$-invariant. Moreover, $E_a = aE_1$, so $m(E_a) = a^{2d} m(E_1)$. Following this we have that

$$\begin{aligned} m_*((D_a \cap \mathbb{H}^+) \times E) &= m(E_a) \\ &= \frac{2\pi}{d}(a^{2d})\alpha(E) \\ &= \alpha(E)\frac{1}{2}\int_{D_a} |w|^{2d-2} dw \\ &= \rho \times \alpha((D_a \cap \mathbb{H}^+) \times E). \end{aligned}$$

Fix $E \in \mathcal{B}_{S_1}$ and let $\mathcal{A}_E$ be the collection of finite disjoint unions of sets of the form $(D_a \cap \mathbb{H}^+) \times E$. Then $\mathcal{A}_E$ is an algebra on $\mathbb{H}^+ \times E$ that generates the σ-algebra $\mathcal{M}_E = \{D \times E \mid D \in \mathcal{B}_{\mathbb{H}^+}\}$. Since $m_* = \rho \times \alpha$ on $\mathcal{A}_E$, $m_* = \rho \times \alpha$ on $\mathcal{M}_E$. But $\bigcup\{\mathcal{M}_E \mid E \in \mathcal{B}_{S_1}\}$ is a set of all Borel rectangles on $\mathbb{H}^+ \times S_1$. Thus $m_* = \rho \times \alpha$ on all Borel sets. Hence equation (4.2) holds when f is a characteristic function of a Borel set and it follows for general f by the usual linearity and approximation argument. □

The measure α in Theorem 4.4 is uniquely determined and can be given explicitly. First, let us recall some definition and theorem about Lie groups and the complex unit sphere S_1.

Theorem 4.5 ([Kn] Theorem 8.36). *Let G be a Lie group, let H be a closed subgroup, and let Δ_G and Δ_H be the respective modular functions. Then a necessary and sufficient condition for G/H to have a nonzero G-invariant Borel measure is that the restriction to H of Δ_G equal Δ_H. In this case such a measure $d\mu(gH)$ is* *unique up to a scalar, and it can be normalized so that*

$$\int_G f(g)d_l g = \int_{G/H} \Big[\int_H f(gh)d_l h \Big] \, d\mu(gH) \tag{4.3}$$

for all $f \in C_c(G)$ where d_l is a left-Haar measure on G.

Notation. Denote by $T^*(S^{d-1})$ the cotangent bundle of the real unit sphere S^{d-1}, which we can think of as

$$T^*(S^{d-1}) = \{(\mathbf{x}, \mathbf{p}) \in \mathbb{R}^d \times \mathbb{R}^d \mid |\mathbf{x}| = 1, \, \mathbf{x} \cdot \mathbf{p} = 0\}.$$

There is a diffeomorphism between $T^*(S^{d-1})$ and the complex unit sphere S_1 given by the formula

$$\mathbf{a}(\mathbf{x}, \mathbf{p}) = \cosh(p) \, \mathbf{x} + \frac{i}{p} \sinh(p) \, \mathbf{p}$$

for any $\mathbf{x} \in S^{d-1}$ and $\mathbf{x} \cdot \mathbf{p} = 0$, where $p = |\mathbf{p}|$. See [HM] for more details.

Lemma 4.6. *The measure*

$$\Big(\frac{\sinh 2p}{2p}\Big)^{d-2} 2^{d-1} \, d\mathbf{p} \, d\mathbf{x}$$

is invariant under the action of $SO(d, \mathbb{C})$ on $S_1 \cong T^*(S^{d-1})$.

Proof. See [HM], Lemma 3. □

Using this coordinate, we can write the measure α explicitly as follows:

Lemma 4.7. *The measure α is given by*

$$d\alpha(z) = a_0 \Big(\frac{\sinh 2p}{2p}\Big)^{d-2} 2^{d-1} \, d\mathbf{p} \, d\mathbf{x},$$

where a_0 is a constant which is unique. Here $z = \mathbf{a}(\mathbf{x}, \mathbf{p})$, $d\mathbf{x}$ is the surface area measure on S^{d-1} and $d\mathbf{p}$ is Lebesgue measure on $\mathbb{R}^d$.

Proof. The measure α and the measure $\big(\frac{\sinh 2p}{2p}\big)^{d-2} 2^{d-1} \, d\mathbf{p} \, d\mathbf{x}$ are both $SO(d, \mathbb{C})$-invariant and finite on compact sets. Thus, by Proposition 2.12 and Theorem 4.5, these two measures must agree up to a constant. □

Chapter 5

Pointwise bound for a function in $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$

In this chapter, we consider the subspace of the Segal-Bargmann space which is invariant under the action of the special complex orthogonal group. Our objective is to find a pointwise bound of a function in this space.

Notation. We write

$$\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})} = \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})} \cap L^2(\mathbb{C}^d, \mu_t).$$

It is easy to see that it is a closed subspace of $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$, and hence is a Hilbert space.

By averaging over each orbit, we obtain the following pointwise bound:

Proposition 5.1. *For any $F \in \mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$ and for any $z \in \mathbb{C}^d$*

$$|F(z)|^2 \le e^{|(z,z)|/t} ||F||^2_{L^2(\mathbb{C}^d,\mu_t)}. \tag{5.1}$$

Proof. Note that $|(z,z)| = |(Az,Az)| \le |Az|^2$ for any $z \in \mathbb{C}^d$ and $A \in SO(d,\mathbb{C})$. If $z \notin S_0$, we have that $(\sqrt{(z,z)}, 0, \ldots, 0) \in \{Az \mid A \in SO(d,\mathbb{C})\}$, because $SO(d,\mathbb{C})$

acts transitively on S_w where $w = \sqrt{(z,z)}$, and thus

$$|(z,z)| = \inf\left\{|Az|^2 : A \in SO(d,\mathbb{C})\right\}.$$

But S_0^c is dense in $\mathbb{C}^d$, so this equation is also true for all $z \in \mathbb{C}^d$. This immediately gives (5.1). □

This simple technique yields an improvement from the Bargmann's pointwise bound (3.1). However, we will establish a polynomially-better bound than the bound in (5.1). Our strategy is to construct a non-Gaussian measure λ on $\mathbb{C}$ so that we can express $\mathcal{H}L^2(\mathbb{C}^d,\mu_t)^{SO(d,\mathbb{C})}$ in terms of the space $\mathcal{H}L^2(\mathbb{C},\lambda)^e$ of holomophic even functions on $\mathbb{C}$ that are square-integrable with respect to λ and then estimate the reproducing kernel of the latter space.

Henceforth, we will write each nonzero complex number w in a general exponential form $w = |w|e^{i\theta}$ where θ is the principal value of $\arg(w)$ $(-\pi < \theta \le \pi)$. Denote by $\mathcal{B}_d$ the Borel σ-algebra in $\mathbb{C}^d$ and by $\mathcal{B}$ the Borel σ-algebra in $\mathbb{C}$. Define $\Phi_i\colon (\mathbb{C}^d,\mathcal{B}_d,\mu_t) \to (\mathbb{C},\mathcal{B})$, $i = 1,2$ to be the branch of $\sqrt{(z,z)}$ with a smaller and larger argument respectively and for each $E \in \mathcal{B}$ define

$$\lambda_i(E) = \mu_t(\Phi_i^{-1}(E)).$$

Then define $\lambda = (\lambda_1+\lambda_2)/2$. It is easy to check that λ is a Borel measure on $\mathbb{C}$ and for any measurable function g and any $E \in \mathcal{B}$

$$\int_E g\,d\lambda \;=\; \frac{1}{2}\int_{\Phi_1^{-1}(E)} g\circ\Phi_1\,d\mu_t + \frac{1}{2}\int_{\Phi_2^{-1}(E)} g\circ\Phi_2\,d\mu_t.$$

Theorem 5.2. *$\mathcal{H}L^2(\mathbb{C}^d,\mu_t)^{SO(d,\mathbb{C})}$ and $\mathcal{H}L^2(\mathbb{C},\lambda)^e$ are unitarily equivalent.*

Proof. From Proposition 3.9 we have that the function

$$\psi\colon \mathcal{H}(\mathbb{C})^e \to \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$$

is a linear isomorphism. We consider the restriction of ψ to the space $\mathcal{H}L^2(\mathbb{C},\lambda)^e$. Let $g \in \mathcal{H}(\mathbb{C})^e$ and $f \in \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$ be such that $f = \psi(g)$. Thus

$$\begin{aligned}\int_{\mathbb{C}} |g|^2 d\lambda &= \frac{1}{2}\int_{\Phi_1^{-1}(\mathbb{C})} |g \circ \Phi_1(z)|^2\, \mu_t(z)\, dz + \frac{1}{2}\int_{\Phi_2^{-1}(\mathbb{C})} |g \circ \Phi_2(z)|^2\, \mu_t(z)\, dz \\ &= \int_{\mathbb{C}^d} |g(\sqrt{(z,z)}\,)|^2\, \mu_t(z)\, dz \\ &= \int_{\mathbb{C}^d} |\psi(g)\,(z)|^2\, \mu_t(z)\, dz \\ &= \int_{\mathbb{C}^d} |f(z)|^2\, \mu_t(z)\, dz.\end{aligned}$$

So we have $\|g\|_{L^2(\mathbb{C},\lambda)} = \|f\|_{L^2(\mathbb{C}^d,\mu_t)}$. Hence, $f \in \mathcal{H}L^2(\mathbb{C}^d,\mu_t)^{SO(d,\mathbb{C})}$ if and only if $g \in \mathcal{H}L^2(\mathbb{C},\lambda)^e$. This shows that ψ is a unitary map from $\mathcal{H}L^2(\mathbb{C},\lambda)^e$ onto $\mathcal{H}L^2(\mathbb{C}^d,\mu_t)^{SO(d,\mathbb{C})}$. □

We next show that the measure λ is absolutely continuous with respect to Lebesgue measure and then we approximate the density of λ. This will yield a pointwise bound for a function in $\mathcal{H}L^2(\mathbb{C}^d,\lambda)^e$. Then a pointwise bound for a function in $\mathcal{H}L^2(\mathbb{C}^d,\mu_t)^{SO(d,\mathbb{C})}$ is obtained by a unitary map.

Theorem 5.3. *The measure λ is absolutely continuous with respect to Lebesgue measure on $\mathbb{C}$ with density given by*

$$\Lambda(w) = \frac{|w|^{2d-2}}{(\pi t)^d}\int_{S_1} e^{-|wz|^2/t} d\alpha(z). \tag{5.2}$$

Proof. Let E be a complex Borel set in $\mathbb{C}$. For any $z \in \mathbb{C}$ with $z = wz'$, $w \in \mathbb{C}$ and $z' \in S_1$, we have that $z \in \Phi_1^{-1}(E)$ if and only if $w \in E$. Similarly $z \in \Phi_2^{-1}(E)$

if and only if $-w \in E$. Therefore by Theorem 4.4

$$\begin{aligned}
\lambda(E) &= \frac{1}{2}\int_{\Phi_1^{-1}(E)} \frac{e^{-|z|^2/t}}{(\pi t)^d}\,dz + \frac{1}{2}\int_{\Phi_2^{-1}(E)} \frac{e^{-|z|^2/t}}{(\pi t)^d}\,dz \\
&= \frac{1}{2}\int_{\mathbb{C}}\int_{S_1} \chi_{\Phi_1^{-1}(E)}(wz')\frac{|w|^{2d-2}}{(\pi t)^d}\,e^{-|wz'|^2/t}\,d\alpha(z')\,dw \\
&+ \frac{1}{2}\int_{\mathbb{C}}\int_{S_1} \chi_{\Phi_2^{-1}(E)}(wz')\frac{|w|^{2d-2}}{(\pi t)^d}\,e^{-|wz'|^2/t}\,d\alpha(z')\,dw \\
&= \frac{1}{2}\int_{\mathbb{C}}\int_{S_1} \chi_E(w)\frac{|w|^{2d-2}}{(\pi t)^d}\,e^{-|wz'|^2/t}\,d\alpha(z')\,dw \\
&+ \frac{1}{2}\int_{\mathbb{C}}\int_{S_1} \chi_E(-w)\frac{|w|^{2d-2}}{(\pi t)^d}\,e^{-|wz'|^2/t}\,d\alpha(z')\,dw \\
&= \frac{1}{2}\int_{\mathbb{C}}\int_{S_1} \chi_E(w)\frac{|w|^{2d-2}}{(\pi t)^d}\,e^{-|wz'|^2/t}\,d\alpha(z')\,dw \\
&+ \frac{1}{2}\int_{\mathbb{C}}\int_{S_1} \chi_E(w)\frac{|w|^{2d-2}}{(\pi t)^d}\,e^{-|wz'|^2/t}\,d\alpha(z')\,dw \\
&= \int_E \frac{|w|^{2d-2}}{(\pi t)^d}\int_{S_1} e^{-|wz'|^2/t}\,d\alpha(z')\,dw \\
&= \int_E \Lambda(w)\,dw
\end{aligned}$$

where Λ is given by (5.2). □

Next, we will approximate the density Λ of λ and show that on holomorphic functions, the L^2-norm with respect to λ is equivalent to the L^2-norm with respect to the measure $\beta(w)dw$ where

$$\beta(w) = \frac{e^{-|w|^2/t}}{\pi t}|w|^{d-1}$$

for all $w \in \mathbb{C}$.

Proposition 5.4. *There exist constants $m, M > 0$, depending on d and t, such that the density function Λ of λ satisfies*

$$m\beta(w) \le \Lambda(w) \le M\beta(w) \tag{5.3}$$

for all $w \in C$ with $|w| \ge 1$.

Proof. From Lemma 4.7, for any $w \in \mathbb{C}$

$$\begin{aligned}
\int_{S_1} e^{-|wz|^2/t}\, d\alpha(z) &= a_0 \int_{S^{d-1}} \int_{\{\mathbf{p}\in\mathbb{R}^d |\, \mathbf{x}\cdot\mathbf{p}=\mathbf{0}\}} e^{-|w\,\mathbf{a}(\mathbf{x},\mathbf{p})|^2/t} \Big(\frac{\sinh(2p)}{2p}\Big)^{d-2} 2^{d-1}\, d\mathbf{p}\, d\mathbf{x} \\
&= a_0\sigma(S^{d-1}) \int_{\mathbb{R}^{d-1}} e^{-\cosh(2p)\,|w|^2/t} \Big(\frac{\sinh(2p)}{2p}\Big)^{d-2} 2^{d-1}\, d\mathbf{p} \\
&= a_d \int_0^\infty e^{-\cosh(2r)\,|w|^2/t} \Big(\frac{\sinh(2r)}{2r}\Big)^{d-2} 2^{d-1}\, r^{d-2}\, dr \\
&= a_d \int_0^\infty e^{-\cosh(2r)\,|w|^2/t} \sinh^{d-2}(2r)\, 2dr \\
&= a_d \int_0^\infty e^{-\cosh(u)|w|^2/t} \sinh^{d-2} u\, du \\
&= a_d e^{-|w|^2/t} \int_0^\infty e^{-(\cosh(u)-1)|w|^2/t} (\cosh^2 u - 1)^{(d-3)/2} d(\cosh u) \\
&= a_d e^{-|w|^2/t} \int_0^\infty e^{-|w|^2 x/t} (x^2+2x)^{(d-3)/2}\, dx, \qquad (5.4)
\end{aligned}$$

with $a_d = a_0\sigma(S^{d-1})\sigma(S^{d-2})$ where σ is the surface measure. The last equality follows from the change of variables $\cosh(u) = x+1$. But then

$$(x^2+2x)^{(d-3)/2} = \left\{ \sum_{k=0}^{d-3} \binom{d-3}{k} 2^{d-3-k} x^{d-3+k} \right\}^{1/2} = \Big\{ \sum_{k=0}^{d-3} a_k x^{d-3+k} \Big\}^{1/2}, \quad (5.5)$$

where

$$a_k = \binom{d-3}{k} 2^{d-3-k}.$$

Now let us consider the case $d \geq 3$. Applying the inequality

$$\frac{1}{\sqrt{n}}(\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n}\,) \leq \sqrt{a_1 + a_2 + \cdots + a_n} \leq \sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n},$$

for any $a_1,\, a_2,\, \ldots,\, a_n \geq 0$, we see that

$$\frac{1}{\sqrt{d-2}} \sum_{k=0}^{d-3} a_k^{1/2} x^{(d-3+k)/2} \ \leq\ \left\{ \sum_{k=0}^{d-3} a_k x^{d-3+k} \right\}^{1/2} \ \leq\ \sum_{k=0}^{d-3} a_k^{1/2} x^{(d-3+k)/2}.$$

Thus

$$\frac{1}{\sqrt{d-2}} \sum_{k=0}^{d-3} a_k^{1/2} x^{(d-3+k)/2} \leq (x^2+2x)^{(d-3)/2} \leq \sum_{k=0}^{d-3} a_k^{1/2} x^{(d-3+k)/2}. \qquad (5.6)$$

By using the formula for the Gamma function we have that

$$\int_0^\infty e^{-|w|^2x/t}x^{(d-3+k)/2}\,dx \;=\; \Gamma\left(\frac{d-3+k}{2}+1\right)\left(\frac{t}{|w|^2}\right)^{(d-1+k)/2}.$$

It follows that

$$\frac{1}{\sqrt{d-2}}P\left(\frac{\sqrt{t}}{|w|}\right)\le\int_0^\infty e^{-|w|^2x/t}\,(x^2+2x)^{(d-3)/2}\,dx\le P\left(\frac{\sqrt{t}}{|w|}\right),$$

where

$$P(x)=\sum_{k=0}^{d-3}a_k^{1/2}\Gamma\left(\frac{d-1+k}{2}\right)x^{d-1+k}.$$

This shows that

$$\frac{a_d}{\sqrt{d-2}}P\left(\frac{\sqrt{t}}{|w|}\right)e^{-|w|^2/t}\;\le\;\int_{S_1}e^{-|wz|^2/t}\,d\alpha(z)\;\le\;a_dP\left(\frac{\sqrt{t}}{|w|}\right)e^{-|w|^2/t}.$$

It follows from (5.2) that

$$\frac{e^{-|w|^2/t}}{t\pi\sqrt{d-2}}Q\left(\frac{|w|}{\sqrt{t}}\right)\;\le\;\Lambda(w)\;\le\;\frac{e^{-|w|^2/t}}{t\pi}Q\left(\frac{|w|}{\sqrt{t}}\right) \tag{5.7}$$

where

$$Q(x)\;=\;\frac{a_d}{\pi^{d-1}}\sum_{k=0}^{d-3}a_k^{1/2}\Gamma\left(\frac{d-1+k}{2}\right)x^{d-1-k}\;=\;\sum_{k=2}^{d-1}b_kx^k.$$

Since we are interested in the behavior of Λ, we will keep only the monomials of the highest degrees x^{d-1} of Q in our estimate. Hence we establish (5.3) for the case $d\ge 3$.

Meanwhile in the $d=2$ case we have

$$\begin{aligned}\int_{S_1}e^{-|wz|^2/t}\,d\alpha(z)&=a_2\,e^{-|w|^2/t}\int_0^\infty\frac{e^{-x|w|^2/t}}{\sqrt{x^2+2x}}\,dx\\&=\frac{a_2}{\sqrt{2}}\,e^{-|w|^2/t}\int_0^\infty e^{-u}\sqrt{\left(\frac{|w|^2}{tu}-\frac{|w|^2}{2|w|^2+tu}\right)}\,\frac{t}{|w|^2}\,du\\&\ge\frac{a_2\sqrt{t}}{\sqrt{2}|w|}\,e^{-|w|^2/t}\left(\int_0^\infty\frac{e^{-u}}{\sqrt{u}}\,du-\int_0^\infty\frac{e^{-u}}{\sqrt{2|w|^2/t+u}}\,du\right).\end{aligned}$$

The function

$$\phi(r)=\int_0^\infty\frac{e^{-u}}{\sqrt{r+u}}\,du\qquad(r\ge 0)$$

is a strictly decreasing function. Hence, if we let $\delta = 2/t$ and $\varepsilon = \phi(0) - \phi(\delta)$, then $\phi(0) - \phi\left(2|w|^2/t\right) \geq \phi(0) - \phi(\delta) = \varepsilon$ for any w with $2|w|^2/t \geq \delta$. It follows that

$$\begin{aligned}\Lambda(w) &= \frac{|w|^2}{(\pi t)^2}\int_{S_1} e^{-|wz|^2/t} d\alpha(z)\\ &\geq \frac{\varepsilon a_2}{\pi\sqrt{2t}}\,\frac{e^{-|w|^2/t}}{\pi t}|w|\end{aligned}$$

for any $w \in \mathbb{C}$ with $|w| \geq 1$.

On the other hand,

$$\begin{aligned}\int_{S_1} e^{-|wz|^2/t}\, d\alpha(z) &\leq a_2\, e^{-|w|^2/t}\int_0^\infty \frac{e^{-x|w|^2/t}}{\sqrt{2x}}\, dx\\ &= \frac{a_2\sqrt{t\pi}}{\sqrt{2}|w|}\, e^{-|w|^2/t}.\end{aligned}$$

Hence

$$\Lambda(w) \leq \frac{a_2}{\sqrt{2\pi t}}\,\frac{e^{-|w|^2/t}}{\pi t}|w|.$$

Thus the theorem is proved. □

Lemma 5.5. *The norms* $\|\cdot\|_{L^2(\mathbb{C},\beta)}$ *and* $\|\cdot\|_{L^2(\mathbb{C},\lambda)}$ *are equivalent, i.e., there are constants* $k, K > 0$*, depending on* d *and* t*, such that*

$$k\|f\|_{L^2(\mathbb{C},\beta)} \leq \|f\|_{L^2(\mathbb{C},\lambda)} \leq K\|f\|_{L^2(\mathbb{C},\beta)}, \tag{5.8}$$

for all $f \in \mathcal{H}L^2(\mathbb{C}, \lambda)$*.*

Proof. First, we will show that there is a constant $D > 0$, depending on d and t, such that

$$\|f\|^2_{L^2(\mathbb{C},\beta)} \leq D\,\|f\|^2_{L^2(\mathbb{C}-\mathbb{D},\lambda)}$$

for any $f \in \mathcal{H}L^2(\mathbb{C}, \lambda)$, where $\mathbb{D} = \{w \in \mathbb{C} : |w| \leq 1\}$.

Let $w \in \mathbb{D}$. Denote by $A(w)$ the annulus $\{z \in \mathbb{C} \mid 2 \leq |z-w| \leq 3\}$. For any $v \in A(w)$, we use the polar coordinates with the origin at w so that $v - w = re^{i\theta}$. If $f \in \mathcal{H}L^2(\mathbb{C}, \lambda)$, then we expand f as a power series around $v = w$:

$$f(v) = f(w) + \sum_{n=1}^{\infty} a_n (v-w)^n.$$

Hence,

$$\int_{A(w)} f(v)\, dv = f(w)(9\pi - 4\pi) + \sum_{n=1}^{\infty} \int_2^3 \int_0^{2\pi} a_n r^n e^{in\theta}\, d\theta dr = 5\pi f(w).$$

Therefore

$$\begin{aligned} f(w) = & \frac{1}{5\pi} \int_{A(w)} f(v)\, dv \\ = & \frac{1}{5\pi} \int_{\mathbb{C}-\mathbb{D}} \chi_{A(w)}(v) \frac{1}{\Lambda(v)} f(v) \Lambda(v)\, dv \\ = & \frac{1}{5\pi} \langle \chi_{A(w)} \frac{1}{\Lambda}, f \rangle_{L^2(\mathbb{C}-\mathbb{D},\lambda)}. \end{aligned}$$

By Cauchy-Schwarz inequality we have that

$$|f(w)| \leq \frac{1}{5\pi} \Big\| \chi_{A(w)} \frac{1}{\Lambda} \Big\|_{L^2(\mathbb{C}-\mathbb{D},\lambda)} \|f\|_{L^2(\mathbb{C}-\mathbb{D},\lambda)}.$$

Since Λ is strictly positive and continuous on $A(w)$, $\frac{1}{\Lambda}$ is bounded on $A(w)$. Thus the first L^2-norm is finite. However, for each $w \in \mathbb{D}$,

$$\frac{1}{5\pi} \Big\| \chi_{A(w)} \frac{1}{\Lambda} \Big\|_{L^2(\mathbb{C}-\mathbb{D},\lambda)} \leq \frac{1}{5\pi} \Big\| \chi_{A^*} \frac{1}{\Lambda} \Big\|_{L^2(\mathbb{C}-\mathbb{D},\lambda)} < \infty$$

where $A^* = \{z \in \mathbb{C} \mid 1 < |z| < 4\}$, which contains each $A(w)$, $w \in \mathbb{D}$. It follows that there exists a constant c such that for any $w \in \mathbb{D}$

$$|f(w)| \leq c \|f\|_{L^2(\mathbb{C}-\mathbb{D},\lambda)}.$$

It now follows from Proposition 5.4 that

$$\begin{aligned} \int_{\mathbb{C}} |f(w)|^2 \beta(w)\, dw &= \int_{\mathbb{D}} |f(w)|^2 \beta(w) dw + \int_{\mathbb{C}-\mathbb{D}} |f(w)|^2 \beta(w)\, dw \\ &\leq c^2 \|f\|^2_{L^2(\mathbb{C}-\mathbb{D},\lambda)} \int_{\mathbb{D}} \beta(w) dw + \frac{1}{m} \|f\|^2_{L^2(\mathbb{C}-\mathbb{D},\lambda)} \\ &\leq D \|f\|^2_{L^2(\mathbb{C}-\mathbb{D},\lambda)} \end{aligned}$$

for some constant $D > 0$ depending on d and t. This give the first inequality in (5.8). The second inequality in (5.8) can be proved in the same way. $\square$

Having established Lemma 5.5, it is remains only to obtain pointwise bounds for elements in $\mathcal{H}L^2(\mathbb{C}, \lambda)$. We do this by reducing to the standard Segal-Bargmann space when d is odd and to the space $\mathcal{H}L^2(\mathbb{C}, (t\pi)^{-1}|w|e^{-|w|^2/t}dw)$ when d is even. We now establish pointwise bound in the latter space.

Lemma 5.6. *The set* $\left\{ \frac{w^n}{\left(t^{(2n+1)/2}\Gamma(n+\frac{3}{2})\right)^{1/2}} \right\}_{n=0}^{\infty}$ *forms an orthonormal basis for* $\mathcal{H}L^2(\mathbb{C}, \nu_t(w)dw)$*, where* $\nu_t(w) = |w|\frac{e^{-|w|^2/t}}{t\pi}$ *for all* $w \in \mathbb{C}$.

Proof. For each $n \in \mathbb{N} \cup \{0\}$, define $g_n\colon \mathbb{C} \to \mathbb{C}$ by

$$g_n(w) = w^n \qquad \text{for all } w \in \mathbb{C}.$$

It is clear that $g_n \in \mathcal{H}(\mathbb{C})$. Claim that $\{g_n\}_{n=0}^{\infty}$ is an orthogonal subset of $\mathcal{H}L^2\big(\mathbb{C}, \nu_t\big)$. To prove the claim, let σ be a positive real number. Define

$$D_\sigma := \{\, w \in \mathbb{C} \mid |w| \le \sigma \,\}.$$

Next, let

$$\begin{aligned} M_\sigma(j,k) &= \int_{D_\sigma} w^j \bar{w}^k \, |w| e^{-w^2/t} (\pi t)^{-1} \, dw \\ &= \int_0^{2\pi} \int_0^{\sigma} e^{i\theta(j-k)} r^{j+k+2} e^{-r^2/t} (\pi t)^{-1} \, dr \, d\theta. \end{aligned}$$

It follows that

(i) $M_\sigma(j,k) = 0$ if $j \neq k$,

(ii) $M_\sigma(k,k) \to t^{k+1/2}\Gamma(k+3/2)$ as $\sigma \to \infty$.

For any $m, n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} \int_{\mathbb{C}} g_m(w)\overline{g_n(w)}\nu_t(w)dw &= \int_{\mathbb{C}} w^m \overline{w^n} |w| \frac{e^{-|w|^2/t}}{t\pi} \, dw \\ &= \lim_{\sigma\to\infty} M_\sigma(m,n) \end{aligned}$$

Thus it follows from (i) that

$$\int_{\mathbb{C}} g_m(w)\overline{g_n(w)}\nu_t(w)dw = 0, \quad \text{if } m \neq n.$$

If $n = m$, we have

$$\begin{aligned}\int_{\mathbb{C}} |g_n(w)|^2\nu_t(w)dw &= \lim_{\sigma\to\infty} M_\sigma(n,n)\\ &= t^{n+1/2}\Gamma(n+3/2).\end{aligned}$$

Thus for any $n \in \mathbb{N}\cup\{0\}$,

$$\|g_n\|^2_{L^2(\mathbb{C},\nu_t)} = t^{n+1/2}\Gamma\Big(n+\frac{3}{2}\Big). \tag{5.9}$$

Hence, $\{g_n\}_{n=0}^{\infty}$ is an orthogonal subset of $\mathcal{H}L^2(\mathbb{C},\nu_t)$. Next, we will show that $\{g_n\}_{n=0}^{\infty}$ is an orthogonal basis of $\mathcal{H}L^2(\mathbb{C},\nu_t)$. Let $g \in \mathcal{H}L^2(\mathbb{C},\nu_t)$. Then $g(w) = \sum\limits_{n=0}^{\infty} a_n w^n$ for each $w \in \mathbb{C}$, where $a_n \in \mathbb{C}$ for all $n \in \mathbb{N}\cup\{0\}$. Since this power series converges uniformly on the set D_σ, we have

$$\begin{aligned}\int_{D_\sigma} |g(w)|^2\nu_t(w)dw &= \int_{D_\sigma} \sum_{n=0}^{\infty} a_n w^n \sum_{m=0}^{\infty} \overline{a_m w^m}|w|\frac{e^{-|w|^2/t}}{t\pi}dw\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n\overline{a_m}\int_{D_\sigma} w^n\overline{w^m}|w|\frac{e^{-|w|^2/t}}{t\pi}dw\\ &= \sum_{n=0}^{\infty} |a_n|^2 M_\sigma(n,n).\end{aligned}$$

Next, using the monotone convergence theorem, we have

$$\begin{aligned}\int_{\mathbb{C}} |g(w)|^2\nu_t(w)dw &= \lim_{\sigma\to\infty}\int_{D_\sigma} |g(w)|^2|w|\frac{e^{-|w|^2/t}}{t\pi}dw\\ &= \lim_{\sigma\to\infty}\sum_{n=0}^{\infty} |a_n|^2 M_\sigma(n,n)\\ &= \sum_{n=0}^{\infty} |a_{2n}|^2\, t^{n+\frac{1}{2}}\Gamma\Big(n+\frac{3}{2}\Big).\end{aligned}$$

Therefore, if g is square-integrable with respect to ν_t, then

$$\sum_{n=0}^{\infty} |a_{2n}|^2\, t^{n+\frac{1}{2}}\Gamma\Big(n+\frac{3}{2}\Big) < \infty.$$

Define a sequence (F_n) to be

$$F_n(w) := \sum_{m=0}^{n} a_m\, w^m$$

for any $w \in \mathbb{C}$. Then (F_n) is a Cauchy sequence in $L^2(\mathbb{C}, \nu_t)$. To see this, notice that for $m > n$,

$$\begin{aligned} \|F_m - F_n\|_{L^2(\mathbb{C},\nu_t)} &= \int_{\mathbb{C}} |F_m - F_n|^2\, |w| \frac{e^{-|w|^2/t}}{t\pi} dw \\ &= \int_{\mathbb{C}} \left| \sum_{k=n+1}^{m} a_k w^k \right|^2 |w| \frac{e^{-|w|^2/t}}{t\pi} dw \\ &= \sum_{k=n+1}^{m} |a_k|^2 t^{k+\frac{1}{2}} \Gamma\Big(k + \frac{3}{2}\Big) \to 0 \quad \text{ as } m, n \to \infty. \end{aligned}$$

Thus (F_n) converges in $L^2(\mathbb{C}, \nu_t)$ to some function h, and hence it has a subsequence which converges pointwise almost everywhere to h. But (F_n) converges pointwise to g, so we have that $h = g$ a.e. $[\nu_t]$. Therefore $h = g$ in $L^2(\mathbb{C}, \nu_t)$, so (F_n) converges to g in $L^2(\mathbb{C}, \nu_t)$. Thus $g \in \overline{\operatorname{span}\{g_n \mid n \in \mathbb{N} \cup \{0\}\}}$. This show that $\left\{ \frac{w^n}{\left(t^{(2n+1)/2}\Gamma(n+\frac{3}{2})\right)^{1/2}} \right\}_{n=0}^{\infty}$ forms an orthonormal basis for $\mathcal{H}L^2\big(\mathbb{C}, \nu_t\big)$. □

Corollary 5.7. *For any* $g \in \mathcal{H}L^2\big(\mathbb{C}, \nu_t\big)$,

$$|g(w)|^2 \le \frac{e^{|w|^2/t}}{|w|} \operatorname{erf}\Big(\frac{|w|}{\sqrt{t}}\Big)\, ||g||^2 \qquad (w \in \mathbb{C} - \{0\}),$$

where the ***error function*** erf *is defined by*

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2}\, dy = e^{-x^2} \sum_{n=0}^{\infty} \frac{x^{2n+1}}{\Gamma(n + \frac{3}{2})}.$$

Proof. By using Lemma 5.6, the reproducing kernel for the space $\mathcal{H}L^2\big(\mathbb{C}, \nu_t\big)$ is given by

$$K(x, y) = \sum_{n=0}^{\infty} \frac{x^n\, \overline{y}^n}{t^{(2n+1)/2}\Gamma(n + \frac{3}{2})}.$$

Thus the pointwise bound for a function g in the space $\mathcal{H}L^2\big(\mathbb{C},\nu_t\big)$ is

$$\begin{aligned} |g(w)|^2 &\leq K(w,w)\|g\|^2 \\ &\leq \sum_{n=0}^{\infty} \frac{|w|^{2n}}{t^{(2n+1)/2}\Gamma(n+\frac{3}{2})}\|g\|^2 \\ &= \frac{e^{|w|^2/t}}{|w|}\,\mathrm{erf}(\frac{|w|}{\sqrt{t}})\,\|g\|^2 \end{aligned}$$

for any $w \in \mathbb{C}$. □

Theorem 5.8. *There is a constant B, depending on d and t, such that for any $f \in \mathcal{H}L^2(\mathbb{C},\lambda)$ and any $w \in \mathbb{C}-\{0\}$,*

$$|f(w)|^2 \leq \frac{B}{|w|^{d-1}}\,e^{|w|^2/t}\|f\|^2_{L^2(\mathbb{C},\lambda)}. \tag{5.10}$$

Proof. Let $f \in \mathcal{H}L^2(\mathbb{C},\lambda)^e$. Then $f \in \mathcal{H}L^2(\mathbb{C},\beta)$, and thus

$$\int_{\mathbb{C}} |w|^{d-1}|f(w)|^2\frac{e^{-|w|^2/t}}{\pi t}\,dw < \infty.$$

If $d-1$ is an even number, then

$$w^{(d-1)/2}f(w) \in \mathcal{H}L^2(\mathbb{C},\frac{e^{-|w|^2/t}}{t\pi}\,dw).$$

This is the one-dimensional Segal-Bargmann space. Using Bargmann's pointwise bound (3.1) for this space, we obtain

$$|w|^{d-1}|f(w)|^2 \leq e^{|w|^2/t}\,||f||^2_{L^2(\mathbb{C},\beta)} \leq \frac{e^{|w|^2/t}}{k^2}||f||^2_{L^2(\mathbb{C},\lambda)}$$

for all $w \in \mathbb{C}$, where k is the constant in Corollary 5.8. On the other hand if $d-1$ is an odd number, then

$$w^{(d-2)/2}f(w) \in \mathcal{H}L^2(\mathbb{C},|w|\frac{e^{-|w|^2/t}}{t\pi}dw).$$

Following Lemma 5.6, we have

$$\begin{aligned} |w|^{d-2}|f(w)|^2 &\leq \|f\|^2_{L^2(\mathbb{C},\beta)}\frac{e^{|w|^2/t}}{|w|}\,\mathrm{erf}\,(\frac{|w|}{\sqrt{t}}) \\ &\leq \frac{1}{k^2}\|f\|^2_{L^2(\mathbb{C},\lambda)}\frac{e^{|w|^2/t}}{|w|} \end{aligned}$$

for all $w \in \mathbb{C} - \{0\}$. In either case we obtain the pointwise (5.10) with $B = 1/k^2$. $\square$

Proof of Theorem 1.1. We will transform the pointwise bound (5.10) to a function in $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$. Let $F \in \mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$. Then $F(w, 0, \ldots, 0) \in \mathcal{H}L^2(\mathbb{C}, \lambda)^e$, which implies

$$|F(z)|^2 = |F(w, 0, \ldots, 0)|^2 \leq B \frac{e^{|w|^2/t}}{|w|^{d-1}} \|F\|^2_{L^2(\mathbb{C}^d, \mu_t)}$$

where $w = \sqrt{(z,z)}$ for any $z \in \mathbb{C}^d$ with $(z,z) \neq 0$. In particular,

$$|F(z)|^2 \leq \frac{Be^{|(z,z)|/t}}{|(z,z)|^{(d-1)/2}} \|F\|^2_{L^2(\mathbb{C}^d, \mu_t)}$$

for any $z \in \mathbb{C}^d$ with $(z,z) \neq 0$. On the other hand, from Proposition 5.1,

$$|F(z)|^2 \leq e^{|(z,z)|/t} \|F\|^2_{L^2(\mathbb{C}^d, \mu_t)} \quad \text{for any } z \in \mathbb{C}^d.$$

Applying the inequality

$$\min\left\{1, \frac{1}{x}\right\} \leq \frac{2}{x+1} \quad \text{for each } x > 0,$$

we have

$$|F(z)|^2 \leq \frac{C\, e^{|(z,z)|/t}}{|(z,z)|^{(d-1)/2} + 1} \|F\|^2_{L^2(\mathbb{C}^d, \mu_t)}$$

for each $z \in \mathbb{C}^d$, where C is a constant depending on d and t. This completes the proof of Theorem 1.1. $\square$

Chapter 6

Reproducing kernel formula

In this last chapter, we obtain the formula for the reproducing kernel of $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$ that will yield an optimal pointwise bound for a function in $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$. This bound is asymptotically equal to the bound on the previous chapter for large arguments. First, we recall some definitions and properties of modified Bessel functions that will be central in the formula for the reproducing kernel of $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$.

6.1 Bessel Functions

In this section, we give some definitions and properties of the Bessel functions that will be used in the next section. All of substances of this section are in [T] and [Wat].

Bessel functions first defined by the Swiss mathematician Daniel Bernoulli and named after Friedrich Bessel, are canonical solutions $y(x)$ of Bessel's differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)y = 0$$

for an arbitrary real number α (the order). The most common and important special case is where α is an integer.

Since this is a second-order differential equation there must be two linearly independent solutions. The Bessel functions of the first kind, denoted by $J_\alpha(x)$, can be defined by its Taylor series expansion around $x = 0$ that is

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}.$$

The graphs of Bessel functions look roughly like oscillating sine or cosine functions that decay proportionally to $1/\sqrt{x}$, although their roots are not generally periodic except asymptotically for large x.

The second linearly independent solution is then found to be the Bessel function of the second kind, denoted by $Y_\alpha(x)$. They are singular (infinite) at $x = 0$. Then $Y_\alpha(x)$ is related to $J_\alpha(x)$ by

$$Y_\alpha(x) = \frac{J_\alpha(x)\cos(\alpha\pi) - J_{-\alpha}(x)}{\sin(\alpha\pi)}$$

where the case of integer α is handled by taking the limit.

The Bessel functions are valid even for complex arguments x, and an important special case is that of a purely imaginary argument. In this case, the solutions to the Bessel equation are called the **modified Bessel functions** of the first and second kind which are defined by,

$$\begin{aligned} \text{BesselI}(\alpha, x) := I_\alpha(x) &= i^{-\alpha} J_\alpha(ix) \\ \text{BesselK}(\alpha, x) := K_\alpha(x) &= \frac{\frac{\pi}{2} I_{-\alpha}(x) - I_\alpha(x)}{\sin(\alpha\pi)}. \end{aligned}$$

These are chosen to be real-valued for real arguments x. They are the two linearly independent solutions to the modified Bessel's equations

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2 + \alpha^2)y = 0.$$

Unlike the ordinary Bessel functions, which are oscillating, $I_\alpha(x)$ and $K_\alpha(x)$ are exponentially growing and decaying functions respectively. Like the ordinary Bessel function $J_\alpha(x)$, the function $I_\alpha(x)$ goes to zero at $x = 0$ for $\alpha > 0$ and is finite at $x = 0$ for $\alpha = 0$. Analogously $K_\alpha(x)$ diverges at $x = 0$.

We next give some properties of the modified Bessel functions which will be used in the next section.

1. If $\mathrm{Re}(\alpha) > -\frac{1}{2}$

$$I_\alpha(z) = \frac{(z/2)^\alpha}{\sqrt{\pi}\Gamma(1/2+\alpha)} \int_{-1}^{1} (1-t^2)^{\alpha-1/2} \cosh(t\alpha) dt.$$

2. For any α

$$I_\alpha(x) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\alpha+1)} \Big(\frac{x}{2}\Big)^{2m+\alpha}.$$

3. If $\mathrm{Re}(\alpha) > -\frac{1}{2}$ and $|\arg z| < \frac{1}{2}\pi$

$$\begin{aligned}\Gamma(1/2+\alpha)K_\alpha(z) &= \sqrt{\pi}\Big(\frac{z}{\alpha}\Big)^\alpha \int_1^\infty (1-t^2)^{\alpha-1/2} e^{-zt} dt \\ &= \sqrt{(\pi)}\Big(\frac{z}{\alpha}\Big)^\alpha \int_0^\infty \exp(-z\cosh(\theta)) \sinh(\theta)^{2\alpha} d\theta.\end{aligned}$$

4. If n is a positive integer (or zero)

$$K_{n+\frac{1}{2}} = \Big(\frac{\pi}{2z}\Big)^{1/2} e^{-z} \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!}(2z)^{-k}.$$

5. $K_\alpha(z) = K_{-\alpha}(z)$.

6. $\displaystyle\int_0^\infty K_\alpha(t) t^{v-1} dt = 2^{v-2}\Gamma\Big(\frac{v-\alpha}{2}\Big)\Gamma\Big(\frac{v+\alpha}{2}\Big)$.

7. The asymptotic approximations for $I_\alpha(x)$ and $K_\alpha(x)$ for large x, given by

$$I_\alpha(x) \sim \frac{e^x}{\sqrt{2\pi x}}$$
$$K_\alpha(x) \sim e^{-x}\sqrt{\frac{\pi}{2x}}$$

for all $\alpha \in \mathbb{R}$.

6.2 Reproducing kernel of $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$

Theorem 6.1. *The set* $\{x^{2n}\}_{n=0}^{\infty}$ *forms an orthogonal basis for* $\mathcal{H}L^2(\mathbb{C}, \lambda)^e$.

Proof. For each $n \in \mathbb{N} \cup \{0\}$, define $g_n \colon \mathbb{C} \to \mathbb{C}$ by

$$g_n(x) = x^{2n} \qquad \text{for all } x \in \mathbb{C}.$$

It is clear that $g_n \in \mathcal{H}(\mathbb{C})$. Claim that $\{g_n\}_{n=0}^{\infty}$ is an orthogonal subset of $\mathcal{H}L^2(\mathbb{C}, \lambda)^e$. To prove the claim, let σ be a positive real number. Define

$$\begin{aligned} D_\sigma &:= \{z \in \mathbb{C}^d \mid |z_j| \leq \sigma \text{ for all } j = 1, \ldots, d\} \qquad \text{and} \\ D_\sigma^1 &:= \{x \in \mathbb{C} \mid |x| \leq \sigma\}. \end{aligned}$$

Next, let

$$\begin{aligned} M_\sigma(j,k) &= \int_{D_\sigma^1} x^{2j} \bar{x}^{2k} \, e^{-x^2/t} (\pi t)^{-1} \, dx \\ &= \int_0^{2\pi} \int_0^{\sigma} e^{2i\theta(j-k)} r^{2j+2k+1} e^{-r^2/t} (\pi t)^{-1} \, dr \, d\theta. \end{aligned}$$

It follows that

(i.) $M_\sigma(j,k) = 0$ if $j \neq k$,

(ii.) $M_\sigma(k,k) \to t^{2k}(2k)!$ as $\sigma \to \infty$.

In other words,

$$\int_{\mathbb{C}} x^{2j} \bar{x}^{2k} \, d\mu_t(z) = \delta_{jk} t^{2k} (2k)!.$$

For any $m, n \in \mathbb{N} \cup \{0\}$,

$$
\begin{aligned}
\int_{\mathbb{C}} g_m \cdot \overline{g_n}\, d\lambda &= \int_{\mathbb{C}^d} (g_m \cdot \overline{g_n}) \circ \Phi(z)\, \mu_t(z)\, dz \\
&= \int_{\mathbb{C}^d} g_m \left(\sqrt{(z,z)}\right) \overline{g_n \left(\sqrt{(z,z)}\right)}\, \mu_t(z)\, dz \\
&= \int_{\mathbb{C}^d} (z,z)^m\, \overline{(z,z)^n}\, \mu_t(z)\, dz \\
&= \int_{\mathbb{C}^d} (z_1^2 + \cdots + z_d^2)^m\, (\bar{z}_1^2 + \cdots + \bar{z}_d^2)^n\, \mu_t(z)\, dz \\
&= \int_{\mathbb{C}^d} \Bigg(\sum_{j_1+\cdots+j_d=m} z_1^{2j_1} \cdots z_d^{2j_d} \Bigg) \Bigg(\sum_{k_1+\cdots+k_d=n} \bar{z}_1^{2k_1} \ldots \bar{z}_d^{2k_d} \Bigg)\, \mu_t(z)\, dz \\
&= \sum_{j_1+\cdots+j_d=m}\ \sum_{k_1+\cdots+k_d=n} \int_{\mathbb{C}^d} (z_1^{j_1} \bar{z}_1^{k_1})^2 \cdots (z_d^{j_d} \bar{z}_d^{k_d})^2\, \frac{e^{-|z|^2/t}}{(\pi t)^d}\, dz \\
&= \sum_{j_1+\cdots+j_d=m}\ \sum_{k_1+\cdots+k_d=n} M_\sigma(j_1, k_1) \cdots M_\sigma(j_d, k_d).
\end{aligned}
$$

Thus it follows from (i) that

$$\int_{\mathbb{C}} g_m \cdot \overline{g_n}\, d\lambda = 0, \qquad \text{if } m \neq n.$$

Hence, $\{g_n\}_{n=0}^{\infty}$ is an orthogonal subset of $\mathcal{H}L^2(\mathbb{C}, \lambda)^e$. Next, we will show that $\{g_n\}_{n=0}^{\infty}$ is an orthogonal basis of $\mathcal{H}L^2(\mathbb{C}, \lambda)^e$. Let $g \in \mathcal{H}(\mathbb{C})^e$. Then $g(x) = \sum_{n=0}^{\infty} a_n x^n$ for each $x \in \mathbb{C}$, where $a_n \in \mathbb{C}$ for all $n \in \mathbb{N} \cup \{0\}$. But $g(x) = g(-x)$, so $a_n = 0$ for each odd positive integer n. Thus

$$g(x) = \sum_{n=0}^{\infty} a_{2n}\, x^{2n}.$$

Since this power series converges uniformly on the set D_σ, we have

$$\begin{aligned}
&\int_{D_\sigma} |g(\sqrt{(z,z)})|^2 \, \mu_t(z)\, dz \\
&= \int_{D_\sigma} \sum_{n=0}^{\infty} a_{2n}(z,z)^n \sum_{m=0}^{\infty} \overline{a_{2m}(z,z)^m}\, \mu_t(z)\, dz \\
&= \int_{D_\sigma} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{2n}\overline{a_{2m}}(z,z)^n \overline{(z,z)^m}\, \mu_t(z)\, dz \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{2n}\overline{a_{2m}} \int_{D_\sigma} (z_1^2 + \cdots + z_d^2)^n\, (\bar{z}_1^2 + \cdots + \bar{z}_d^2)^m\, \mu_t(z)\, dz \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{2n}\overline{a_{2m}} \int_{D_\sigma} \sum_{j_1+\cdots+j_d=n} z_1^{2j_1} \cdots z_d^{2j_d} \sum_{k_1+\cdots+k_d=m} \bar{z}_1^{2k_1} \cdots \bar{z}_d^{2k_d}\, \frac{e^{-|z|^2/t}}{(\pi t)^d}\, dz \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{2n}\overline{a_{2m}} \sum_{j_1+\cdots+j_d=n} \sum_{k_1+\cdots+k_d=m} \int_{D_\sigma} (z_1^{j_1}\bar{z}_1^{k_1})^2 \cdots (z_d^{j_d}\bar{z}_d^{k_d})^2\, \frac{e^{-|z|^2/t}}{(\pi t)^d}\, dz \\
&= \sum_{n=0}^{\infty} |a_{2n}|^2 \sum_{k_1+\cdots+k_d=n} M_\sigma(k_1,k_1)\cdots M_\sigma(k_d,k_d).
\end{aligned}$$

Next, using the monotone convergence theorem, we have

$$\begin{aligned}
\int_{\mathbb{C}} |g|^2\, d\lambda &= \int_{\mathbb{C}^d} |g\circ\Phi(z)|^2\, \mu_t(z)\, dz \\
&= \lim_{\sigma\to\infty} \int_{D_\sigma} |g(\sqrt{(z,z)})|^2\, \mu_t(z)\, dz \\
&= \lim_{\sigma\to\infty} \sum_{n=0}^{\infty} |a_{2n}|^2 \sum_{k_1+\cdots+k_d=n} M_\sigma(k_1,k_1)\cdots M_\sigma(k_d,k_d) \\
&= \sum_{n=0}^{\infty} |a_{2n}|^2 \sum_{k_1+\cdots+k_d=n} (t^{2k_1}(2k_1)!)\cdots(t^{2k_d}(2k_d)!) \\
&= \sum_{n=0}^{\infty} \Big(|a_{2n}|^2\, t^{2n} \sum_{k_1+\cdots+k_d=n} (2k_1)!\cdots(2k_d)! \Big).
\end{aligned}$$

Therefore, if g is square-integrable with respect to λ, then

$$\sum_{n=0}^{\infty} \Big(|a_{2n}|^2\, t^{2n} \sum_{k_1+\cdots+k_d=n} (2k_1)!\cdots(2k_d)! \Big) = \int_{\mathbb{C}} |g|^2\, d\lambda < \infty.$$

Define a sequence (F_n) to be

$$F_n(x) := \sum_{m=0}^{n} a_{2m}\, x^{2m}$$

for any $x \in \mathbb{C}$. Then (F_n) is a Cauchy sequence in $L^2(\mathbb{C}, \lambda)$. To see this, notice that for $m > n$,

$$\begin{aligned}
\|F_m - F_n\|_{L^2(\mathbb{C}\lambda)} &= \int_{\mathbb{C}} |F_m - F_n|^2 \, d\lambda \\
&= \int_{\mathbb{C}^d} \left| \sum_{k=n+1}^{m} a_{2k}(z,z)^k \right|^2 \mu_t(z)\, dz \\
&= \sum_{k=n+1}^{m} |a_{2k}|^2\, t^{2k} \sum_{k_1+\cdots+k_d=k} (2k_1)! \cdots (2k_d)! \to 0 \qquad \text{as } m, n \to \infty.
\end{aligned}$$

Thus (F_n) converges in $L^2(\mathbb{C}, \lambda)$ to some function h, and hence it has a subsequence which converges pointwise almost everywhere to h . But (F_n) converges pointwise to g, so we have that $h = g$ a.e.$[\lambda]$. Therefore $h = g$ in $L^2(\mathbb{C}, \lambda)$, so (F_n) converges to g in $L^2(\mathbb{C}, \lambda)$. Thus $g \in \overline{\text{span}\{g_n \mid n \in \mathbb{N}\}}$. This show that $\{x^{2n}\}_{n=0}^{\infty}$ forms an orthogonal basis for $\mathcal{H}L^2(\mathbb{C}, \lambda)$. □

Corollary 6.2. *The set* $\{(z,z)^n\}_{n=0}^{\infty}$, *forms an orthogonal basis for* $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$.

Proof. This follows from Theorem 5.2 and Theorem 6.1. □

Proposition 6.3. *The reproducing kernel* K *of* $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$ *is*

$$K(z,z) = \frac{\Gamma(d/2)}{a_0 2^{d/2+1}} \,\text{BesselI}\left(\frac{d-2}{2}, \frac{|(z,z)|}{t}\right)\left(\frac{t}{|(z,z)|}\right)^{d/2-1}$$

for all $z \in \mathbb{C}^d$, *where* a_0 *is a constant in Lemma 4.7.*

Proof. From Proposition 5.3, the density function of the measure λ is

$$\begin{aligned}
\Lambda(w) &= \frac{|w|^{2d-2}}{(\pi t)^d} \int_{S_1} e^{-|wz|^2/t} d\alpha(z) \\
&= a_0 \frac{|w|^{2d-2}}{(\pi t)^d} \int_{S^{d-1}} \int_{\mathbf{x}\cdot\mathbf{p}=\mathbf{0}} e^{-|w\,\mathbf{a}(\mathbf{x},\mathbf{p})|^2/t} \left(\frac{\sinh 2p}{2p}\right)^{d-2} p^{d-2} 2^{d-1}\, d\mathbf{p}\, d\mathbf{x} \\
&= a_d \frac{|w|^{2d-2}}{(\pi t)^d} \int_0^{\infty} e^{-(\cosh x)|w|^2/t} (\sinh x)^{d-2}\, dx \\
&= a_d \frac{|w|^{2d-2}}{(\pi t)^d}\, \Gamma\Big(\frac{d-1}{2}\Big) \,\text{BesselK}\left(\frac{d-2}{2}, \frac{|w|^2}{t}\right)\left(\frac{t}{|w|^2}\right)^{(d-2)/2} \frac{2^{(d-2)/2}}{\sqrt{\pi}}
\end{aligned}$$

for all $w \in \mathbb{C}$. So for any $z \in \mathbb{C}^d$ and $n \geq 0$, we we have that

$$\begin{aligned}
\|(z,z)^n\|^2 &= \int_{\mathbb{C}^d} \|(z,z)^n\|^2 \mu_t(z)\, dz \\
&= \int_{\mathbb{C}} |w^{2n}|^2 \, d\lambda(w) \\
&= \frac{a_d 2^{(d-2)/2}}{\pi^{(d+1/2)}} \Gamma\Big(\frac{d-1}{2}\Big) \int_{\mathbb{C}} \frac{|w|^{2d-2+4n}}{t^d} K_{\frac{d-2}{2}}\Big(\frac{|w|^2}{t}\Big)\Big(\frac{t}{|w|^2}\Big)^{(d-2)/2} dw \\
&= \frac{a_d 2^{(d-2)/2}}{\pi^{(d-1/2)}} \Gamma(\frac{d-1}{2}) \int_0^\infty t^{2n} x^{2n+d/2} K_{\frac{d-2}{2}}(x)\, dx \\
&= \frac{a_d 2^{(d-2)/2}}{\pi^{(d-1/2)}} t^{2n} \Gamma\Big(\frac{d-1}{2}\Big) \Gamma(n+1)\Gamma(n+d/2) 2^{2n-1+d/2} \\
&= \frac{a_0}{\Gamma(d/2)} 2^{2n+d} t^{2n} \Gamma(n+1)\Gamma(n+d/2).
\end{aligned}$$

Since $\{(z,z)^n\}_0^\infty$ is an orthogonal basis for $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$, the reproducing kernel K of $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$ is

$$\begin{aligned}
K(z,z) &= \sum_{n=0}^{\infty} \frac{|(z,z)^n|^2}{\|(z,z)^n\|^2} \\
&= \frac{\Gamma(d/2)}{a_0} \sum_{n=0}^{\infty} \frac{|(z,z)|^{2n}}{t^{2n} 2^{2n+d} \Gamma(n+1)\Gamma(n+d/2)} \\
&= \frac{\Gamma(d/2)}{a_0 2^{d/2+1}} \operatorname{BesselI}\Big(\frac{d-2}{2}, \frac{|(z,z)|}{t}\Big)\Big(\frac{t}{|(z,z)|}\Big)^{d/2-1}
\end{aligned}$$

for all $z \in \mathbb{C}^d$. The last equality follows from the property number (2) of the modified Bessel functions in the previous section. □

However, $\operatorname{BesselI}(d/2-1, x) \sim \frac{e^x}{\sqrt{2\pi x}}$ asymptotically if x is large enough. Thus for any $z \in \mathbb{C}^d$ such that $|(z,z)|$ is large enough we have that

$$K(z,z) \sim \frac{t^{(d-1)/2}\Gamma(d/2)}{a_0\sqrt{\pi}\, 2^{(d+3)/2}} \frac{e^{|(z,z)|/t}}{|(z,z)|^{(d-1)/2}}.$$

This implies that for all $F \in \mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$

$$|F(z)|^2 \leq C \frac{e^{|(z,z)|/t}}{|(z,z)|^{(d-1)/2}} \|F\|^2_{L^2(C^2,\mu_t)}$$

for any $z \in \mathbb{C}^d$ such that $|(z,z)|$ is large enough, where $C > 0$ is a constant depending on d and t. Hence the bound in Theorem 1.1 is sharp.

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