ปัญหาการระเบิดในหลายมิติที่เกิดจากแหล่งกำเนิดแบบเข้มข้นไม่เชิงเส้นซึ่งมีลักษณะเฉพาะที่และไม่เฉพาะที่

นายอาทิตย์ ศิริรูป

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
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MULTI-DIMENSIONAL BLOW-UP PROBLEM DUE TO CONCENTRATED NONLINEAR SOURCE HAVING LOCAL AND NONLOCAL FEATURES

Mr. Artit Siriroop

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By	Mr. Artit Siriroop
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Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

. . . . Dean of the Faculty of Science

(Professor Supot Hannongbua, Dr.rer.nat.)

THESIS COMMITTEE

. . . . Chairman

(Associate Professor Anusorn Chonwerayuth, Ph.D.)

. . . . Thesis Advisor

(Ratinan Boonklurb, Ph.D.)

. . . . Examiner

(Sujin Khomrutai, Ph.D.)

. . . . External Examiner

(Assistant Professor Tawikan Treeyaprasert, Ph.D.)

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วิทยานิพนธ์ฉบับนี้ ศึกษาปัญหาการระเบิดในหลายมิติที่เกิดจากแหล่งกำเนิดแบบเข้มข้นไม่เชิงเส้นซึ่งมีลักษณะเฉพาะที่และไม่เฉพาะที่ จากการศึกษาพบว่าผลเฉลยของปัญหา u มีเพียงคำตอบเดียวก่อนที่การระเบิดจะเกิดขึ้น เงื่อนไขพร้อมทั้งให้สำหรับ u ที่จะทำให้เกิดการระเบิดในเวลาจำกัด ถ้า u เกิดการการระเบิดในเวลาจำกัดแล้ว u จะระเบิดทุกๆ จุดบริเวณขอบของทรงกลมซึ่งแหล่งกำเนิดแบบเข้มข้นไม่เชิงเส้นซึ่งมีลักษณะเฉพาะที่ซึ่งได้ถูกกำหนดไว้

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This thesis studies a multi-dimensional parabolic problem with a concentrated nonlinear source having local and nonlocal features. It is shown that there is a unique solution u before a blow-up occurs. A criterion for u to blow-up in a finite time is given. If u exists in a finite time only, then u blows up everywhere on the boundary of a ball where the concentrated local source is situated.

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CONTENTS

CHAPTER I

INTRODUCTION

Let $\varsigma = (\varsigma_1, \varsigma_2, \varsigma_3, ..., \varsigma_n)$ be a point in $n-$dimensional Euclidean space in $\mathbb{R}^n$, $\hat{T}(> 0)$ and $m(> 1)$ be any real numbers, $\hat{D}$ be a bounded domain in $\mathbb{R}^n, \partial\hat{D}$ be a boundary of $\hat{D}, \hat{B}$ be an $n-$dimensional ball, $\{\varsigma \in \mathbb{R}^n : |\varsigma - \hat{b}| < \hat{R}\}$, centered at a given point $\hat{b}$ with radius $\hat{R}, \bar{\hat{B}} \subset \hat{D}, \partial\hat{B}$ be the boundary of $\hat{B}, v(\varsigma)$ denote the unit inward normal vector at $\varsigma \in \partial\hat{B}$, and

$$\chi_{\hat{B}}(\varsigma) = \begin{cases} 1 \text{ for } \varsigma \in \hat{B}, \\ 0 \text{ for } \varsigma \in \hat{D}\backslash\hat{B}, \end{cases}$$

be the characteristic function. Without loss of generality, let $\hat{b}$ be the origin. We would like to study the following multi-dimensional nonlinear parabolic problem.

$$u_\gamma - \Delta_\varsigma u = \frac{\partial\chi_{\hat{B}}(\varsigma)}{\partial v} F(u(\varsigma, \gamma))Z^m(\gamma) \text{ in } \hat{D} \times (0, \hat{T}],$$

where $u(\varsigma, 0) = \psi(\varsigma)$ on $\bar{\hat{D}}$ and $u(\varsigma, \gamma) = 0$ for $\varsigma \in \partial\hat{D}, 0 < \gamma \leq \hat{T}$ are initial and boundary condition, respectively. We note that $\Delta_\varsigma = \sum_{i=1}^{n} \partial^2/\partial\varsigma_i^2$, and $Z(\gamma) = \int_{\hat{D}} S(u(\varsigma, \gamma))d\varsigma$ and F and S are given functions. This problem has a nonlinear source which is a product of a local contribution $(\partial\chi_B(\varsigma)/\partial v)F(u(\varsigma, \gamma))$ and a global contribution $Z^m(\gamma)$.

In order to study the behavior of the solution over a unit domain, we consider the domain having the same shape [2]. If the shape is given, then a domain can be uniquely determined by its size. For example, in one- dimensional space, all intervals $[a, b]$ have the same shape. In two-dimensional space, all circular domains have the same shape. Let D be a bounded $n-$dimensional domain having the same shape as $\hat{D}$. Then, there is $x_0 \in \hat{D} \cap D$ and a positive constant

A such that

$$\hat{D} = \{\varsigma : \varsigma = x_0 + A(x - x_0) \text{ for } x \in D\}.$$

Let the size $|D| = \int_D dx = 1$. Without lost of generality, we can let x_0 be the origin. We note that from the above transformation,

$$A = \left(\frac{\text{size of } \hat{D}}{\text{size of } D}\right)^{\frac{1}{n}} = \left(\frac{\int_{\hat{D}} d\varsigma}{\int_D dx}\right)^{\frac{1}{n}} = \left(\frac{|\hat{D}|}{|D|}\right)^{\frac{1}{n}} = |\hat{D}|^{\frac{1}{n}}.$$

Let $\gamma = A^2 t$ and $\varsigma = Ax$. Then,

$$\frac{\partial u}{\partial \gamma} = \frac{\partial u}{\partial t}\frac{\partial t}{\partial \gamma} = \frac{1}{A^2}\frac{\partial u}{\partial t} \text{ and } \frac{\partial^2 u}{\partial \varsigma_i^2} = \frac{1}{A^2}\frac{\partial^2 u}{\partial x_i^2}.$$

Let $B = \{x \in \mathbb{R}^n : |x| < R\}$ with ∂B be its boundary and $R = \hat{R}/A$, and ν denote the inward normal at $x \in \partial B$. Let $\varphi(x)$ be the infinitely differentiable function with compact support and $\delta(x)$ denote the usual Dirac delta function.

We can rewrite $\partial\chi_{\hat{B}}(\varsigma)/\partial v$ by using the spherical coordinates [12] as follows

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \frac{\partial \chi_{\hat{B}}(Ax)}{\partial v}\varphi(x)dx \\
&\quad = \int_0^\infty \int_{-\pi}^{\pi} \int_0^{\pi} ... \int_0^{\pi} \delta(Ar-\hat{R})\varphi(r,\omega_1,...,\omega_{n-1})r^{n-1} \\
&\quad \times \prod_{i=1}^{n-2} \sin^{n-1-i}\omega_i d\omega_1...d\omega_{n-1}dr \\
&\quad = \int_0^\infty \int_{-\pi}^{\pi} \int_0^{\pi} ... \int_0^{\pi} \delta(\sigma-\hat{R})\varphi\left(\frac{\sigma}{A},\omega_1,...,\omega_{n-1}\right)\left(\frac{\sigma}{A}\right)^{n-1} \\
&\quad \times \prod_{i=1}^{n-2} \sin^{n-1-i}\omega_i d\omega_1...d\omega_{n-1}\frac{1}{A}d\sigma \\
&\quad = \frac{1}{A}\int_{-\pi}^{\pi} \int_0^{\pi} ... \int_0^{\pi} \varphi\left(\frac{\hat{R}}{A},\omega_1,...,\omega_{n-1}\right)\left(\frac{\hat{R}}{A}\right)^{n-1} \\
&\quad \times \prod_{i=1}^{n-2} \sin^{n-1-i}\omega_i d\omega_1...d\omega_{n-1} \\
&\quad = \frac{1}{A}\int_0^\infty \int_{-\pi}^{\pi} \int_0^{\pi} ... \int_0^{\pi} \delta\left(r-\frac{\hat{R}}{A}\right)\varphi\left(\frac{\hat{R}}{A},\omega_1,...,\omega_{n-1}\right)\left(\frac{\hat{R}}{A}\right)^{n-1} \\
&\quad \times \prod_{i=1}^{n-2} \sin^{n-1-i}\omega_i d\omega_1...d\omega_{n-1}dr \\
&\quad = \frac{1}{A}\int_{\mathbb{R}^n} \frac{\partial \chi_B(x)}{\partial \nu}\varphi(x)dx.
\end{aligned}
$$

Hence,

$$
\frac{\partial \chi_{\hat{B}}(Ax)}{\partial v} = \frac{1}{A}\frac{\partial \chi_B(x)}{\partial \nu}
$$

and

$$
Z(\gamma) = \int_{\hat{D}} S(u(\varsigma,\gamma))d\varsigma = A^n \int_D s(u(x,t))dx,
$$

where $S(u(\varsigma,\gamma)) = s(u(x,t))$. Let $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2, H = \partial/\partial t - \Delta$, $F(u(\varsigma,\gamma)) = f(u(x,t)), U(t) = \int_D s(u(x,t))dx, T = \hat{T}/A^2, \partial D$ be the boundary of $D, \bar{D}$ be the closure of D, and $\Omega = D \times (0,T]$. Then, the original problem becomes

$$\left.\begin{aligned} Hu = |\hat{D}|^{m+\frac{1}{n}}\frac{\partial \chi_B(x)}{\partial \nu} f(u(x,t))U^m(t) \text{ in } \Omega, \\ u(x,0) = \psi(x) \text{ on } \bar{D}, u(x,t) = 0 \text{ for } x \in \partial D, 0 < t \le T. \end{aligned}\right\} \tag{1.1}$$

We assume that $f(0)$, $f'(0)$, $s(0)$ and $s'(0)$ are nonnegative while $f(u)$, $f'(u)$, $f''(u)$, $s(u)$, $s'(u)$ and $s''(u)$ are positive for $u > 0$. We further assume for construction later that

$$\Delta\psi(x) + |\hat{D}|^{m+\frac{1}{n}}\frac{\partial \chi_B(x)}{\partial \nu} f(\psi(x)) \left(\int_D s(\psi(x))dx\right)^m \ge 0. \tag{1.2}$$

The inequality (1.2) intuitively means that at the beginning, a temperature will rise up.

This model describes a temperature u due to a nonlinear source having local and nonlocal features subject to the initial condition $\psi(x)$ and zero temperature on the lateral boundary. Instead of studying $u(b,t)$ for any point $b \in B$, we would like to investigate a solution $u(x,t)$ of (1.1) for $(x,t) \in \Omega$. Since the term $\partial\chi_B(x)/\partial\nu$ intuitively means that we have Dirac delta functions on each point of ∂B, a solution of (1.1) that are we looking for is in distributional sense and at most a continuous function on $\bar{\Omega}$ satisfies (1.1).

A solution u of (1.1) is said to blow up at a point (x, t_b) if there exists a sequence $\{u(x_n, t_n)\} \to \infty$ as $(x_n, t_n) \to (x, t_b)$.

We also assume that Ω has the property that for any point $P \in \partial D \times (0,T]$, there exists an $(n+1)-$ dimensional neighborhood Σ such that $\Sigma \cap \partial D \times (0,T]$ can be represented, for some $i = 1, 2, ..., n$ in the form,

$$x_i = \beta(x_1, x_2, ...x_{i-1}, x_{i+1}, ...x_n, t),$$

where $\beta, D_x\beta$ and $D_1^2\beta$ are Hölder continuous of exponent $\alpha \in (0,1)$ while $D_xD_t\beta$ and $D_t^2\beta$ are continuous.

This type of problem in $1-$dimension was studied by Chan and Tian [6]. They proved that there is a unique continuous solution before the blow-up occurs and also giving a blow-up criterion. For $n-$dimensional problem, Chan and Tian [5] studied a blow-up problem with nonlinear source of the form

$$\frac{\partial \chi_B(x)}{\partial \nu} f(u(x,t)).$$

They showed that the problem has a unique solution before the blow-up occurs on the boundary of a ball.

Quenching and blow-up are closely related. As an illustration, and explosion described by a quenching model occurs at a finite temperature [4] while if it is described by a blow-up model, it occurs at an infinitely high temperature. Boonklurb [2] studied an $n-$dimension quenching problem with having local and nonlocal features,

$$\frac{\partial \chi_B(x)}{\partial \nu} f(u(x,t)) \left(\int_D s(u(x,t))dx \right)^m.$$

He showed that there is a unique solution before the quenching occurs and gave quenching criterion. We modify and extend their study to $n-$dimensional blow-up problem with having the same local and nonlocal features.

In chapter 2, some of preliminary theorems are listed for ease of reference.

In chapter 3, we show that the integral equation derived from (1.1) has a unique continuous solution $u,$ which is nondecreasing function of $t.$ Then, we prove that u is a unique solution of problem (1.1).

In chapter 4, we give sufficient conditions for blow-up to occur and the blow-up set of (1.1).

CHAPTER II

PRELIMINARIES

The followings are some theorems that we frequently use thoughout this thesis.

Theorem 2.1. *(Leibniz's rule; [1], p. 201) Let $f : \mathbb{R} \times X \to \mathbb{R}$ and X be a closed interval in $\mathbb{R}$ satisfying*

(i) The function $(x,t) \to f(x,t)$ is continuous on $\mathbb{R} \times X$.

(ii) The partial derivative $(x,t) \to f_t(x,t)$ exists and is continuous on $\mathbb{R} \times X$.

(iii) The function $u, v : X \to \mathbb{R}$ are differentiable on X.

Then, the function $G : X \to \mathbb{R}$, defined by

$$G(t) = \int_{u(t)}^{v(t)} f(x,t)dx,$$

exists and is differentiable on X. Moreover, its derivative is given by

$$G'(t) = \int_{u(t)}^{v(t)} f_t(x,t)dx - f(u(t),t)u'(t) + f(v(t),t)v'(t).$$

Theorem 2.2. *(Strong maximum principle; [8], p. 34) Let H be a heat operator, $\partial/\partial t - \Delta$ and if u has a positive maximum in D which is attained at a point (x_0, t_0) then $u(x,t) = u(x_0,t_0)$ for all a point (x,t) that path connected to (x_0,t_0).*

Theorem 2.3. *(Second Mean Value Theorem; [1], pp. 193-195) If f is an integrable function and g is monotone on $I := [a,b]$, then there exist $\xi \in I$ such that*

$$\int_a^b fg = g(a)\int_a^{\xi} f + g(b)\int_{\xi}^b f.$$

Theorem 2.4. *(Divergence theorem; [7], p. 627) Suppose that $u \in C^1(\bar{U})$. Then*

$$\int_U u_{x_i} dx = \int_{\partial U} u\nu_i dS, \qquad (i \in \{1,2,3,...,n\})$$

where $\nu = (\nu_1, ..., \nu_n)$ is outward pointing unit normal vector field.

Theorem 2.5. *(Green's Formulas; [7], p. 628) Let $u, v \in C^2(\bar{U})$. Then*

$$(i) \int_U \Delta u dx = \int_{\partial U} \frac{\partial u}{\partial \nu} dS,$$
$$(ii) \int_U Dv \cdot Du dx = -\int_U u \Delta v dx + \int_U \frac{\partial u}{\partial \nu} u dS,$$
$$(iii) \int_U u\Delta v - v\Delta u dx = \int_{\partial U} u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} dS,$$

where $Du = (u_{x_1}, ..., u_{x_n})$ and Δ is the Laplace operator.

Theorem 2.6. *(Monotone convergence theorem; [10], p. 21) Assume that the functions $\{f_k\}_{k=1}^{\infty}$ are measurable, with*

$$f_1 \leq f_2 \leq ... \leq f_k \leq f_{k+1} \leq ....$$

Then

$$\int_{\mathbb{R}^n} \lim_{k\to\infty} f_k dx = \lim_{k\to} \int_{\mathbb{R}^n} f_k dx.$$

Theorem 2.7. *(Schwarz's inequality; [10], p. 63) Let p and q be positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Let X be a measurable space with measure μ. Let f and g be measurable functions on X, with range in $[0, \infty)$. Then*

$$\int_X fg dx \leq \left\{ \int_X f^p d\mu \right\}^{\frac{1}{p}} \left\{ \int_X g^q d\mu \right\}^{\frac{1}{q}} \tag{2.1}$$

and

$$\int_X (f+g) dx \leq \left\{ \int_X f^p d\mu \right\}^{\frac{1}{p}} + \left\{ \int_X g^q d\mu \right\}^{\frac{1}{q}}. \tag{2.2}$$

The inequality (2.1) is Hölder's; (2.2) is Minkowski's. If $p = q = 2$, (2.1) is known as the Schwarz inequality.

Theorem 2.8. *(Fubini's theorem; [9], p. 416) Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be two $\sigma-$finite measurable spaces and f is nonnegative $(\mu \times \nu)-$measurable function on $X \times$*

Y. If $0 \leq f \leq \infty$, and if

$$\phi(x) = \int_Y f_x d\nu, \psi(y) = \int_X f^y d\mu, \qquad (x \in X, y \in Y)$$

then ϕ is $\mu-$ measurable, ψ is $\nu-$ measurable, and

$$\int_X \psi d\mu = \int_{X \times Y} f d(\mu \times \nu) = \int_Y \psi d\nu,$$

where $f_x = \int_X f d\mu$ and $f^y = \int_Y f d\nu$.

CHAPTER III
EXISTENCE AND UNIQUENESS

In this chapter, we establish the existence and uniqueness result locally for the integral equation corresponding to (1.1). The solution of this integral equation is a continuous function and an increasing function of t. Next, we will prove that the solution of this integral equation is a unique solution of (1.1).

Green's function ([11], p. 198) $G(x,t;\xi,\tau)$ corresponding to (1.1) is determined by

$$HG(x,t;\xi,\tau) = \delta(x-\xi)\delta(t-\tau) \text{ for } x \text{ and } \xi \text{ in } D, \text{ and } t \text{ and } \tau \text{ in } (-\infty,\infty),$$
$$G(x,t;\xi,\tau) = 0 \text{ for } x \text{ and } \xi \text{ in } D, \text{ and } t < \tau,$$
$$G(x,t;\xi,\tau) = 0 \text{ for } x \in \partial D \text{ and } \xi \in D, \text{ and } t \text{ and } \tau \text{ in } (-\infty,\infty).$$

As we need this Green's function to have some useful properties ([8], pp. 82-83), we let Ω have the property that for every point $P \in \bar{S}$, there exists an $(n+1)-$dimensional neighborhood such that $N \cap \bar{S}$ can be represented in the form $x_i = z(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n, t)$, for some $i \in \{1, 2, 3, ..., n\}$. Here, $z, D_x z, D_x^2 z$ and $D_t z$ are Hölder's continuous of exponent $\alpha \in (0,1)$, while $D_x D_t z$ and $D_t^2 z$ are continuous. For ease of reference, let us state those useful properties in Lemma 3.1 as follow:

Lemma 3.1. *([8], pp. 82-83) (i) There exists a unique $G(x,t;\xi,\tau)$ that continuous in $\bar{\Omega} \times (D \times [0,T)), t > \tau$. Furthermore, $D_x G, D_x^2 G$ and $D_t G$ are continuous functions of $(x,t;\xi,\tau)$ in $\Omega \times (D \times [0,T)), t > \tau$.*

(ii) For each $(\xi,\tau) \in D \times [0,T), G(x,t;\xi,\tau) = 0$ on $\partial D \times (\tau, T]$ and $G(x,t;\xi,\tau)$ is positive in $D \times (\tau, T]$.

(iii) For any fixed $(\xi,\tau) \in D \times [0,T)$ and any $\epsilon > 0$, $D_x G, D_x^2 G$ and $D_t G$ are uniformly continuous functions of $(x,t) \in \Omega$ with $t - \tau > \epsilon$.

Consider the eigenvalue problem of the form,

$$\Delta\phi + \lambda\phi = 0 \text{ in } D, \qquad \phi = 0 \text{ on } \partial D.$$

This problem has a sequence of positive eigenvalues, $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ...$, which are listed in an increasing order with due regard to multiplicity. Their corresponding eigenfunctions, $\phi_i, i \in \mathbb{N}$ can be chosen to form a complete orthonormal set with the fundamental eigenfunction $\phi_1 > 0$ in D. Thus, we have ([11], pp. 213-214)

$$G(x,t;\xi,\tau) = \sum_{i=1}^{\infty} \phi_i(x)\bar{\phi}_i(\xi)e^{-\lambda_i(t-\tau)},$$

where $\bar{\phi}_i$ denoted the complex conjugate of ϕ_i.

Let us consider the adjoint operator of H, which is given by $H^*u = -u_t - \Delta u$. Using Theorem 2.4(iii), the Green's Formulas, we obtain the integral equation from (1.1) by first consider the equation

$$\begin{aligned}
&\int_0^T \int_D (G^*(x,t;\xi,\tau)Hu(x,t) - u(x,t)H^*G^*(x,t;\xi,\tau))dxdt \\
&\qquad = \int_0^T \int_D \left(u(x,t)\frac{\partial}{\partial t}G^*(x,t;\xi,\tau) + G^*(x,t;\xi,\tau)\frac{\partial}{\partial t}u(x,t) \right) dx \\
&\qquad + \int_{\partial D\times(0,T]} \left(G^*(x,t;\xi,\tau)\frac{\partial}{\partial \nu}u(x,t) - u(x,t)\frac{\partial}{\partial \nu}G^*(x,t;\xi,\tau) \right) dS_x,
\end{aligned}$$

where $G^*(x,t;\xi,\tau) = G(\xi,\tau;x,t)$. Using the fact that $H^*G^*(x,t;\xi,\tau) = \delta(x-\xi)\delta(t-\tau)$, we obtain

$$\begin{aligned}
u(\xi,\tau) = &\int_0^T \int_D G^*(x,t;\xi,\tau)Hu(x,t)dxdt \\
&- \int_0^T \int_D \left(u(x,t)\frac{\partial}{\partial t}G^*(x,t;\xi,\tau) + G^*(x,t;\xi,\tau)\frac{\partial}{\partial t}u(x,t) \right) dx \\
&- \int_{\partial D\times(0,T]} \left(G^*(x,t;\xi,\tau)\frac{\partial}{\partial \nu}u(x,t) - u(x,t)\frac{\partial}{\partial \nu}G^*(x,t;\xi,\tau) \right) dS_x.
\end{aligned}$$

Changing the roles between ξ and x and between τ and t, the above equation becomes

$$
\begin{aligned}
u(x,t) = &\int_0^T \int_D G(x,t;\xi,\tau) H u(\xi,\tau) dx dt \\
&- \int_0^T \int_D \left(u(x,t) \frac{\partial}{\partial t} G(x,t;\xi,\tau) + G(x,t;\xi,\tau) \frac{\partial}{\partial t} u(x,t) \right) dx \\
&- \int_{\partial D \times (0,T]} \left(G(x,t;\xi,\tau) \frac{\partial}{\partial \nu} u(x,t) - u(x,t) \frac{\partial}{\partial \nu} G(x,t;\xi,\tau) \right) dS_x.
\end{aligned}
$$

Since G and u are zero on the boundary of $D \times (0,T]$, we obtain

$$
\begin{aligned}
u(x,t) = &\int_0^T \int_D G(x,t;\xi,\tau) H u(\xi,\tau) d\xi d\tau \\
&- \int_0^T \int_D \left(u(x,t) \frac{\partial}{\partial t} G(x,t;\xi,\tau) + G(x,t;\xi,\tau) \frac{\partial}{\partial t} u(x,t) \right) d\xi d\tau.
\end{aligned}
$$

Using the fundamental theorem of calculus and the fact that $G(x,t;\xi,\tau) = 0$, for all $\tau > t$, we have

$$
\begin{aligned}
u(x,t) &= \int_0^T \int_D G(x,t;\xi,\tau) H u(\xi,\tau) d\xi d\tau \\
&\quad - \int_D G(x,t;\xi,T) u(\xi,T) d\xi + \int_D G(x,t;\xi,0) u(\xi,0) d\xi \\
&= \int_0^T \int_D G(x,t;\xi,\tau) H u(\xi,\tau) d\xi d\tau + \int_D G(x,t;\xi,0) \psi(\xi) d\xi.
\end{aligned}
$$

Now, taking $T \to t$, we have

$$
\begin{aligned}
u(x,t) = &\int_D G(x,t;\xi,0) \psi(\xi) d\xi \\
&+ |\hat{D}|^{m+\frac{1}{n}} \int_0^t \int_D G(x,t;\xi,\tau) \frac{\partial \chi_B(\xi)}{\partial \nu} f(u(\xi,\tau)) U^m(\tau) d\xi d\tau.
\end{aligned}
$$

Using integration by parts and Theorem 2.3, the divergence theorem, we have

$$\begin{aligned}
&\int_D G(x,t;\xi,\tau)\frac{\partial \chi_B(\xi)}{\partial \nu} f(u(\xi,\tau))U^m(\tau)d\xi \\
&= \int_D G(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)[\nu(\xi)\cdot\nabla\chi_B(\xi)]d\xi \\
&= -\int_D \chi_B(\xi)\sum_{i=1}^n \frac{\partial}{\partial \xi_i}(G(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)\nu_i(\xi))d\xi \\
&= -\int_B \nabla\cdot(G(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)\nu(\xi))d\xi \\
&= \int_{\partial B} G(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi.
\end{aligned}$$

Finally, the integral representation of (1.1) becomes

$$u(x,t) = \int_D G(x,t;\xi,0)\psi(\xi)d\xi + |\hat{D}|^{m+\frac{1}{n}}\int_0^t\int_{\partial B} G(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau. \tag{3.1}$$

Lemma 3.2. [5] *On* $\bar{\Omega}$, $\int_0^t\int_{\partial B} G(x,t;\xi,\tau)d\xi d\tau$ *is continuous.*

Proof. Since

$$G(x,t;\xi,\tau) \leq \frac{(t-\tau)^{-n/2}}{2^n\pi^{n/2}}e^{|x-\xi|^2/(4(t-\tau))},$$

and $\int_{-\infty}^{\infty} e^{-\xi_i^2}d\xi_i = \pi^{1/2}$, it follows that

$$\int_{\partial B} G(x,t;\xi,\tau)d\xi \leq \frac{1}{2\pi^{1/2}(t-\tau)^{1/2}}.$$

Let w_n be the surface area of an $n-$dimensional unit sphere. Since Green's function is continuous, given $x, x_0 \in \mathbb{R}^n$ and $t, t_0 \geq 0$ for $t, t_0 \geq \tau$. Given $\epsilon > 0$. We can choose $\delta > 0$ such that whenever $|(x,t)-(x_0,t_0)| < \delta$, we have

$$|G(x,t;\xi,\tau) - G(x_0,t_0;\xi,\tau)| < \frac{\epsilon}{2w_N R^{n-1}t_0}.$$

If we choose $\tilde{\delta} = \min\{\delta, \pi\epsilon^2/4\}$, then

$$
\begin{aligned}
&\left| \int_0^t \int_{\partial B} G(x,t;\xi,\tau) d\xi d\tau - \int_0^{t_0} \int_{\partial B} G(x_0,t_0;\xi,\tau) d\xi d\tau \right| \\
&= \left| \int_{t_0}^t \int_{\partial B} G(x,t;\xi,\tau) d\xi d\tau + \int_0^t \int_{\partial B} G(x,t;\xi,\tau) - G(x_0,t_0;\xi,\tau) d\xi d\tau \right| \\
&\leq \left| \int_{t_0}^t \int_{\partial B} G(x,t;\xi,\tau) d\xi d\tau \right| + \left| \int_0^{t_0} \int_{\partial B} G(x,t;\xi,\tau) - G(x_0,t_0;\xi,\tau) d\xi d\tau \right| \\
&\leq \frac{2\sqrt{t-t_0}}{2\sqrt{\pi}} + \frac{\epsilon w_n R^{n-1} t_0}{2 w_n R^{n-1} t_0} \\
&\leq \frac{2\sqrt{\delta}}{2\sqrt{\pi}} + \frac{\epsilon}{2} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}
$$

This completes the proof. □

We use the technique of proving theorem 3 of Chan and Tian [5] to prove the theorems 3.3 and 3.4.

Theorem 3.3. *There exists some t_b such that for $0 \leq t < t_b$, then the integral equation (3.1) has a unique continuous solution u such that $u \geq \psi(x)$, and u is a nondecreasing function of t. If t_b is finite, then u is unbound in $[0, t_b)$.*

Proof. First, construct a sequence $\{u_i\}$ in $\Omega = D \times (0, T]$ by $u_0 \equiv \psi(x)$, and for $i \in \{0, 1, 2, ...\}$,

$$
\begin{aligned}
&H u_{i+1} = |\hat{D}|^{m+\frac{1}{n}} \frac{\partial \chi_B(x)}{\partial \nu} f(u_i(x,t)) U_i^m(t) \text{ in } \Omega, \\
&u_{i+1}(x,0) = \psi(x) \text{ on } \bar{D}, u_{i+1}(x,t) = 0 \text{ on } \partial D \times (0,T].
\end{aligned}
$$

From (3.1),

$$
\begin{aligned}
u_{i+1}(x,t) = &\int_D G(x,t;\xi,0) \psi(\xi) d\xi \\
&+ |\hat{D}|^{m+\frac{1}{n}} \int_0^t \int_{\partial B} G(x,t;\xi,\tau) f(u_i(\xi,\tau)) U_i^m(\tau) d\xi d\tau. \qquad (3.2)
\end{aligned}
$$

Let $\partial\Omega$ denote the parabolic boundary $(\bar{D} \times \{0\}) \cup (\partial D \times (0,T])$ of Ω. The condition (1.2) implies

$$\begin{aligned}
H(u_1 - u_0) =& Hu_1 - Hu_0 \\
=& Hu_1 - H\psi(x) \\
=& Hu_1 + \Delta\psi(x) \\
\geq& |\hat{D}|^{m+\frac{1}{n}} \frac{\partial \chi_B(x)}{\partial \nu} f(u_0(x,t)) U_0^m(t) - |\hat{D}|^{m+\frac{1}{n}} f(\psi(x)) \Big(\int_D s(\psi(x)) dx \Big)^m \\
=& |\hat{D}|^{m+\frac{1}{n}} \frac{\partial \chi_B(x)}{\partial \nu} \Big(f(u_0(x,t) U_0^m(t) - f(\psi(x)) \Big(\int_D s(\psi(x)) dx \Big)^m \Big) \\
=& |\hat{D}|^{m+\frac{1}{n}} \frac{\partial \chi_B(x)}{\partial \nu} \Big(f(\psi(x)) \Big(\int_D s(\psi(x)) dx \Big)^m \\
& - f(\psi(x)) \Big(\int_D s(\psi(x)) dx \Big)^m \Big) \\
=& 0
\end{aligned}$$

and $u_1 - u_0 = 0$ on $\partial\Omega$.

By Lemma 3.1(ii), $G(x,t;\xi,\tau)$ is positive in $\{(x,t;\xi,\tau) : x \text{ and } \xi \text{ are in } D, T \geq t > \tau \geq 0\}$. From (3.1), we consider the representation of the above parabolic problem, we obtain that

$$u_1 - u_0 = \int_D G(x,t;\xi,\tau) h(x,t) d\xi d\tau,$$

for some $h(x,t) : \Omega \to [0,\infty)$. Thus, we can conclude that $u_1(x,t) \geq u_0(x)$ in Ω. Assume that for some positive j,

$$\psi \leq u_1 \leq ... \leq u_{j-1} \leq u_j \text{ in } \Omega.$$

Since f and s are increasing functions, and $u_j \geq u_{j-1}$, we have

$$H(u_{j+1} - u_j) = |\hat{D}|^{m+\frac{1}{n}} \frac{\partial \chi_B(x)}{\partial \nu} \Big(f(u_j) U_j^m(t) - f(u_{j-1}) U_{j-1}^m(t) \Big) \geq 0 \text{ in } \Omega,$$

$$u_{j+1} - u_j = 0 \text{ on } \partial\Omega.$$

From (3.1), we have

$$u_{j+1} - u_j = |\hat{D}|^{m+1/n} \int_0^t \int_{\partial B} G(x,t;\xi,\tau) \times \left(f(u_j(x,t))U_j^m(t) - f(u_{j-1}(x,t))U_{j-1}^m(t) \right) d\xi d\tau,$$

which is nonnegative. So, we have $u_{j+1} \geq u_j$. By the principle of mathematical induction,

$$\psi \leq u_1 \leq u_2 \leq ... \leq u_{i-1} \leq u_i \text{ on } \bar{\Omega}, \tag{3.3}$$

for all positive integer i. Next, we show that each u_i is a nondecreasing function of t. Let us consider a sequence $\{w_i\}$ such that for $i \in \{0,1,2,...\}, w_i(x,t) = u_i(x,t+h) - u_i(x,t)$, where $0 < h < T$. Then, $w_0(x,t) = 0$ and

$$Hw_1 = 0 \text{ in } D \times (0, T-h].$$

By construction and (3.3),

$$w_1(x,0) \geq 0 \text{ on } \bar{D}, w_1(x,t) = 0 \text{ on } \partial D \times (0,T].$$

By (3.1), $w_1 \geq 0$ for $0 < t \leq T-h$. Next, assume that for some positive integer $j, w_j \geq 0$ for $0 < t \leq T-h$. Then, in $D \times (0, T-h]$,

$$\begin{aligned}
Hw_{j+1} = & |\hat{D}|^{m+\frac{1}{n}} \frac{\partial \chi_B(x)}{\partial \nu} \left(f(u_j(x,t+h))U_j^m(t+h) - f(u_j(x,t))U_j^m(t) \right) \\
= & |\hat{D}|^{m+\frac{1}{n}} \frac{\partial \chi_B(x)}{\partial \nu} \Big(f(u_j(x,t+h)) - f(u_j(x,t))U_j^m(t+h) \\
& + f(u_j(x,t))(U_j^m(t+h) - U_j^m(t)) \Big) \\
= & |\hat{D}|^{m+\frac{1}{n}} \frac{\partial \chi_B(x)}{\partial \nu} \Big(f'(u_j(x,t_1))U_j^m(t+h)w_j(x,t) \\
& + mf(u_j(x,t))U_j^{m-1}(t_2) \int_D (s(u_j(x,t+h)) - s(u_j(x,t))dx \Big)
\end{aligned}$$

for some t_1 and t_2 in $(t, t+h)$. Hence, $Hw_{j+1} \geq 0$ in $D \times (0, T-h]$. Since $w_{j+1} \geq 0$ on $\bar{D}$, and $w_{j+1}(x,t) = 0$ on $\partial D \times (0,T]$ for all $0 < t \leq T-h$. It follows from (3.1) that

$$w_{j+1}(x,t) = \int_D G(x,t;\xi,0)w_{j+1}(\xi,0)d\xi$$
$$+ \int_0^t \int_{\partial B} G(x,t;\xi,\tau)\Big(f'(u_j(\xi,\tau_1))U_j^m(\tau+h)w_j(\xi,\tau)$$
$$+ mf(u_j(\xi,\tau))U_j^{m-1}(\tau_2)\int_D (s(u_j(\xi,\tau+h)) - s(u_j(\xi,\tau)))d\xi\Big)d\xi d\tau,$$

for some τ_1 and τ_2 in $(\tau, \tau+h)$. This give us, $w_{j+1} \geq 0$ for all $0 < t < T-h$. By the principle of mathematical induction, $w_i \geq 0$ for $0 < t \leq T-h$ for all positive integer i. Thus, each u_i is a nondecreasing function of t.

Now, consider the problem

$$Hv = 0 \text{ in } \Omega,$$
$$v(x,0) = \psi(x) \text{ on } \bar{D}, v = 0 \text{ on } \partial D \times (0,T].$$

By Theorem 2.2, the strong maximum principle, $v > 0$ in Ω, and v attains its maximum $\max_{x\in\bar{D}} \psi(x)(:= k_0)$ somewhere in $D \times \{0\}$. Since

$$v(x,t) = \int_D G(x,t;\xi,0)\psi(\xi)d\xi,$$

it follows that the first term in (3.1) has a maximum value in $D \times \{0\}$. Thus, for any given constant $M > k_0$, it follows from (3.2) and u_i being a nondecreasing function of t that there exists some t_1 such that $u_i \leq M$ for $0 \leq t \leq t_1$ and $i \in \{0, 1, 2, ...\}$. This t_1 satisfies

$$\int_D G(x,t;\xi,0)\psi(\xi)d\xi + |\hat{D}|^{m+\frac{1}{n}} f(M)s^m(M) \int_0^{t_1} \int_{\partial B} G(x,t;\xi,\tau)d\xi d\tau \leq M.$$

Let u denote $\lim_{i\to\infty} u_i$. From (3.2) and the monotone convergence theorem ([9], p. 83), we have (3.1) for $0 \leq t \leq t_1$. To prove uniqueness, assume that the integral equation (3.1) has two distinct solutions u and $\tilde{u}$ on the interval $[0, t_1]$.

Let $\Theta = \sup_{\bar{D}\times[0,t_1]} |u-\tilde{u}|$. From (3.1),

$$\begin{aligned} u(x,t)-\tilde{u}(x,t) =& |\hat{D}|^{m+\frac{1}{n}} \int_0^t \int_{\partial B} G(x,t;\xi,\tau) \\ & \times [f(u(\xi,\tau))U^m(\tau) - f(\tilde{u}(\xi,\tau))\tilde{U}^m(\tau)] d\xi d\tau. \end{aligned}$$

Since f, f', s and s' are increasing functions, it follows from the mean value theorem that

$$|f(u(\xi,\tau))U^m(\tau) - f(\tilde{u}(\xi,\tau))\tilde{U}^m(\tau)| \leq (f(M)ms^{m-1}(M)s'(M) + f'(M)s^m(M))\Theta.$$

Then,

$$\begin{aligned} |u(x,t)-\tilde{u}(x,t)| =& |\hat{D}|^{m+\frac{1}{n}} \int_0^t \int_{\partial B} G(x,t;\xi,\tau) \\ & \times [f(u(\xi,\tau))U^m(\tau) - f(\tilde{u}(x,t))\tilde{U}^m(\tau)] d\xi d\tau \\ \leq & |\hat{D}|^{m+\frac{1}{n}} (f(M)ms^{m-1}(M)s'(M) + f'(M)s^m(M))\Theta \\ & \times \int_0^t \int_{\partial B} G(x,t;\xi,\tau) d\xi d\tau. \end{aligned}$$

Thus,

$$\begin{aligned} \Theta \leq & |\hat{D}|^{m+\frac{1}{n}} (f(M)ms^{m-1}(M)s'(M) + f'(M)s^m(M))\Theta \\ & \times \int_0^t \int_{\partial B} G(x,t;\xi,\tau) d\xi d\tau. \end{aligned} \tag{3.4}$$

By Lemma 3.2, there exists some $t_2(\leq t_1)$ such that

$$|\hat{D}|^{m+\frac{1}{n}} (f(M)ms^{m-1}(M)s'(M) + f'(M)s^m(M)) \left(\int_0^t \int_{\partial B} G(x,t;\xi,\tau) d\xi d\tau \right) < 1, \tag{3.5}$$

for $0 \leq t \leq t_2$. This is a contradiction. Thus, we have uniqueness of a solution for $t \in [0, t_2]$. If $t_2 < t_1$, then

$$\Theta \leq |\hat{D}|^{m+\frac{1}{n}} (f(M)ms^{m-1}(M)s'(M) + f'(M)s^m(M)) \left(\int_{t_2}^t \int_{\partial B} G(x,t;\xi,\tau) d\xi d\tau \right)\Theta, \tag{3.6}$$

for $t_2 \leq t \leq t_1$.

From the definition of G, $G(x,t;\xi,\tau) = G(x,t-\tau;\xi,0)$, we have for $t \in [t_2, \min\{2t_2, t_1\}]$ that

$$|D|^{m+\frac{1}{n}}((f(M)ms^{m-1}(M)s'(M) + s^m(M)f'(M))\left(\int_{t_2}^{t}\int_{\partial B} G(x,t;\xi,\tau)d\xi d\tau\right) < 1, \tag{3.7}$$

which is a contradiction. Hence, we have uniqueness of the solution for $0 \le t \le \min\{2t_2, t_1\}$. By proceeding in this way, the integral equation (3.1) has a unique solution u for $t \in [0, t_1]$.

Next, we prove that u is continuous on $\bar{D} \times [0, t_1]$. Since

$$\lim_{t\to 0^+} \int_D G(x,t;\xi,0)\psi(\xi)d\xi = \psi(x) \text{ for } x \in \bar{D}$$

and

$$\begin{aligned} 0 < \int_D G(x,t;\xi,0)\psi(\xi)d\xi \le& \frac{1}{2^n \pi^{n/2}} \int_{\mathbb{R}^n} t^{-n/2} e^{-|x-\xi|^2/4t}\psi(\xi)d\xi \\ &\left(\max_{\bar{D}} \psi(x)\right)\frac{1}{\pi^{n/2}}\int_{\mathbb{R}^n} e^{-((x-\xi)/2t)^2} d\frac{(x-\xi)}{2t} \\ &= \max_{\bar{D}} \psi(\xi). \end{aligned}$$

It follows from Lemma 3.1(i) that the first term on the right-hand side of (3.2) is continuous. Since $f(u_i(\xi,\tau))$ is bounded by $f(M)$ and $U^m(\tau)$ is bounded by $s^m(M)$, the second term is continuous by Lemma 3.2. Thus, each $u_{i+1}(x,t)$ given by (3.2) is continuous on $\bar{D} \times [0, t_1]$. From (3.2),

$$\begin{aligned} u_{i+1}(x,t) - u_i(x,t) = &\int_0^t \int_{\partial B} G(x,t;\xi,\tau) \\ &\times [f(u_i(x,t))U_i^m(\tau) - f(u_{i-1}(\xi,\tau))U_{i-1}^m(\tau)]d\xi d\tau. \end{aligned}$$

Using the Mean Value Theorem and f, f', s and s' are increasing functions, we have

$$\begin{aligned} f(u_i(\xi,\tau))U_i^m(\tau) - f(u_{i-1}(\xi,\tau))U_{i-1}^m(\tau) \le &(f(M)ms^{m-1}(M)s'(M) + f'(M)s^m(M)) \\ &\times (u_i(\xi,\tau) - u_{i-1}(\xi,\tau)). \end{aligned}$$

Let $S_i = \max_{\bar{D}\times[0,t_1]}(u_i - u_{i-1})$. As in derivation of (3.4), we obtain

$$S_{i+1} \leq |\hat{D}|^{m+\frac{1}{n}}(f(M)ms^{m-1}(M)s'(M) + f'(M)s^m(M)) \times \Big(\int_0^t \int_{\partial B} G(x,t;\xi,\tau)d\xi d\tau\Big) S_i.$$

For $0 \leq t \leq t_2$, it follows from (3.5) that the sequence $\{u_i(x,t)\}$ converges uniformly to $u(x,t)$ for $0 \leq t \leq t_2$. Therefore, u is continuous there. If $t_2 < t_1$, then for $t \in [t_2, t_1]$, we use $u(\xi, t_2)$ to replace $\psi(\xi)$ in (3.2). As in the derivation of (3.6), we obtain

$$S_{i+1} \leq |\hat{D}|^{m+\frac{1}{n}}(f(M)ms^{m-1}(M)s'(M) + f'(M)s^m(M)) \times \Big(\int_{t_2}^t \int_{\partial B} G(x,t;\xi,\tau)d\xi d\tau\Big) S_i.$$

For $t \in [t_2, \min\{2t_2, t_1\}]$, it follows form (3.7) that $S_{i+1} < S_i$. Hence, u is continuous there. By proceeding in this way, the integral equation (3.1) has a unique continuous solution u for $0 \leq t \leq t_1$.

Let t_b be the supremum of the interval for which the integral equation (3.1) has a unique continuous solution u. If t_b is finite, and $u(x,t)$ is bounded in $[0, t_b)$, then we can replace an initial condition of (1.1) by any positive constant greater than $\max_{\bar{D}} u(x, t_b)$ and repeat a proof similar to the above shows that there exists an interval $[t_b, t_3]$ such that the integral equation (3.1) has a unique continuous solution u. This contradicts the definition of t_b. Hence, if t_b is finite, then u is unbounded in $[0, t_b)$. From (3.3), $\psi(x) \leq u(x,t)$. Since u_i is a nondecreasing function of t, u is a nondecreasing function of t. □

The following theorem is to prove that the solution of the integral equation (3.1) is also the solution of the original problem (1.1).

Theorem 3.4. *The problem (1.1) has a unique solution for* $0 \leq t < t_b$.

Proof. Since $\int_0^t \int_{\partial B} G(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau$ is finite for $x \in \bar{D}$ and t in any compact subset of $[0, t_b)$, it follows that for any $t_5 \in (0, t)$,

$$\begin{aligned}
&\int_0^t \int_{\partial B} G(x,t;\xi,\tau) f(u(\xi,\tau)) U^m(\tau) d\xi d\tau \\
&= \lim_{n\to\infty} \int_0^{t-\frac{1}{n}} \int_{\partial B} G(x,t;\xi,\tau) f(u(\xi,\tau)) U^m(\tau) d\xi d\tau \\
&= \lim_{n\to\infty} \int_{t_5}^t \frac{\partial}{\partial \zeta}\left(\int_0^{\zeta-\frac{1}{n}} \int_{\partial B} G(x,\zeta;\xi,\tau) f(u(\xi,\tau)) U^m(\tau) d\xi d\tau \right) d\zeta \\
&\quad + \lim_{n\to\infty} \int_0^{t_5-\frac{1}{n}} \int_{\partial B} G(x,t_5;\xi,\tau) f(u(\xi,\tau)) U^m(\tau) d\xi d\tau.
\end{aligned}$$

Since $G_\zeta(x,\zeta;\xi,\tau) f(u(\xi,\tau)) U^m(\tau) = G_\zeta(x,\zeta-\tau;\xi,0) f(u(\xi,\tau)) U^m(\tau)$, from lemma 3.1(iii) and the Leibniz's rule, we have

$$\begin{aligned}
&\frac{\partial}{\partial \zeta}\left(\int_0^{\zeta-1/n} \int_{\partial B} G(x,\zeta;\xi,\tau) f(u(\xi,\tau)) U^m(\tau) d\xi d\tau \right) \\
&= \int_{\partial B} G\left(x,\zeta;b,\zeta-\frac{1}{n}\right) f\left(u\left(\xi,\zeta-\frac{1}{n}\right)\right) U^m\left(\zeta-\frac{1}{n}\right) d\xi \\
&\quad + \int_0^{\zeta-\frac{1}{n}} \int_{\partial B} G_\zeta(x,\zeta;\xi,\tau) f(u(\xi,\tau)) U^m(\tau) d\xi d\tau.
\end{aligned}$$

Since $G(x,\zeta;\xi,\zeta-1/n) = G(x,1/n;\xi,0)$, which is independent of ζ, we have

$$\begin{aligned}
&\int_0^t \int_{\partial B} G(x,t;\xi,\tau) f(u(\xi,\tau)) U^m(\tau) d\xi d\tau \\
&= \lim_{n\to\infty} \int_{t_5}^t \int_D G\left(x,\zeta;\xi,\zeta-\frac{1}{n}\right) \frac{\partial \chi_B(\xi)}{\partial \nu} f\left(u\left(\xi,\zeta-\frac{1}{n}\right)\right) U^m\left(\zeta-\frac{1}{n}\right) d\xi d\zeta \\
&\quad + \lim_{n\to\infty} \int_{t_5}^t \int_0^{\zeta-1/n} \int_{\partial B} G_\zeta(x,\zeta;\xi,\tau) f(u(\xi,\tau)) U^m(\tau) d\xi d\tau d\zeta \\
&\quad + \lim_{n\to\infty} \int_0^{t-1/n} \int_{\partial B} G(x,t_5;\xi,\tau) f(u(\xi,\tau)) U^m(\tau) d\xi d\tau \\
&= \int_{t_5}^t \int_D \delta(x-\xi) \frac{\partial \chi_B(\xi)}{\partial \nu} f(u(\xi,\zeta)) U^m(\zeta) d\xi d\zeta \\
&\quad + \lim_{n\to\infty} \int_{t_5}^t \int_0^{\zeta-1/n} \int_{\partial B} G_\zeta(x,\zeta;\xi,\tau) f(u(\xi,\tau)) U^m(\tau) d\xi d\tau d\zeta \\
&\quad + \int_0^{t_5} \int_{\partial B} G(x,t_5;\xi,\tau) f(u(\xi,\tau)) U^m(\tau) d\xi d\tau
\end{aligned}$$

$$
\begin{aligned}
&= \int_{t_5}^{t} \frac{\partial \chi_B(x)}{\partial \nu} f(u(x,\zeta))U^m(\zeta)d\zeta \\
&+ \lim_{m\to\infty} \int_{t_5}^{t}\int_0^{\zeta-1/n}\int_{\partial B} G_\zeta(x,\zeta;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau d\zeta \\
&+ \int_0^{t_5}\int_{\partial B} G(x,t_5;\xi,\tau)f(u(\xi;\tau))U^m(\tau)d\xi d\tau. \qquad (3.8)
\end{aligned}
$$

Let

$$
g_n(x,t) = \int_0^{t-1/n}\int_{\partial B} G_t(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau.
$$

Without loss of generality, let $n > l$, we have

$$
g_n(x,\zeta) - g_l(x,\zeta) = \int_{\zeta-1/l}^{\zeta-1/n}\int_{\partial B} G_\zeta(x,\zeta;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau.
$$

Since $G_\zeta(x,\zeta;\xi,\tau) \in C(\bar{D}\times(\tau,T])$ and $f(u(\xi,\tau))U^m(\tau)$ is monotone function of τ, it follows from Theorem 2.3, the Second Mean Value Theorem for integral, that there exists a real number γ such that $\zeta-\gamma \in (\zeta-1/l, \zeta-1/n)$ and

$$
\begin{aligned}
g_n(x,\zeta) - g_l(x,\zeta) =& \int_{\partial B} f\Big(u\Big(\xi,\zeta-\frac{1}{l}\Big)\Big)U^m\Big(\zeta-\frac{1}{l}\Big)\int_{\zeta-1/l}^{\zeta-\gamma} G_\zeta(x,\zeta;\xi,\tau)d\tau d\xi \\
&+ \int_{\partial B} f\Big(u\Big(\xi,\zeta-\frac{1}{n}\Big)\Big)U^m\Big(\zeta-\frac{1}{n}\Big)\int_{\zeta-\gamma}^{\zeta-1/n} G_\zeta(x,\zeta;\xi,\tau)d\tau d\xi.
\end{aligned}
$$

From $G_\zeta(x,\zeta;\xi,\tau) = -G_\tau(x,\zeta;\xi,\tau)$, we have

$$
\begin{aligned}
&g_n(x,\zeta) - g_l(x,\zeta) \\
&= -\int_{\partial B} f\Big(u\Big(\xi,\zeta-\frac{1}{l}\Big)\Big)U^m\Big(\zeta-\frac{1}{l}\Big)\int_{\zeta-1/l}^{\zeta-\gamma} G_\tau(x,\zeta;\xi,\tau)d\tau d\xi \\
&- \int_{\partial B} f\Big(u\Big(\xi,\zeta-\frac{1}{n}\Big)\Big)U^m\Big(\zeta-\frac{1}{n}\Big)\int_{\zeta-\gamma}^{\zeta-1/n} G_\tau(x,\zeta;\zeta,\tau)d\tau d\xi
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\partial B} f\Big(u\Big(\xi,\zeta-\frac{1}{l}\Big)\Big)U^m\Big(\zeta-\frac{1}{l}\Big)G\Big(x,\zeta;\xi,\zeta-\frac{1}{l}\Big)d\xi \\
&\quad - \int_{\partial B} f\Big(u\Big(\xi,\zeta-\frac{1}{l}\Big)\Big)U^m\Big(\zeta-\frac{1}{l}\Big)G(x,\zeta;\xi,\zeta-\gamma)d\xi \\
&\quad \int_{\partial B} f\Big(u\Big(\xi,\zeta-\frac{1}{n}\Big)\Big)U^m\Big(\zeta-\frac{1}{n}\Big)G(x,\zeta;\xi,\zeta-\gamma)d\xi \\
&\quad - \int_{\partial B} f\Big(u\Big(\xi,\zeta-\frac{1}{n}\Big)\Big)U^m\Big(\zeta-\frac{1}{n}\Big)G\Big(x,\zeta;\xi,\zeta-\frac{1}{n}\Big)d\xi \\
&= \int_{\partial B} \Big[f\Big(u\Big(\xi,\zeta-\frac{1}{n}\Big)\Big)U^m\Big(\zeta-\frac{1}{n}\Big) - f\Big(u\Big(\xi,\zeta-\frac{1}{l}\Big)\Big)U^m\Big(\zeta-\frac{1}{l}\Big)\Big] \\
&\quad \times G(x,\zeta;\xi,\zeta-\gamma)d\xi \\
&\quad + \int_{\partial B} f\Big(u\Big(\xi,\zeta-\frac{1}{l}\Big)\Big)U^m\Big(\zeta-\frac{1}{l}\Big)G\Big(x,\zeta;\xi,\zeta-\frac{1}{l}\Big)d\xi \\
&\quad - \int_{\partial B} f\Big(u\Big(\xi,\zeta-\frac{1}{n}\Big)\Big)U^m\Big(\zeta-\frac{1}{n}\Big)G\Big(x,\zeta;\xi,\zeta-\frac{1}{n}\Big)d\xi.
\end{aligned}
$$

We note that $f(u)$ and $s(u)$ are bounded on compact sets of $(0,\infty)$, $G(x,\zeta;\xi,\zeta-\epsilon) = G(x,\epsilon;\xi,0)$, which converges uniformly with respect to ζ as $\epsilon \to 0$. Therefore $\{g_n\}$ is a Cauchy sequence, and hence, $\{g_n\}$ convergence uniformly with respect to ζ in any compact subset of $(0,t_b)$. We obtain

$$
\begin{aligned}
&\lim_{n\to\infty} \int_{t_5}^{t}\int_0^{\zeta-1/n}\int_{\partial B} G_\zeta(x,\zeta;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau d\zeta \\
&\qquad = \int_{t_5}^{t}\int_0^{\zeta}\int_{\partial B} G_\zeta(x,\zeta;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau d\zeta.
\end{aligned}
$$

From (3.8), we have

$$
\begin{aligned}
&\int_0^t\int_{\partial B} G(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau \\
&\qquad = \int_{t_5}^{t} \frac{\partial\chi_B(x)}{\partial\nu} f(u(\xi,\tau))U^m(\tau)d\zeta \\
&\qquad\quad + \int_{t_5}^{t}\int_0^{\zeta}\int_{\partial B} G_\zeta(x,\zeta;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau d\zeta \\
&\qquad\quad + \int_0^{t_5}\int_{\partial B} G(x,t_5;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau.
\end{aligned}
$$

Thus,

$$\frac{\partial}{\partial t}\int_0^t\int_{\partial B}G(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau$$
$$=\frac{\partial\chi_B(x)}{\partial\nu}f(u(x,t))U^m(t)+\int_0^t\int_{\partial B}G_t(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau.$$

By using Lemma 3.1(iii), we have that

$$\int_0^{t-\epsilon}\int_{\partial B}G_{x_i}(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau$$

and

$$\int_0^{t-\epsilon}\int_{\partial B}G_{x_ix_i}(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau$$

exist. By using Theorem 2.1, Leibnitz rule, we have for any x in any compact subset of D and t in any compact subset of $(0,t_b)$

$$\frac{\partial}{\partial x_i}\int_0^{t-\epsilon}\int_{\partial_B}G(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau$$
$$=\int_0^{t-\epsilon}\int_{\partial B}G_{x_i}(x,t;\xi,\tau)f(u(\xi,t))U^m(\tau)d\xi d\tau$$

and

$$\frac{\partial}{\partial x_i}\int_0^{t-\epsilon}\int_{\partial_B}G_{x_i}(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau$$
$$=\int_0^{t-\epsilon}\int_{\partial B}G_{x_ix_i}(x,t;\xi,\tau)f(u(\xi,t))U^m(\tau)d\xi d\tau.$$

For any $(x_1, ..., x_{i-1}, \tilde{x}_i, x_{i+1}, ..., x_n) \in D$, we obtain

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_0^{t-\epsilon}\int_{\partial B} G(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau \\
&= \lim_{\epsilon\to 0}\int_{\tilde{x}_i}^{x_i}\left(\frac{\partial}{\partial x_i}\int_0^{t-\epsilon}\int_{\partial B} G(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau\right)dx_i \\
&\quad + \int_0^{t-\epsilon}\int_{\partial B} G(x_1,...,x_{i-1},\tilde{x}_i,x_{i+1}...,x_n,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau \\
&= \lim_{\epsilon\to 0}\int_{\tilde{x}_i}^{x_i}\int_0^{t-\epsilon}\int_{\partial B} G_{x_i}(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau dx_i \\
&\quad + \int_0^{t}\int_{\partial B} G(x_1,...,x_{i-1},\tilde{x}_i,x_{i+1},...,x_n,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau. \qquad (3.9)
\end{aligned}$$

We would like to show that

$$\lim_{\epsilon\to 0}\int_{\tilde{x}_i}^{x_i}\int_0^{t-\epsilon}\int_{\partial B} G_{x_i}(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau dx_i$$

$$= \int_{\tilde{x}_i}^{x_i}\lim_{\epsilon\to 0}\int_0^{t-\epsilon}\int_{\partial B} G_{x_i}(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau dx_i. \qquad (3.10)$$

By Theorem 2.8, we have

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{\tilde{x}_i}^{x_i}\int_0^{t-\epsilon}\int_{\partial B} G_{x_i}(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau dx_i \\
&= \lim_{\epsilon\to 0}\int_0^{t-\epsilon}\int_{\partial B}\left(f(u(\xi,\tau))U^m(\tau)\int_{\tilde{x}_i}^{x_i} G_{x_i}(x,t;\xi,\tau)dx_i\right)d\xi d\tau \\
&= \lim_{\epsilon\to 0}\int_0^{t-\epsilon}\int_{\partial B} f(u(\xi,\tau))U^m(\tau) \\
&\quad \times (G(x,t;\xi,\tau) - G(x_1,...,x_{i-1},\tilde{x}_i,x_{i+1},...,x_n,t;\xi,\tau))d\xi d\tau,
\end{aligned}$$

which exists by Lemma 3.2, f is bounded on $\partial B \times [0,t]$ and U is bounded on $[0,t]$. Therefore,

$$\begin{aligned}
&\int_0^t\int_{\partial_B} f(u(\xi,\tau))U^m(\tau)(G(x,t;\xi,\tau) - G(x_1,...,x_{i-1},\tilde{x}_i,x_{i+1},...,x_n,t;\xi,\tau))d\xi d\tau \\
&= \int_{\tilde{x}_i}^{x_i}\int_0^t\int_{\partial B} G_{x_i}(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau dx_i,
\end{aligned}$$

and we have

$$\frac{\partial}{\partial x_i}\int_0^t\int_{\partial B} G(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau$$
$$=\int_0^t\int_{\partial_B} G_{x_i}(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau.$$

For any $(x_1,...,x_{i-1},\tilde{x}_i,x_{i+1},...,x_n)\in D$,

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_0^{t-\epsilon}\int_{\partial B} G_{x_i}(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau\\
&\qquad=\lim_{\epsilon\to 0}\int_{\tilde{x}_i}^{x_i}\frac{\partial}{\partial x_i}\bigg(\int_0^{t-\epsilon}\int_{\partial B} G_{x_i}(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau\bigg)dx_i\\
&\qquad\quad+\lim_{\epsilon\to 0}\int_0^{t-\epsilon}\int_{\partial B} G_{x_i}(x_1,...,x_{i-1},\tilde{x}_i,x_{i+1},...,x_n,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau\\
&\qquad=\lim_{\epsilon\to 0}\int_{x_2}^{x}\int_0^{t-\epsilon}\int_{\partial B} G_{x_ix_i}(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau dx_i\\
&\qquad\quad+\int_0^t\int_{\partial_B} G_{x_i}(x_1,...,x_{i-1},\tilde{x}_i,x_{i+1},...,x_n,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau.
\end{aligned}\tag{3.11}$$

We would like to show that

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{\tilde{x}_i}^{x_i}\int_0^{t-\epsilon}\int_{\partial B} G_{x_ix_i}(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau dx_i\\
&=\int_{\tilde{x}_i}^{x_i}\lim_{\epsilon\to 0}\int_0^{t-\epsilon}\int_{\partial B} G_{x_ix_i}(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau dx_i.
\end{aligned}\tag{3.12}$$

By theorem 2.1 and theorem 2.8, we obtain

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{\tilde{x}_i}^{x_i}\int_0^{t-\epsilon}\int_{\partial B} G_{x_ix_i}(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau dx_i\\
&\quad=\int_0^t\int_{\partial B}(G_{x_i}(x,t;\xi,\tau)-G_{x_i}(x_1,...,x_{i-1},\tilde{x}_i,x_{i+1},...,x_n,t;\xi,\tau))f(u(\xi,\tau))d\xi d\tau
\end{aligned}$$

which exists. Therefore, by using theorem 2.8, we have

$$\begin{aligned}
&\int_0^t\int_{\partial B}(G_{x_i}(x,t;\xi,\tau)-G_{x_i}(x_1,...,x_{i-1},\tilde{x}_i,x_{i+1},...,x_n,t;\xi,\tau))f(u(\xi,\tau))U^m(\tau)d\xi d\tau\\
&\quad=\int_{\tilde{x}_i}^{x_i}\int_0^t\int_{\partial B} G_{x_ix_i}(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau dx_i.
\end{aligned}$$

From the above and (3.12), we have

$$\begin{aligned}&\frac{\partial}{\partial x_i}\int_0^t\int_{\partial B} G_{x_i}(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau\\&=\int_0^t\int_{\partial B} G_{x_ix_i}(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau.\end{aligned}$$

It follows from (3.11) that for any x in any compact subset of D and any t in any compact subset of $(0,t_b)$, that

$$\begin{aligned}&\frac{\partial^2}{\partial x_i^2}\int_0^t\int_{\partial B} G(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau\\&=\int_0^t\int_{\partial B} G_{x_ix_i}(x,t;\xi,\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau.\end{aligned}$$

By Theorem 2.1, the Leibnitz's rule and Lemma 3.1(iii), we have for any x in any compact subset of D and any t in any compact subset of $(0,t_b)$,

$$\begin{aligned}\frac{\partial}{\partial t}\int_D G(x,t;\xi,0)\psi(\xi)d\xi&=\int_D G_t(x,t;\xi,0)\psi(\xi)d\xi,\\ \frac{\partial}{\partial x_i}\int_D G(x,t;\xi,0)\psi(\xi)d\xi&=\int_D G_{x_i}(x,t;\xi,0)\psi(\xi)d\xi,\\ \frac{\partial^2}{\partial x_i^2}\int_D G(x,t;\xi,0)\psi(\xi)d\xi&=\int_D G_{x_ix_i}(x,t;\xi,0)\psi(\xi)d\xi.\end{aligned}$$

From the integral equation (3.1), we have for $x\in D$ and $t\in(0,t_b)$ that

$$\begin{aligned}Hu=&\int_D HG(x,t;\xi,0)\psi(\xi)d\xi+|\hat{D}|^{m+\frac{1}{n}}\frac{\partial\chi_B(x)}{\partial\nu}f(u(x,t))U^m(t)\\&+\int_0^t\int_{\partial B}(HG(x,t;\xi,\tau))f(u(\xi,\tau))U^m(\tau)d\xi d\tau\\&=\delta(t)\int_D\delta(x-\xi)\psi(\xi)d\xi+|\hat{D}|^{m+\frac{1}{n}}\frac{\partial\chi_B(x)}{\partial\nu}f(u(x,t))U^m(t)\\&+\lim_{\epsilon\to 0}|\hat{D}|^{m+\frac{1}{n}}\int_0^{t-\epsilon}\int_{\partial B}\delta(x-\xi)\delta(t-\tau)f(u(\xi,\tau))U^m(\tau)d\xi d\tau\\&=|\hat{D}|^{m+\frac{1}{n}}\frac{\partial\chi_B(x)}{\partial\nu}f(u(x,t))U^m(t).\end{aligned}$$

From the integral equation (3.1), we also have

$$\lim_{t\to 0^+}u(x,t)=\lim_{t\to 0^+}\int_D G(x,t;\xi,0)\psi(\xi)d\xi=\psi(x)\text{ for }x\in\bar{D}.$$

From Lemma 3.1(iii), we obtain $u = 0$ on $\partial D \times (0, T]$. Thus, the continuous solution u of the integral equation (3.1) is a solution of (1.1) which has a unique solution before a blow-up occurs. $\square$

CHAPTER IV
SUFFICIENT CONDITIONS FOR BLOW-UP AND BLOW-UP SET

We modify the proof of theorem 4 of Chan [3] to prove the following result.

Lemma 4.1. *If t_b is finite and u attains its maximum on ∂B, then the solution of* (1.1) *u blows up everywhere on ∂B only.*

Proof. We rewrite the problem (1.1) as two initial-boundary value problems:

$$\left.\begin{aligned} Hu &= 0 \text{ in } B \times (0, t_b), \\ u &= g(x,t) \text{ on } \partial B \times (0, t_b), \\ u(x,0) &= \psi(x) \text{ in } \bar{B}, \end{aligned}\right\} \tag{4.1}$$

and

$$\left.\begin{aligned} Hu &= 0 \text{ in } D\backslash\bar{B} \times (0, t_b), \\ u(x,0) &= \psi(x) \text{ on } \bar{D}\backslash B, \\ u(x,t) &= 0 \text{ in } \partial D \times (0, t_b), u = g(x,t) \text{ on } \partial B \times (0, t_b). \end{aligned}\right\} \tag{4.2}$$

First, we will show that u dose not have a relative maximum in $B \times (0, t_b)$ and $D\backslash\bar{B} \times (0, t_b)$.

We differentiate the partial differential equations in the (4.1) and (4.2) with respect to t. We obtain that $Hu_t = 0$ in $B \times (0, t_b)$ and $D\backslash\bar{B} \times (0, t_b)$. Since $u_t \geq 0$ and u is a nontrivial and applying the maximum principle, we have $u_t > 0$ in $B \times (0, t_b)$ and $D\backslash\bar{B} \times (0, t_b)$. Hence, $\Delta u = u_t > 0$ in $B \times (0, t_b)$ and $D\backslash\bar{B} \times (0, t_b)$. This implies that u dose not have a relative maximum in $B \times (0, t_b)$ and in $D\backslash\bar{B} \times (0, t_b)$.

Next, we will show that for any $t > 0$,

$$u(x,t) > u(y,t) \text{ for all } x \in \partial B \text{ and } y \notin \partial B.$$

We assume that there exists a smallest positive number, say t_0, and some $x_0 \notin \partial B$ such that $u(x_0, t_0) = \min_{x\in\partial B} u(x, t_0)$. We suppose that there is a point y_0 in ∂B such that $u(y_0, t_0) > \min_{x\in\partial B} u(x, t_0)$. Since u is continuous, there is a point (y', t_0) in a neighborhood of (y_0, t_0) such that $y' \notin \partial B$ and $u(y', t_0) > \min_{x\in\partial B} u(x, t_0)$. Hence, t_0 is not the smallest number such that $u(x_0, t_0) = \min_{x\in\partial B} u(x, t_0)$. This is a contradiction. We suppose that u attains its maximum at a point $(\bar{y}, t_0)$ for $0 \le t \le t_0$. If $\bar{y} \in B$, then $u \equiv u(\bar{y}, t_0)$ in $B \times (0, t_0]$ by Theorem 2.5, the Strong Maximum Principle. Since u is continuous, we have $u \equiv u(\bar{y}, t_0)$ in $\bar{B} \times [0, t_0]$. Then $u(x,t)$ is a constant in $\bar{B} \times [0, t_0]$. It contradicts to $u_t > 0$. If $\bar{y} \in D\backslash\bar{B}$, then $u \equiv u(\bar{y}, t_0)$ in $D\backslash\bar{B} \times (0, t_0]$. Since u is continuous, we have $u \equiv u(\bar{y}, t_0)$ in $\bar{D}\backslash B \times [0, t_0]$. It contradicts with $u(x,t) = 0$ for $x \in \partial D$. Thus, for any $t > 0$,

$$u(x,t) > u(y,t) \text{ for all } x \in \partial B \text{ and } y \notin \partial B. \tag{4.3}$$

Finally, we claim that for each $t > 0$, u attains the same value for $x \in \partial B$. Suppose that there is a point $y_0 \in \partial B$ and $t > 0$ such that $u(y_0, t) > \min_{x\in\partial B} u(x,t)$. Since u is continuous, there is a point (y', t) in a neighbourhood of $u(y_0, t)$ such that $y' \notin \partial B$ and $u(y', t) > \min_{x\in\partial B} u(x, t)$ for all $t > 0$. This contradicts with (4.3). □

Let $|D|$ denote the volume of D, ω denote the $n-$dimensional solid angle and

$$\mu(t) = \int_D \phi(x) u(x,t)\,dx,$$

where ϕ is the normalized fundamental eigenfunction of the problem,

$$\Delta\phi + \lambda\phi = 0 \text{ in } D, \phi = 0 \text{ on } \partial D,$$

with λ denoting its corresponding eigenvalue.

Finally, we can modify the technique of proving theorem 5 in [6] to obtain sufficient conditions for (1.1) to blow-up in a finite time.

Theorem 4.2. *Let* $\sup_{x\in\bar{D}} u(x,t) = M(t)$ *and* ψ *attains its maximum at every point on* ∂B. *Assume that*

$$\mu(0) > \left(\frac{\lambda}{|\hat{D}|^{m+\frac{1}{n}} R^{n-1}\omega}\right)^{\frac{1}{(2m-1)}},$$

$\phi(x)f(u(x,t)) \geq u^m(x,t)$ *for all* $x \in \partial B$ *and for all* $t > 0$,

$s(u(x,t)) \geq u(x,t)$ *for all* $x \in D$ *and for all* $t > 0$.

Then the solution u *of* (1.1) *blows up everywhere on* ∂B *in a finite time.*

Proof. Multiplying the normalized eigenvalue to (1.1) and integrating over D, we obtain that

$$\begin{aligned}
\mu_t(t) + \lambda\mu(t) &= \int_D \phi(x)|\hat{D}|^{m+\frac{1}{n}} \frac{\partial \chi_B}{\partial \nu} f(u(x,t))U^m(t)dx \\
&= \int_D |\hat{D}|^{m+\frac{1}{n}} \phi(x) f(u(x,t))U^m(t)\Big(\nu(x)\cdot\nabla\chi_B(x)\Big)dx \\
&= -\int_B \sum_{i=1}^{n} |\hat{D}|^{m+\frac{1}{n}} \frac{\partial}{\partial \nu_i}\phi(x)f(u(x,t))U^m(t)\nu_i dx \\
&= \int_{\partial B} |\hat{D}|^{m+\frac{1}{n}} \phi(x)f(u(x,t))U^m(t)dx.
\end{aligned}$$

Since $s(u(x,t)) \geq u(x,t)$ for all x in D,for all $t > 0$, using the supremum property, we obtain that $s(u(x,t)) \geq M(t)$ for all $t > 0$ and

$$\int_D s(u(x,t))dx \geq \int_D M(t)dx = M(t), \text{ for all } t > 0.$$

We use the conditions prescribed in the theorem and the above argument to obtain that

$$\begin{aligned}\int_{\partial B} |\hat{D}|^{m+\frac{1}{n}}\phi(x)f(u(x,t))U^m(t)dx &\geq \int_{\partial B} |\hat{D}|^{m+\frac{1}{n}}u^m(x,t)U^m(t)dx \\ &= |\hat{D}|^{m+\frac{1}{n}}M^m(t)U^m(t)\int_{\partial B} dx \\ &= |\hat{D}|^{m+\frac{1}{n}}M^m(t)U^m(t)R^{n-1}\omega \\ &\geq |\hat{D}|^{m+\frac{1}{n}}M^{2m}(t)R^{n-1}\omega,\end{aligned}$$

where $R^{n-1}\omega$ is an $n-$dimensional surface area of a sphere. Thus,

$$\mu_t(t) + \lambda\mu(t) \geq |\hat{D}|^{m+\frac{1}{n}}M^{2m}(t)R^{n-1}\omega, \text{ for all } t > 0. \tag{4.2}$$

By Theorem 2.7, the Schwarz's inequality,

$$\begin{aligned}\mu(t) &= \int_D \phi(x)u(x,t)dx \\ &\leq M(t)\int_D \phi(x)dx \\ &\leq M(t)\Big(\int_D dx\Big)^{\frac{1}{2}}\Big(\int_D \phi(x)^2dx\Big)^{\frac{1}{2}}.\end{aligned}$$

Since $\int_D \phi(x)^2dx = 1$ and $|D| = 1$, we have

$$\mu(t) \leq M(t)|D|^{\frac{1}{2}} = M(t), \text{ for all } t > 0.$$

Thus, (4.2) becomes

$$\mu_t(t) + \lambda\mu(t) \geq |\hat{D}|^{m+\frac{1}{n}}R^{n-1}\omega\mu^{2m}(t).$$

That is,

$$\frac{1}{\mu^{2m}(t)}\mu_t(t) + \frac{1}{\mu^{2m}(t)}\lambda\mu(t) \geq |\hat{D}|^{m+\frac{1}{n}}R^{n-1}\omega.$$

Multiplying the above equation by $(1-2m)e^{(1-2m)\lambda t}$ and rewritting in the following form.

$$e^{(1-2m)\lambda t}\frac{d(\mu^{1-2m}(t))}{dt}+\frac{de^{(1-2m)\lambda t}}{dt}\mu^{1-2m}(t)\leq e^{(1-2m)\lambda t}(1-2m)|\hat{D}|^{m+\frac{1}{n}}R^{n-1}\omega.$$

That is

$$\frac{d(e^{(1-2m)\lambda t})\mu^{(1-2m)}(t)}{dt}\leq(1-2m)e^{(1-2m)\lambda t}|\hat{D}|^{m+\frac{1}{n}}R^{n-1}\omega.$$

Integrating both sides from 0 to t, we have

$$\mu^{(1-2m)}(t)\leq\frac{|\hat{D}|^{m+\frac{1}{n}}R^{n-1}\omega}{\lambda}+\frac{1}{e^{(1-2m)\lambda t}}\left(\mu^{(1-2m)}(0)-\frac{|\hat{D}|^{m+\frac{1}{n}}R^{n-1}\omega}{\lambda}\right).$$

Finally,

$$\mu^{(2m-1)}(t)\geq\frac{\lambda e^{(1-2m)t}}{|\hat{D}|^{m+\frac{1}{n}}e^{(1-2m)\lambda t}+(\lambda\mu^{(1-2m)}(0)-|\hat{D}|^{m+\frac{1}{n}}R^{n-1}\omega)}.$$

Hence, we can see that $\mu^{(2m-1)}(t)$ tends to infinity whenever

$$|\hat{D}|^{m+\frac{1}{n}}e^{(1-2m)\lambda t}+(\lambda\mu^{(1-2m)}(0)-|\hat{D}|^{m+\frac{1}{n}}R^{n-1}\omega)\to 0.$$

Thus, we obtains that if

$$t\to\frac{1}{(1-2m)\lambda}\ln\left(\frac{-\lambda\mu^{(1-2m)}(0)+|\hat{D}|^{m+\frac{1}{n}}R^{n-1}\omega}{|\hat{D}|^{m+\frac{1}{n}}}\right),$$

then $\mu^{2m-1}(t)\to\infty$. From our assumption, we have $\mu^{1-2m}(0)<|\hat{D}|^{m+\frac{1}{n}}R^{n-1}\omega/\lambda$. Hence, μ tends to infinity for some finite time t_b, where

$$t_b\leq\frac{1}{(1-2m)\lambda}\ln\left(\frac{-\lambda\mu^{(1-2m)}(0)+|\hat{D}|^{m+\frac{1}{n}}R^{n-1}\omega}{|\hat{D}|^{m+\frac{1}{n}}}\right).$$

That is, u tends to infinity for some finite time t_b on the boundary of the concentrated ball B.

□

CHAPTER V
CONCLIUSION AND DISCUSSION

We consider a mutidimensional nonlinear parabolic problem:

$$\left.\begin{aligned} Hu &= |\hat{D}|^{m+\frac{1}{n}} \frac{\partial \chi_B(x)}{\partial \nu} f(u(x,t)) \left(\int_D s(u(x,t))dx \right)^m \text{ in } \Omega, \\ u(x,0) &= \psi(x) \text{ on } \bar{D}, u(x,t) = 0 \text{ for } x \in \partial D, 0 < t \leq T. \end{aligned}\right\}$$

We can show that it has a unique continuous solution, which is an increasing function of t. We explore that the blow-up occurs everywhere on the boundary of a ball, ∂B, where the local source is situated. Finally, we also find sufficient conditions for blow-up to occurs in a finite time which are

$$\begin{aligned} &\int_D \phi(x)u(x,0)dx > \left(\frac{\lambda}{|\hat{D}|^{m+\frac{1}{n}} R^{n-1}\omega} \right)^{\frac{1}{2m-1}}, \\ &\phi(x)f(u(x,t)) \geq u^m(x,t) \text{ for all } x \in \partial B \text{ for all } t > 0, \\ &s(u(x,t)) \geq u(x,t) \text{ for all } x \in D \text{ and for all } t > 0, \end{aligned}$$

where $\phi(x)$ is the normalized eigenfunction of $\Delta\phi + \lambda\phi = 0$ in D, $\phi = 0$ on ∂D, with λ denoting its corresponding eigenvalue, $m > 1$ and $R^{n-1}\omega$ is the surface area of an $n-$dimensional sphere.

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VITA

Name Mr. Artit Siriroop

Date of Birth 14 October 1984

Place of Birth Trat, Thailand

Education B.Sc. (Applied Mathematics),
King Mongkut Institute of Technology Lardkrabang, 2006