CHAPTER III
MODELING

Pressure difference across an interface that isolates a pair of immiscible fluids arises from interfacial tension and the pressure difference can be calculated from the shape and size of the interface. Consider the static interface (Figure 3.1), static pressure is the only stress acting on both sides of the surface. Let p_o and $p_i(r)$ be the pressure on both sides of the interface, which depends on position. F_σ is the net force due to interfacial tension. R_1 and R_2 are radii of the curve, which describe the curvature of a surface at a point.

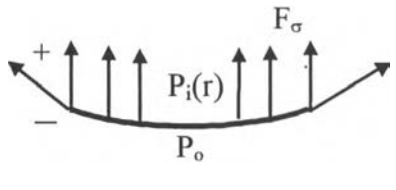
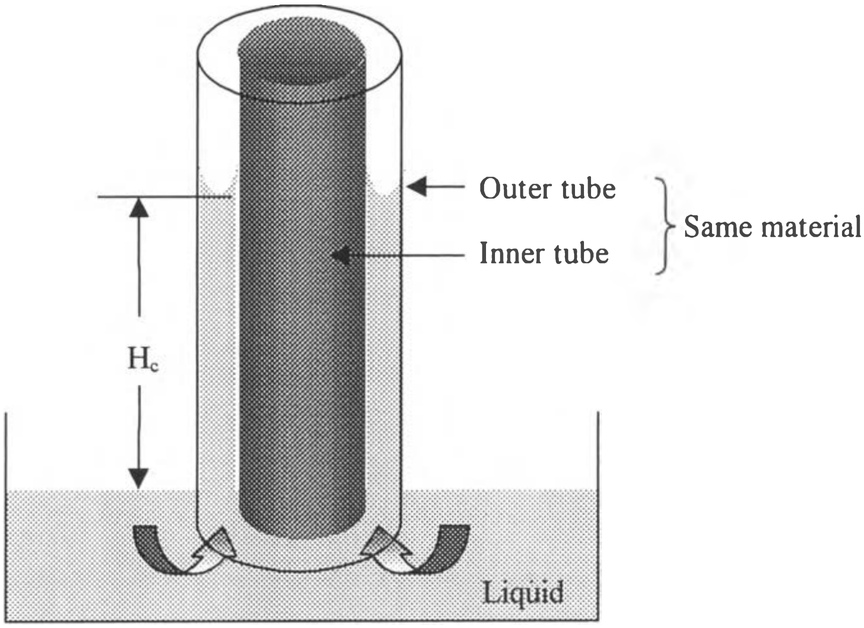


Figure 3.1 Interface between two fluids.

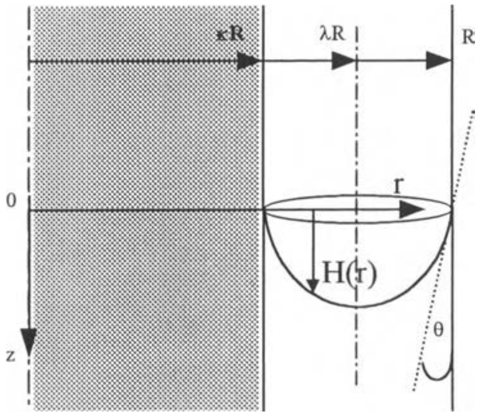
Using force balance for the condition of equilibrium of normal stresses across a static interface as shown in the figure above yields the so-called Young-Laplace equation (Middleman, 1998).

$$p_i(r) - p_o = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \qquad (3.1)$$

In order to relate the liquid height to the static pressures, $p_i(r)$ is the pressure just inside the liquid at any position along r axis and $p_{\ell o}$ is the pressure on the liquid side at the tip of the meniscus (the point where $r = \lambda R$, $z = H(r = 0)$) (Figure 3.2).



(a) The schematic diagram of the capillary rise method in annular tube.



(b) Boundaries of the system to be used in modeling.

Figure 3.2 Sketch of configuration of annular tube and meniscus.

Applying hydrostatic law to the static interface between two immiscible fluids yields

$$p_i - p_{\ell o} = \rho g \,[H(r) - H(r = \lambda R)] \tag{3.2}$$

Despite the fact that the local curvature can be approximated as shown in Equation (3.3), it can be reduced to Equation (3.4) by neglecting the slopes dH/dr in denominators with the assumption of a small slope, which corresponds to a large contact angle near the wall.

$$\left(\frac{1}{R_1}+\frac{1}{R_2}\right)=\frac{d^2H/dr^2}{\left[1+\left(\frac{dH}{dr}\right)^2\right]^{3/2}}+\frac{1}{r}\frac{dH/dr}{\left[1+\left(\frac{dH}{dr}\right)^2\right]^{1/2}} \tag{3.3}$$

$$\left(\frac{1}{R_1}+\frac{1}{R_2}\right)=\frac{d^2H}{dr^2}+\frac{1}{r}\frac{dH}{dr} \tag{3.4}$$

The height of the meniscus can be defined as the difference between meniscus height at any position r and a fixed position at the tip of the meniscus, that is

$$h(r) = H(r) - Ho(r = \lambda R) \tag{3.5}$$

Combining Equations (3.1), (3.2), (3.4) and (3.5) gives the second-order linear nonhomogeneous partial differential equation as follows

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right)+\left(\frac{\rho g}{\sigma}\right)h=\frac{p_o - p_{\ell o}}{\sigma} \tag{3.6}$$

The model to be used in this study will be expressed in terms of H(r) and hence H(r = λR) have to be specified by arbitrarily setting one more boundary condition. Solving Equation (6) gives,

$$h(r) = c_1J_0(ar) + c_2Y_0(ar) + h_p \tag{3.7}$$

where $J_0(ar)$ = Bessel function of the first kind of order zero

$Y_0(ar)$ = Bessel function of the first kind of order zero

$a = (\rho g/\sigma)^{1/2}$

The constants of the above equations, c_1, c_2 and h_p can be found as follows,

1. At the tip of the meniscus, the slope of the surface curve is equal to zero.

$$\frac{dh}{dr} = 0 \quad \text{at} \quad r = \lambda R \tag{3.8}$$

From Equation (3.7),

$$\left.\frac{dh}{dr}\right|_{r=\lambda R} = 0 = \frac{d}{dr}\left[c_1 J_0(ar) + c_2 Y_0(ar) + h_p\right]_{r=\lambda R} \tag{3.9}$$

Differentiating and rearranging the above equation for c_1,

$$c_1 = -\frac{c_2 Y_1(a\lambda R)}{J_1(a\lambda R)} \tag{3.10}$$

2. At the wall of the outer tube

$$\frac{dh}{dr} = \cot\theta \quad \text{at} \quad r = R \tag{3.11}$$

The Equation (3.7) becomes

$$\left.\frac{dh}{dr}\right|_{r=R} = \cot\theta = \frac{d}{dr}\left[c_1 J_0(ar) + c_2 Y_0(ar) + h_p\right]_{r=R} \tag{3.12}$$

Substituting c_1 from Equation (3.10) in the above equation and solving for c_2, yields

$$c_2 = \frac{\cot\theta J_1(a\lambda R)}{a\left[Y_1(aR)J_1(a\lambda R) - Y_1(a\lambda R)J_1(aR)\right]} \tag{3.13}$$

Substituting c_2 from Equation (3.13) to Equation (3.10), gives

$$c_1 = \frac{-\cot\theta Y_1(a\lambda R)}{a\left[Y_1(aR)J_1(a\lambda R) - Y_1(a\lambda R)J_1(aR)\right]} \tag{3.14}$$

3. The other boundary condition is

$$h(r) = 0 \text{ at } r = \lambda R \tag{3.15}$$

From Equation (3.7),

$$h(r) = 0 = c_1J_0(a\lambda r) + c_2Y_0(a\lambda r) + h_p \tag{3.16}$$

So,

$$h_p = -[c_1J_0(a\lambda r) + c_2Y_0(a\lambda r)] \tag{3.17}$$

where c_2 and c_1 are given by Equations (3.13) and (3.14), respectively.

4. H_o can be found arbitrarily by setting H(r) = 0 at r = R. From Equation (3.5),

$$Ho = -h(r) \tag{3.18}$$

So,

$$Ho = -[c_1J_0(aR) + c_2Y_0(aR) + h_p] \tag{3.19}$$

Substituting Equations (3.2) and (3.7) in Equation (3.1), gives

$$H(r) = c_1[J_o(ar) - J_o(aR)] + c_2[Y_o(ar) - Y_o(aR)] \tag{3.20}$$

The above equation is the analytical model to be used in this study. It implies that factors on which the height of the rising liquid depend are the radii of the inner and outer tubes (that is the gap distance between the tubes), the density of liquid, contact angle and surface tension. Noting that the Young-Laplace equation indicates an influence of pressure difference across the interface (Equation (3.6)) but Equation (3.20) does not contain the pressure difference term. The reason is that it is accounted by the h_p term.

Equation (3.20) has been derived from the fact that we avoided to set the pressure on the liquid side of the meniscus, $p_{\ell o}$, equal to the gas side pressure, p_o, because the interface is curved. In addition, the slopes dH/dr in denominators are neglected by assuming that a contact angle near the wall is large. To avoid this assumption, Equations (3.1), (3.2), and (3.3) are combined resulting the following equation.

$$\frac{d^2H}{dr^2} = -a^2 H\left[1+\left(\frac{dH}{dr}\right)^2\right]^{\frac{3}{2}} - \frac{dH/dr}{r}\left[1+\left(\frac{dH}{dr}\right)^2\right] \qquad (3.21)$$

Equation (3.21) is a non-linear ordinary differential equation and to be simulated numerically to predict a shape of a meniscus.

H_c, Figure 3.2(a), devotes the height of the liquid rise in the annular tube from the flat liquid surface (for which pressure drop must be zero) to the bottom of the meniscus. By performing force balance, H_c is then obtained as shown below,

$$H_c = \frac{2\sigma}{\rho g(R_1 - R_2)} \qquad (3.22)$$

where R_1 and R_2 are outer and inner annular tube radius, respectively.