CHAPTER III

EUCLIDEAN N-SPACE

The main purpose of this chapter is to introduce the notion of Euclidean n-space, linear manifold, and Euclidean motion in R^n.

The materials of this chapter are drawn from reference [2], [5], and [6] .

1. Vector space and subspace

3.1.1 Definition. A vector space consist of an abelian group V under addition and a field F, together with an operation of scalar multiplication of each element of V by each element of F on the left, such that for all a, b $\in$ F and u, v $\in$ V the following conditions are satisfied :

(i) $au \in V$

(ii) $a(bu) = (ab)u$

(iii) $(a+b)u = au + bu$

(iv) $a(u+v) = au + av$

(v) $1u = u$.

The elements of V are called vectors and the elements of F are called scalars. We shall say that V is a vector space over F.

3.1.2 Definition. Let V be a vector space over R. Suppose to each pair of vectors u, v $\in$ V there is assigned a scalar (u,v) $\in$ R. This mapping is called an inner product in V if it satisfies the following axioms

(i) $(au_1 + bu_2, v) = a(u_1, v) + b(u_2, v)$

(ii) $(u, v) = (v, u)$

(iii) $(u,u) \geqslant 0$; and $(u,u) = 0$ if and only if $u = 0$.

The vector space V with an inner product is called a real inner product space.

3.1.3 Example. The set of all n-tuples of elements of R with vector addition and scalar multiplication defined by

$$(u_1,u_2,\ldots,u_n) + (v_1,v_2,\ldots,v_n) = (u_1+v_1,u_2+v_2,\ldots,u_n+v_n)$$

and

$$a(u_1,u_2,\ldots,u_n) = (au_1,au_2,\ldots au_n) ,$$

where the u_i, v_i and a belong to R, is a vector space over R ; we denote this space by V_n .

Consider the dot product in V_n

$$U \cdot V = u_1v_1 + u_2v_2 + \ldots + u_nv_n,$$

where $U = (u_1,u_2,\ldots u_n)$ and $V = (v_1,v_2,\ldots v_n)$.

This is an inner product on V_n, and V_n with this inner product is usually refered to as Euclidean n-space and is denoted by R^n.

3.1.4 Definition. Let W be a subset of a vector space over a field F. W is called a subspace of V if W is itself a vector space over F with respect to the operations of vector addition and scalar multiplication on V.

3.1.5 Definition A subset L of a vector space V is said to be linear variety of V if L = u + W for some u in V and some subspace W of V.

The subspace W is called base space of the linear variety L.

2. Dimension of vector space.

3.2.1 Definition. Let V be a vector space over a field F and let $v_1, v_2, \ldots v_m \in V$. Any vector in V of the form

$$a_1v_1 + a_2v_2 + \ldots + a_mv_m$$

where the $a_i \in F$, is called a linear combination of $v_1, v_2, \ldots, v_m$.
The following theorem can be easily verified.

3.2.2 Theorem. Let S be a nonempty subset of V. The set of all linear combinations of vectors in S, denoted by L(S), is a subspace of V containing S. It is called the subspace spanned or generated by S.

3.2.3 Definition. Let V be a vector space over a field F. The vector $v_1, v_2, \ldots, v_m$ are said to be linearly independent, if for every choice of scalars $a_1, a_2, \ldots, a_m \in F$,

$$a_1u_1 + a_2u_2 + \ldots + a_mu_m = 0$$

implies

$$a_1 = a_2 = \ldots = a_m = 0$$

3.2.4 Definition. A vector space V is said to be of dimension n, if there exists linearly independent vectors $u_1, u_2, \ldots, u_n$ which span V. The sequence $u_1, u_2, \ldots, u_n$ is then called a basis of V.

3.2.5 Definition. The dimension of a linear variety is defined to be the dimension of the base space of the linear variety.

3.2.6 Definition. Let $U = \{u_1, u_2, \ldots, u_n\}$ be a subset of a real inner product space V. The set U is said to be orthonormal if

$$(u_i, u_j) = \begin{cases} 0 & \text{for } i \neq j, \\ 1 & \text{for } i = j \end{cases}$$

3.2.7 Theorem. (Gram-Schmidt theorem). Let

$$\mathcal{U} = \{ U_1, U_2, \ldots, U_m \} \qquad (1)$$

be any finite set of linear independent vectors of R^n. Then R^n contains a set of vectors

$$\mathcal{V} = \{ V_1, V_2, \ldots, V_m \}$$

such that

(i) The set $\mathcal{V}$ is orthonormal ;

(ii) Every vector V_k is a linear combination

$$V_k = a_{k1}U_1 + a_{k2}U_2 + \ldots + a_{kk}U_k$$

of the vectors $U_1, U_2, \ldots, U_k$;

(iii) Every vector U_k is a linear combination

$$U_k = b_{k1}V_1 + b_{k2}V_2 + \dots b_{kk}V_k$$

of the vector $V_1, V_2, \dots V_k$ for $k = 1,2,\dots,m$.
Moreover, every vector of $\mathcal{V}$ is uniquely determined by these conditions to within a factor of ± 1.

Proof. First we construct V_1. Setting

$$V_1 = a_{11}U_1,$$

we determine a_{11} from the condition

$$(V_1, V_1) = a_{11}^2(U_1, U_1) = 1,$$

which implies

$$a_{11} = \frac{1}{b_{11}} = \frac{1}{\sqrt{(U_1, U_1)}} .$$

This obviously determines V_1 uniquely (except for sign).

Next suppose vectors $V_1, V_2, \dots, V_{k-1}$ satisfying the conditions of theorem have already been constructed. Then U_k can be written in the form

$$U_k = b_{k1}V_1 + \dots + b_{kk-1}V_{k-1} + W_k \qquad (2)$$

where

$$(W_k, V_j) = 0 \qquad (j = 1,2,\dots, k-1).$$

In fact, the coefficients b_{kj} and hence the vector W_k are uniquely determined by the conditions

$$(W_k, V_j) = (U_k - b_{k1}V_1 - \ldots - b_{kk-1}V_{k-1},\ V_j)$$

$$= (U_k, V_j) - b_{kj}(V_j, V_j) = 0,$$

i.c., $b_{kj} = (U_k, V_j) \quad (j = 1,2,\ldots,k-1).$

Clearly $(W_k, W_k) > 0$, if $(W_k, W_k) = 0$ by Definition 3.1.2,

$W_k = \theta$, and thus

$$U_k - b_{k1}V_1 - \ldots - b_{kk-1}V_{k-1} = \theta, \qquad \ldots\ldots (3)$$

where θ is the zero vector.

By induction hypothesis, we can write V_j on the left hand side of Equ.(3), interm of $U_1, U_2, \ldots, U_j$ $(j = 1,2,\ldots, k-1)$.

Then the zero vector in Equ.(3), can be written as a linear conbination of $U_1, U_2, \ldots, U_k$ for which not all coefficients of U_k are zero (since the coefficient of $U_k = 1$). This contradicts the assumed linear independence of the vectors (1). Let

$$V_k = \frac{W_k}{\sqrt{(W_k, W_k)}}. \qquad \ldots\ldots (4)$$

Using (2) and (4), we express W_k and hence V_k in terms of the functions $U_1, U_2, \ldots, U_k$, i.e.,

$$V_k = a_{k1}U_1 + a_{k2}U_2 + \ldots + a_{kk}U_k,$$

where

$$a_{kk} = \frac{1}{\sqrt{(W_k, W_k)}}.$$

Moreover

$$(V_k, V_j) = 0 \quad (j = 1,2,\ldots, k-1),$$

$$(V_k, V_k) = 1$$

and

$$U_k = b_{k1}V_1 + b_{k2}V_2 + \ldots + b_{kk}V_k ,$$

where

$$b_{kk} = \sqrt{(W_k, W_k)}$$

Thus, start from the vectors $V_1, V_2, \ldots, V_{k-1}$ satisfying the conditions of the theorem, we have constructed vectors $V_1, V_2, \ldots, V_{k-1}, V_k$ satisfying the same conditions.

The proof now follows by mathematical induction.

3. Linear manifold in R^n.

3.3.1 Definition. A set in R^n is a k-dimensional linear manifold if the corresponding vectors form a k-dimensional linear variety of R^n.

3.3.2 Remark. From definition 3.1.5, let L be a linear manifold of dimension k. Then the vectors of the linear variety L are all vectors of the form

$$U = V_0 + t_1W_1 + t_2W_2 + \ldots + t_kW_k ,$$

where $t_1, t_2, \ldots, t_k$ are arbitrary scalars, V_0 is a constant vector and $W_1, W_2, \ldots, W_k$ are linearly independent vectors in L.

4. Euclidean motion of R^n.

3.4.1 Definition. A mapping $F : R^n \longrightarrow R^n$ is called a linear mapping (or linear transformation) if it satisfies the following two conditions.

(i) For any $U, V \in R^n$, $F(U+V) = F(U) + F(V)$.

(ii) For any $a \in R$ and any $U \in R^n$, $F(aU) = aF(U)$.

3.4.2 Definition. A translation R in R^n is any transformation of the form

$$R(U) = U + C ,$$

where $U \in R^n$ and C is a constant vector in R^n.

3.4.3 Definition. An affine transformation T of R^n is any transformation of the form

$$T(U) = F(U) + C ,$$

where F is a linear mapping of R^n and C is a fixed vector in R^n.

3.4.4 Definition. A Euclidean motion of R^n is an affine map $T(V) = F(V) + D$, where F is a linear mapping and D is a constant vector in R^n, such that

$$(F(U), F(V)) = (U, V) , \quad U, V \in R^n.$$

3.4.5 Theorem Given two orthonormal basis of $R^n (E_1, E_2, \ldots, E_n)$ and $(F_1, F_2, \ldots, F_n)$. Then there exists a unique Euclidean motion T such that

$$T(E_i) = F_i, \quad i = 1, 2, \ldots, n .$$

Proof. Given any two vectors p_1, p_2 in R^n, then clearly there is a unique translation R in R^n such that

$$R(p_1) = p_2$$

So from now we can assume that $(E_1, E_2, \ldots, E_n)$ and $(F_1, F_2, \ldots, F_n)$ have the same initial point, which we can take as the origin.

Claim that there is a unique linear map F such that

$$F(E_i) = F_i, \quad i = 1,2,\ldots, n.$$

Define a mapping $F : R^n \longrightarrow R^n$ as follow :

Let $V \in R^n$. Since $(E_1, E_2, \ldots, E_n)$ is a basis of R^n, there exist unique scalars $a_1, a_2, \ldots, a_n \in R$ for which $V = a_1E_1 + a_2E_2 + \ldots + a_nE_n$. We define $F : R^n \longrightarrow R^n$ by

$$F(V) = a_1F_1 + a_2F_2 + \ldots + a_nF_n .$$

(Since the a_i are unique, the mapping F is well-defined.) Now, for $i = 1,2,\ldots, n$,

$$E_i = 0E_1 + \ldots + 1E_i + \ldots + 0E_n .$$

Hence

$$F(E_i) = 0F_1 + \ldots + 1F_i + \ldots + 0F_i = F_i$$

To prove F is linear. Suppose $V = a_1E_1 + a_2E_2 + \ldots + a_nE_n$ and $W = b_1E_1 + b_2E_2 + \ldots + b_nE_n$. Then

$$V + W = (a_1 + b_1)E_1 + (a_2 + b_2)E_2 + \ldots + (a_n + b_n)E_n$$

and, for any $k \in R$, $kV = ka_1E_1 + ka_2E_2 + \ldots + ka_nE_n$.

By definition of the mapping F,

$$F(V) = a_1F_1 + a_2F_2 + \ldots + a_nF_n \text{ and } F(W) = b_1F_1 + b_2F_2 + \ldots + b_nF_n .$$

Hence $F(V+W) = (a_1+b_1)F_1+(a_2+b_2)F_2+\dots+(a_n+b_n)F_n$

$= (a_1F_1+a_2F_2+\dots+a_nF_n)+(b_1F_1+b_2F_2+\dots+b_nF_n)$

$= F(V) + F(W)$

and $F(kV) = k(a_1F_1+a_2F_2+\dots+a_nF_n) = kF(V)$.

Thus F is linear .

Now suppose $G : R^n \longrightarrow R^n$ is linear and $G(E_i) = F_i$, $i = 1,2,\dots, n$ if $V = a_1E_1+a_2E_2+\dots+a_nE_n$, then

$$G(V) = G(a_1E_1+a_2E_2+\dots+a_nE_n) = a_1G(E_1)+a_2G(E_2)+\dots+a_nG(E_n)$$

$$= a_1F_1 + a_2F_2+\dots+a_nF_n = F(V)$$

Since $G(V) = F(V)$ for every $V \in R^n$, $G = F$. Thus F is unique. Thus our claim is proved.

Finally we shall show that T is inner product preserving. Suppose $V = a_1E_1+a_2E_2+\dots+a_nE_n$ and $W = b_1E_1+b_2E_2+\dots+b_nE_n$. Then

$$(V,W) = (a_1E_1+a_2E_2+\dots+a_nE_n,\ b_1E_1+b_2E_2+\dots+b_nE_n)$$

$$= \sum_{i=1}^{n} a_ib_i(E_i,E_i) + \sum_{i\neq j} a_ib_j(E_i,E_j)$$

and $(F(V), F(W)) = (a_1F_1+a_1F_2+\dots+a_nF_n,\ b_1F_1+b_2F_2+\dots+b_nF_n)$

$$= \sum_{i=1}^{n} a_ib_i(F_i,F_i) + \sum_{i\neq j} a_ib_j(F_i,F_j)$$

By orthonormality of $(E_1,E_2,\dots,E_n)$ and $(F_1,F_2,\dots,F_n)$ the last two equation becomes

$$(V,W) = \sum_{i=1}^{n} a_ib_i$$

and $$(F(V), F(W)) = \sum_{i=1}^{n} a_i b_i$$

Which gives

$$(V,W) = (F(V), F(W)) , \quad V,W \in R^n.$$

The required Euclidean motion is the mapping $T = R \circ F$, and T is unique because of the uniqueness of R and F.

Hence the theorem is proved.