

ปัญหาการเลือกในโครงสร้างโอมินิมอล

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SELECTION PROBLEMS IN O-MINIMAL STRUCTURES

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Department of Mathematics and Computer Science

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ในวิทยานิพนธ์ฉบับนี้เราได้ปรับปรุงทฤษฎีบทการเลือกที่นิยามได้ของไมเคิลในโครงสร้างขยายของเรียลโคลสฟิลด์ที่โอมินิมอล จากนั้นเราแสดงการประยุกต์ใช้ทฤษฎีบทนี้ เช่น เราพิสูจน์ข้อความต่อไปนี้ “กำหนดให้ \mathcal{R} เป็นโครงสร้างขยายของ \mathbb{R} ที่โอมินิมอล และ $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ เป็นการส่งค่าเซตที่นิยามได้โดยที่ $n = 1$ หรือ $m = 1$ ถ้า T มีการเลือกแบบต่อเนื่องแล้ว T มีการเลือกแบบต่อเนื่องที่นิยามได้” ยิ่งไปกว่านั้นเรายังพิสูจน์ข้อความต่อไปนี้ “กำหนดให้ \mathcal{R} เป็นโครงสร้างขยายของเรียลโคลสฟิลด์ R ที่โอมินิมอล และ E เป็นสับเซตปิดของ R^n ถ้า $T : E \rightrightarrows R^m$ เป็นการส่งค่าเซตแบบต่อเนื่องที่นิยามได้และสำหรับทุกสมาชิก x ในขอบของเซต E $T(x)$ มีขอบเขตแล้ว T มีการส่งภาคขยายแบบต่อเนื่องที่นิยามได้”

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In this dissertation, we improve the Definable Michael's Selection Theorem in o-minimal expansions of real closed fields. Then applications of this theorem are established; for instance, we prove the following statement: *Let \mathfrak{R} be an o-minimal expansion of \mathbb{R} and $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a definable set-valued map where $n = 1$ or $m = 1$. If T has a continuous selection, then T has a definable continuous selection.* Moreover, we prove the statement: *Let \mathfrak{R} be an o-minimal expansion of a real closed field R and E be a closed subset of R^n . If $T : E \rightrightarrows R^m$ is a definable continuous set-valued map and $T(x)$ is bounded for each x in the boundary of E , then T has a definable continuous extension.*

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CONTENTS

	page
ABSTRACT IN THAI	iv
ABSTRACT IN ENGLISH	v
ACKNOWLEDGEMENTS	vi
CONTENTS	vii
CHAPTER	
I INTRODUCTION	1
1.1 Set-valued maps	1
1.2 O-minimal structures	2
1.2.1 Definitions	3
1.2.2 Examples	4
1.2.3 Geometry of definable sets	8
1.3 Outline	17
1.4 Conventions and notations	18
II GENERRALIZATION OF DEFINABLE MICHAEL SELECTION	
THEOREM	20
2.1 Preliminaries	20
2.2 Proof of Theorem 2.1	23
III GLEASER REFINEMENT	27
IV APPLICATIONS	34
4.1 Existence of selections and definable selections	34
4.2 Existence of definable ϵ -approximate selections	37
4.3 Existence of fixed points of definable set-valued maps	39
4.4 Approximation of upper semi-continuous set-valued maps by using	
C^p -maps	41
V DEFINABLE EXTENSION OF CONTINUOUS SET-VALUED MAPS	44
REFERENCES	49

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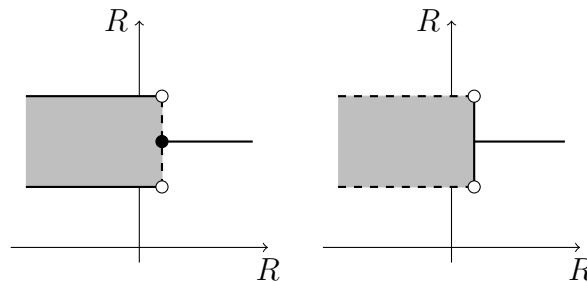
CHAPTER I

INTRODUCTION

1.1 Set-valued maps

In this section, we let X and Y be sets. A **set-valued map** is a map T from X to the power set of Y and we denote such a map by $T: X \rightrightarrows Y$. The concept of set-valued maps is widely used in optimal control theory and mathematical economics (see [1]).

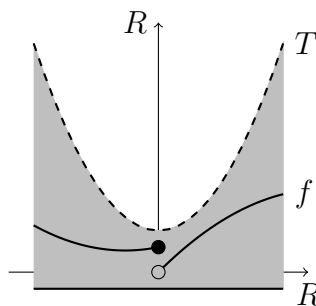
For the rest of this section, let R be an ordered field, $X \subseteq R^n, Y \subseteq R^m$, and $T: X \rightrightarrows Y$ be a set-valued map. The maximum R -norm $\|x\|$ on R^n is defined by $\|x\| := \max_{i=1, \dots, n} \{|x_i|\}$ where $|\cdot|$ is the absolute value function on R . We equip X and Y with maximum R -norms. Let $x \in R^n, S \subseteq R^n$, and $\epsilon > 0$ ($\epsilon \in R$). We let $B_n(x, \epsilon) := \{y \in R^n : \|x - y\| < \epsilon\}$ and $B_n(S, \epsilon) := \bigcup_{s \in S} B_n(s, \epsilon)$. We say that T is **lower semi-continuous** if for each $x_0 \in X, y_0 \in T(x_0)$ and $\epsilon > 0$, there is $\delta > 0$ such that for every $x \in B_n(x_0, \delta) \cap X, T(x) \cap B_m(y_0, \epsilon) \neq \emptyset$; **upper semi-continuous** if for each $x_0 \in X$ and $\epsilon > 0$, there is $\delta > 0$ such that for all $x \in B_n(x_0, \delta) \cap X, T(x) \subseteq B_m(T(x_0), \epsilon)$; and **continuous** if it is both lower semi-continuous and upper semi-continuous. Note that if f is a continuous map, then f is continuous as a set-valued map.



Examples of a lower semi-continuous set-valued map (left)
and an upper semi-continuous set-valued map (right).

A **selection of T** is a map $f: X \rightarrow Y$ such that $f(x) \in T(x)$ for every $x \in X$. In [19], E. Michael studied the existence of continuous selections of set-valued maps and proved the following statement: *if $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is lower semi-continuous and each $T(x)$ is nonempty closed and convex, then T has a continuous selection.* This statement, which is known as Michael’s Selection Theorem, has applications in various fields of mathematics (see [23], [25], and [17]).

In this dissertation, we make no distinction between set-valued maps and their graphs.



An example of a selection f of a set-valued map T .

1.2 O-minimal structures

The beginning of the study of o-minimal structures arose from Tarski’s question on the decidability of the theory of the real field with the exponential function, called “the real exponential field.” While the quantifier eliminability of the real field is obtained by Tarski’s Theorem, L. van den Dries showed that the theory of the real exponential field does not admit quantifier elimination (see [5]). However, he argued that the theory of the real field and the theory of the real exponential field share many nice finiteness properties. Then he proved that these finiteness properties can be deduced from a certain axiom, which he called “finite type”. Later, this axiom is known as “o-minimality”. This term was introduced by A. Pillay and C. Steinhorn in [22].

A. Grothendieck has an influence in this subject. One of his aims is to study topological objects that satisfy finiteness properties and have tame behaviors (ex-

cluding objects like the topologist's sine curve, space-filling curves, fractals, etc.). This field is known as tame topology (or *topologie modérée* in French). See [14] for more details. It turns out that the o-minimality is the right framework for tame topology.

1.2.1 Definitions

First, we introduce the concept of structures (*cf.* [7]). Let R be a set. Let $\mathfrak{R} := (\mathfrak{R}_n)_{n \in \mathbb{N}}$ and $\mathfrak{R}' := (\mathfrak{R}'_n)_{n \in \mathbb{N}}$ be sequences such that $\mathfrak{R}_n, \mathfrak{R}'_n$ are subsets of the power set of R^n . We say that $\mathfrak{R} \subseteq \mathfrak{R}'$ if $\mathfrak{R}_n \subseteq \mathfrak{R}'_n$ for each $n \in \mathbb{N}$. Let $\{f_i\}_{i \in I}$ be a collection of functions $f_i : R^{n_i} \rightarrow R$ and $\{r_j\}_{j \in J}$ be a collection of relations $r_j \subseteq R^{m_j}$. A **structure on R generated by $\{f_i\}_{i \in I}, \{r_j\}_{j \in J}$** is the smallest sequence $\mathfrak{R} := (\mathfrak{R}_n)_{n \in \mathbb{N}}$ with respect to \subseteq such that for each natural number n :

1. \mathfrak{R}_n is a boolean algebra of subsets of R^n , that is, \mathfrak{R}_n is closed under finite intersection and relative complement.
2. The diagonal sets $\Delta_{ij} := \{x \in R^n : x_i = x_j\} \in \mathfrak{R}_n$ for $1 \leq i < j \leq n$;
3. If $A \in \mathfrak{R}_{n+1}$, then $\pi A \in \mathfrak{R}_n$ where $\pi : R^{n+1} \rightarrow R^n$ is the projection map onto the first n coordinates;
4. If $A \in \mathfrak{R}_n$, then $A \times R \in \mathfrak{R}_{n+1}$ and $R \times A \in \mathfrak{R}_{n+1}$;
5. The singleton $\{r\} \in \mathfrak{R}_1$ for each $r \in R$;
6. For each $i \in I, j \in J$, there exist $m, k \in \mathbb{N}$ such that $f_i \in \mathfrak{R}_m, r_j \in \mathfrak{R}_k$.

We denote such a structure by $(R; \{f_i\}_{i \in I}, \{r_j\}_{j \in J})$ and let $\mathfrak{L}_{\mathfrak{R}} := \{f_i\}_{i \in I} \cup \{r_j\}_{j \in J}$ be the “**language**” of this structure. We say that \mathfrak{R} is a **structure on R** if there exist a collection of functions and a collection of relations such that \mathfrak{R} is a structure on R generated by these collections. Let $\mathfrak{R} := (R; \{f_i\}_{i \in I}, \{r_j\}_{j \in J})$ be a structure and $\{f_\alpha\}_{\alpha \in \Lambda}$ be a collection of functions $f_\alpha : R^{k_\alpha} \rightarrow R$. Then a structure $(R; \{f_i\}_{i \in I} \cup \{f_\alpha\}_{\alpha \in \Lambda}, \{r_j\}_{j \in J})$ is denoted by $\mathfrak{R} \cup \{f_\alpha\}_{\alpha \in \Lambda}$.

Next, let A be a subset of R^n . We say that A is **definable in \mathfrak{R}** or **\mathfrak{R} defines A** if $A \in \mathfrak{R}_n$. We simply say A is **definable** if the structure is clear from the context. A map $f: A \rightarrow R^m$ is **definable in \mathfrak{R}** if $f \in \mathfrak{R}_{n+m}$. Similarly, a set-valued map $T: A \rightrightarrows R^m$ is **definable in \mathfrak{R}** if $T \in \mathfrak{R}_{n+m}$.

Moreover, let \mathfrak{R} and \mathfrak{R}' be structures on R such that $\mathfrak{R} \subseteq \mathfrak{R}'$. We call \mathfrak{R}' an **expansion** of \mathfrak{R} and call \mathfrak{R} a **reduct** of \mathfrak{R}' . Note that the notion of structures is, in fact, a geometric point of view of the collection of all sets which is “definable in \mathfrak{R} with parameters” in first order logic. In other words, a definable set is a set that can be described by a first-order $\mathfrak{L}_{\mathfrak{R}}$ -formula with parameters. We can prove that if A is definable in \mathfrak{R} , then $\text{int } A$ the interior, $\text{cl } A$ the closure, and $\text{fr } A := \text{cl } A \setminus A$ the frontier of A are also definable.

Now, we are ready to give the definition of o-minimal structures. Let \mathfrak{R} be an expansion of $(R; <)$ where $<$ is a linear order on R . Then \mathfrak{R} is **o-minimal** if every definable subset of R is a finite union of points and open intervals. Hence, every definable subset of R is also definable in $(R; <)$. In other words, every definable subset of R can be described by a quantifier free formula in language $\{<\}$ with parameters from R . Intuitively, the letter “o” in “o-minimal” stands for “order”.

1.2.2 Examples of o-minimal structures

First, we give examples of o-minimal structures over the real field \mathbb{R} .

Example 1.1. A subset of \mathbb{R}^n is **semialgebraic** if it is a finite union of sets of the form

$$\{x \in \mathbb{R}^n : f(x) = 0, g_1(x) > 0, \dots, g_k(x) > 0\}$$

where f and g_1, \dots, g_k are real polynomials in n variables. Clearly, each semialgebraic set is definable in $\mathbb{R}_{\text{alg}} := (\mathbb{R}; <, +, \cdot)$. In addition, Tarski–Seidenberg Theorem implies that the projection of semialgebraic set is also semialgebraic. Hence, each definable set in \mathbb{R}_{alg} is semialgebraic. Moreover, \mathbb{R}_{alg} is o-minimal since each semialgebraic subset of \mathbb{R} is a finite union of points and intervals. For more details on the structure \mathbb{R}_{alg} , see [7, Chapter 2].

Example 1.2. A set $E \subseteq \mathbb{R}^n$ is called **semianalytic** if for every $x \in \mathbb{R}^n$, there is an open neighborhood U of x such that $E \cap U$ is a finite union of sets of the form

$$\{x \in U : f(x) = 0, g_1(x) > 0, \dots, g_k(x) > 0\}$$

where f and g_1, \dots, g_k are analytic functions on U . Note that the projections of semianalytic sets need not be semianalytic. A set $E \subseteq \mathbb{R}^n$ is called **subanalytic** in \mathbb{R}^n if for every $x \in \mathbb{R}^n$, there is an open neighborhood U of x and a bounded semianalytic set $S \subseteq \mathbb{R}^{n+m}$ such that $U \cap E = \pi S$, where $\pi : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$ is the projection on the first n coordinates. In [6], L. van den Dries introduced a notion called finitely subanalytic sets. A set $E \subseteq \mathbb{R}^n$ is called **finitely subanalytic** (also known as **globally subanalytic**) if its image under the analytic isomorphism map from \mathbb{R}^n to \mathbb{R}^n :

$$(x_1, \dots, x_n) \mapsto (x_1/\sqrt{1+x_1^2}, \dots, x_n/\sqrt{1+x_n^2})$$

is subanalytic in \mathbb{R}^n . Let $\mathfrak{R}_n := \{E \subseteq \mathbb{R}^n : E \text{ is globally subanalytic}\}$ and $\mathfrak{R} := (\mathfrak{R}_n)_{n \in \mathbb{N}}$. Then L. van den Dries proved that \mathfrak{R} is an o-minimal structure.

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$. Then f is **restricted analytic** if f vanishes outside $[-1, 1]^n$ and there exist an open neighborhood U of $[-1, 1]^n$ and an analytic function $g : U \rightarrow \mathbb{R}$ such that $f = g$ on $[-1, 1]^n$. We denote the real field with restricted analytic functions by

$$\mathbb{R}_{\text{an}} := \mathbb{R}_{\text{alg}} \cup \{f : \mathbb{R}^n \rightarrow \mathbb{R} : f \text{ is restricted analytic and } n \in \mathbb{N}\}.$$

Note that $\mathbb{R}_{\text{an}} = \mathfrak{R}$ and hence \mathbb{R}_{an} is o-minimal.

Example 1.3. An **exponential set** in \mathbb{R}^n is a set of the form:

$$\{(x_1, \dots, x_n) : P(x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n}) = 0\}$$

where P is a real polynomial in $2n$ variables; a **subexponential set** is the image

of an exponential set under a coordinate projection. In 1991, A. J. Wilkie showed in his preprint that the real exponential field is model complete (see [30]). Let

$$\mathbb{R}_{\text{exp}} := \mathbb{R}_{\text{alg}} \cup \{\text{exp}\}.$$

Hence, every definable set in \mathbb{R}_{exp} is a subexponential set. Since every exponential set has finitely many connected components (this fact is called Khovanskii's Theorem), \mathbb{R}_{exp} is an o-minimal structure.

Example 1.4. Next, we consider an expansion of \mathbb{R}_{an} and \mathbb{R}_{exp} , called the real exponential field with restricted analytic functions. Let

$$\mathbb{R}_{\text{an,exp}} := \mathbb{R}_{\text{an}} \cup \{\text{exp}\}.$$

The o-minimality of this structure was first proved by L. van den Dries and C. Miller in 1994 (see [9]) and a simpler proof was given by L. van den Dries, A. Macintyre and D. Marker in [8].

Example 1.5. A **pfaffian chain** is a finite sequence of C^1 -functions $g_1(x), \dots, g_k(x)$ on an open set $U \subset \mathbb{R}^n$ such that for each $i = 1, \dots, k$, $j = 1, \dots, n$,

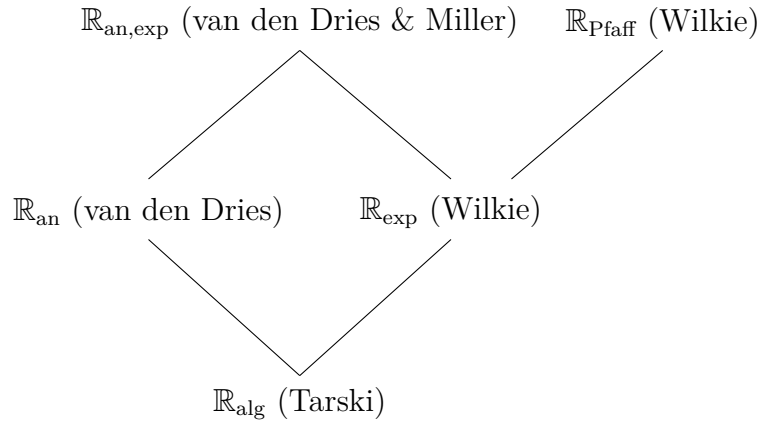
$$\frac{\partial g_i}{\partial x_j}(x) = P_{ij}(x, g_1(x), \dots, g_i(x))$$

where P_{ij} is a real polynomial in $n + i$ variables. A **pfaffian function** is a function which belongs to a pfaffian chain. Note that polynomials and exponential functions are pfaffian while the sine function is not. Let $g_1(x), \dots, g_k(x)$ be a pfaffian chain on open set U containing $[-1, 1]^n$ and let $f_1(x), \dots, f_k(x)$ be the corresponding truncations to $[-1, 1]^n$. In [29], A. Wilkie proved the model completeness of the structure $(\mathbb{R}; <, +, \cdot, f_1, \dots, f_k)$. Combining with the result of A. Khovanskii in [16], $(\mathbb{R}; <, +, \cdot, f_1, \dots, f_k)$ is o-minimal. Moreover, A. Wilkie proved the o-minimality of

$$\mathbb{R}_{\text{pfaff}} := \mathbb{R}_{\text{alg}} \cup \{f : \mathbb{R}^n \rightarrow \mathbb{R} : f \text{ is pfaffian and } n \in \mathbb{N}\}$$

in 1999 (see [30]). Since the exponential function is pfaffian, $\mathbb{R}_{\text{pfaff}}$ is an expansion of \mathbb{R}_{exp} .

From the above examples, we get the following diagram:



Examples of o-minimal structures over the real field.

The reader may ask whether there is the largest o-minimal expansion \mathcal{M} of the real field. In 2003, J.-P. Rolin, P. Speissegger, and A. Wilkie introduced a new construction of o-minimal expansions of the real field and proved the following:

Theorem 1.6 (J.-P. Rolin, P. Speissegger, and A. Wilkie [24, Theorem 2(1)]). *Let $f : U \rightarrow \mathbb{R}$ be a C^∞ -function where U is an open neighborhood of $[-1, 1]^n$. Then there exist o-minimal expansions of the real field $\mathfrak{R}_1, \mathfrak{R}_2$ and functions $f_1, f_2 : [-1, 1]^n \rightarrow \mathbb{R}$ such that for $i = 1, 2$, f_i is definable in \mathfrak{R}_i and $f = f_1 + f_2$ on $[-1, 1]^n$.*

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by

$$f(x) := \begin{cases} e^{-\frac{1}{x^2}} \sin(1/x), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Note that f is a C^∞ -function and $\{x \in [-1, 1] : f(x) = 0 \text{ and } x \neq 0\}$ is an infinite discrete set. By Theorem 1.6, if there is the largest o-minimal expansion of the real field, then $f \upharpoonright [-1, 1]$ and $\{x \in [-1, 1] : f(x) = 0 \text{ and } x \neq 0\}$ are definable in this structure, which is absurd.

In [26], P. Speissegger introduced the pfaffian closure and proved that the pfaffian closure of an o-minimal expansion of the real field is also o-minimal.

Example 1.7. Let \mathfrak{R} be an o-minimal expansion of $(\mathbb{N}; <)$ where $<$ is a discrete linear order on \mathbb{N} . In [21], A. Pillay and C. Steinhorn proved that all definable one-variable functions in \mathfrak{R} are translations. Therefore, the theory of \mathfrak{R} is well-understood.

Example 1.8. Let R be a dense linearly ordered set without endpoints. The o-minimality of $(R; <)$ is obtained from the quantifier elimination property of the theory of dense linear orderings without endpoints. See [7] for more details.

Next, we introduce more definitions. Let R be a field. We say that R is **formally real** if -1 is not a sum of squares; and we say that R is a **real closed field** if R is formally real and each polynomial over R with single variable has an intermediate value property. Hence, by Intermediate Value Theorem, \mathbb{R} is a real closed field. Note that if R is a real closed field, then $(R; +, \cdot)$ defines a linear order $<$ that makes R an ordered field. Hence, R is a dense linearly ordered set without endpoints.

Example 1.9. Let R be a real closed field. Since the theory of real closed fields admits quantifier elimination, $(R; <, +, \cdot)$ is o-minimal.

Theorem 1.10 (L. van den Dries [7, Chapter 1, Proposition 4.6]). *If R is an ordered ring and there exists an o-minimal expansion of $(R; <, +, \cdot)$, then R is a real closed field.*

We say that a structure \mathfrak{R} is an **expansion of a real closed field** R if \mathfrak{R} is an expansion of $(R; <, +, \cdot)$. In this thesis, we only focus on o-minimal expansions of a real closed field.

1.2.3 Geometry of definable sets

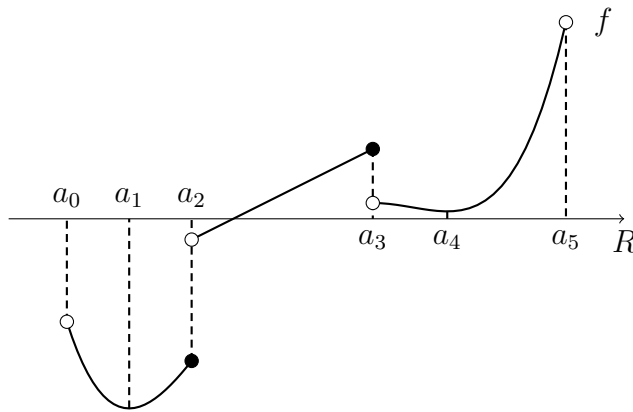
In this section, we recall important results in o-minimal structures, for example, Monotonicity Theorem, Cell Decomposition Theorem, and C^m -Cell Decomposition

Theorem. These theorems tell us that definable sets in o-minimal structures can be partitioned into finitely many “well-behaved” sets. We also state Definable Choice, Triangulation Theorem, Definable Tietze’s Extension, and Trivialization Theorem. For more details and proofs, see [7]. Throughout, let \mathfrak{R} be an o-minimal expansion of a real closed field R and $E \subseteq R^n$.

Monotonicity Theorem

Consider a polynomial in one variable $f(x)$ over the real field. Observe that, as a function, f is piecewise monotone and piecewise continuous. In fact, every one-variable function definable in \mathfrak{R} has this property.

Theorem 1.11 (Monotonicity Theorem). *Let $a \in R \cup \{-\infty\}$ and $b \in R \cup \{+\infty\}$. Let $f : (a, b) \rightarrow R$ be a definable function on the interval (a, b) . Then there are points $a = a_0 < \dots < a_{k+1} = b$ such that for each $0 < i < k + 1$, the restriction of f to (a_i, a_{i+1}) is either constant, or strictly monotone and continuous.*



An example of a partition of (a, b) that satisfies the above conditions.

Let $a \in R \cup \{-\infty\}$, $b \in R \cup \{+\infty\}$, $c \in (a, b)$, and $f : (a, b) \rightarrow R$ be a definable function. By Monotonicity Theorem, $\lim_{x \rightarrow a^+} f(x)$, $\lim_{x \rightarrow b^-} f(x)$, $\lim_{x \rightarrow c^+} f(x)$, and $\lim_{x \rightarrow c^-} f(x)$ exist in $R \cup \{-\infty, +\infty\}$.

Cell Decomposition Theorem

In this part, we introduce the concept of “cells” and state that each definable set in o-minimal structures is a finite union of cells.

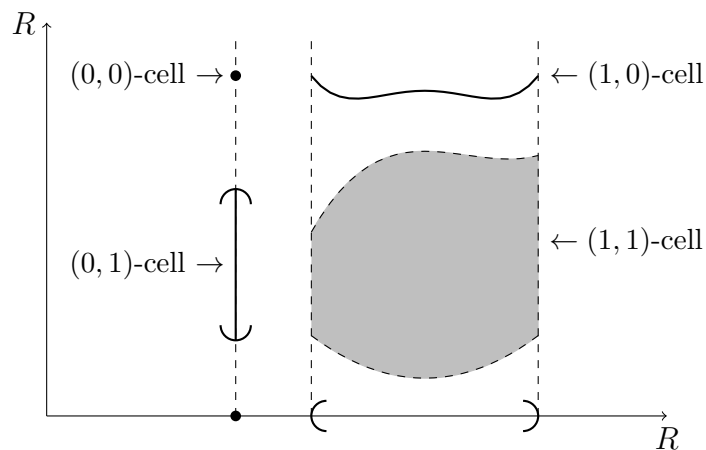
Let E be a definable subset of R^n . We define $C(E)$ to be the collection of all definable continuous functions $f : E \rightarrow R$; and $C_\infty(E) := C(E) \cup \{-\infty, +\infty\}$ where $-\infty$ and $+\infty$ are considered as constant functions on E . Let (i_1, \dots, i_n) be a sequence of zeroes and ones of length n . An (i_1, \dots, i_n) -**cell** is a definable subset of R^n defined inductively on n as follows:

- (i) a (0)-**cell** is a singleton set $\{r\} \subseteq R$, a (1)-**cell** is an open interval $(a, b) \subseteq R$;
- (ii) suppose (i_1, \dots, i_n) -cells are already defined;
 - (a) an $(i_1, \dots, i_n, 0)$ -**cell** is a function in $C(E)$ where E is an (i_1, \dots, i_n) -cell;
 - (b) an $(i_1, \dots, i_n, 1)$ -**cell** is a set

$$(f, g)_E := \{(x, y) \in R^n \times R : x \in E, f(x) < y < g(x)\}$$

where E is an (i_1, \dots, i_n) -cell and $f, g \in C_\infty(E)$ such that $f < g$.

We call such f and g defining functions.



Examples of cells in R^2 .

A **cell** is an (i_1, \dots, i_n) -cell, for some $(i_1, \dots, i_n) \in \{0, 1\}^n$. Note that a cell in R^n is open if and only if it is a $(1, \dots, 1)$ -cell. Moreover, the interior of non-open cells is empty.

We say that a definable set A is **definably connected** if A is not a disjoint union of two nonempty definable open subsets of A .

Proposition 1.12. *Each cell is definably connected.*

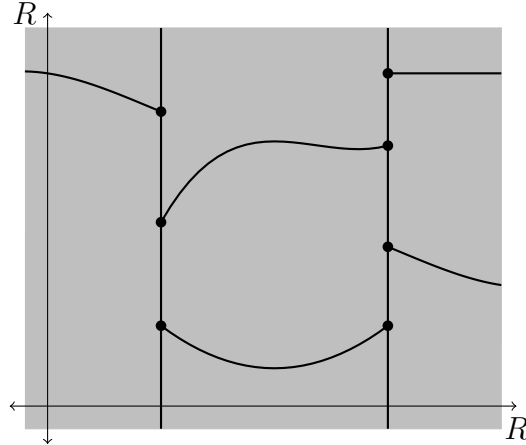
A **cell decomposition** of R^n is a special kind of partitions of R^n into finitely many cells. We define cell decompositions of R^n inductively as follows:

- (i) a cell decomposition of R is a collection

$$\{(-\infty, a_1), (a_1, a_2), \dots, (a_k, +\infty), \{a_1\}, \dots, \{a_k\}\}$$

where $a_1 < \dots < a_k$ are points in R ;

- (ii) a cell decomposition of R^{n+1} is a finite partition \mathcal{C} of R^{n+1} into cells such that $\pi\mathcal{C} := \{\pi C : C \in \mathcal{C}\}$ is also a cell decomposition of R^n where $\pi : R^{n+1} \rightarrow R^n$ is the projection to the first n coordinates.



An example of a cell decomposition of R^2 .

Let $A \subseteq R^n$ and \mathcal{C} be a cell decomposition of R^n . We say that \mathcal{C} **partitions** A if each cell $C \in \mathcal{C}$ is either $C \subseteq A$ or $C \cap A = \emptyset$. The next theorem tells us that cells are the building blocks of definable sets in o-minimal structures.

Theorem 1.13 (Cell Decomposition Theorem).

- (I) Given definable sets $A_1, \dots, A_k \subseteq R^n$, there is a cell decomposition \mathcal{C} of R^n such that for each $i = 1, \dots, k$, \mathcal{C} partitions A_i .
- (II) For each definable function $f : A \rightarrow R$ where $A \subseteq R^n$, there is a cell decomposition \mathcal{C} of R^n such that \mathcal{C} partitions A and for each cell $C \in \mathcal{C}$ with $C \subseteq A$, the restriction $f|_C : C \rightarrow R$ is continuous.

This theorem provides more information on the geometry of definable sets. As a consequence, each definable set has finitely many definably connected components (a definably connected component of a definable set E is a maximal definably connected subset of E) and each definable function is piecewise continuous. Moreover, we get this result:

Theorem 1.14 (Uniform Finiteness Theorem). *Let A be a definable subset of R^{n+m} . If for each $x \in R^n$, $\{y \in R^m : (x, y) \in A\}$ is finite, then there is $N \in \mathbb{N}$ such that $|\{y \in R^m : (x, y) \in A\}| \leq N$ for all $x \in R^n$.*

C^m -Decomposition Theorem

The concept of differentiability makes sense in o-minimal context. Let $m \in \mathbb{N}$ and $f : E \rightarrow R^k$ be definable. Then f is C^m if f is the restriction of a definable C^m -map $g : U \rightarrow R^k$ where U is a definable open set containing E . For each $m \in \mathbb{N}$, let $C^m(E)$ be the collection of all definable C^m -functions $f : E \rightarrow R$ and $C_\infty^m(E) := C^m(E) \cup \{-\infty, +\infty\}$.

Now we can define C^m -cells and C^m -cell decompositions in the same manner as the above, by replacing $C(E)$ and $C_\infty(E)$ by $C^m(E)$ and $C_\infty^m(E)$, respectively. Then we get this modified version of the Cell Decomposition Theorem:

Theorem 1.15 (C^m -Cell Decomposition Theorem). *Let $m \in \mathbb{N}$. Then*

- (I) *Given definable sets $A_1, \dots, A_k \subseteq R^n$, there is a C^m -cell decomposition \mathcal{C} of R^n such that for each $i = 1, \dots, k$, \mathcal{C} partitions A_i .*

(II) For each definable function $f : A \rightarrow R$ where $A \subseteq R^n$, there is a C^m -cell decomposition \mathcal{C} of R^n such that \mathcal{C} partitions A and for each cell $C \in \mathcal{C}$ with $C \subseteq A$ the restriction $f|_C : C \rightarrow R$ is C^m .

Since the above theorem holds for every natural number m , the smooth version of the Cell Decomposition Theorem (where smooth cells and smooth cell decomposition are defined in the same fashion) looks promising. However, in [15], O. Le Gal and J.-P. Rolin showed that there is an o-minimal expansion of a real closed field that does not admit smooth cell decomposition.

Invariant properties

We introduce concepts of “dimension” and “Euler characteristic” of definable sets and state that they are invariant under definable bijection. We define the **dimension** of a nonempty definable set $A \subseteq R^n$ by

$$\dim(A) := \max\{i_1 + \dots + i_n : A \text{ contains an } (i_1, \dots, i_n)\text{-cells}\}$$

(equivalently,

$$\dim(A) := \max\{k \in \mathbb{N} : \text{there is a coordinate projection } \pi : R^n \rightarrow R^k \text{ such that } \text{int } \pi A \neq \emptyset\}$$

and assign the dimension of the empty set to be $-\infty$. Then we have the following theorem:

Theorem 1.16. *Let $A \subseteq R^n$ be nonempty and definable. Then $\dim(\text{cl } A \setminus A) < \dim A$. In particular, $\dim \text{cl } A = \dim A$.*

This theorem implies that the topologist’s sine curve and space-filling curves are not definable in o-minimal structures.

For each cell C of dimension d , we define the **Euler characteristic** of C to be

$E(C) := (-1)^d$. Let $A \subseteq R^n$. We define the **Euler characteristic** of A to be

$$E(A) := \sum_{C \in \mathcal{P}} E(C)$$

where \mathcal{P} is a finite partition of A into cells. Note that the Euler characteristic of A does not depend on \mathcal{P} . Moreover, we get this result:

Proposition 1.17. *Let $A \subseteq R^n$ and $B \subseteq R^m$ be definable sets. Then there exists a definable bijection from A to B if and only if $\dim A = \dim B$ and $E(A) = E(B)$.*

For more details on properties of dimension and Euler characteristic, we refer to [7, Chapter 3].

Definable Choice

Next is a definable analog of the Axiom of Choice in o-minimal context.

Theorem 1.18 (Definable Choice). *If $A \subseteq R^{n+m}$ is definable and $\pi : A \rightarrow R^n$ is the projection to the first n coordinates, then there is a definable map $f : \pi A \rightarrow R^m$ such that $f \subseteq A$. Moreover, if $x, x' \in \pi A$ such that $\{y \in R^m : (x, y) \in A\} = \{y \in R^m : (x', y) \in A\}$, then $f(x) = f(x')$.*

The following are consequences of the Definable Choice.

Corollary 1.19 (Curve Selection). *Let X be a definable set and $a \in \text{cl } X \setminus X$. Then there exist $\epsilon > 0$ and a definable continuous injective map $\gamma : (0, \epsilon) \rightarrow X$ such that $\lim_{t \rightarrow 0} \gamma(t) = a$.*

By the Definable Choice, we can show that closed bounded definable sets in o-minimal structures have many properties resemble compact sets.

Proposition 1.20. *Let $f : E \rightarrow R^m$ be a definable continuous map on a nonempty closed bounded set E . Then*

1. $f(E)$ is closed and bounded in R^m .
2. f has a maximum and a minimum values.
3. f is uniformly continuous.

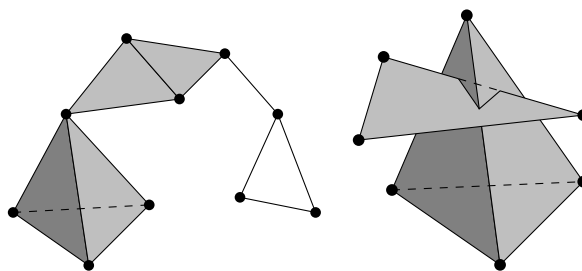
Triangulation Theorem

In this part, we introduce the concept of “simplices” and “complexes”. We also state that each definable set can be triangulated. Hence, the topology of definable sets can be completely described via finite unions of simplices. For more details, see [7, Chapter 6].

Let $a_0, \dots, a_k \in R^n$ be **affinely independent**, that is, $a_1 - a_0, \dots, a_k - a_0$ are linearly independent. Then a k -**simplex** in R^n spanned by $\{a_0, \dots, a_k\}$ is the set

$$(a_0, \dots, a_k) := \left\{ \sum_{i=0}^k t_i a_i : t_i \geq 0, \sum_{i=0}^k t_i = 1 \right\}.$$

We say that a set $S \subseteq R^n$ is a **simplex** if there exist $k \in \{0, \dots, n\}$ and affinely independent elements a_0, \dots, a_k in R^n such that $S = (a_0, \dots, a_k)$. A **face** of (a_0, \dots, a_k) is a simplex spanned by a nonempty subset of $\{a_0, \dots, a_k\}$. A **complex** in R^n is a finite collection K of simplices in R^n such that for all $\sigma_1, \sigma_2 \in K$, either $\text{cl } \sigma_1 \cap \text{cl } \sigma_2 = \emptyset$, or there is a face σ of σ_1 and σ_2 such that $\text{cl } \sigma_1 \cap \text{cl } \sigma_2 = \text{cl } \sigma$. Let K be a finite collection of simplices in R^n . Then we denote the union of the simplices of K by $|K|$.



An example of $|K|$ where K is a complex in R^3 (left) and an example of $|K|$ where K is not a complex in R^3 (right).

Theorem 1.21 (Triangulation Theorem). *Let A be a definable subset of R^n . Then there is a complex K in R^n and a definable homeomorphism $h : A \rightarrow |K|$.*

There are many useful consequences of this theorem. For example,

Lemma 1.22. *Let E_0, E_1 be disjoint definable closed subsets of a definable set E . Then there exist disjoint definable open subsets U_0, U_1 of E with $E_i \subseteq U_i, i = 0, 1$.*

Lemma 1.23. *Let E_0, E_1 be disjoint definable closed subsets of a definable set E . Then there is a definable continuous function $f: E \rightarrow [0, 1]$ such that $f^{-1}(0) = E_0$ and $f^{-1}(1) = E_1$.*

Let $A \subseteq E \subseteq R^n$. We say that a map $r: E \rightarrow A$ is a **retraction** if r is continuous and the restriction $r|_A$ is an identity map on A .

Corollary 1.24. *Let A be a definable closed subset of a definable set E . Then there are a definable open subset U of E containing A and a definable retraction $r: \text{cl}U \cap E \rightarrow A$.*

Theorem 1.25 (Definable Tietze's Extension). *Let A be a closed subset of a definable set E and $f: A \rightarrow R^m$ be a definable continuous map. Then there is a definable continuous map $g: E \rightarrow R^m$ such that $g = f$ on A .*

Trivialization Theorem

Throughout the rest of this section, let $A \subseteq R^m, S \subseteq R^n$ be definable sets and $f: S \rightarrow A$ be a definable continuous map. A **definable trivialization of f** is a pair (F, λ) consisting of a definable set $F \subseteq R^N$ and a definable map $\lambda: S \rightarrow F$ such that $(f, \lambda): S \rightarrow A \times F$ is a homeomorphism. If f has a definable trivialization (F, λ) , then f behaves like a projection map $A \times F \rightarrow A$; more precisely, let π be the projection on the first m coordinates, then the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{(f, \lambda)} & A \times F \\ & \searrow f & \swarrow \pi \\ & & A \end{array}$$

Note that for each $a \in A, f^{-1}(a)$ are homeomorphic to F . We call f **definably trivial** if f has a definable trivialization. Given a definable subset $A' \subseteq A$, we say that f is **definably trivial over A'** if the restriction $f|_{f^{-1}(A')}: f^{-1}(A') \rightarrow A'$ is

definably trivial. Note that if (F, λ) is a definable trivialization of f and $A' \subseteq A$, then $(F, \lambda|_{f^{-1}(A)})$ is a definable trivialization of $f|_{f^{-1}(A)} : f^{-1}(A) \rightarrow A'$.

Now we can state this theorem:

Theorem 1.26 (Trivialization Theorem [7, Chapter 9, Theorem 1.2]).

We can partition A into definable subsets A_1, \dots, A_k , such that for each $i = 1, \dots, k$, f is definably trivial over A_i .

1.3 Outline

In this dissertation, we mainly work on the selection problems in o-minimal expansion of a real closed field R . Fixed point problems and extension problems in o-minimal context are also discussed.

In Chapter 2, we focus on the Definable Michael's Selection Theorem. Let $E \subseteq R^n$ and $T : E \rightrightarrows R^m$. In [19, 20], E. Michael gave a sufficient condition for the existence of continuous selections of T when $R = \mathbb{R}$. In [2], A. Thamrongthanyalak and M. Aschenbrenner proved a definable version of Michael's Selection Theorem: *if E is a closed subset of R^n and $T : E \rightrightarrows R^m$ is a definable lower semi-continuous set-valued map such that $T(x)$ is nonempty, closed, and convex for each $x \in E$, then T has a definable continuous selection.* In addition, E. Michael showed that the closedness of E and the closedness of $T(x)$ are not necessary for the existence of continuous selections. Hence, we want to improve the Definable Michael's Selection Theorem in the same manner. By using the Definable Michael's Selection Theorem, we first prove a special case and will use it to prove an improved version of the Definable Michael's Selection Theorem.

In Chapter 3, we introduce the concept of the Glaeser refinement. Roughly speaking, the Glaeser refinement is a tool that we use to shave off "bad points" in set-valued maps (in the sense that set-valued maps are not lower semi-continuous because of these points). We also prepare some results for Chapter 4. For instance, we prove that if each value of a definable set-valued map is convex, then each value of its Glaeser refinement is also convex. Moreover, we prove that definable set-

valued maps will be stable after we apply Glaeser refinement finitely many times. We finish this chapter by showing that the argument in the proof of stability is sharp. More precisely, for each $n \in \mathbb{N}$, we provide an example of a definable set-valued map T_n such that the m^{th} -refinement of T_n is stable if and only if $m \geq n$.

Chapter 4 contains applications of the main theorem in Chapter 2. In Section 4.1, we ask this question: *Let T be a definable set-valued map. Is the existence of continuous selections a sufficient condition for the existence of definable continuous selection of T ?* By using results in Chapter 3, we give a partial answer to this question. Section 4.2 and 4.3 are consequences of Section 4.1. In section 4.2, we want to characterize definable almost lower semicontinuous set-valued maps by using the concept of definable ϵ -approximate selections. In section 4.3, we study the lower semi-continuous version of the Definable Kakutani's Fixed Point Theorem. In the last section, we give an example of a definable upper semi-continuous set-valued map that does not have a continuous selection. Then we prove the following statement: *for each $\epsilon > 0$, if $T : E \rightrightarrows R^m$ is a definable upper semi-continuous set-valued map such that $T(x)$ is nonempty and convex for each $x \in E$, then there is a definable continuous map $f : E \rightarrow R^m$ such that $f \subseteq B_{n+m}(T, \epsilon)$.*

The content of Chapter 5 is independent from the rest of this dissertation. By the Definable Tietze's Extension, we know that each definable continuous map on a closed subset of R^n can be extended to the whole space. Hence, if $T : E \rightrightarrows R^m$ is a definable continuous set-valued map and E is a closed subset of R^n , it is natural to ask whether T has a definable continuous extension. We give a partial answer to this question.

1.4 Conventions and notations

Throughout this dissertation, m, n and k will range over $\mathbb{N} = \{0, 1, 2, 3, \dots\}$. We let R be a real closed field and we equip R^n with the Euclidean R -norm $\| \cdot \|$ defined by $\|(x_1, \dots, x_n)\| :=$ the positive square root of $\sum_{i=1, \dots, n} x_i^2$. We also equip R^n with

the corresponding topology.

Let $\epsilon > 0$, $x \in R^n$ and $S, K \subseteq R^n$. Let $d(x, S) := \inf\{\|x - y\| : y \in S\}$, $B_n(x, \epsilon) := \{y \in R^n : \|x - y\| < \epsilon\}$ and $B_n(S, \epsilon) := \bigcup_{s \in S} B_n(s, \epsilon)$. Also, we just use $B(x, \epsilon)$ and $B(S, \epsilon)$ if its ambient space is clear from the context. Note that $\{B_n(x, \epsilon) : x \in R^n \text{ and } \epsilon > 0\}$ is a topological basis of R^n .

Let $\text{Aff } S := \{\sum_{i=0}^n \alpha_i x_i : x_i \in S, \alpha_i \in R, \sum_{i=0}^n \alpha_i = 1\}$ be the affine hull of S , $\text{int}_K S$ be the relative interior of S in K , $\text{cl}_K S$ be the relative closure of S in K , and $\text{bd } S$ be the boundary of S .

CHAPTER II

GENERALIZATION OF DEFINABLE MICHAEL SELECTION THEOREM

Throughout this chapter, let \mathfrak{R} be an o-minimal expansion of a real closed field R and let $E \subseteq R^n$ and $T : E \rightrightarrows R^m$ be definable. In 1956, E. Michael proved that: ([19, Theorem 3.1''']) *if T is lower semi-continuous and each $T(x)$ is nonempty convex, then T has a continuous selection.* However, the given construction may produce a continuous selection that is far more complicated than how T arose (see [28] for a discussion on this problem). Hence, we study this statement in the o-minimal context. The main goal of this chapter is to prove the following theorem:

Theorem 2.1. *Assume T is lower semi-continuous and $T(x)$ is nonempty and convex for every $x \in E$. Then T has a definable continuous selection.*

2.1 Preliminaries

Now we begin working towards the proof of Theorem 2.1. First we prove the following facts:

Lemma 2.2. *Suppose that T is lower semi-continuous. Then the set-valued map $F : E \rightrightarrows R^m$ defined by $F(x) := \text{cl}T(x)$ is definable and lower semi-continuous.*

Proof. Obviously, F is definable. Let $x_0 \in E$ and $y_0 \in F(x_0)$. Let $\epsilon > 0$. Then there is $y \in B_m(y_0, \epsilon) \cap T(x_0)$. Since T is lower semi-continuous, there is $\delta > 0$ such that for each $x \in B_n(x_0, \delta) \cap E$, $\emptyset \neq B_m(y, \epsilon - \|y - y_0\|) \cap T(x) \subseteq B_m(y_0, \epsilon) \cap F(x)$. □

Lemma 2.3. *Let A be a definable closed subset of E and $f : A \rightarrow R^m$ be a definable selection of $T|_A$. Suppose that T is lower semi-continuous set-valued map and f*

is continuous. Then the set-valued map $F : E \rightrightarrows R^m$ defined by

$$F(x) := \begin{cases} T(x), & \text{if } x \in E \setminus A; \\ \{f(x)\}, & \text{if } x \in A \end{cases}$$

is definable and lower semi-continuous.

Proof. It is clear that F is definable. Let $x_0 \in E$, $y \in F(x_0)$ and $\varepsilon > 0$. Since T is lower semi-continuous, there is $\delta_1 > 0$ such that for each $x \in B(x_0, \delta_1) \cap E$, $T(x) \cap B(y, \varepsilon) \neq \emptyset$. Suppose that $x_0 \in E \setminus A$. Since $E \setminus A$ is open, there exists $\delta_2 > 0$ such that for each $B(x_0, \delta_2) \subseteq E \setminus A$. Choose $\delta = \min\{\delta_1, \delta_2\}$. Then, for each $x \in B(x_0, \delta) \cap E$, $F(x) \cap B(y, \varepsilon) \neq \emptyset$. Next, we assume $x_0 \in A$. Then $y = f(x_0)$. Since f is continuous, there is $\delta_2 > 0$ such that for each $x \in B(x_0, \delta_2) \cap A$, $f(x) \in B(f(x_0), \varepsilon)$. Choose $\delta = \min\{\delta_1, \delta_2\}$. Hence, for each $x \in B(x_0, \delta) \cap E$, $F(x) \cap B(f(x_0), \varepsilon) \neq \emptyset$. \square

Theorem 2.4 (Definable Michael Selection Theorem, M. Aschenbrenner and A. Thamrongthanyalak [2, Theorem 2.2.1]). *Let E be closed and $T : E \rightrightarrows R^m$ be lower semi-continuous. Assume $T(x)$ is nonempty, closed, and convex for each $x \in E$. Then T has a definable continuous selection.*

We can prove that the closedness of E in the above theorem can be omitted.

Theorem 2.5. *Suppose T is lower semi-continuous and $T(x)$ is nonempty, closed and convex for every $x \in E$. Then T has a definable continuous selection.*

To prove this theorem, we recall these lemmas:

Lemma 2.6 (M. Aschenbrenner and A. Thamrongthanyalak [2, Theorem 2.1.4]). *There is a cell decomposition \mathcal{C} of R^n such that \mathcal{C} partitions E and for each $C \in \mathcal{C}$, $T|_C$ is lower semi-continuous and closed in $C \times R^m$.*

Lemma 2.7 (M. Aschenbrenner and A. Thamrongthanyalak [2, Lemma 2.2.2]). *Let $T : (0, 1) \rightrightarrows R^m$ be a definable set-valued map. Let $(0, y) \in \text{cl}T$. Then there are $\epsilon \in (0, 1)$ and a definable continuous map $f : (0, \epsilon) \rightarrow R^m$ such that $f(t) \in T(t)$ for all $t \in (0, \epsilon)$ and $\lim_{t \rightarrow 0^+} f(t) = y$.*

To prove Theorem 2.5, we follow the proof of the Definable Michael's Selection Theorem in [2].

Proof of Theorem 2.5. We proceed by induction on $\dim E$. If $\dim E = 0$, then E is finite and the above statement is trivial. Suppose $\dim E > 0$ and the result holds for every set with dimension less than $\dim E$. By Lemma 2.6, there is a cell decomposition \mathcal{C} partitions E such that for each $C \in \mathcal{C}$, $T|_C$ is lower semi-continuous and closed in $C \times R^m$. By the Cell Decomposition Theorem, there is a definable open subset $S \subseteq E$ such that $\dim E \setminus S < \dim E$ and $T|_S$ is lower semi-continuous and closed in $S \times R^m$. By the induction hypothesis, there is a definable continuous selection $f : E \setminus S \rightarrow R^m$. Since $E \setminus S$ is closed in E , by the Definable Tietze's Extension Theorem, there is a definable continuous function $g : E \rightarrow R^m$ such that $g = f$ on $E \setminus S$. Since $T(x)$ is closed and convex for all $x \in E$, we can define $h : E \rightarrow R^m$ by

$$h(x) := \text{the unique } y \in T(x) \text{ such that } \|x - y\| = d(x, T(x)).$$

Since T is definable, so is h . To show that h is continuous, let $x_0 \in E$ and $\gamma : (0, 1) \rightarrow E$ such that $\lim_{t \rightarrow 0^+} \gamma(t) = x_0$; we need to show that $\lim_{t \rightarrow 0^+} h(\gamma(t)) = h(x_0)$. First, we prove the following claim:

Claim. Let $\epsilon > 0$. Then

$$\|g(x_0) - h(\gamma(t))\| \leq \|g(x_0) - h(x_0)\| + \epsilon \quad \text{as } t \rightarrow 0^+.$$

Proof of Claim. Since T is lower semi-continuous, by Lemma 2.7, there exist $0 < \epsilon < 1$ and a definable continuous function $\phi : (0, \epsilon) \rightarrow R^m$ such that $\phi(t) \in T(\gamma(t))$ for each $0 < t < \epsilon$ and $\lim_{t \rightarrow 0^+} \phi(t) = h(x_0)$. By continuity of g at x_0 , there is $\delta > 0$ such that for all $x \in B_n(x_0, \delta) \cap E$, $\|g(x) - g(x_0)\| < \frac{1}{3}\epsilon$. Let $t_0 > 0$ be such that for each $0 < t \leq t_0$, $\gamma(t) \in B_n(x_0, \delta)$ and $\|\phi(t) - h(x_0)\| < \frac{1}{3}\epsilon$. By the

definition of h ,

$$\|h(\gamma(t)) - g(\gamma(t))\| \leq \|\phi(t) - g(\gamma(t))\| \quad \text{for all } t \in (0, 1).$$

Moreover, for $0 < t \leq t_0$ we have

$$\begin{aligned} \|\phi(t) - g(\gamma(t))\| &\leq \|\phi(t) - h(\gamma(t))\| + \|h(\gamma(t)) - g(x_0)\| + \|g(x_0) - g(\gamma(t))\| \\ &\leq \|g(x_0) - h(\gamma(t))\| + \frac{2}{3}\epsilon \end{aligned}$$

and therefore

$$\begin{aligned} \|g(x_0) - h(\gamma(t))\| &\leq \|g(x_0) - g(\gamma(t))\| + \|g(\gamma(t)) - h(\gamma(t))\| \\ &\leq \frac{1}{3}\epsilon + \|g(\gamma(t)) - \phi(t)\| \\ &\leq \|g(x_0) - h(\gamma(t))\| + \epsilon \end{aligned}$$

as required. \square

By the Monotonicity Theorem and this claim, $y_0 := \lim_{t \rightarrow 0^+} h(\gamma(x))$ exists in R^m and $\|g(x_0) - y_0\| \leq \|g(x_0) - h(x_0)\| + \epsilon$ for every $\epsilon > 0$. Hence, $\|g(x_0) - y_0\| \leq \|g(x_0) - h(x_0)\|$. If $x_0 \in E \setminus S$, then $g(x_0) = h(x_0)$ and hence $y_0 = g(x_0) = h(x_0)$. Suppose $x_0 \in S$. Since $T \upharpoonright S$ is closed in $S \times R^m$, we have $y_0 \in T(x_0)$. By the definition of h , $y_0 = h(x_0)$. Therefore, h is continuous at x_0 . \square

2.2 Proof of Theorem 2.1

First, we will prove the following special case of Theorem 2.1.

Lemma 2.8. *Suppose T is lower semi-continuous, and $T(x)$ is a nonempty and convex subset of $B_m(0, 1)$ for every $x \in E$, then T has a definable continuous selection.*

Proof. We proceed by induction on the dimension of E . If $\dim E = 0$, then E is finite and the result holds trivially. Suppose $\dim E > 0$ and the result holds

for every set with dimension less than $\dim E$. Define $T_0: E \rightrightarrows \mathbb{R}^m$ by $T_0(x) := \text{int}_{\text{Aff}T(x)} T(x)$. Obviously, T_0 is definable and $T_0(x) \subseteq T(x)$ for every $x \in E$. By the Definable Choice and the Cell Decomposition Theorem, there exist $S \subseteq E$ open in E and a definable map $f: E \rightarrow \mathbb{R}^m$ such that $\dim E \setminus S < \dim E$, $f|_S$ is continuous and $f(x) \in T_0(x)$ for every $x \in T_0(x)$. Inductively, let $g_0: E \setminus S \rightarrow \mathbb{R}^m$ be a definable continuous selection of $T|_{E \setminus S}$. Define $T_1: E \rightrightarrows \mathbb{R}^m$ by

$$T_1(x) := \begin{cases} \text{cl} T(x), & \text{if } x \in S; \\ \{g_0(x)\}, & \text{if } x \in E \setminus S. \end{cases}$$

By Lemma 2.2 and 2.3, we have T_1 is lower semi-continuous and definable, and $T_1(x)$ is closed and convex for every $x \in E$. By Theorem 2.5, let g_1 be a definable continuous selection of T_1 .

By Lemma 1.22 and 1.23, there exist $U \subseteq E$ definable open and $t: E \rightarrow [0, 1]$ definable continuous such that $E \setminus S \subseteq U \subsetneq E$, $t^{-1}(0) = E \setminus S$, and $t^{-1}(1) = E \setminus U$. Next, we define $h: E \rightarrow \mathbb{R}^m$ by

$$h(x) := (1 - t(x))g_1(x) + t(x)f(x).$$

Obviously, h is definable. To prove that h is a selection of T , it suffices to prove the following claim:

Claim. If $C \subseteq \mathbb{R}^m$ is convex, $a \in \text{int}_{\text{Aff}C} C$, $b \in \text{cl} C$, and $0 < t < 1$, then $(1 - t)a + tb \in \text{int}_{\text{Aff}C} C$.

Proof of Claim. Let $c = (1 - t)a + tb$. First, assume $b \in \text{int}_{\text{Aff}C} C$. Since $a, b \in \text{int}_{\text{Aff}C} C$, there is $\epsilon > 0$ such that $B_m(a, \epsilon) \cap \text{Aff}C \subseteq C$ and $B_m(b, \epsilon) \cap \text{Aff}C \subseteq C$. We will show that $B_m(c, \epsilon) \cap \text{Aff}C \subseteq C$. Let $d \in B_m(c, \epsilon) \cap \text{Aff}C$. Then $a + d - c \in B_m(a, \epsilon) \cap \text{Aff}C$ and $b + d - c \in B_m(b, \epsilon) \cap \text{Aff}C$. Therefore, $a + d - c, b + d - c \in C$. Since C is convex and $d = (1 - t)(a + d - c) + t(b + d - c)$, $d \in C$.

Now let $b \in \text{cl} C$. Then there is $u \in B_m(b, (1 - t)\epsilon/t) \cap \text{int}_{\text{Aff}C} C$. Since $a \in \text{int}_{\text{Aff}C} C$, there is $\epsilon > 0$ such that $B_m(a, \epsilon) \cap \text{Aff}C \subseteq C$. Let $v := t(b - u)/(1 - t) + a \in \text{Aff}C$. Then $\|v - a\| < \epsilon$. Hence, $v \in \text{int}_{\text{Aff}C} C$. Since $u, v \in \text{int}_{\text{Aff}C} C$

and $tu + (1 - t)v = ta + (1 - t)b = c$, by the above case, $c \in \text{int}_{\text{Aff}C} C$. \square

Therefore, it remains to prove that h is continuous. Since $E \setminus S$ is closed in E , it suffices to show that h is continuous at $x_0 \in E \setminus S$. Let $x_0 \in E \setminus S$ and $\epsilon > 0$. Since g_1 is continuous at x_0 , there is $\delta > 0$ such that for all $x \in B_n(x_0, \delta)$, $\|g_1(x_0) - g_1(x)\| < \epsilon/2$ and $\|t(x)\| = \|t(x_0) - t(x)\| < \epsilon/4$. Let $x \in B_n(x_0, \delta)$. Hence,

$$\begin{aligned} \|h(x_0) - h(x)\| &\leq \|g_1(x_0) - g_1(x)\| + \|t(x)\| \cdot \|g_1(x)\| + \|t(x)\| \cdot \|f(x)\| \\ &< \frac{\epsilon}{2} + \frac{2\epsilon}{4} \\ &= \epsilon. \end{aligned}$$

Therefore, h is continuous at x_0 . \square

Now that we have the above result, one of the most intuitive ideas to finish the proof of Theorem 2.1 is to apply semialgebraic homeomorphisms from $B_m(0, 1)$ to R^m to reduce to the bounded case. However, such homeomorphisms do not preserve the convexity. Therefore, we propose the following:

Proof of Theorem 2.1. We proceed by induction on $\dim E$. The case $\dim E = 0$ is trivial. Suppose that $\dim E > 0$ and the result holds for sets with dimension less than $\dim E$. Suppose T is lower semi-continuous such that $T(x)$ is nonempty and convex for every $x \in E$. Similar to the proof of Lemma 2.8, there exists a definable selection f of T and $S \subseteq E$ definable such that $E \setminus S$ is closed in E , $\dim E \setminus S < \dim E$, and $f|_S$ is continuous. By the inductive hypothesis and Lemma 1.25, let $g: E \rightarrow R^m$ be definable and continuous such that $g(x) \in T(x)$ for every $x \in E \setminus S$. Define $T - g: E \rightrightarrows \mathbb{R}^m$ by

$$T - g(x) := \{y - g(x) : y \in T(x)\}.$$

Observe that $T - g$ is definable and lower semi-continuous, and $T - g(x)$ is nonempty and convex for every $x \in E$. Note that if h is a definable continuous selection of

$T - g$, then $h + g$ is a definable continuous selection of T . Hence, it suffices to assume that $g = 0$.

Let $W = \{x \in E : B_m(0, 1) \cap T(x) \neq \emptyset\}$. Then W is open in E and $E \setminus S \subseteq W$. By Lemma 1.22, there are definable $U, V \subseteq E$ such that U, V are open in E and $E \setminus S \subseteq V \subseteq \text{cl}_E V \subseteq U \subseteq \text{cl}_E U \subseteq W$.

Define $T_0: \text{cl}_E U \rightrightarrows R^m$ by $T_0(x) = B_m(0, 1) \cap T(x)$. It is routine to check that T_0 is definable and lower semi-continuous. By Lemma 2.8 and Lemma 1.25, let $s: E \rightarrow R^m$ be a definable and continuous such that $s(x) \in T_0(x)$ for every $x \in \text{cl}_E U$. Since $\text{cl}_E V \subseteq U$, there is a definable continuous function $t: E \rightarrow [0, 1]$ such that $t^{-1}(0) = \text{cl}_E V$ and $t^{-1}(1) = E \setminus U$. Define $h: E \rightarrow R^n$ by

$$h(x) := (1 - t(x))s(x) + t(x)f(x).$$

Note that h is definable, $h|_V = s|_V$ and $h|_{E \setminus U} = f|_{E \setminus U}$. Since $s(x), f(x) \in T(x)$ for every $x \in U \setminus V$, h is a selection of T . It remains to show that h is continuous. Since s, f and t are continuous on S , $h|_S$ is continuous. The restriction $h|_V$ is continuous because $h|_V = s|_V$. Since $V, E \setminus A$ are open and $V \cap (E \setminus A) \neq \emptyset$, h is continuous. \square

In Chapter 4, we will show some applications of Theorem 2.1.

CHAPTER III

GLAESER REFINEMENT

In this chapter, we study the concept of the Glaeser refinement which is introduced in [13] (this concept is also called derived mapping in [3]). This concept plays an important role in Chapter 4. Throughout this chapter, assume that \mathfrak{R} is an o-minimal expansion of a real closed field R and let $E \subseteq R^n$ and $T: E \rightrightarrows R^m$ be definable.

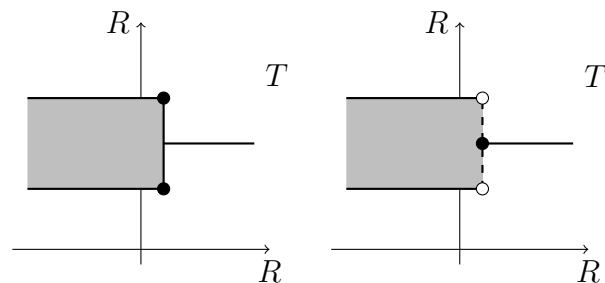
The **Glaeser refinement** of T is a set-valued map $T': E \rightrightarrows R^m$ given by for each $x_0 \in E$,

$$T'(x_0) := \{y \in T(x_0) : \forall \epsilon > 0 \exists \delta > 0 \forall x \in B_n(x_0, \delta) \cap E, B_m(y, \epsilon) \cap T(x) \neq \emptyset\}$$

(equivalently,

$$T'(x_0) := \{y \in T(x_0) : d(y, T(x)) \rightarrow 0 \text{ as } x \rightarrow x_0 \text{ where } x \in E\}.$$

Readers may think that the Glaeser refinement is a tool that we use to shave off “bad points” in T . Obviously, if T is definable, then so is T' . Since $T' \subseteq T$, selections of T' are also selections of T .



An example of the Glaeser refinement T' of a set-valued map T .

The following lemmas follow immediately from the definitions of lower semi-

continuity and the Glaeser refinement.

Lemma 3.1. *If T is lower semi-continuous, then $T' = T$.*

Proof. Let $(x_0, y_0) \in T$ and $\epsilon > 0$. Since T is lower semi-continuous, there is $\delta > 0$ such that for each $x \in B_n(x_0, \delta)$, $B_m(y_0, \epsilon) \cap T(x) \neq \emptyset$. Hence, $(x_0, y_0) \in T'$. \square

Lemma 3.2. *If $U \subseteq E$ is open in E , then $T' \upharpoonright U = (T \upharpoonright U)'$.*

Proof. Let $(x_0, y_0) \in T' \upharpoonright U$ and $\epsilon > 0$. Then there is $\delta > 0$ such that for all $x \in B_n(x_0, \delta)$, $B_m(y_0, \epsilon) \cap T(x) \neq \emptyset$. Hence, $(x_0, y_0) \in (T \upharpoonright U)'$. Next, let $(x_0, y_0) \in (T \upharpoonright U)'$ and $\epsilon > 0$. Then there is $\delta > 0$ such that $B_n(x_0, \delta) \subseteq U$ and for all $x \in B_n(x_0, \delta)$, $B_m(y_0, \epsilon) \cap T(x) = B_m(y_0, \epsilon) \cap T \upharpoonright U(x) \neq \emptyset$. Hence, $(x_0, y_0) \in T' \upharpoonright U$. \square

Moreover,

Lemma 3.3. *Suppose $T(x_0)$ is convex for every $x_0 \in E$. Then $T'(x_0)$ is convex for every $x_0 \in E$.*

Proof. Let $x_0 \in E$. If $T'(x_0) = \emptyset$, then we are done. Suppose $T'(x_0) \neq \emptyset$. Let $a, b \in T'(x_0)$, and $t \in [0, 1]$. We will show that $c = (1-t)a + tb \in T'(x_0)$. Let $\epsilon > 0$. Then there is $\delta > 0$ such that for each $x \in B_n(x_0, \delta)$ there exist $a_x \in B_m(a, \epsilon/2) \cap T(x)$ and $b_x \in B_m(b, \epsilon/2) \cap T(x)$. Hence, for each $x \in B_n(x_0, \delta)$,

$$\begin{aligned} \|((1-t)a + tb) - ((1-t)a_x + tb_x)\| &= \|(1-t)(a - a_x) + t(b - b_x)\| \\ &\leq (1-t)\|a - a_x\| + t\|b - b_x\| \\ &< \frac{(1-t)\epsilon}{2} + \frac{t\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence, $ta + (1-t)b \in T'(x_0)$. \square

We inductively define a sequence $(T^{(l)})_{l \in \mathbb{N}}$ of set-valued maps by $T^{(0)} := T$ and $T^{(l+1)} = (T^{(l)})'$ for every $l \in \mathbb{N}$. We say that T is **stable (under the Glaeser refinement)** if $T' = T$. It is natural to ask whether there exists a natural number l such that $T^{(l)}$ is stable.

Lemma 3.4. *For all $l \geq \dim E$, $T^{(l)} = T^{(\dim E)}$.*

Proof of Lemma 3.4. It is enough to prove that for each $d \leq \dim E$, there exists a definable open subset U of E such that $\dim(E \setminus U) < \dim E - d$ and $T^{(d)} \upharpoonright U$ is lower semi-continuous. We proceed by induction on d . The case $d = 0$ is immediate from Lemma 2.6. Suppose the result holds for d . Let $V \subseteq E$ be definable and open in E such that $\dim(E \setminus V) < \dim E - d$ and $T^{(d)} \upharpoonright V$ is lower semi-continuous. Then $T^{(d)}(x) = T^{(d+1)}(x)$ for every $x \in V$. By Lemma 2.6 and the Cell Decomposition Theorem, let \mathcal{C} be a cell decomposition of $E \setminus V$ such that for each $C \in \mathcal{C}$, $T^{(d+1)} \upharpoonright C$ is lower semi-continuous and $\text{fr } C$ is a finite union of cells in \mathcal{C} . Let $U = V \cup \bigcup \{C \in \mathcal{C} : \dim C = \dim(E \setminus V)\}$. It is routine to check that U is definable and open in E , and $\dim(E \setminus U) < \dim E - d - 1$. It remains to prove that $T^{(d+1)} \upharpoonright U$ is lower semi-continuous. Note that $T^{(d)} \upharpoonright V$ is lower semi-continuous. Let $x_0 \in U \setminus V$ and $y_0 \in T^{(d+1)}(x_0)$. Let $\epsilon > 0$. Then there exists $\delta_1 > 0$ such that for every $x \in B_n(x_0, \delta_1) \cap U$, $B_m(y_0, \epsilon) \cap T^{(d)}(x) \neq \emptyset$. Since $T^{(d)}(x) = T^{(d+1)}(x)$ for every $x \in B_n(x_0, \delta_1) \cap V$, we have $B_m(y_0, \epsilon) \cap T^{(d+1)}(x) \neq \emptyset$ for every $x \in B_n(x_0, \delta_1) \cap V$. Observe that $T^{(d+1)} \upharpoonright U \setminus V$ is lower semi-continuous. Hence, there exists a box $0 < \delta_2 < \delta_1$ such that $B_m(y_0, \epsilon) \cap T^{(d+1)}(x) \neq \emptyset$ for every $x \in B_n(x_0, \delta_2) \cap (U \setminus V)$. Thus, $T^{(d+1)} \upharpoonright U$ is lower semi-continuous. \square

As a consequence of Lemma 3.4, we have $T^{(\dim E)}$ is stable. For convenience, we write $T^{(*)} = T^{(\dim E)}$. Note that, by the definition of the Glaeser refinement, $T^{(*)}$ is lower semi-continuous.

Next, we will show that the stabilization method in Lemma 3.4 is sharp. Let $n \in \mathbb{N} \setminus \{0\}$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. For $i \in \{1, \dots, n-1\}$, let

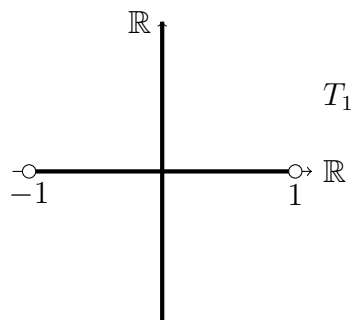
$$y_{ij}(x) := \begin{cases} 1, & \text{if } j = i; \\ x_i x_{i+1} / |x_{i+1}|, & \text{if } j = i + 1 \text{ and } x_{i+1} \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

and let $v_i^n(x) = (y_{i1}(x), \dots, y_{in}(x))$.

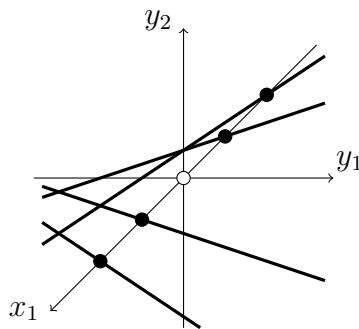
Let $U := (-1, 0) \cup (0, 1)$. A sequence $(E_n)_{n \in \mathbb{N}}$ is recursively defined by $E_0 := \mathbb{R}^0$ and $E_{n+1} := U^{n+1} \cup (E_n \times \{0\}) \subseteq \mathbb{R}^{n+1}$.

Example 3.5. For each $n \in \mathbb{N} \setminus \{0\}$, define $T_n: E_n \rightrightarrows \mathbb{R}^n$ by

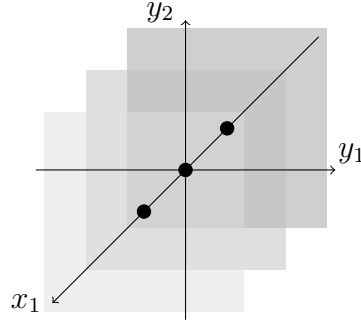
$$T_n(x) := \begin{cases} \text{span}\{v_1^n(x), \dots, v_{n-1}^n(x)\}, & \text{if } x \in U^n; \\ \mathbb{R}^n, & \text{otherwise.} \end{cases}$$



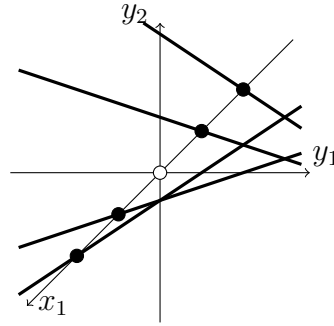
The graph of T_1 .



Examples of $T_2(x_1, x_2)$ when $(x_1, x_2) \in E_2$ and $x_2 < 0$.



Examples of $T_2(x_1, 0)$ when $(x_1, 0) \in E_2$.



Examples of $T_2(x_1, x_2)$ when $(x_1, x_2) \in E_2$ and $x_2 > 0$.

Let $n \in \mathbb{N} \setminus \{0\}$ and $k \in \mathbb{N}$. Then for every $x \in E_n$, $T_n^{(k)}(x) = T_n^{(k+1)}(x)$ if and only if $k \geq n$.

Proof. We proceed by induction on $n \geq 1$. It is easy to see that $T_1'(0) = \{0\} \neq \mathbb{R} = T_1(0)$. Assume for every $x \in E_n$, $T_n^{(k)}(x) = T_n^{(k+1)}(x)$ if and only if $k \geq n$. It suffices to show that for every $x \in E_{n+1}$,

$$T'_{n+1}(x) = \begin{cases} T_{n+1}(x), & \text{if } x \in U^{n+1}; \\ \text{span}\{v_1^{n+1}(x), \dots, v_{n-1}^{n+1}(x)\}, & \text{if } x \in U^n \times \{0\}; \\ \mathbb{R}^n \times \{0\}, & \text{otherwise.} \end{cases}$$

Note that for each i , v_i^{n+1} is continuous on U^{n+1} ; therefore, T_{n+1} is lower semi-continuous on U^{n+1} . If $x \in U^{n+1}$, then $T'_{n+1}(x) = T_{n+1}(x)$. Suppose $x =$

$(x_1, \dots, x_n, 0) \in U^n \times \{0\}$. We will prove that $T'_{n+1}(x) = \text{span}\{v_1^{n+1}(x), \dots, v_{n-1}^{n+1}(x)\}$.

Since $\lim_{U^{n+1} \ni z \rightarrow x} v_i^{n+1}(z) = v_i^{n+1}(x)$ for each $i \in \{1, \dots, n-1\}$,

$$\text{span}\{v_1^{n+1}(x), \dots, v_{n-1}^{n+1}(x)\} \subseteq T'_{n+1}(x).$$

Let $y \in T'_{n+1}(x)$. Let $v_1 = v_n^{n+1}(x_1, \dots, x_n, 1/2)$ and $v_2 = v_n^{n+1}(x_1, \dots, x_n, -1/2)$.

Note that for every $t \in (0, 1)$ and $i \in \{1, \dots, n-1\}$, we have

$$\begin{aligned} v_i^{n+1}(x_1, \dots, x_n, t) &= v_i^{n+1}(x) = v_i^{n+1}(x_1, \dots, x_n, -t), \\ v_n^{n+1}(x_1, \dots, x_n, t) &= v_1, \\ v_n^{n+1}(x_1, \dots, x_n, -t) &= v_2. \end{aligned}$$

Then for every $\epsilon > 0$,

$$\begin{aligned} d(y, \text{span}\{v_1^{n+1}(x), \dots, v_{n-1}^{n+1}(x), v_1\}) &< \epsilon, \\ d(y, \text{span}\{v_1^{n+1}(x), \dots, v_{n-1}^{n+1}(x), v_2\}) &< \epsilon. \end{aligned}$$

Therefore,

$$\begin{aligned} y &\in \text{span}\{v_1^{n+1}(x), \dots, v_{n-1}^{n+1}(x), v_1\} \cap \text{span}\{v_1^{n+1}(x), \dots, v_{n-1}^{n+1}(x), v_2\} \\ &= \text{span}\{v_1^{n+1}(x), \dots, v_{n-1}^{n+1}(x)\}. \end{aligned}$$

Let $x \in E_{n-1} \times \{0\}^2$. Then there is an integer $k \leq n-1$ such that $x \in U^k \times \{0\}^{n+1-k}$. Since $\dim T'_{n+1}(x) \leq \liminf_{z \rightarrow x} \dim T_{n+1}(z)$, it suffices to prove that $v_i^{n+1}(x) \in T'_{n+1}(x)$ for every $i \in \{1, \dots, n\}$. Observe that

$$\lim_{U^{n+1} \ni z \rightarrow x} v_i^{n+1}(z) = v_i^{n+1}(x)$$

for every $i \in \{1, \dots, n\} \setminus \{k\}$. For each $z = (z_1, \dots, z_{n+1}) \in U^{n+1}$, let $C_z :=$

$z_k z_{k+1} / |z_{k+1}|$. Therefore,

$$\lim_{U^{n+1} \ni z \rightarrow x} (v_k^{n+1}(z) - C_z v_{k+1}^{n+1}(z)) = v_k^{n+1}(x).$$

Hence, $T'_{n+1}(x) = \mathbb{R}^n \times \{0\}$. □

Note that, for each $n \in \mathbb{N}$, T_n is definable in the real field $(\mathbb{R}; <, +, \cdot)$. Hence, the stabilization method in Lemma 3.4 is sharp. Moreover, as far as we are concerned, T_3 is the first example of a set-valued map which is not stable after applying the Glaeser refinement twice.

CHAPTER IV

APPLICATIONS

Unless indicated otherwise, let \mathfrak{R} be an o-minimal expansion of a real closed field R . Let $E \subseteq R^n$ and $T : E \rightrightarrows R^m$ be a set-valued map. In this chapter, we study applications of Theorem 2.1 in various problems.

4.1 Existence of selections and definable selections

In this section, we study the correspondence between the existence of selections and definable selections in \mathfrak{R} . Obviously, the existence of definable continuous selections implies the existence of continuous selections. Hence, this question arises:

If T has a continuous selection, is there a continuous selection of T definable in $(R; <, +, \cdot, T)$?

Roughly speaking, if T has a continuous selection, is there a continuous selection of T that can be defined by *only* $+$, \cdot and T ?

However, the answer is “no”.

Example 4.1. Let R be the set of real algebraic numbers and $\mathfrak{M} := (R; <, +, \cdot)$.

Let $T : R \rightrightarrows R$ be defined by

$$T(x) := \begin{cases} \{0\}, & \text{if } x < 0; \\ \{0, 1\}, & \text{if } 0 \leq x \leq 4; \\ \{1\}, & \text{if } x > 4. \end{cases}$$

The structure \mathfrak{M} is o-minimal and T is definable in \mathfrak{M} . Observe that $f : R \rightarrow R$

defined by

$$f(x) := \begin{cases} 0, & \text{if } x < \pi; \\ 1, & \text{if } x > \pi, \end{cases}$$

is a continuous selection of T . However, by o-minimality, T has no continuous selection definable in \mathfrak{M} .

The problem is that definably connected sets need not be connected. Therefore, it is natural to ask this question in structures that definably connected sets are connected. Note that definably connected sets in o-minimal expansion of the real field are connected. Moreover, let us consider the following examples:

1. If $(\mathbb{R}; <, +, \cdot, T)$ defines \mathbb{Z} , then every Borel set is definable in $(\mathbb{R}; <, +, \cdot, T)$ (see [18, Exercise 37.6]). Therefore, if E is Borel and $(\mathbb{R}; <, +, \cdot, T)$ defines \mathbb{Z} , then every continuous selection of T is definable.
2. Let M be the double helix defined in [12] and define $T: \mathbb{R} \rightrightarrows \mathbb{R}^2$ by $T(z) = \{(x, y) \in \mathbb{R}^2 : (x, y, z) \in M\}$. Then $(\mathbb{R}; <, +, \cdot, T)$ is not o-minimal and a continuous selection of T must be interdefinable with a connected component of M , which is not definable in $(\mathbb{R}; <, +, \cdot, T)$.

These give rise to the following question:

If T has a continuous selection and $(\mathbb{R}; <, +, \cdot, T)$ is o-minimal, is there a continuous selection of T definable in $(\mathbb{R}; <, +, \cdot, T)$?

Throughout this section, let \mathfrak{R} be an o-minimal expansion of the real field, $E \subseteq \mathbb{R}^n$ and $T: E \rightrightarrows \mathbb{R}^m$ be definable in \mathfrak{R} . Here, we will give a partial answer to the above question:

Theorem 4.2. *Suppose that $n = 1$ or $m = 1$. Then T has a continuous selection if and only if T has a definable continuous selection.*

Now, we work towards the proof of Theorem 4.2. First, we prove this useful theorem:

Theorem 4.3. *Suppose for each $x \in E$, each connected component of $T(x)$ is convex, and T has a continuous selection. Then T has a definable continuous selection.*

Proof. Let $T: E \rightrightarrows \mathbb{R}^n$ be definable such that for each $x \in E$, connected components of $T(x)$ are convex, and $f: E \rightarrow \mathbb{R}^n$ be a continuous selection of T . By the Trivialization Theorem and the Cell Decomposition Theorem, there exists a cell decomposition \mathcal{C} of E , $(A_C)_{C \in \mathcal{C}}$ and $(h_C)_{C \in \mathcal{C}}$ such that for each $C \in \mathcal{C}$, $A_C \subseteq \mathbb{R}^n$ is definable and $h_C: T|_C \rightarrow C \times A_C$ is a definable homeomorphism, and for each $x \in C$, $h_C(\{x\} \times T(x)) = \{x\} \times A_C$. Let $C \in \mathcal{C}$. Note that $f|_C$ is connected. Therefore, $h_C(f|_C)$ is contained in a unique connected component of $C \times A_C$. Observe that every connected component of $C \times A_C$ is the set $C \times X$ where X is a connected component of A_C . For each $C \in \mathcal{C}$, let X_C be a connected component of A_C such that $h_C(f|_C) \subseteq C \times X_C$.

Let $T_0 = \bigcup \{h_C^{-1}(C \times X_C) : C \in \mathcal{C}\}$. Then $T_0 \subseteq T$ is a definable set-valued map $E \rightrightarrows \mathbb{R}^n$ such that $T_0(x)$ is convex for every $x \in E$.

Note that f is a continuous selection of T_0 . Hence, $T_0^{(*)}(x)$ is nonempty for every $x \in E$. Moreover, by Lemma 3.3, each $T_0^{(*)}(x)$ is convex. Since $T_0^{(*)}$ is stable, $T_0^{(*)}$ is lower semi-continuous. Hence, by Theorem 2.1, $T_0^{(*)}$ has a definable continuous selection, which is also a selection of T_0 . Therefore, T has a definable continuous selection. \square

Note that the last part of this proof is also valid in an o-minimal expansion of a real closed field. Therefore, we have:

Corollary 4.4. *Let \mathfrak{R} be an o-minimal expansion of a real closed field R and $E \subseteq R^n$. Let $T: E \rightrightarrows R^m$ be definable. Suppose for each $x \in E$, $T(x)$ is convex. If T has a continuous selection, then T has a definable continuous selection.*

Let $T: E \rightrightarrows \mathbb{R}$ be definable. Since every connected subset of \mathbb{R} is convex, T has a continuous selection if and only if T has a definable continuous selection. To finish the proof of Theorem 4.2, it suffices to consider the case $n = 1$.

In [4], M. Czapla and W. Pawłucki proved the following theorem:

Theorem 4.5 (M. Czapla and W. Pawłucki [4]). *Let $T : E \rightrightarrows \mathbb{R}^m$ be a definable and lower semi-continuous set-valued map with nonempty connected values and $\dim E = 1$. Then T has a definable continuous selection.*

We are now ready to prove a slight modification of the case $n = 1$.

Theorem 4.6. *Suppose $\dim E = 1$. If T has a continuous selection, then T has a definable continuous selection.*

Proof. Let $f : E \rightarrow \mathbb{R}^m$ be a continuous selection of T . By the same argument as in the Proof of Theorem 4.3, there exists a definable set-valued map $T_0 : E \rightrightarrows \mathbb{R}^m$ such that $f \subseteq T_0$ and for every $x \in E$, $T_0(x)$ is connected. By Lemma 2.6, let \mathcal{C} be a cell decomposition of E such that $T_0|_C$ is lower semi-continuous for every $C \in \mathcal{C}$. Let $E_0 = \bigcup\{C \in \mathcal{C} : \dim C = 0\}$. Then E_0 is finite. Define $T_1 : E \rightrightarrows \mathbb{R}^m$ by

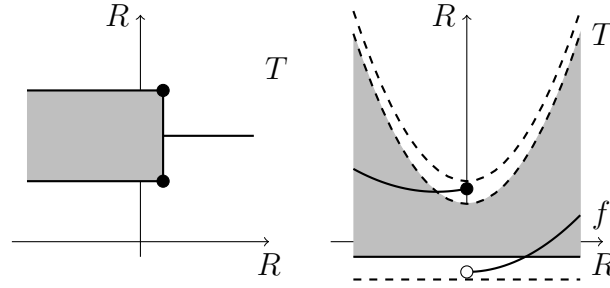
$$T_1(x) := \begin{cases} \{f(x)\}, & \text{if } x \in E_0; \\ T_0(x), & \text{if } x \notin E_0. \end{cases}$$

By Lemma 2.3, T_1 is lower semi-continuous. By Theorem 4.5, this completes the proof. \square

4.2 Existence of definable ϵ -approximate selections

In this section, we assume that \mathfrak{R} is an o-minimal expansion of a real closed field R .

We say that T is **almost lower semi-continuous** if for every $x_0 \in E$ and $\epsilon > 0$ there is $\delta > 0$ such that $\bigcap_{x \in B_n(x_0, \delta) \cap E} B_m(T(x), \epsilon) \neq \emptyset$; for each $\epsilon > 0$, a map $f : E \rightarrow R^m$ is an **ϵ -approximate selection** of T if $f(x) \in B_m(T(x), \epsilon)$, for each $x \in E$. In [10], F. Deutsch and P. Kenderov studied the concepts of almost lower semi-continuous and ϵ -approximate selections, and showed a relation between these two concepts.



Examples of an almost lower semi-continuous set-valued map (left) and an ϵ -approximate selection f of a set-valued map T (right).

Theorem 4.7 (F. Deutsch and P. Kenderov [10, Theorem 2.4]). *Suppose that $T(x)$ is nonempty and convex for each $x \in E$. Then T is almost lower semi-continuous iff T admits a continuous ϵ -approximate selection for each $\epsilon > 0$.*

If each $T(x)$ is nonempty and convex, then this theorem provides an alternative definition of almost lower semi-continuity of T . Hence, this question arises:

Let T be definable such that each $T(x)$ is nonempty and convex. Is the almost lower semi-continuity of T equivalent to the existence of definable ϵ -approximate selections?

The answer is yes. Next, we prove a definable version of Theorem 4.7 in o-minimal context.

Theorem 4.8. *Suppose that T is definable and $T(x)$ is nonempty and convex for each $x \in E$. Then the following are equivalent:*

- (a) T is almost lower semi-continuous;
- (b) T admits a continuous ϵ -approximate selection for each $\epsilon > 0$;
- (c) T admits a definable continuous ϵ -approximate selection for each $\epsilon > 0$.

Proof. By Theorem 4.7, (a) and (b) are equivalent. To show that (b) and (c) are equivalent, by Corollary 4.4, it suffices to show that if C is a convex subset of R^m , then $B(C, \epsilon)$ is convex. Let $a, b \in B(C, \epsilon)$ and $t \in [0, 1]$. Let $c := ta + (1-t)b$. Then

there exist $a', b' \in C$ such that $\|a - a'\|, \|b - b'\| < \epsilon$. Define $c' := ta' + (1 - t)b'$, then

$$\|c - c'\| \leq t\|a - a'\| + (1 - t)\|b - b'\| < \epsilon.$$

Hence, $c \in B(C, \epsilon)$. □

4.3 Existence of fixed points of definable set-valued maps

In this section, we assume that \mathfrak{R} is an o-minimal expansion of a real closed field R .

Let $X \subseteq R^n$. We say that a function $f : X \rightarrow X$ has a **fixed point** $x_0 \in X$ if $x_0 = f(x_0)$. Moreover, a set-valued map $T : X \rightrightarrows X$ has a **fixed point** $x_0 \in X$ if $x_0 \in T(x_0)$. The main result of this section is a lower semi-continuous version of the Definable Kakutani's Fixed Point Theorem:

Theorem 4.9. *Let $X \subseteq R^n$ be closed bounded convex and definable. Let $T : X \rightrightarrows X$ be a definable lower semi-continuous set-valued map such that $T(x)$ is nonempty and convex for each $x \in X$. Then T has a fixed point.*

We will prove the above theorem by using the following results:

Theorem 4.10 (M. J. Edmundo [11, Corollary 3.6]). *Let $X \subseteq R^n$ be definable closed and bounded and $f : X \rightarrow X$ be definable and continuous. If the Euler characteristic $E(X) \neq 0$ and there exists a definable continuous function $h : X \times [0, 1] \rightarrow X$ such that $h(x, 0) = f(x)$ and $h(x, 1) = x$ for each $x \in X$, then f has a fixed point.*

Lemma 4.11. *Let S be a simplex. Then $E(\text{cl } S) = 1$.*

Proof. Let S be a k -simplex and \mathcal{S} be the collection of faces of S . By the definition of the Euler characteristic of simplices, we have

$$E(\text{cl } S) = \sum_{\sigma \in \mathcal{S}} E(\sigma) = \sum_{i=1}^{k+1} (-1)^{i+1} \binom{k+1}{i}.$$

By the Binomial Theorem, $\sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} = 0$. Therefore,

$$E(\text{cl } S) = \sum_{i=1}^{k+1} (-1)^{i+1} \binom{k+1}{i} = 1 + \sum_{i=0}^{k+1} (-1)^{i+1} \binom{k+1}{i} = 1.$$

This completes the proof. \square

We are ready to prove Theorem 4.9.

Proof of Theorem 4.9. Define $F : X \times [0, 1] \rightrightarrows X$ by

$$F(x, t) := \begin{cases} T(x), & \text{if } t = 0; \\ \{x\}, & \text{if } t = 1; \\ X, & \text{otherwise.} \end{cases}$$

Clearly, F is definable. Since X and $T(x)$ are convex for each $x \in X$, $F(x, t)$ is convex for each $x \in X$ and $t \in [0, 1]$. Moreover, F is lower semi-continuous because T is lower semi-continuous. By Theorem 2.1, F has a definable continuous selection f . Define $g : X \rightarrow X$ by $g(x) := f(x, 0)$. If g has a fixed point, so is T . Hence, by Theorem 4.10, it suffices to show that $E(X) = 1$. By the Triangulation Theorem, there exists a complex K in R^n such that X is definably homeomorphic to $|K|$. Since X is closed, bounded, and definably connected, $|K|$ is closed and definably connected. Note that, by the Cell Decomposition Theorem, for all definable sets A and B , $E(A \cup B) = E(A) + E(B) - E(A \cap B)$. Hence, we reduce to the case where $|K|$ is the closure of a simplex. By Lemma 4.11, $E(|K|) = 1$. Since the Euler characteristic is preserved under definable bijection, $E(X) = 1$. \square

The convex condition of X in this theorem is necessary. Consider this example, let $S := \{x \in R^n : \|x\| = 1\}$ and $T : S \rightrightarrows S$ be defined by $T(x) := \{-x\}$. Then T has no fixed points and T is definable and continuous.

4.4 Approximation of upper semi-continuous set-valued maps by using C^p -maps

Note that upper semi-continuous set-valued maps do not necessarily have continuous selections even their values are closed, bounded, and convex;

Example 4.12. Let $T : \mathbb{R} \rightrightarrows \mathbb{R}$ be a set-valued map defined by

$$T(x) := \begin{cases} \{0\}, & \text{if } x < 0; \\ [0, 1], & \text{if } x = 0; \\ \{1\}, & \text{if } x > 0. \end{cases}$$

Then T does not have continuous selections.

However, in [1, Chapter 9], we have this result: *let $\epsilon > 0$ and $T : E \rightarrow R^m$ be an upper semi-continuous set-valued map such that for each $x \in E$, $T(x)$ is nonempty and convex. Then there exists a continuous function $f : E \rightarrow R^m$ such that $f \subseteq B_{n+m}(T, \epsilon)$.*

Hence, this question arises:

Let $\epsilon > 0$. If $T : E \rightrightarrows R^m$ is a definable upper semi-continuous set-valued map and $T(x)$ is nonempty convex for each $x \in E$, is there a definable continuous function $f : E \rightarrow R^m$ such that $f \subseteq B_{n+m}(T, \epsilon)$?

The answer is yes.

Theorem 4.13. *Suppose T is upper semi-continuous and definable such that $T(x)$ is nonempty and convex for each $x \in E$. Then for each $\epsilon > 0$, there is a definable continuous map $f : E \rightarrow R^m$ such that $f \subseteq B_{n+m}(T, \epsilon)$.*

Proof. Let $\epsilon > 0$. We proceed by induction on $\dim E$. The case $\dim E = 0$ is trivial. Suppose the theorem holds for all set-valued maps whose domain has dimension less than $\dim E$. By Lemma 2.6, there is a definable set S open in E such that $T|_S$ is lower semi-continuous and $\dim E \setminus S < \dim E$. Then there is a definable continuous function $g_0 : E \setminus S \rightarrow R^m$ such that $g_0 \subseteq B_{n+m}(T|_{E \setminus S}, \epsilon/4)$. By

the Definable Tietze's Extension Theorem, there is a definable continuous function $g : E \rightarrow R^m$ such that $g|_{E \setminus S} = g_0$. Next, by Theorem 2.1, there exists a definable continuous selection h of $T|_S$. Define $r : E \setminus S \rightarrow R$ by

$$r(x) := \sup\{r' \in (0, \frac{\epsilon}{2}] : \forall x' \in B_n(x, r') \cap E, T(x') \subseteq B_m(T(x), \frac{\epsilon}{4}) \text{ and } g(x') \in B_m(g(x), \frac{\epsilon}{4})\}.$$

Obviously, r is definable. Since T is upper semi-continuous and g is continuous, $r(x) > 0$ for every $x \in E \setminus S$. Let $U := \bigcup_{x \in E \setminus S} B_n(x, r(x))$. Then U is a definable open subset of E containing $E \setminus S$. By Lemma 1.22, there exists a definable open subset V of E such that $E \setminus S \subseteq V \subseteq \text{cl}_E V \subseteq U \subseteq E$. By Lemma 1.23, there exists a definable continuous function $t : E \rightarrow [0, 1]$ such that $t^{-1}(0) = \text{cl}_E V$ and $t^{-1}(1) = E \setminus U$. Define $f : E \rightarrow R^n$ by

$$f(x) := \begin{cases} (1 - t(x))g(x) + t(x)h(x), & \text{if } x \in S; \\ g(x), & \text{if } x \in E \setminus S. \end{cases}$$

Clearly, f is definable. To show that f is continuous, let $x_0 \in E$. Since S is open in E and $g(x), h(x)$, and $t(x)$ are continuous on S , f is continuous on E . Therefore, we may assume $x_0 \in E \setminus S$. Since $t^{-1}(0) = \text{cl}_E V$, $f(x) = g(x)$ for every $x \in V$. Since g is continuous on V and $x_0 \in V$, f is continuous at x_0 .

Next, we will show that $f \subseteq B_{n+m}(T, \epsilon)$. Let $x_0 \in E$. If $x_0 \in E \setminus U$, then $f(x_0) = h(x_0) \in T(x_0)$. Suppose that $x_0 \in U$. Then there is $x \in E \setminus S$ such that $x_0 \in B_n(x, r(x))$, $T(x_0) \subseteq B_m(T(x), \epsilon/4)$, and $g(x_0) \in B_m(g(x), \epsilon/4)$. Since $B_m(T(x), \epsilon/2)$ is a convex set containing $g(x_0)$ and $h(x_0)$, $B_m(T(x), \epsilon/2)$ also contains $f(x_0)$. Hence, there is $y \in T(x)$ such that $\|y - f(x_0)\| < \epsilon/2$. Therefore,

$$\|(x, y) - (x_0, f(x_0))\| \leq \|x - x_0\| + \|y - f(x_0)\| < \epsilon/2 + \epsilon/2 = \epsilon.$$

This completes the proof. □

In fact, if E is open, then for each $\epsilon > 0$ and $p \in \mathbb{N}$, there exists a definable C^p -map $f : E \rightarrow R^m$ satisfies $f \subseteq B_{n+m}(T, \epsilon)$.

Corollary 4.14. *Suppose T is upper semi-continuous and definable such that $T(x)$ is nonempty and convex for each $x \in E$. If E is open, then for each $\epsilon > 0$ and $p \in \mathbb{N}$, there is a definable C^p -map $f : E \rightarrow R^m$ such that $f \subseteq B_{n+m}(T, \epsilon)$.*

To prove this result we recall:

Theorem 4.15 (A. Thamrongthanyalak [27, Corollary 4.1]). *Let E be open and $f : E \rightarrow R^m$ be a definable continuous map. Let $\epsilon > 0$ and $p \in \mathbb{N}$. Then there exists a definable C^p -map $g : E \rightarrow R^m$ such that $\|g(x) - f(x)\| < \epsilon$ for each $x \in E$.*

This theorem tells us that we can approximate definable continuous maps by definable C^p -maps.

Proof of Corollary 4.14. Suppose that E is open. Let $\epsilon > 0$ and $p \in \mathbb{N}$. By Theorem 4.13, we obtain a definable continuous map $f : E \rightarrow R^m$ such that $f \subseteq B_{n+m}(T, \epsilon/2)$. By Theorem 4.15, there is a C^p -map $g : E \rightarrow R^m$ such that $\|g(x) - f(x)\| < \epsilon/2$ for each $x \in E$. Hence, $g \subseteq B_{n+m}(T, \epsilon)$. \square

CHAPTER V

DEFINABLE EXTENSION OF CONTINUOUS SET-VALUED MAPS

Throughout this chapter, assume that \mathfrak{R} is an o-minimal expansion of a real closed field R and let $E \subseteq R^n$ and $T: E \rightrightarrows R^m$ be definable. We say a set-valued map $F: R^n \rightrightarrows R^m$ is an **extension** of T if $F(x) = T(x)$ for each $x \in E$. By the Definable Tietze's Extension Theorem, we know that definable continuous maps on closed sets have definable continuous extensions. Hence, we are interested in this question:

Does a definable continuous set-valued map on a closed domain have a definable continuous extension?

Recall that a set-valued map is continuous if it is lower semi-continuous and upper semi-continuous. In this chapter, we partially answer this question. First, we prove these lemmas.

Lemma 5.1. *Let $Y \subseteq R^m$. If $f: E \rightarrow Y$ is definable and continuous and $T: Y \rightrightarrows R^k$ is definable and continuous, then $T \circ f: E \rightrightarrows R^k$ defined by $T \circ f(x) := T(f(x))$ is definable and continuous.*

Proof. Obviously, $T \circ f$ is definable. To show that $T \circ f$ is lower semi-continuous, let $x_0 \in E$, $z_0 \in T \circ f(x_0)$ and $\epsilon > 0$. Since T is lower semi-continuous, there is $\delta_1 > 0$ such that for each $y \in B_m(f(x_0), \delta_1) \cap Y$, $T(y) \cap B_k(z_0, \epsilon) \neq \emptyset$. Since f is continuous, there is $\delta > 0$ such that for each $x \in B_n(x_0, \delta) \cap E$, $f(x) \in B_m(f(x_0), \delta_1)$. Hence, for each $x \in B_n(x_0, \delta) \cap E$, $T \circ f(x) \cap B_k(z_0, \epsilon) \neq \emptyset$.

To prove that $T \circ f$ is upper semi-continuous, let $x_0 \in E$ and $\epsilon > 0$. Since T is upper semi-continuous, there is $\delta_1 > 0$ such that $T(y) \subseteq B_k(T \circ f(x_0), \epsilon)$ for all $y \in B_m(f(x_0), \delta_1) \cap Y$. Since f is continuous, there is $\delta > 0$ such that

for each $x \in B_n(x_0, \delta) \cap E$, $f(x) \in B_m(f(x_0), \delta_1)$. Hence, for each $x \in B_n(x_0, \delta)$, $T \circ f(x) \subseteq B_k(T \circ f(x_0), \epsilon)$. \square

Next, we show that the Cartesian product of two continuous set-valued maps is continuous:

Lemma 5.2. *Let $T : E \rightrightarrows R^m$ and $T' : E' \rightrightarrows R^k$ be definable continuous set-valued maps where $E, E' \subseteq R^n$. Define $T \times T' : E \times E' \rightrightarrows R^{m+k}$ by $T \times T'(x, x') := T(x) \times T'(x')$. Then $T \times T'$ is definable and continuous.*

Proof. Obviously, $T \times T'$ is definable. We will show that $T \times T'$ is lower semi-continuous. Let $(x_0, x'_0) \in E \times E'$, $(y_0, y'_0) \in T \times T'(x_0, x'_0)$ and V be an open neighborhood of (y_0, y'_0) . Then there is $\epsilon > 0$ such that $B_m(y_0, \epsilon) \times B_k(y'_0, \epsilon) \subseteq V$. Since T and T' are lower semi-continuous, there exists $\delta > 0$ such that for each $x \in B_n(x_0, \delta)$ and $x' \in B_n(x'_0, \delta)$, $T(x) \cap B_m(y_0, \epsilon) \neq \emptyset$ and $T'(x') \cap B_k(y'_0, \epsilon) \neq \emptyset$. Hence, for each $(x, x') \in B_{2n}((x_0, x'_0), \delta)$, $T \times T'(x, x') \cap V \neq \emptyset$.

To prove that $T \times T'$ is upper semi-continuous, let $(x_0, x'_0) \in E \times E'$ and $\epsilon > 0$. Then there is $\epsilon' > 0$ such that $B_m(T \times T'(x_0), \epsilon') \times B_k(T \times T'(x'_0), \epsilon') \subseteq B_{m+k}(T \times T'(x_0, x'_0), \epsilon)$. Since T and T' are upper semi-continuous, there exists $\delta > 0$ such that for each $x \in B_n(x_0, \delta)$ and $x' \in B_n(x'_0, \delta)$, $T(x) \subseteq B_m(T(x_0), \epsilon')$ and $T'(x') \subseteq B_k(T(x'_0), \epsilon')$. Therefore, for each $(x, x') \in B_{2n}((x_0, x'_0), \delta)$,

$$T \times T'(x, x') \subseteq B_m(T \times T'(x_0), \epsilon') \times B_k(T \times T'(x'_0), \epsilon') \subseteq B_{m+k}(T \times T'(x_0, x'_0), \epsilon).$$

This completes the proof. \square

Next is a version of the Pasting Lemma for definable set-valued maps.

Lemma 5.3. *Let E, E' be closed subsets of R^n . Let $T : E \rightrightarrows R^m$ and $T' : E' \rightrightarrows R^m$ be definable continuous set-valued maps such that $T = T'$ on $E \cap E'$. Define $F : E \cup E' \rightrightarrows R^m$ by*

$$F(x) := \begin{cases} T(x), & \text{if } x \in E; \\ T'(x), & \text{if } x \in E'. \end{cases}$$

Then F is definable and continuous.

Proof. Obviously, F is definable. Next, we will prove that F is continuous. Let $x_0 \in E \cup E'$. Since $E \cap E'$ is closed, $(E \cup E') \setminus (E \cap E')$ is open in $E \cup E'$. Since $F = T$ on $E \setminus E'$ and $F = T'$ on $E' \setminus E$, F is continuous on $(E \cup E') \setminus (E \cap E')$. We may assume $x_0 \in E \cap E'$. Let $y_0 \in F(x_0)$ and $\epsilon > 0$. Since T and T' are continuous, there is $\delta > 0$ such that for each $x \in B_n(x_0, \delta) \cap (E \cup E')$, $F(x) \cap B_m(y_0, \epsilon) \neq \emptyset$ and $F(x) \subseteq B_m(F(x_0), \epsilon)$. Therefore, F is continuous. \square

Now, we ready to prove the main theorem of this chapter:

Theorem 5.4. *Let E be a closed subset of R^n . If $T : E \rightrightarrows R^m$ is a definable continuous set-valued map such that for each $x \in \text{bd } E$, $T(x)$ is bounded, then T has a definable continuous extension.*

Proof. Let $T : E \rightrightarrows R^m$ be a definable continuous set-valued map where E is a closed subset of R^n . By Corollary 1.24, there is a definable open set U containing $\text{bd } E$ and a definable retraction $r : \text{cl } U \rightarrow \text{bd } E$. Define $r' : \text{cl } U \cup E \rightarrow E$ by

$$r'(x) := \begin{cases} x, & \text{if } x \in E; \\ r(x), & \text{otherwise.} \end{cases}$$

Clearly, r' is a definable continuous map such that for each $x \in \text{cl } U \setminus \text{int } E$, $r'(x) \in \text{bd } E$. Let $V = U \cup E$. Since $U \cup E = U \cup \text{int } E$, V is an open neighborhood of E . By Lemma 1.22, there is a definable continuous function $t : R^n \rightarrow [0, 1]$ such that $t^{-1}(1) = E$ and $t^{-1}(0) = R^n \setminus V$. Note that every function can be considered as a set-valued map. Define $\phi : R^m \times [0, 1] \rightarrow R^m$ by $\phi(x_1, \dots, x_m, s) := (sx_1, \dots, sx_m)$ and define $F : R^n \rightrightarrows R^m$ by

$$F(x) := \begin{cases} \phi(T \circ r' \times t(x)), & \text{if } x \in \text{cl } V; \\ \{0\}, & \text{otherwise.} \end{cases}$$

We claim that F is a definable continuous extension of T . Obviously, F is definable. Next, we will show that F is continuous. Clearly, $F|_{R^n \setminus V}$ is continuous. By

Lemma 5.3, it suffices to show that F is continuous on $\text{cl } V$. Since r' and t are continuous and T is a continuous set-valued map, by Lemma 5.1 and 5.2, $T \circ r' \times t$ is continuous on $\text{cl } V$. To show that F is lower semi-continuous on $\text{cl } V$, let $x_0 \in \text{cl } V$, $z_0 \in F(x_0)$ and $\epsilon > 0$. Then there is $y_0 \in T \circ r'(x_0)$ such that $\phi(y_0, t(x_0)) = z_0$. Since ϕ is continuous, there exists $\epsilon_1 > 0$ such that $\phi(B_{m+1}((y_0, t(x_0)), \epsilon_1)) \subseteq B_m(z_0, \epsilon)$. Since $T \circ r' \times t$ is lower semi-continuous on $\text{cl } V$, there is $\delta > 0$ such that for each $x \in B_n(x_0, \delta) \cap \text{cl } V$,

$$\begin{aligned} F(x) \cap B_m(z_0, \epsilon) &\supseteq \phi(T \circ r' \times t(x)) \cap B_m(z_0, \epsilon) \\ &\supseteq \phi(T \circ r' \times t(x) \cap B_{m+1}((y_0, t(x_0)), \epsilon_1)) \\ &\neq \emptyset. \end{aligned}$$

Hence, F is lower semi-continuous on $\text{cl } V$. To show that F is upper semi-continuous on $\text{cl } V$, let $\epsilon > 0$ and $x_0 \in \text{cl } V$. Since $F(x) = T \circ r'(x)$ for each $x \in \text{int } E$, F is upper semi-continuous on $\text{int } E$. We may assume $x_0 \in \text{cl } V \setminus \text{int } E$. Since $r'(x_0) \in E$, $T \circ r'(x_0)$ is bounded. Then there is $M > 1$ such that $T \circ r'(x_0) \subseteq B_m(0, M)$. We claim that $\phi(B_{m+1}(T \circ r' \times t(x_0), \epsilon/2M)) \subseteq B_m(F(x_0), \epsilon)$. Let $(y, s) \in B_{m+1}(T \circ r' \times t(x_0), \epsilon/2M)$. Then there is $y' \in T \circ r'(x_0)$ such that $\|y - y'\| < \epsilon/2M$. Since $T \circ r'(x_0) \subseteq B_m(0, M)$, we have

$$\begin{aligned} \|\phi(y, s) - \phi(y', t(x_0))\| &\leq \|\phi(y, s) - \phi(y', s)\| + \|\phi(y', s) - \phi(y', t(x_0))\| \\ &\leq \|s \cdot (y - y')\| + \|(s - t(x_0)) \cdot y'\| \\ &\leq \|y - y'\| + \frac{\epsilon}{2M} \cdot \|y'\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Since $T \circ r' \times t$ is upper semi-continuous on $\text{cl } V$, there is $\delta > 0$ such that for every $x \in B_n(x_0, \delta) \cap \text{cl } V$, $T \circ r' \times t(x) \subseteq B_{m+1}(T \circ r' \times t(x_0), \epsilon/2M)$. Hence,

$$F(x) \subseteq \phi(B_{m+1}(T \circ r' \times t(x_0), \epsilon/2M)) \subseteq B_m(F(x_0), \epsilon).$$

Therefore, F is upper semi continuous on $\text{cl}V$. To show that F is an extension of T , let $x \in E$. Then $F(x) = \phi(T \circ r'(x) \times \{t(x)\}) = \phi(T(x) \times \{1\}) = T(x)$. Therefore, F is a definable continuous extension of T as desired. \square

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