รูปแบบของเส้นที่ตัดกันบนระนาบ

นางสาววงศ์ตะวัน ชำนาญ

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2560

PATTERNS OF LINES INTERSECTING ON PLANE

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A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Mathematics
Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Academic Year 2017

Thesis Title	PATTERNS OF LINES INTERSECTING ON PLANE
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Field of Study	Mathematics
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Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

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วงศ์ตะวัน ชำนาญ : รูปแบบของเส้นที่ตัดกันบนระนาบ. (PATTERNS OF LINES INTERSECTING ON PLANE) อ.ที่ปรึกษาวิทยานิพนธ์หลัก : รองศาสตราจารย์ ดร.วัชรินทร์ วิชิรมาลา, 57 หน้า.

ในการศึกษานี้เราสนใจรูปแบบของการจัดเรียงของเส้นตรง n เส้นบนระนาบที่ก่อให้เกิดจุดตัด M จุด ปัญหานี้คล้ายกับ ปัญหา 17 เส้น ของฟูเรียร์ แม้ว่าจะทราบบางเงื่อนไขจำเป็นสำหรับการตรวจสอบการวาดได้จริงของการจัดเรียงของเส้น แต่บางการจัดเรียงก็อาจจะไม่สามารถวาดได้จริง เราจึงจะหาเงื่อนไขจำเป็นเพิ่มเติมสำหรับการจัดเรียงของเส้น n เส้นที่ก่อให้เกิดจุดตัด M จุดเพื่อหาการจัดเรียงของเส้นที่เป็นไปได้และหาวิธีการตรวจสอบการวาดได้จริงของการจัดเรียงของเส้นที่เป็นไปได้ สำหรับการตรวจสอบการวาดได้จริงของการจัดเรียงของเส้นที่เป็นไปได้ เราใช้วิธีการวาดโดยตรง ทำให้ได้บทตั้ง บทแทรกเกี่ยวกับการวาด ทฤษฎีบทประกอบและทฤษฎีบทเกี่ยวกับการวาดได้จริงของการจัดเรียงบางแบบของเส้น นอกจากนี้เราเสนอวิธีการตรวจสอบการวาดได้จริงของการจัดเรียงของเส้นโดยวิธีหาคำตอบของระบบสมการที่สอดคล้องกับแต่ละการจัดเรียงของเส้น

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต ....

สาขาวิชา คณิตศาสตร์ ลายมือชื่อ อ.ที่ปรึกษาหลัก ....

ปีการศึกษา 2560

5872097923 : MAJOR MATHEMATICS
KEYWORDS : INTERSECTION OF LINES, FOURIER'S 17 LINE PROBLEM, PLANAR ARRANGEMENT, PATTERN

WONGTAWAN CHAMNAN : PATTERNS OF LINES INTERSECTING ON-PLANE
ADVISOR : ASSOCIATE PROFESSOR WACHARIN WICHIRAMALA, Ph.D., 57 pp.

In this work, we are interested in patterns of n lines on the plane such that lines produce M intersecting points. One famous related problem is called Fourier's 17 line problem. Although there are only a few known necessary conditions for realization, some arrangements may not even drawable. Consequently, we try to find more necessary conditions for this problem that can lead to possible arrangements and realize the arrangements. We provide a systematic algorithm to draw the possible arrangements. We establish lemmas for the direct drawing and find some certain cases of arrangements which are drawable. We also provide another approach to this problem by determining the existence of solutions that correspond to the systems of equations with conditions for each arrangement.

Department : ...Mathematics and.....	Student's Signature : ....
...Computer Science...	Advisor's Signature : ....
Field of Study : ......Mathematics......	
Academic Year : ....2017............	

ACKNOWLEDGEMENTS

I am greatly indebted to Associate Professor Wacharin Wichiramala, my thesis advisor for his suggestion, friendly encouragement and helpful advice in preparing and writing this thesis. I am also grateful to express my special thanks to Associate Professor Nataphan Kitisin, Assistant Professor Sujin Khomruthai, and Assistant Professor Wittawat Kositwattanarerk, my thesis committees, for their valuable comment and suggestions. Moreover, I would like to thank all of my teachers who have taught me.

In particular, I wish to express my thankfulness to Development Promotion of Science and Technology Talents Project (DPST) for financial support and for granting me great opportunity to study Mathematics. I wish to thanks my dear friends for giving me good experiences at Chulalongkorn University.

Finally, I would like to express my deep gratitude to my beloved family for their love and encouragement throughout my graduate study.

CONTENTS

CHAPTER I
INTRODUCTION

In 1788, Fourier introduced a question about the arrangements of 17 lines on the plane so that they formed 101 points of intersection, and no point of intersection belonged to more than two lines. In 1980, Turner [2] and Webster [3] found four different arrangements as in Figure 1.1 and there was a further study about this problem with the unrestricted number of lines passing each point of intersection which gave combinatorial formulas related to the geometry of lines and points on the plane.

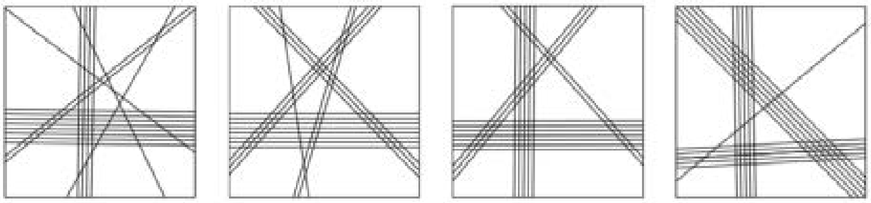

Figure 1.1: Four different arrangements of Fourier's 17 line problem

Definition 1.1. Given n lines on the plane, a **parallel family** is a group of all lines from the n lines which are all parallel to each other, and a line having no other line in its parallel family is called a **single line**. Let W be a point of intersection. The **degree** of W is the number of lines from the n lines passing W, and we call W a **multiple point** if there are more than two lines passing W.

Definition 1.2. Given $n \geq 1$ lines with $M \geq 0$ points of intersection, a **code** $\mathcal{A}$ of an arrangement of these lines is a representation of the arrangement.
For $M \geq 1$, $\mathcal{A}$ is denoted by

$$\langle\langle \lambda_1, \lambda_2, ..., \lambda_M | \mu_1, \mu_2, ..., \mu_P \rangle\rangle,$$

where

1. λ_i is the degree of the i^{th} point in $\mathcal{A}$ for $i = 1, ..., M$, and $\lambda_j \geq \lambda_{j+1}$ for all $j = 1, 2, ..., M - 1$,

2. μ_i is the number of lines in the i^{th} parallel family in $\mathcal{A}$ for $i = 1, ..., P$, and $\mu_j \geq \mu_{j+1}$ for all $j = 1, 2, ..., P - 1$.

Otherwise, $\mathcal{A}$ is denoted by

$$\langle\langle\ |\mu_1, \mu_2, ..., \mu_P\rangle\rangle,$$

where μ_i is the number of lines in the i^{th} parallel family in $\mathcal{A}$ for $i = 1, ..., P$, and $\mu_j \geq \mu_{j+1}$ for all $j = 1, 2, ..., P - 1$.

Notice that for $\mathcal{A}$ in the Definition 1.2, P is the number of all parallel families. Let m be the number of multiple points and p be the number of nontrivial parallel families. We have that if $\mathcal{A}$ has multiple points then $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_m \geq 3$ and $\lambda_{m+1} = \lambda_{m+2} = ... = \lambda_M = 2$ where $1 \leq m \leq M$, and if $\mathcal{A}$ has nontrivial parallel families then $\mu_1 \geq \mu_2 \geq ... \geq \mu_p \geq 2$ and $\mu_{p+1} = \mu_{p+2} = ... = \mu_P = 1$ where $1 \leq p \leq M$. For a code of an arrangement, we define m, P, and p in the same way as for $\mathcal{A}$ in the Definition 1.2. For the code with many copies of λ_j's and μ_i's we sometimes substitute ${\lambda_j}^{k_1}$ and ${\mu_i}^{k_2}$ for k_1 copies of λ_j's and k_2 copies of μ_i's respectively. For example, the code $\langle\langle 3, 3, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2|3, 2, 1, 1\rangle\rangle$ of 7 lines forming 13 points can be written as $\langle\langle 3^2, 2^{11}|3, 2, 1^2\rangle\rangle$.

Remark 1.3. A code can be shortened by omitting λ_j for all j with $\lambda_j = 2$ and omitting μ_i for all i with $\mu_i = 1$. We use single brackets for the shortened code so it can be written as

$$\langle\lambda_1, \lambda_2, ..., \lambda_m|\mu_1, \mu_2, ..., \mu_p\rangle.$$

In 2012, Alexanderson and Wetzel [1] studied the perplexities related to Fourier's 17 line problem by dividing this problem into no parallel case and no restriction case and gave some necessary conditions of possible arrangements which are

$$\sum_{j=1}^{m}\left[\binom{\lambda_j}{2} - 1\right] + \sum_{i=1}^{p}\binom{\mu_i}{2} = 35, \tag{1.1}$$

$$\sum_{i=1}^{p} \mu_i \leq 17, \tag{1.2}$$

$$\lambda_1 \leq p + 17 - \sum_{i=1}^{p} \mu_i. \tag{1.3}$$

Supplementary to [1], Lichtblau and Wichiramala [4], by using computational programming, found that there are 924 codes for Fourier's 17 line problem satisfying the necessary conditions (1.1), (1.2), and (1.3).

The problem is that whether all 924 codes are realizable and, moreover, Alexanderson and Wetzel also mentioned the remark by Douglas West on the difficulty of finding all different ways that 10 lines in general position can be arranged which is a special case for the generalized problem.

As all problems above are still open, we are interested in the problem of the arrangement of n lines intersecting on the plane and forming M points. The aims of this work are to find more necessary conditions and to provide a systematic algorithm to draw the possible arrangements and to find a way to obtain all different patterns.

CHAPTER II
PRELIMINARIES

In this section, we present some definitions, notations, lemmas, and remarks which will be useful for this work.

Here we may assume that the ends of all straight lines meet together at infinity and the number of points of intersection does not include the infinity point.

2.1 Pictures and Patterns

Definition 2.1. Suppose there are n lines on the plane. The **picture** of the lines is a set of point $(x, y) \in \mathbb{R}^2$ that lies on some of these lines.

Notice that there are one-to-one correspondences between pictures and sets of lines on the plane.

Definition 2.2. Two pictures are said to be **the same** if there is a composition of a translation, a rotation, and possibly a reflection that maps one picture to be the other. Otherwise, they are said to be **different**.

Example 2.3. According to Figure 2.1, pictures A and B are the same because there is a rotation $R(x, y) = (x \cos \frac{\pi}{6} - y \sin \frac{\pi}{6}, x \sin \frac{\pi}{6} + y \cos \frac{\pi}{6})$ and a translation $T(x, y) = (x + 3, y + 2)$ such that $T \circ R(A) = B$.

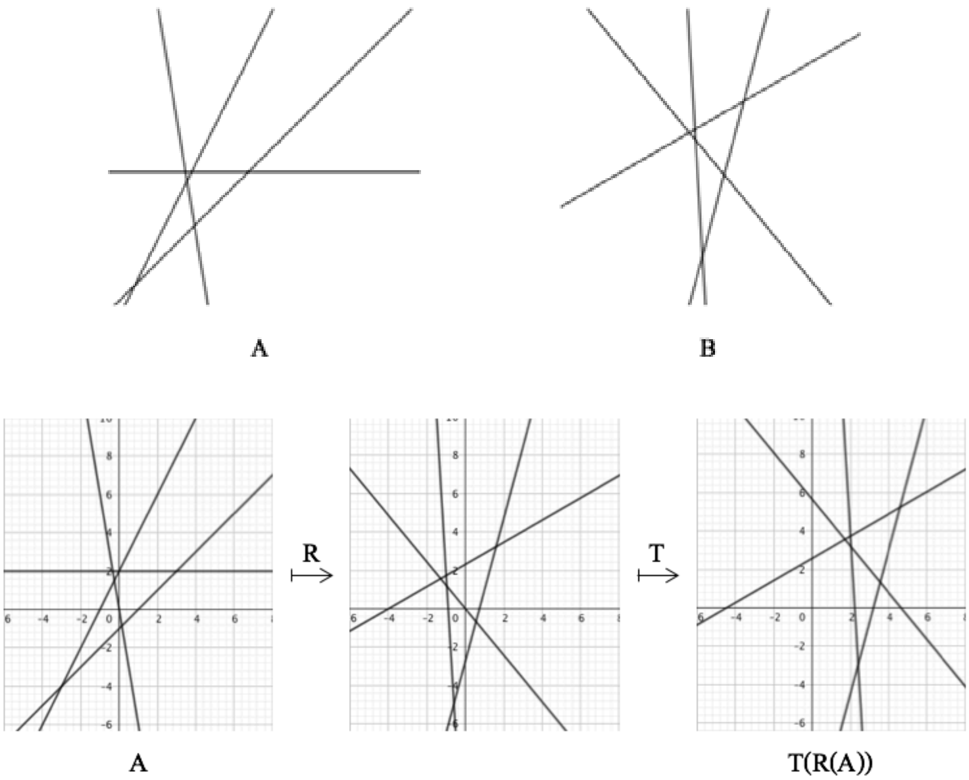

Figure 2.1: The same pictures A and B

Definition 2.4. Let A be a picture. The **pattern** of A is a planar graph where every vertex corresponds to a point of intersection of the lines including the point of infinity and every edge corresponds to a segment of any line partitioned by points of intersection.

Definition 2.5. Two patterns are said to be **the same** if they are graph isomorphic. Otherwise, they are said to be **different**.

Remark 2.6.

1. A code may have more than one corresponding patterns.

2. Since the ends of lines meet together at infinity, each pattern has a vertex at infinity having the maximum degree, twice the number of all lines

We can observe that the same pictures will have the same patterns and two different pictures may possibly have the same pattern.

Example 2.7. Let A and B be pictures. Let G_A and G_B be patterns of A and B respectively as illustrated in Figure 2.2.

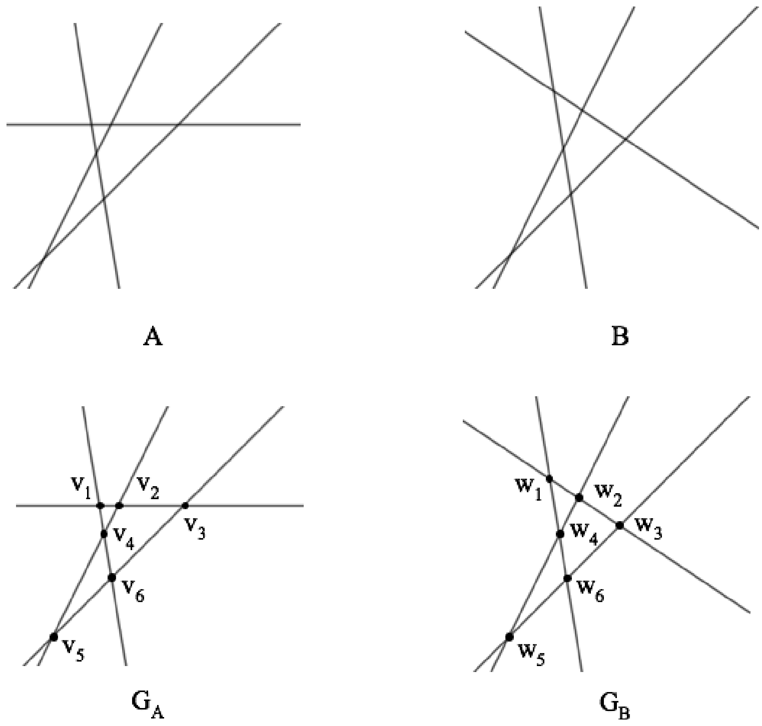

Figure 2.2: A and B (top) with their patterns G_A and G_B (bottom) respectively

Let v_0 and w_0 be vertices at infinity of G_A and G_B respectively. We can see that pictures A and B are different but patterns G_A and G_B are the same because there is a graph isomorphism f between the vertex sets of G_A and G_B with $f(v_i) = (w_i)$, $i = 0, 1, ..., 6$.

2.2 Codes of an Arrangement

As we define codes of arrangements in Chapter I, we will give some formulas for codes related to their graphs.

Robert's Formulas. [1] Suppose n lines has an arrangement $\mathcal{A} = \langle\langle \lambda_1, \lambda_2, ..., \lambda_M | \mu_1, \mu_2, ..., \mu_P \rangle\rangle$ on the plane. Then $\mathcal{A}$ forms

$$C = 1 + n + \binom{n}{2} - \sum_{j=1}^{M} \binom{\lambda_j - 1}{2} - \sum_{i=1}^{P} \binom{\mu_i}{2} \quad \text{regions,}$$

$$E = n^2 - \sum_{j=1}^{M} \lambda_j(\lambda_j - 2) - 2\sum_{i=1}^{P} \binom{\mu_i}{2} \text{ segments and rays, and}$$

$$V = \binom{n}{2} - \sum_{j=1}^{M} \left[\binom{\lambda_j}{2} - 1\right] - \sum_{i=1}^{P} \binom{\mu_i}{2} \text{ points.} \tag{2.1}$$

2.3 Manipulation of pictures

In this work, we define a (rigid) **motion** to be a composition of rotations and translations of some lines, and we define a **perturbation** to be a motion (of some lines) without changing the pattern.

Definition 2.8. Let A and B be pictures. We said that picture A and B are **combined** to be a picture C if there is a motion ϕ_1 and ϕ_2 such that C is the union of $\phi_1(A)$ and $\phi_2(B)$.

Affine Transformation. A transformation f of the plane of the form $f(\vec{x}) = A\vec{x} + \vec{b}$ where A is an invertible matrix is called an **affine transformation** of the plane. The inverse of an affine transformation of the plane is also an affine transformation of the plane. And a composition of affine transformations is an affine transformation.

Theorem 2.9. *Let $f(\vec{x}) = A\vec{x} + \vec{b}$ be an affine transformation. Then*

(1) f maps a line to a line,

(2) f maps a line segment to a line segment,

(3) f preserves the property of parallelism among lines and line segments,

(4) f maps an n-gon to an n-gon,

(5) f maps a parallelogram to a parallelogram,

(6) f preserves the ratio of lengths of two parallel segments, and

(7) f preserves the ratio of areas of two figures.

Examples of affine transformations include translation, scaling, similarity transformation, reflection, rotation, shearing (as Figure 2.3), and compositions of them in any combination.

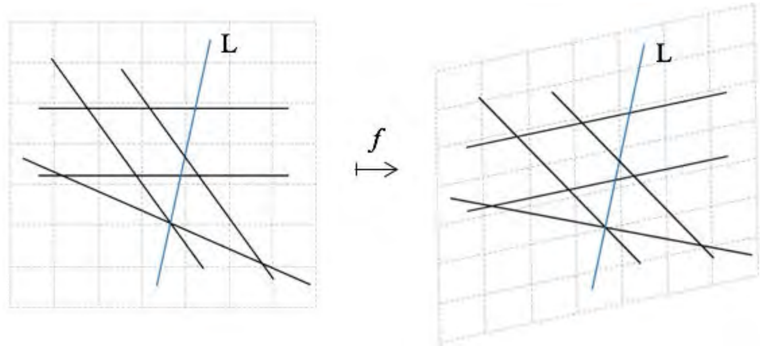

Figure 2.3: A picture mapped by shear mapping with the fixed line L

Theorem 2.10. *Given two ordered sets of three non-colinear points each, there exists a unique affine transformation f mapping one set onto the other.*

2.4 Implicit Function Theorem

Theorem 2.11. *Let $f : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a continuously differentiable function, and let $\mathbb{R}^{n+m}$ have coordinates $(x, y) = (x_1, x_2, ..., x_n, y_1, y_2, ..., y_m)$. Fix a point $(a, b) = (a_1, ..., a_n, b_1, ..., b_m)$ with $f(a, b) = (f_1(a, b), f_2(a, b), ..., f_m(a, b)) = 0$, where $0 \in \mathbb{R}^m$ is the zero vector. If the Jacobian matrix $J_{f,y}(a, b) = \left[\frac{\partial f_i}{\partial y_j}(a, b)\right]$ is invertible, then there exist an open set U of $\mathbb{R}^n$ containing a, and a unique continuously differentiable function $g : U \to \mathbb{R}^m$ such that*

$$g(a) = b$$

and for all $x \in U$,

$$f(x, g(x)) = 0.$$

Moreover, the partial derivative of g in U are given by

$$\frac{\partial g}{\partial x_j}(x) = -\sum_i (J_{f,y}(x, g(x))^{-1})_{ji} \frac{\partial f}{\partial x_i}(x, g(x)).$$

Example 2.12. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2 + y^2$ for $(x, y) \in \mathbb{R}^2$.

Consider the equation $f(x, y) = 1$ which is a unit circle. We cannot represent a unit circle as a graph of a function of one variable $y = g(x)$ because for $x \in (-1, 1)$, $y = \pm\sqrt{1 - x^2}$.

We will use the implicit function theorem to represent part of the circle as the graph of one variable. At $(x_0, y_0) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $f(x_0, y_0) = 1$ and the Jacobian matrix $J_{f,y}(x_0, y_0) = \left[\frac{\partial f}{\partial y}(x_0, y_0)\right] = \left[2y(\frac{1}{2}, \frac{\sqrt{3}}{2})\right] = \left[\sqrt{3}\right]$ which is invertible. By implicit function theorem, there exist an open set U of $\mathbb{R}^2$ containing x_0, and a unique continuously differentiable function $g : U \to \mathbb{R}$ such that

$$g(x_0) = y_0$$

and for all $x \in U$,

$$f(x, g(x)) = 1.$$

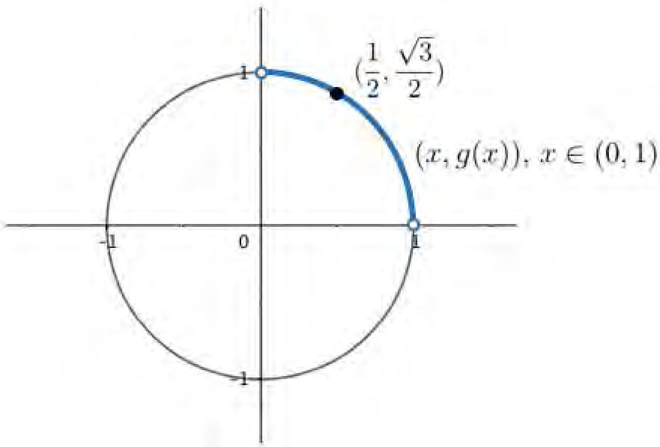

Figure 2.4: Set of part of a unit circle represented as a graph of the function g

In Figure 2.4, we may choose U to be the set $(0, 1)$ which contains $\frac{1}{2}$ and g is a unique continuously differentiable function $g(x) = \sqrt{1 - x^2}$ for all $x \in (0, 1)$ such that $g(\frac{1}{2}) = \frac{\sqrt{3}}{2}$ and $f(x, g(x)) = 1$ for all $x \in (0, 1)$.

CHAPTER III
NECESSARY CONDITIONS FOR CODES OF ARRANGEMENTS

Using (2.1), we may easily derive the following generalized necessary conditions to (1.1), (1.2), and (1.3).

Lemma 3.1. *Let* $\mathcal{A} = \langle\langle \lambda_1, \lambda_2, ..., \lambda_m, ..., \lambda_M | \mu_1, \mu_2, ..., \mu_p, ..., \mu_P \rangle\rangle$ *be the code of an arrangement of* n *lines forming* M *points. Then*

$$\sum_{j=1}^{M} \left[\binom{\lambda_j}{2} - 1 \right] + \sum_{i=1}^{P} \binom{\mu_i}{2} = \binom{n}{2} - M, \tag{3.1}$$

$$\sum_{i=1}^{P} \mu_i = n, \tag{3.2}$$

$$\lambda_1 \leq P \tag{3.3}$$

or, equivalently,

$$\sum_{j=1}^{m} \left[\binom{\lambda_j}{2} - 1 \right] + \sum_{i=1}^{p} \binom{\mu_i}{2} = \binom{n}{2} - M, \tag{3.4}$$

$$\sum_{i=1}^{p} \mu_i + P - p = n, \tag{3.5}$$

$$\lambda_1 \leq p + n - \sum_{i=1}^{p} \mu_i \tag{3.6}$$

Proof.

Equation (3.1) can be directly derived from Roberts' fomulas, where $V = M$. Equation (3.2) is obvious by definition of codes of arrangements. Inequality (3.3) is clear since the point with highest degree is formed by λ_1 lines with distinct slopes. Equation (3.4) is equivalent to (3.1) since $\sum_{j=m+1}^{M} \left[\binom{\lambda_j}{2} - 1 \right]$ and $\sum_{i=p+1}^{P} \binom{\mu_i}{2}$ are equal to 0 because $\lambda_j = 2$ for $m < j \leq M$, $\mu_i = 1$ for $p < i \leq P$, and $\binom{1}{2} = 0$. Equation (3.5) is equivalent to (3.2) by the definition of codes of arrangements. Inequality (3.6) is equivalent to (3.3) because $n = \sum_{i=1}^{p} \mu_i + \sum_{i=p+1}^{P} \mu_i = \sum_{i=1}^{p} \mu_i + (P-p)$ since $\mu_i = 1$ for $p < i \leq P$.

Here we give the another proof of an equation (3.1) and (3.4) which is different from the proof of (2.1) in Robert's formula.

Let $\mathcal{A} = \langle\langle \lambda_1, \lambda_2, ..., \lambda_m, ..., \lambda_M | \mu_1, \mu_2, ..., \mu_p, ..., \mu_P \rangle\rangle$ be the code of an arrangement of n lines forming M points. Note that each two lines that are not parallel has a meeting and there is no meeting of each two lines in the same parallel family. We have that the number of meetings is $\binom{n}{2} - \sum_{i=1}^{p} \binom{\mu_i}{2}$ which is equal to $\sum_{j=1}^{M} \binom{\lambda_j}{2}$. Then

$$\binom{n}{2} - \sum_{i=1}^{p} \binom{\mu_i}{2} - M = \sum_{j=1}^{M} \binom{\lambda_j}{2} - M. \tag{3.7}$$

Since $\displaystyle\sum_{j=1}^{M} \binom{\lambda_j}{2} = \sum_{j=1}^{m} \binom{\lambda_j}{2} + \sum_{j=m+1}^{M} \binom{\lambda_j}{2} = \sum_{j=1}^{m} \binom{\lambda_j}{2} + (M-m)$ and (3.7),

$$\begin{aligned}\binom{n}{2} - \sum_{i=1}^{p} \binom{\mu_i}{2} - M &= \sum_{j=1}^{m} \binom{\lambda_j}{2} + (M-m) - M \\ &= \sum_{j=1}^{m} \binom{\lambda_j}{2} - m \\ &= \sum_{j=1}^{m} \left[\binom{\lambda_j}{2} - 1 \right].\end{aligned}$$

Hence we get (3.4).

To prove (3.1), we have that p in (3.7) can be replaced by P because $\binom{\mu_i}{2} = 0$ for all $p+1 \leq i \leq P$, so $\binom{n}{2} - \sum_{i=1}^{P} \binom{\mu_i}{2} - M = \sum_{j=1}^{M} \binom{\lambda_j}{2} - M = \sum_{j=1}^{M} \left[\binom{\lambda_j}{2} - 1 \right]$. Hence we get (3.1). □

Example 3.2. Considering the arrangement of 4 lines, we get 8 different patterns corresponding to the codes in Figure 3.1. We found that $\langle\langle 3, 3 | 1, 1, 1, 1 \rangle\rangle$ also satisfies the necessary conditions (3.1), (3.2), and (3.3) but cannot be drawn as it needs at least 5 lines to form 2 points of degree 3. This shows that there are more necessary conditions for codes of arrangements to be found.

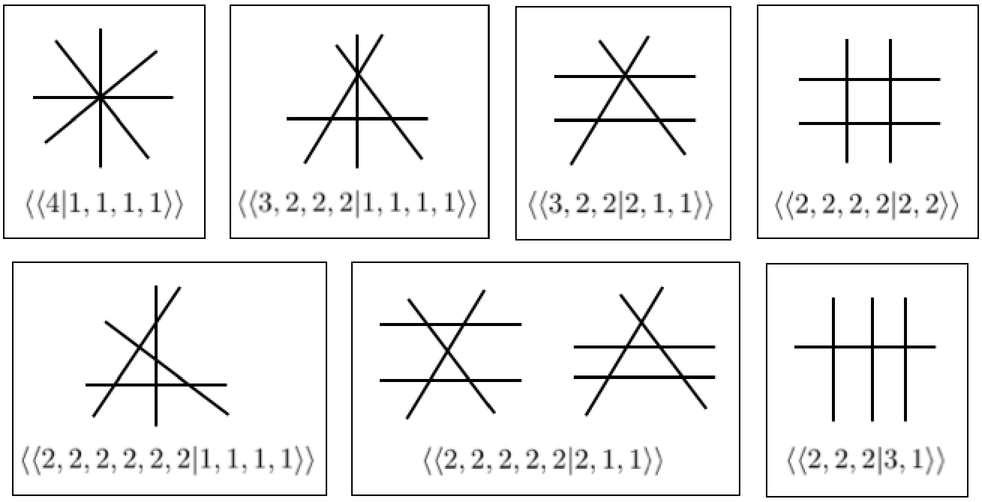

Figure 3.1: Patterns of 4 lines and their corresponding codes

Theorem 3.3. *For the code of an arrangement of n lines forming M points, we have*

$$\sum_{j=1}^{M} max\{\lambda_j - j + 1, 0\} \leq n \tag{3.8}$$

Proof. Let a code $\mathcal{A}$ be given.

Case 1 : $\mathcal{A}$ has no multiple point

For $M = 1$, we have $n = 2$. For $M \geq 2$, we have $n \geq 3$ and $max\{\lambda_j - j + 1, 0\} = 0$ for all $j = 3, 4, ..., M$. Thus $\sum_{j=1}^{M} max\{\lambda_j - j + 1, 0\} \leq n$.

Case 2 : $\mathcal{A}$ has multiple points.

We will consider the least number of lines needed to form all points satisfying this code.

i) To form the first point with degree λ_1, we need λ_1 lines with distinct slopes.

ii) To form the second point with degree λ_2, we need at least new $\lambda_2 - 1$ lines with distinct slopes since the second point may lie on a same line of the first point.

iii) To form each next point $j = 3, 4, ..., M$, we do not need more lines if that point has already formed by many former lines, or we may add only $\lambda_j - (j-1)$ lines pass through some other point of degree $j-1$ formed by many former lines.

Note that $\lambda_j - j + 1$ can be positive, zero, or negative. That is lines may be needed or may not for forming each new point. From i)-iii), the number of lines we need to form all points is $\sum_{j=1}^{M} max\{\lambda_j - j + 1, 0\}$ which must not be more than n. Thus we get (3.8). □

We found that (3.8) cannot be derived from (3.1), (3.2), and (3.3) because the code $\langle\langle 3, 3|1, 1, 1, 1\rangle\rangle$, which are not drawable (see Example 3.2), satisfies (3.1), (3.2), and (3.3) but fails (3.8). There may be more necessary conditions for codes but we did not find an example where a code satisfies these four conditions but cannot be drawn.

CHAPTER IV
REALIZATION OF AN ARRANGEMENT BY DIRECTLY DRAWING

As it is too difficult to find necessary and sufficient conditions for a code to be drawn, we then find a direct method to realize a code. One method to check whether a code is drawable is to directly draw.

Notation We will give some note for this chapter.

(1) We use $\overline{\mathbb{R}}$ for the set $\mathbb{R} \cup \{\infty\}$ and we use p_j for $(x_j, y_j) \in \mathbb{R}^2$.

(2) Point refers to point of intersection. Segment refers to both segment from given lines and segment on the plane not shown in a picture.

(3) The slope of a vertical line is ∞.

(4) For convenience, we assume that the given lines are not vertical. And the equation of a line on the plane is in the form $x = c$ or $y = mx + c$, where $(x, y) \in \mathbb{R}^2$, $m \in \mathbb{R}$, and $c \in \mathbb{R}$.

4.1 Adding Lines

Lemma 4.1. *Let $p_1, p_2, ..., p_k$ be given points and $L_1, L_2, ..., L_r$ given lines on the plane. Then*

(1) There exists a line L parallel to a certain line L_i for some $1 \leq i \leq r$ and not passing through any point p_j for all $1 \leq j \leq k$.

(2) There exists a line L not parrallel to any line L_i for all $1 \leq i \leq r$ and any segment formed by p_j for all $1 \leq j \leq k$, and not passing through any point p_j for all $1 \leq j \leq k$.

(3) There exists a line L passing only one point $p_{j_0}, 1 \leq j_0 \leq k$ and not parallel to any line L_i for all $1 \leq i \leq r$ and any segment formed by p_j, $1 \leq j \leq k$.

(4) Let $\mu \in \mathbb{Z}^+$. Then there exist distinct parallel lines $L_1', L_2', ..., L_\mu'$ not passing through any point p_j for all $1 \leq j \leq k$ and parallel to a certain line L_i for some $1 \leq i \leq r$.

(5) Let $\mu_0, \mu \in \mathbb{Z}^+$ be such that $\mu_0 \leq \mu$ and $\mu_0 \leq k$. Then there exists distinct parallel lines $L_1', L_2', ..., L_\mu'$ not parallel to any line L_i for all $1 \leq i \leq r$ such that each L_l', $l = 1, 2, ..., \mu_0$, passes only one distinct point $p_{j_l}, 1 \leq l \leq \mu_0$, and each L_l', $l = \mu_0 + 1, \mu_0 + 2, ..., \mu$, does not pass any point $p_j, 1 \leq j \leq k$.

Proof.

(1) For $j = 1, 2, ..., k$, let $p_j = (x_j, y_j)$. Let L_{i_0} be a certain line having slope m_0. Let $C = \{-m_0x_j + y_j | j \in \{1, 2, ..., k\}\}$. Then C is finite. There is $c \in \mathbb{R} \backslash C$. Then $c \neq -m_0x_j + y_j$ for all $j \in \{1, 2, ..., k\}$. Let L be a line with equation $y = m_0x + c$. To claim that L does not pass any point, suppose L passes a point $p_{j_0} = (x_{j_0}, y_{j_0})$ for some $j_0 \in \{1, 2, ..., k\}$. Then $y_{i_0} = m_0x_{i_0} + c$. So $c = -m_0x_{i_0} + y_{i_0}$, a contradiction. Thus there exists a line L parallel to a certain line L_{i_0} and not passing through any point p_j for all $1 \leq j \leq k$.

(2) For each $i = 1, 2, ..., r$, let m_i be slope of L_i. For each $j, l = 1, 2, ..., k$ with $j \neq l$, let $\tilde{m}_{jl}$ be a slope of segment formed by p_j and p_l. Let $S = \{m_i | i = 1, ..., r\} \cup \{\tilde{m}_{jl} | j, l = 1, 2, ..., k, j \neq l\}$. Since S is finite, there is $m_0 \in \mathbb{R} \backslash S$. Similar to (1), we can choose $c \neq -m_0x_j + y_j$ for all $j \in \{1, 2, ..., k\}$, and then let L be a line with equation $y = m_0x + c$. Hence L is not parallel to any line L_i for all $1 \leq i \leq r$ and any segment formed by p_j for all $1 \leq j \leq k$, and does not pass through any point p_j for all $1 \leq j \leq k$.

(3) For each $i = 1, 2, ..., r$, let m_i be slope of L_i. Let $p_{j_0} = (x_{j_0}, y_{j_0}) \in \{p_1, p_2, ..., p_k\}$ be a point such that a line L passes through. For each $j, l = 1, 2, ..., k$ with $j \neq l$, let $\tilde{m}_{jl}$ be a slope of segment formed by p_j and p_l. Let $S = \{\tilde{m}_{jl} | j, l = 1, 2, ..., k, j \neq l\} \cup \{m_1, m_2, ..., m_r\}$. Then S is finite, so there is $m \in \mathbb{R} \backslash S$. Let L be a line with equation $y = mx - mx_{j_0} + y_{j_0}$. Thus L passes only one point p_{j_0}

and it is not parallel to any line L_i for all $1 \leq i \leq r$ and any segment formed by p_j, $1 \leq j \leq k$.

(4) By (1), there exists a line L'_1 parallel to a certain line L_i for some $1 \leq i \leq r$ and not passing through any point p_j for all $j = 1, 2, ..., k$. Let C_1 be the set of points of intersection between L'_1 and L_i, $i = 1, 2, ..., r$.

Now consider the elements in $\{p_1, p_2, ..., p_k\} \cup C_1$ as given points and $L_1, L_2, ..., L_r, L'_1$ as given lines on the plane. By (1), there exists a line L'_2 parallel to a certain line L'_1 and not passing through any point in $\{p_1, p_2, ..., p_k\} \cup C_1$. Let C_2 be the set of points of intersection between L'_2 and L_i, $i = 1, 2, ..., r$.

Now consider the elements in $\{p_1, p_2, ..., p_k\} \cup C_1 \cup C_2$ as given points and $L_1, L_2, ..., L_r, L'_1, L'_2$ as given lines on the plane. By (1), there exists a line L'_3 parallel to a certain line L_i and not passing through any point in $\{p_1, p_2, ..., p_k\} \cup C_1 \cup C_2$.

We can continue this process for the lines $L'_l, l = 4, ..., \mu$. Hence there exist distinct parallel lines $L'_1, L'_2, ..., L'_\mu$ not passing through any point p_j for all $j = 1, 2, ..., k$ and parallel to a certain line L_i for some $1 \leq i \leq r$.

(5) By (3), there exists a line L'_1 passing only one point p_{j_1} and not parallel to any line L_i for all $i = 1, 2, ..., r$ and any segment formed by p_j, $1 \leq j \leq k$. Let m'_1 be the slope of L'_1. For each $l = 2, 3, ..., \mu_0$, let L'_l be a line with equation $y = m'_1 x - x_{j_l} + y_{j_l}$. We get distinct lines $L'_l, l = 1, 2, ..., \mu_0$, each L'_l passes a point p_{j_l}. Let O be the set of all points formed by each L'_l and each L_i, $l = 1, 2, ..., \mu_0$, $i = 1, 2, ..., r$.

Now consider L'_l, $l = 1, 2, ..., \mu_0$, and L_i, $l = 1, 2, ..., \mu_0$, as given lines and the elements in $\{p_1, p_2, ..., p_k\} \cup O$ as given points. By (4), there exist distinct parallel lines $L'_l, l = \mu_0 + 1, \mu_0 + 2, ..., \mu$, not passing through any point in $\{p_1, p_2, ..., p_k\} \cup O$ and parallel to a certain line L'_1. Hence there exist distinct parallel lines L'_l, $l = 1, 2, ..., \mu$, not parallel to any line L_i for all $1 \leq i \leq r$, each L'_l, $l = 1, 2, ..., \mu_0$ passes only one distinct point $p_{j_l}, 1 \leq l \leq \mu_0$ and each L'_l, $l = \mu_0 + 1, \mu_0 + 2, ..., \mu$, does not pass any point $p_j, 1 \leq j \leq k$. □

From now on, we assume that a code satisfies the necessary conditions (3.1), (3.2), (3.3), and (3.8).

Proposition 4.2. *Let* $\mathcal{A} = \langle\langle \lambda_1, \lambda_2, ..., \lambda_m, ..., \lambda_M | \mu_1, \mu_2, ..., \mu_p, ..., \mu_P \rangle\rangle$ *be a code of* $n \geq 3$ *lines forming* $M \geq 3$ *points. Suppose* $\mathcal{A}$ *satisfies*

(1) $M = 1$, $\lambda_1 = n$ *and* $\mu_i = 1$ *for all* $i = 1, 2, ..., P$, *or*

(2) $\lambda_j = 2$ *for all* $j = 1, 2, ..., M$, *or*

(3) $\sum_{j=1}^{m} \lambda_j - m + 1 \leq P - p$, *or*

(4) $\lambda_1 = ... = \lambda_m$, $\binom{P-p}{2} \geq m$, $p \geq \lambda_1 - 2$ *and* $\mu_i \geq m$ *for all* $i = 1, ..., p$.

Then $\mathcal{A}$ *is drawable.*

Proof.

(1) Suppose $\mathcal{A}$ satisfies $M = 1$, $\lambda_1 = n$ and $\mu_i = 1$ for all $i = 1, 2, ..., P$. By Lemma 4.1(3), we can draw a picture having only one point with degree n.

(2) Suppose $\mathcal{A}$ satisfies $\lambda_j = 2$ for all $j = 1, 2, ..., M$. Using Lemma 4.1(5), we can draw each $\mu_i, i = 1, 2, ..., P$, parallel lines not passing any point and not parallel to the previous parallel families. By Lemma 4.1(2), we can draw $P - p$ lines not passing any point and not parallel to any line. Hence we get a pattern of no multiple points.

(3) Suppose $\mathcal{A}$ satisfies $\sum_{j=1}^{m} \lambda_j - m + 1 \leq P - p$. Note that the number of lines not parallel to any other line is $P - p$. To draw all multiple points lying on only one line, for each multiple point p_j, we need additional $\lambda_j - 1$ lines. So we need $1 + \sum_{j=1}^{m}(\lambda_j - 1) = \sum_{j=1}^{m} \lambda_j - m + 1$ lines to form all multiple points. Since $\sum_{j=1}^{m} \lambda_j - m + 1 \leq P - p$, we will form all multiple points using $\sum_{j=1}^{m} \lambda_j - m + 1$ single lines. In particular, we first draw a line L. We will draw all multiple points on the line L. By Lemma 4.1(3) and (2), we draw $\lambda_1 - 1$ lines passing through the same point p_1 and not parallel to any line, then draw $\lambda_2 - 1$ lines passing through the a point $p_2 \neq p_1$ and not parallel to any line, and also we draw $\lambda_j - 1$ lines passing through the a point $p_j \neq p_h$ for all $j, h \in \{1, 2, ..., m\}, j > h$ and not parallel to any line. We get multiple points $p_1, p_2, ..., p_m$ with degree $\lambda_1, \lambda_2, ..., \lambda_m$ respectively. Using Lemma 4.1(5), we draw $\sum_{i=1}^{p} \mu_i$ lines satisfying $\mu_1, \mu_2, ..., \mu_p\rangle$, not passing any point and not parallel to any single line. By Lemma 4.1(2), we

draw $(P-p)-(\sum_{j=1}^{m}\lambda_j-m+1)$ single lines not passing through any point and not parallel to any lines. Hence we get a pattern satisfying $\mathcal{A}$.

(4) Suppose $\mathcal{A}$ satisfies $\lambda_1=...=\lambda_m$, $\binom{P-p}{2}\geq m$, $p\geq\lambda_1-2$ and $\mu_i\geq m$ for all $i=1,...,p$. First, we will draw $P-p$ single lines $L_1,L_2,...,L_{P-p}$ forming no multiple points using Lemma 4.1(2). Then there are $\binom{P-p}{2}$ points of degree 2. Let $p_1,p_2,...,p_m$ be points from those $\binom{P-p}{2}$ points. We will form multiple points at p_j for all $j=1,2,...,m$. Since $\mu_1\geq m$, by Lemma 4.1(5), we can draw μ_1 parallel lines $L_l^{(1)}$, $l=1,2,...,\mu_1$ such that for each $l=1,2,...,m$, $L_l^{(1)}$ passes only one point p_l and for each $l=m+1,m+2,...,\mu_1$, $L_l^{(1)}$ does not pass any point p_j for all $j=1,2,...,\binom{P-p}{2}$. Now consider L_i and $L_l^{(1)}$ for all $i=1,2,...,P-p$ and $l=1,2,...,\mu_1$ as given lines and consider all points formed by those given lines as given points. Since $\mu_2\geq m$, by Lemma 4.1(5), we can draw μ_2 parallel lines $L_l^{(2)}$, $l=1,2,...,\mu_2$ such that for each $l=1,2,...,m$, $L_l^{(2)}$ passes only one point p_l and for each $l=m+1,m+2,...,\mu_2$, $L_l^{(2)}$ does not pass any point. Let $e=\lambda_1-2$. We continue this process until we draw μ_e lines of the e^{th} parallel family such that for each $l=1,2,...,m$, $L_l^{(e)}$ passes only one point p_l and for each $l=m+1,m+2,...,\mu_e$, $L_l^{(e)}$ does not pass any point. We get that there are λ_1-2 parallel families such that for each $i=1,2,...,\lambda_1-2$, the i^{th} family has μ_i parallel lines passing through $p_1,p_2,...,p_m$, hence all $p_1,p_2,...,p_m$ have degree $2+(\lambda_1-2)=\lambda_1$, and the other points have degree 2. Now for each $i=\lambda_1-1,\lambda_1,...,p$, we can draw μ_i parallel lines not passing any point and not parallel all former lines using Lemma 4.1(5). Hence we get a pattern satisfying $\mathcal{A}$. □

Example 4.3. The code $\langle\langle 4,3,3,2^{89}|4,2,1^{9}\rangle\rangle$ of 17 lines forming 101 points is drawable.

Since $\sum_{j=1}^{m}\lambda_j-m+1=8\leq 9=P-p$, by Proposition 4.2, $\langle\langle 4,3,3,2^{89}|4,2,1^{9}\rangle\rangle$ is drawable, and its picture is shown in Figure 4.1

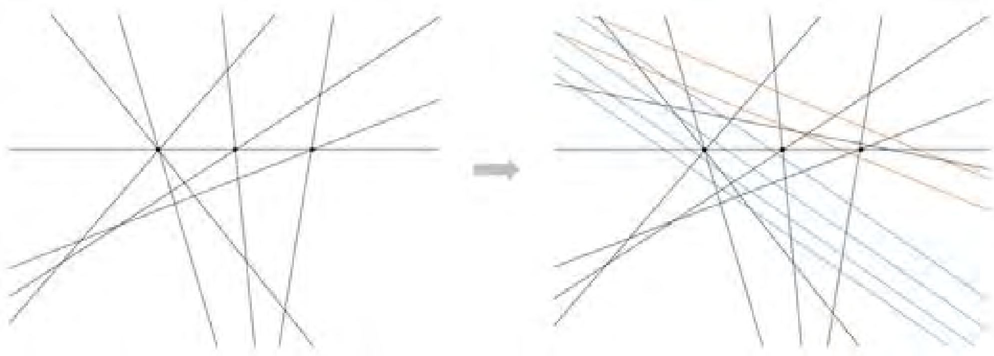

Figure 4.1: A drawable picture for the code $\langle\langle 4,3,3,2^{89}|4,2,1^{9}\rangle\rangle$

4.2 Composing and Manipulation of pictures

In addition to sequentially adding of a line or a whole parallel family, we can manipulate and combine picture to get a pattern satisfying a given code. With every arbitrary picture in proofs, we ignore the case of its reflected picture. In simple cases, without loss of generality, we can combine two pictures by fixing one of them then moving and manipulating only the other. And for convenience during mapping process in proofs, every image of any picture A having the same pattern as A is also called A.

Lemma 4.4. *Let K be a finite subset of $[0,\pi)$. We can rigidly move the whole picture so that every line in the picture has the angle with X-axis not equal to any element in K.*

Proof. Let A be a picture containing lines $L_1, L_2, ..., L_r$ for some $r \in \mathbb{N}$, and $F = \{\alpha_1, \alpha_2, ..., \alpha_k\} \subseteq [0,\pi)$ for some $k \in \mathbb{N}$. For each $j \in \{1,2,...,r\}$, L_j has angle with X-aixs θ_j for some unique $\theta_j \in [0,\pi)$. Note that $\{\delta \in [0,\pi) | \alpha_i - \theta_j \equiv \delta \mod \pi, i \in \{1,2,...,k\}, j \in \{1,2,...,r\}\}$ is finite. There is $\delta_0 \in [0,\pi) \backslash \{\delta \in [0,\pi) | \alpha_i - \theta_j \equiv \delta \mod \pi, i \in \{1,2,...,k\}, j \in \{1,2,...,r\}\}$.

Let ϕ be a rotation of δ_0 couterclockwise. Then $\phi(A)$ is the same picture such that for each $j = 1,2,...,r$, $\phi(L_j)$ is the line having angle with X-axis $\theta'_j \in [0,\pi)$ where $\theta_j + \delta_0 \equiv \theta'_j \mod \pi$.

To claim that every line in $\phi(A)$ does not have angle with X-axis not equal to any element in K, suppose there is L_j such that $\phi(L_j)$ have angle with X-axis α_i for some $i \in \{1, 2, ..., k\}$. Then $\theta_j + \delta_0 \equiv \alpha_i \mod \pi$ so $\delta_0 \equiv \alpha_i - \theta_j \mod \pi$, a contradiction. Hence we can move a picture so that every line in the picture has angle with X-axis not equal to any element in K. □

Lemma 4.5. *Let F be a finite subset of $\mathbb{R}^2$. We can rigidly move the whole picture so that every line in the picture does not pass through any element in F.*

Proof. Let A be a picture containing lines $L_1, L_2, ..., L_r$ for some $r \in \mathbb{N}$, and $F = \{(a_1, b_1), (a_2, b_2), ..., (a_k, b_k)\}$ for some $k \in \mathbb{N}$. Let $J = \{j \in \{1, ..., r\} | L_j$ is a vertical line$\}$. For each $j \in J$, L_j has equation $x = h_j$ for some unique $h_j \in \mathbb{R}$, and for each $j \in \{1, 2, ..., r\} \backslash J$, L_j has equation $y = m_j x + c_j$ for some unique $m_j, c_j \in \mathbb{R}$. Note that $\{a_i - h_j | i \in \{1, 2, ..., k\}, j \in J\}$ is finite. There is $\epsilon_x \in \mathbb{R} \backslash (\{a_i - h_j | i \in \{1, 2, ..., k\}, j \in J\} \cup \{0\})$. Note that $\{b_i - m_j a_i + m_j \epsilon_x - c_j | i \in \{1, 2, ..., k\}, j \in \{1, 2, ..., r\} \backslash J\}$ is finite. There is $\epsilon_y \in \mathbb{R} \backslash (\{b_i - m_j a_i + m_j \epsilon_x - c_j | i \in \{1, 2, ..., k\}, j \in \{1, 2, ..., r\} \backslash J\} \cup \{0\})$.

Let ψ be a translation in the direction of vector $\overrightarrow{(\epsilon_x, \epsilon_y)}$. Then $\psi(A)$ is the same picture as A such that for each $j = 1, 2, ..., r$, if L_j is a vertical line then $\psi(L_j)$ is the line having equation $x = h_j + \epsilon_x$, otherwise, $\psi(L_j)$ is the line having equation $y - \epsilon_y = m_j(x - \epsilon_x) + c_j$ that is $y = m_j x - m_j \epsilon_x + c_j + \epsilon_y$.

To claim that every line in $\psi(A)$ does not pass through any element in F, suppose there is L_j such that $\psi(L_j)$ passing (a_i, b_i) for some $i \in \{1, 2, ..., k\}$. If L_j is a vertical line then $a_i = h_j + \epsilon_x$ so $\epsilon_x = a_i - h_i$, a contradiction. If L_j is not a vertical line then $b_i = m_j a_i - m_j \epsilon_x + c_j + \epsilon_y$ so $\epsilon_y = b_i - m_j a_i + m_j \epsilon_x - c_j$, a contracidtion. Thus for all $j = 1, 2, ..., r$, $\psi(L_j)$ does not pass (a_i, b_i) for all $i \in \{1, 2, ..., k\}$. Hence we can move a picture so that every line in the picture does not pass through any element in F. □

Corollary 4.6. *Let K be a finite subset of $[0, \pi)$ and F a finite subset of $\mathbb{R}^2$. We can rigidly move the whole picture so that every line in the picture has the angle with X-axis not equal to any element in K and every line in the picture does not pass through any element in F.*

Proof. Use a composition of the motion following from Lemma 4.4 and the motion following from Lemma 4.5 to map the picture. □

Note that for two pictures such that every line contains at least two points, if there is no line of one picture passing through any point of the other then it implies that there is no point of one picture lying on any point of the other and there is no line of one picture coincides with any line of the other and hence the degree of points of each picture does not change.

For the case that there is a line in a picture contains only one point, for example in Figure 4.2, it is an easy case to draw using only Proposition 4.2(1), or using Proposition 4.2(1) then Lemma 4.1(1).

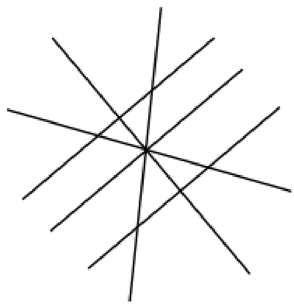

Figure 4.2: A picture with a line containing only one point

From now on, we assume that every line in a picture contains at least two points.

Lemma 4.7. *Let A and B be pictures. We can combine without rotating A and B so that there is no line of one picture passing through any point of the other.*

Proof. Since A and B have finite points of intersection, by Lemma 4.5, we can move B so that every line in B does not pass through all points of intersection of A and then we can move A so that every line in A does not pass through all points of intersection of B. □

We call a combination of A and B in Proposition 4.7 a **basic combination** of A and B.

For two points not lying on a given line, we can say that they are either on the same side or the opposite side with respect to this line. For a picture A having a line L, a transformation of a picture preserving sides of all points with respect to L is a transformation mapping all points in the same side with respect to L to points in the same side with respect to the image of L and mapping all points on L to points on the image of L.

Lemma 4.8. *Let L be a line in a picture containing at most one multiple point. Then we can perturb L. In particular, L is rotated preserving side of all lines with respect to L.*

Proof. If L contains one multiple point, then to preserve the multiple point, we cannot translate L except rotating it about that multiple point. Since the number of points not lying on L are finite, there is a rotation around that multiple point which maps L by preserving the side of all points with respect to L.

If L contains no multiple point, then we can translate L a little bit preserving the side of all points with respect to L.

Clearly, by the definition of patterns and the definition of having the same pattern, if L is rotated preserving side of all lines with respect to L then the pattern does not change. Thus the motions of those two cases preserve pattern, and hence we can perturb L. □

Corollary 4.9. *Let L be a line in a picture such that L has at most one multiple point lying on it. We can perturb all lines in the picture except L.*

Proof. Let A be the picture. By Lemma 4.8, L can be perturbed by some motion ϕ. Consider the inverse of the motion ϕ. We have that the picture after perturbing L by ϕ and the picture after perturbing all lines in A except L by ϕ^{-1} are the same. That is the motion ϕ^{-1} maps all lines in A except L by preserving the side of all points with respect to L. That is we can perturb all lines in a picture except L. □

Lemma 4.10. *Let L be a line in a picture. Suppose each line in the parallel family of L contains at most one multiple point. Then we can perturb the whole parallel family of L.*

Proof. Suppose the parallel family of L has k lines $L_1, ..., L_k$. For each $i = 1, 2, ..., k$, if L_i containing a multiple point then let p_i be such multiple point, otherwise let p_i be any point lying L_i. From the construction Lemma 4.8, all $L_i, i = 1, 2, ..., k$ have a common angle such that the rotation of L_i by that common angle about p_i preserves the side of all points with respect to L_i. Hence we can perturb the whole parallel family of L. □

Remark 4.11. From Lemma 4.10, the composition of each individual rotation of the lines in the parallel family of L may change the distance between those lines. Hence there is no analogy of Corollary 4.9 for perturbation of the whole picture except the parallel family of L.

Next we will show combinations of two pictures in some following possible cases we could identify their codes of the combinations;

- A basic combination with no pair of parallel lines (one from each picture), we call this combination a **simple combination**.
- A certain line from one picture passes a certain point of the other one and the other part is a simple combination
- A certain point from one picture lies on a certain point of the other one and the other part is a simple combination
- A certain line from one picture passes two certain points of the other one and the other part is a simple combination
- Each of two certain points from one picture lies on each of two certain points of the other one and the other part is a simple combination.

Proposition 4.12. *Let A be a picture corresponding to the code $\langle\langle \lambda_1, \lambda_2, ..., \lambda_{M_1} | \mu_1, \mu_2, ..., \mu_{P_1} \rangle\rangle$ and B a picture corresponding to the code $\langle\langle \lambda'_1, \lambda'_2, ..., \lambda'_{M_2} | \mu'_1, \mu'_2, ..., \mu'_{P_2} \rangle\rangle$. Then we can make a simple combination between A and B and the code of the combination of A and B is*

$$\langle\langle \tilde{\lambda}_1, \tilde{\lambda}_2, ..., \tilde{\lambda}_{M_1+M_2}, 2^{\sum_{i=1}^{P_1} \mu_i \sum_{j=1}^{P_2} \mu'_j} | \tilde{\mu}_1, \tilde{\mu}_2, ..., \tilde{\mu}_{P_1+P_2} \rangle\rangle,$$

where

1. $\tilde{\lambda}_i$*'s are nonincreasing list sorted from* λ_q*'s,* $q \in \{1, 2, ..., M_1\}$*, and* λ'_j*'s,* $j \in \{1, 2, ..., M_2\}$*,*

2. $\tilde{\mu}_i$*'s are nonincreasing list sorted from* μ_q*'s,* $q \in \{1, 2, ..., P_1\}$*, and* μ'_j*'s,* $j \in \{1, 2, ..., P_2\}$*.*

Proof. Since the set of angles with X-axis of lines in A is finite, by Lemma 4.4, we move B so that no pair of lines, one from each picture, are parallel. Then make a simple combination between A and B without rotating by Lemma 4.7.

Consider the code of combination of A and B. There are $P_1 + P_2$ distinct parallel families with the same numbers of members since there is no pair of lines, one from each picture, are parallel. Every point has the same degree since it is simple combination. There are new $\sum_{i=1}^{P_1} \mu_i \sum_{j=1}^{P_2} \mu'_j$ meetings of all pairs of lines one from A and the other from B at distinct points of intersection and they all have degree 2. Thus the code of combination is $\langle\langle \tilde{\lambda}_1, \tilde{\lambda}_2, ..., \tilde{\lambda}_{M_1+M_2}, 2^{\sum_{i=1}^{P_1} \mu_i \sum_{j=1}^{P_2} \mu'_j} | \tilde{\mu}_1, \tilde{\mu}_2, ..., \tilde{\mu}_{P_1+P_2} \rangle\rangle$. □

We call a combination of A and B in Proposition 4.12 a **simple combination** of A and B.

Example 4.13. Let A be a picture corresponding to the code $\langle\langle 3, 2^6 | 2, 1^3 \rangle\rangle$ and B a picture corresponding to the code $\langle\langle 2^3 | 1^3 \rangle\rangle$. By Proposition 4.12, we can make a simple combination of A and B as in Figure 4.3 and the code of the combination of A and B is $\langle\langle 3, 2^9, 2^{15} | 2, 1^6 \rangle\rangle = \langle\langle 3, 2^{24} | 2, 1^6 \rangle\rangle$.

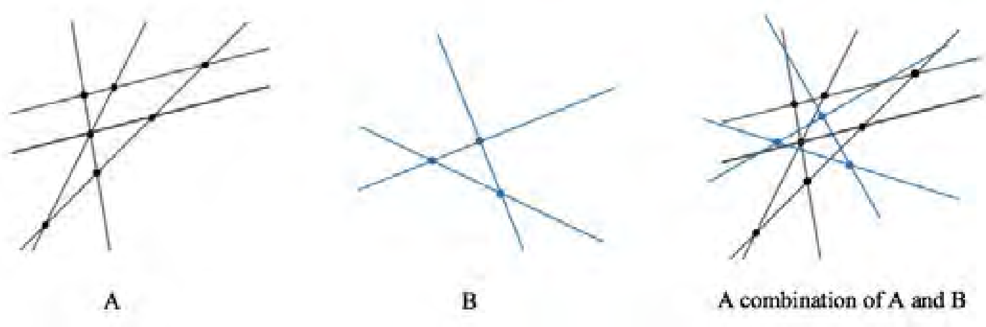

Figure 4.3: A basic combination of A and B

Proposition 4.14. *Let A be a picture corresponding to the code $\langle\langle \lambda_1, \lambda_2, ..., \lambda_{M_1} | \mu_1, \mu_2, ..., \mu_{P_1} \rangle\rangle$ and B a picture corresponding to the code $\langle\langle \lambda'_1, \lambda'_2, ..., \lambda'_{M_2} | \mu'_1, \mu'_2, ..., \mu'_{P_2} \rangle\rangle$. Let p_{j_0} a point in A with degree λ_{j_0} and L a certain line in a parallel family fam_l in B.*

(1) Let L_0 be a line in A and fam_k a parallel family containing L_0. Suppose every line in fam_k does not contain p_{j_0}. Then we can combine A and B so that L passes through p_{j_0}, no point on L lying on p_{j_0}, no line in one picture passes points of the other except for passing p_{j_0} of L, and

a. for the case that every line in fam_k contains at most one multiple point, the lines in fam_k and fam_l are parallel but no other pair of lines from A and B are parallel, so the code of combination is

$$\langle\langle \tilde{\lambda}_1, \tilde{\lambda}_2, ..., \tilde{\lambda}_{M_1+M_2}, 2^{\sum_{i=1}^{P_1} \mu_i \sum_{j=1}^{P_2} \mu'_j - \mu_k \mu'_l - \lambda_{j_0}} | \tilde{\mu}_1, \tilde{\mu}_2, ..., \tilde{\mu}_{P_1+P_2-1} \rangle\rangle,$$

where

1. $\tilde{\lambda}_i$'s are nonincreasing list sorted from λ_q's, $q \in \{1, 2, ..., M_1\} \backslash \{j_0\}$, λ'_j's, $j \in \{1, 2, ..., M_2\}$, and $\lambda_{j_0} + 1$,

2. $\tilde{\mu}_i$'s are nonincreasing list sorted from μ_q's, $q \in \{1, 2, ..., P_1\} \backslash \{k\}$, μ'_j's, $j \in \{1, 2, ..., P_2\} \backslash \{l\}$, and $\mu_k + \mu'_l$,

b. no pair of lines from A and B are parallel, so the code of combination is

$$\langle\langle \tilde{\lambda}_1, \tilde{\lambda}_2, ..., \tilde{\lambda}_{M_1+M_2}, 2^{\sum_{i=1}^{P_1} \mu_i \sum_{j=1}^{P_2} \mu'_j - \lambda_{j_0}} | \tilde{\mu}_1, \tilde{\mu}_2, ..., \tilde{\mu}_{P_1+P_2} \rangle\rangle,$$

where

1. $\tilde{\lambda}_i$'s are nonincreasing list sorted from λ_q's, $q \in \{1, 2, ..., M_1\} \backslash \{j_0\}$, and λ'_j's, $j \in \{1, 2, ..., M_2\}$, and $\lambda_{j_0} + 1$,

2. $\tilde{\mu}_i$'s are nonincreasing list sorted from μ_q's, $q \in \{1, 2, ..., P_1\}$, and μ'_j's, $j \in \{1, 2, ..., P_2\}$.

(2) Let L_{j_0} be a line in a parallel family $fam_{k'}$ in A passing p_{j_0}. Suppose that every line in fam_l contains at most one multiple point. Then we can combine A and B so that L passes through p_{j_0} and coincides with L_{j_0}, no point on L

lying on point on L_{j_0}, no line in one picture passes points of the other except for coincidence of L_{j_0} and L, and no pair of lines from A and B are parallel except for the lines in $fam_{k'}$ and fam_l, so the code of combination is

$$\langle\langle\tilde{\lambda}_1, ..., \tilde{\lambda}_{M_1+M_2}, 2^{\sum_{i=1}^{P_1}\mu_i\sum_{j=1}^{P_2}\mu'_j - \sum_{i=1}^{P_1}\mu_i - \sum_{j=1}^{P_2}\mu'_j - \mu'_l\mu_{k'} + \mu_i + \mu_{k'}}|\tilde{\mu}_1, ..., \tilde{\mu}_{P_1+P_2-1}\rangle\rangle,$$

where

1. $\tilde{\lambda}_i$'s are nonincreasing list sorted from λ_q's, $q \in \{1, 2, ..., M_1\}$, and λ'_j's, $j \in \{1, 2, ..., M_2\}$,

2. $\tilde{\mu}_i$'s are nonincreasing list sorted from μ_q's, $q \in \{1, 2, ..., P_1\}\backslash\{k'\}$, μ'_j's, $j \in \{1, 2, ..., P_2\}\backslash\{l\}$, and $\mu_{k'} + \mu'_l - 1$.

Proof.

(1)a. Frist, we perturb the lines in fam_k to have their slope not equal to any segment formed by p_{j_0} and each point in A by using Lemma 4.10. Then we rotate B so that L parallel to L_0 and then translate B by the vector from L to p_{j_0}. We have that L passes only point p_{j_0}. There may be some points on L lying on p_{j_0} or lying on some lines in A so we can avoid this by translating B in the direction parallel to L. There may be some lines in fam_l coinciding with some lines in fam_k so we can avoid this by directional scaling B using L as a scaling axis as Figure 4.4(ii)-(iii). Then we use a shear map with fixing L to map B so that no lines from A excluding fam_k and B excluding fam_l are parallel and no line in one picture passes points of the other except for passing p_{j_0} of L as Figure 4.4(iii)-(iv). Thus we get the desired combination.

The code of this combination is calculated similarly to the code in Lemma 4.12 but it has a little bit differences. Since the lines in fam_l are parallel to the lines in fam_k, the number of total parallel families decreases 1 and no meeting of each pair from them. Since L passes p_{j_0}, the degree of p_{j_0} increases 1 and the meetings between L and the lines in A containing p_{j_0} occur at a point p_{j_0}. Hence there are new $\sum_{i=1}^{P_1}\mu_i\sum_{j=1}^{P_2}\mu'_j - \mu_k\mu'_l - \lambda_{j_0}$ meetings of all pairs of lines one from A and the other from B at new distinct points of intersection and they all have degree 2. So we get the code as in (1)a.

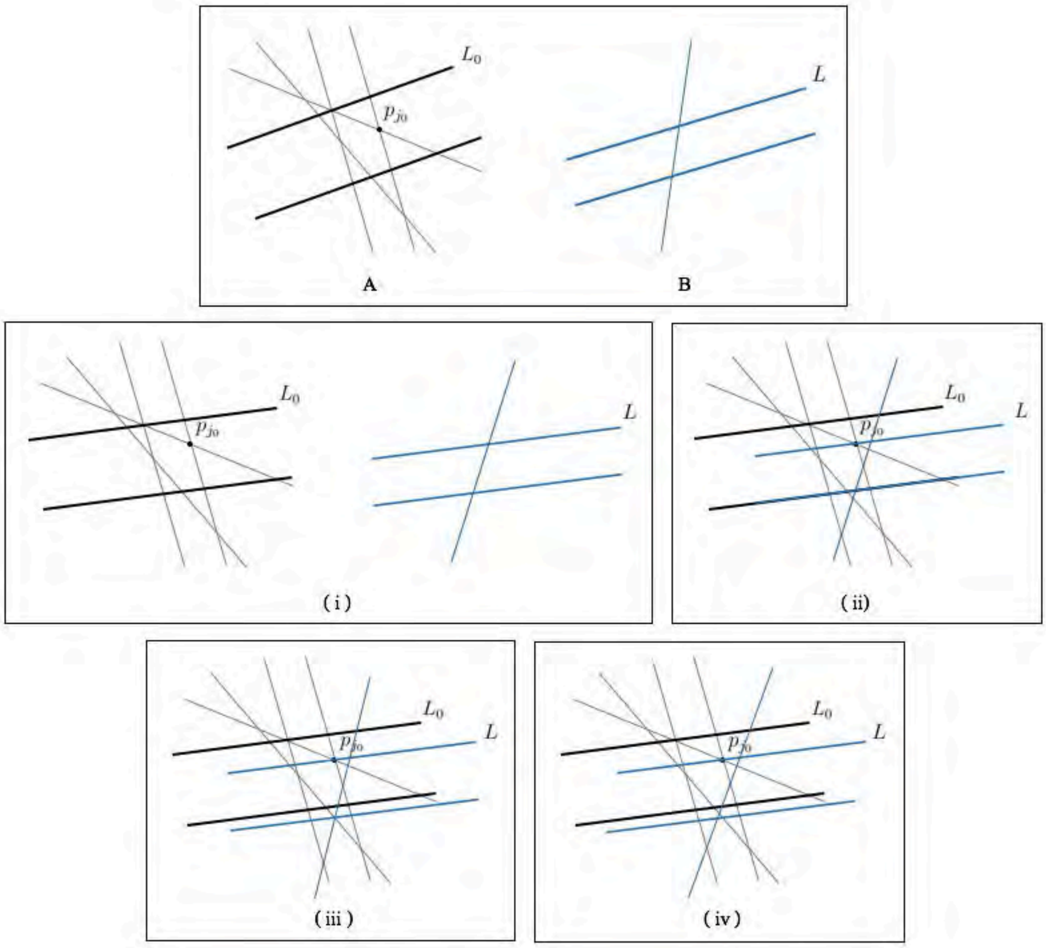

Figure 4.4: A combination of (1)a.

(1)b. We rotate B so that L is not parallel to any segment in A. Similar to (1)a, we can translate B so that L passes p_{j_0} and no point on L lies on p_{j_0} or lies on the lines in A. Then we use a shear map with fixing L to map B so that no lines from A and B are parallel and no line in one picture passes points of the other except for passing p_{j_0} of L. Thus we get the desired combination.

The code of this combination is calculated similarly to the code in Lemma 4.12 but it has a little bit differences. Since L passes p_{j_0}, the degree of p_{j_0} increases 1 and the meetings between L and the lines in A containing p_{j_0} occur at a point p_{j_0}. Hence there are new $\sum_{i=1}^{P_1} \mu_i \sum_{j=1}^{P_2} \mu'_j - \lambda_{j_0}$ meetings of all pairs of lines one from A and the other from B at new distinct points of intersection and they all have degree 2. So we get the code as in (1)b.

(2) Apply the process in (1)a using L_{j_0} and $fam_{k'}$ instead of L_0 and fam_k respectively without perturbing the lines in $fam_{k'}$ at the beginning and we can not avoid passing all points on L_{j_0} of L and passing all points on L of L_{j_0}. We get the desired combination as Figure 4.5.

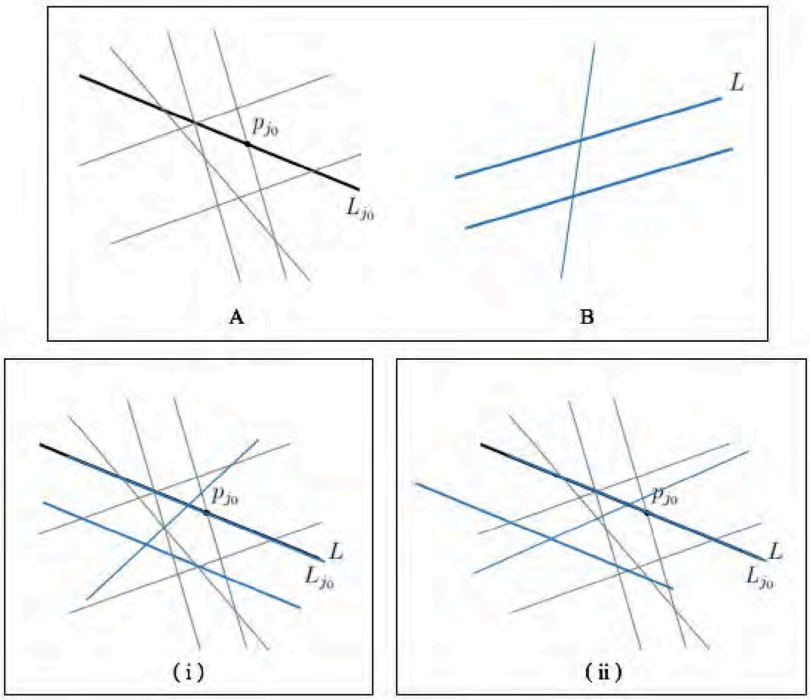

Figure 4.5: A combination of (2)

The code of this combination is calculated similarly to the code in (1)a but it has a little bit differences. Since the lines in fam_l are parallel to the lines in $fam_{k'}$ and L_{j_0} coincides with L, the number of total parallel families decreases 1 and the paralell family containing L has $\mu_{k'} + \mu'_l - 1$ members with no meeting of each pair from them. There are $\sum_{i=1}^{P_1} \mu_i - \mu_{k'}$ meetings between L and each line in A not parallel to L occur at each point on L_{j_0} and also there are $\sum_{j=1}^{P_2} \mu'_j - \mu'_l$ meetings between L_{j_0} and each line in B not parallel to L_{j_0} occur at each point on L. Thus there are new $\sum_{i=1}^{P_1} \mu_i \sum_{j=1}^{P_2} \mu'_j - (\sum_{i=1}^{P_1} \mu_i - \mu_{k'}) - (\sum_{j=1}^{P_2} \mu'_j - \mu'_l) - \mu'_l \mu_{k'}$ meetings of all pairs of lines one from A and the other from B at new distinct points of intersection and they all have degree 2. Since L coincides with L_{j_0}, the degree of each point lying on L and L_{j_0} is still the same. So we get the code as in (2). $\square$

Proposition 4.15. *Let A be a picture corresponding to the code $\langle\langle \lambda_1, \lambda_2, ..., \lambda_{M_1} | \mu_1, \mu_2, ..., \mu_{P_1}\rangle\rangle$ and B a picture corresponding to the code $\langle\langle \lambda'_1, \lambda'_2, ..., \lambda'_{M_2} | \mu'_1, \mu'_2, ..., \mu'_{P_2}\rangle\rangle$. Let p_{j_0} a point in A with degree λ_{j_0}, $p'_{j'_0}$ a point in B with degree $\lambda'_{j'_0}$, and L a certain line in a parallel family fam_l passing $p'_{j'_0}$ in B.*

(1) Let L_0 be a line in A and fam_k a parallel family containing L_0. Suppose every line in fam_k does not contain p_{j_0}. Then we can combine A and B so that $p'_{j'_0}$ lies on p_{j_0}, no line in one picture passes points of the other except for coincidence of p_{j_0} and $p'_{j'_0}$, and

a. for the case that every line in fam_k contains at most one multiple point, the lines in fam_k and fam_l are parallel but no other pair of lines from A and B are parallel, so the code of combination is

$$\langle\langle \tilde{\lambda}_1, \tilde{\lambda}_2, ..., \tilde{\lambda}_{M_1+M_2-1}, 2^{\sum_{i=1}^{P_1} \mu_i \sum_{j=1}^{P_2} \mu'_j - \mu_k \mu'_l - \lambda_{j_0} \lambda'_{j'_0}} | \tilde{\mu}_1, \tilde{\mu}_2, ..., \tilde{\mu}_{P_1+P_2-1}\rangle\rangle,$$

where

1. $\tilde{\lambda}_i$'s are nonincreasing list sorted from λ_q's, $q \in \{1, 2, ..., M_1\}\backslash\{j_0\}$, λ'_j's, $j \in \{1, 2, ..., M_2\}\backslash\{j'_0\}$, and $\lambda_{j_0} + \lambda'_{j'_0}$,

2. $\tilde{\mu}_i$'s are nonincreasing list sorted from μ_q's, $q \in \{1, 2, ..., P_1\}\backslash\{k\}$, μ'_j's, $j \in \{1, 2, ..., P_2\}\backslash\{l\}$, and $\mu_k + \mu'_l$,

b. no pair of lines from A and B are parallel, so the code of combination is

$$\langle\langle \tilde{\lambda}_1, \tilde{\lambda}_2, ..., ..., \tilde{\lambda}_{M_1+M_2-1}, 2^{\sum_{i=1}^{P_1} \mu_i \sum_{j=1}^{P_2} \mu'_j - \lambda_{j_0} \lambda'_{j'_0}} | \tilde{\mu}_1, \tilde{\mu}_2, ..., \tilde{\mu}_{P_1+P_2}\rangle\rangle,$$

where

1. $\tilde{\lambda}_i$'s are nonincreasing list sorted from λ_q's, $q \in \{1, 2, ..., M_1\}\backslash\{j_0\}$, λ'_j's, $j \in \{1, 2, ..., M_2\}\backslash\{j'_0\}$, and $\lambda_{j_0} + \lambda'_{j'_0}$,

2. $\tilde{\mu}_i$'s are nonincreasing list sorted from μ_q's, $q \in \{1, 2, ..., P_1\}$, and μ'_j's, $j \in \{1, 2, ..., P_2\}$.

(2) Let L_{j_0} be a line in a parallel family $fam_{k'}$ in A passing p_{j_0}. Suppose that every line in fam_l contains at most one multiple point. Then we can

combine A and B so that $p'_{j'_0}$ *lies on* p_{j_0} *and* L *coincides with* L_{j_0}*, no point on* L *lying on point on* L_{j_0} *except for coincidence of* p_{j_0} *and* $p'_{j'_0}$*, no line in one picture passes points of the other except for coincidence of* L_{j_0} *and* L*, and no pair of lines from* A *and* B *are parallel except for the lines in* $fam_{k'}$ *and* fam_l*, so the code of combination is*

$$\langle\langle\tilde{\lambda}_1, ..., \tilde{\lambda}_{M_1+M_2-1}, 2^{\sum_{i=1}^{P_1}\mu_i\sum_{j=1}^{P_2}\mu'_j-\sum_{i=1}^{P_1}\mu_i-\sum_{j=1}^{P_2}\mu'_j-\mu'_l\mu_{k'}+\mu_i+\mu_{k'}}|\tilde{\mu}_1, ..., \tilde{\mu}_{P_1+P_2-1}\rangle\rangle,$$

where

1. $\tilde{\lambda}_i$*'s are nonincreasing list sorted from* λ_q*'s,* $q \in \{1, 2, ..., M_1\}\backslash\{j_0\}$*,* λ'_j*'s,* $j \in \{1, 2, ..., M_2\}\backslash\{j'_0\}$*, and* $\lambda_{j_0} + \lambda'_{j'_0}$*,*

2. $\tilde{\mu}_i$*'s are nonincreasing list sorted from* μ_q*'s,* $q \in \{1, 2, ..., P_1\}\backslash\{k'\}$*,* μ'_j*'s,* $j \in \{1, 2, ..., P_2\}\backslash\{l\}$*, and* $\mu_{k'} + \mu'_l - 1$*.*

Proof.

The combinations in (1)a, (1)b, and (2) have the same process as the combination in Proposition 4.14(1)a, (1)b, and (2) respectively but they have the same little bit differences. After rotating B, we translate B by the vector from $p'_{j'_0}$ to p_{j_0} so we have that $p'_{j'_0}$ lies on p_{j_0}, and to avoid coincidence of points on L and lines in A, we cannot translate B in direction parallel to L so we directional scale B using a line passing $p'_{j'_0}$ orthogonal to L as a scaling axis instead.

The codes of the combinations in (1)a and (1)b are similar to the codes of the combination in Proposition 4.14(1)a and (1)b respectively but they have the same little bit differences. Since p_{j_0} lies on $p'_{j'_0}$, they are counted to be one point with degree $\lambda_{j_0} + \lambda'_{j'_0}$ and the meetings between the lines containing p_{j_0} and the lines containing $p'_{j'_0}$ occur at a point p_{j_0}. Hence we get the codes as in (1)a and (1)b. The code of the combination in (2) is similar to the codes of the combination in Proposition 4.14(2) but p_{j_0} and $p'_{j'_0}$ are counted to be one point with degree $\lambda_{j_0} + \lambda'_{j'_0} - 1$. Hence we get the codes as in (2). □

Proposition 4.16. *Let* A *be a picture corresponding to the code* $\langle\langle\lambda_1, \lambda_2, ..., \lambda_{M_1}|\mu_1, \mu_2, ..., \mu_{P_1}\rangle\rangle$ *and* p_a *and* p_b *points in* A *with degree* λ_a *and* λ_b *respectively such that* *no any other point in A is collinear with p_a and p_b. Let B be a picture corresponding to the code $\langle\langle \lambda'_1, \lambda'_2, ..., \lambda'_{M_2} | \mu'_1, \mu'_2, ..., \mu'_{P_2} \rangle\rangle$ and having a line L containing no multiple point. Let fam_k and fam_l be the parallel families belonging to the lines in A parallel to segment $\overline{p_a p_b}$ and the lines in B parallel to L respectively.*

(1) Suppose there is no line in A passing through p_a and p_b. Then we can combine A and B so that L passes through p_a and p_b, no point on L lying on p_a or p_b, no line in one picture passes points of the other except for passing p_a and p_b of L, and

a. the lines in fam_k and the lines in fam_l are parallel but no other pair of lines from A and B are parallel, so the code of combination is

$$\langle\langle \tilde{\lambda}_1, \tilde{\lambda}_2, ..., \tilde{\lambda}_{M_1+M_2}, 2^{\sum_{i=1}^{P_1} \mu_i \sum_{j=1}^{P_2} \mu'_j - \mu_k \mu'_l - (\lambda_a + \lambda_b)} | \tilde{\mu}_1, \tilde{\mu}_2, ..., \tilde{\mu}_{P_1+P_2-1} \rangle\rangle,$$

where

1. $\tilde{\lambda}_i$'s are nonincreasing list sorted from λ_q's, $q \in \{1, 2, ..., M_1\} \backslash \{a, b\}$, λ'_j's, $j \in \{1, 2, ..., M_2\}$, and $\lambda_a + 1$, $\lambda_b + 1$,

2. $\tilde{\mu}_i$'s are nonincreasing list sorted from μ_q's, $q \in \{1, 2, ..., P_1\} \backslash \{k\}$, μ'_j's, $j \in \{1, 2, ..., P_2\} \backslash \{l\}$, and $\mu_k + \mu'_l$,

b. in the case that every line in fam_k contains at most one multiple point, no pair of lines from A and B are parallel, so the code of combination is

$$\langle\langle \tilde{\lambda}_1, \tilde{\lambda}_2, ..., \tilde{\lambda}_{M_1+M_2}, 2^{\sum_{i=1}^{P_1} \mu_i \sum_{j=1}^{P_2} \mu'_j - (\lambda_a + \lambda_b)} | \tilde{\mu}_1, \tilde{\mu}_2, ..., \tilde{\mu}_{P_1+P_2} \rangle\rangle,$$

where

1. $\tilde{\lambda}_i$'s are nonincreasing list sorted from λ_q's, $q \in \{1, 2, ..., M_1\} \backslash \{a, b\}$, λ'_j's, $j \in \{1, 2, ..., M_2\}$, and $\lambda_a + 1$, $\lambda_b + 1$,

2. $\tilde{\mu}_i$'s are nonincreasing list sorted from μ_q's, $q \in \{1, 2, ..., P_1\}$, μ'_j's, $j \in \{1, 2, ..., P_2\}$,

(2) Suppose there is a line L_0 in A passing through p_a and p_b. Then we can combine A and B so that L passes through p_a and p_b, no point on L lying on

p_a or p_b, no line in one picture passes points of the other except for passing p_a and p_b of L and for passing points on L of L_0, and the lines in fam_k and the lines in fam_l are parallel but no other pair of lines from A and B are parallel, so the code of combination is

$$\langle\langle\tilde{\lambda}_1, ..., \tilde{\lambda}_{M_1+M_2}, 2^{\sum_{i=1}^{P_1}\mu_i\sum_{j=1}^{P_2}\mu'_j-\sum_{i=1}^{P_1}\mu_i-\sum_{j=1}^{P_2}\mu'_j-\mu'_l\mu_{k'}+\mu_i+\mu_{k'}}|\tilde{\mu}_1, ..., \tilde{\mu}_{P_1+P_2-1}\rangle\rangle,$$

where

1. $\tilde{\lambda}_i$'s are nonincreasing list sorted from λ_q's, $q \in \{1, 2, ..., M_1\}$, and λ'_j's, $j \in \{1, 2, ..., M_2\}$,

2. $\tilde{\mu}_i$'s are nonincreasing list sorted from μ_q's, $q \in \{1, 2, ..., P_1\}\backslash\{k'\}$, μ'_j's, $j \in \{1, 2, ..., P_2\}\backslash\{l\}$, and $\mu_{k'} + \mu'_l - 1$.

Proof.

(1)a. We rotate B so that L parallel to segment $\overline{p_ap_b}$ and then translate B by the vector from L to p_a. There may be some point on L lying on p_a or p_b so we translate B in direction parallel to L so that every point on L does not lie on p_a or p_b or lying on some lines in A so we can avoid this by translating B in direction parallel to L. There may be some lines in fam_l coinciding with some lines in fam_k so we can avoid this by directional scaling B using L as a scaling axis as Figure 4.6(i)-(ii). Then we use a shear map with fixing L to map B so that no lines from A excluding fam_k and B excluding fam_l are parallel and no line in one picture passes points of the other except for passing p_a and p_b of L as Figure 4.6(ii)-(iii). Thus we get the desired combination.

(1)b. A combination in (1)b is similar to the combination in a. but it has a little bit difference. Note that every line in fam_k contains at most one multiple point, so by Lemma 4.10, we can perturb the lines in fam_k to be not parallel to any line in fam_l in B. We use this perturbation after rotating B as Figure 4.7(i). Then we follow the same step of the combination in a. We get a combiation as Figure 4.7(i)-(ii).

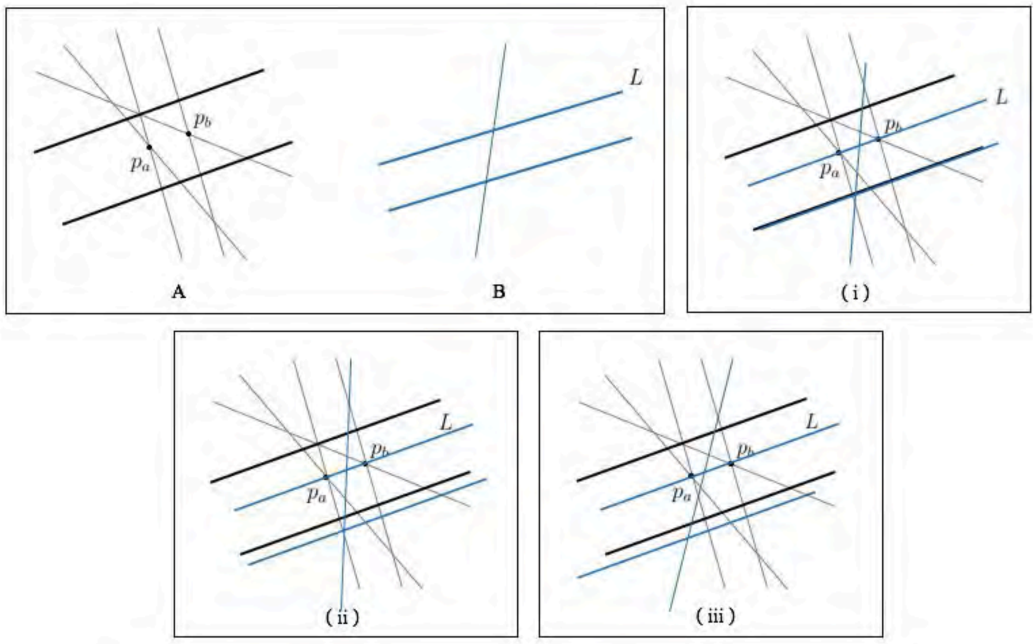

Figure 4.6: A combination of (1)a.

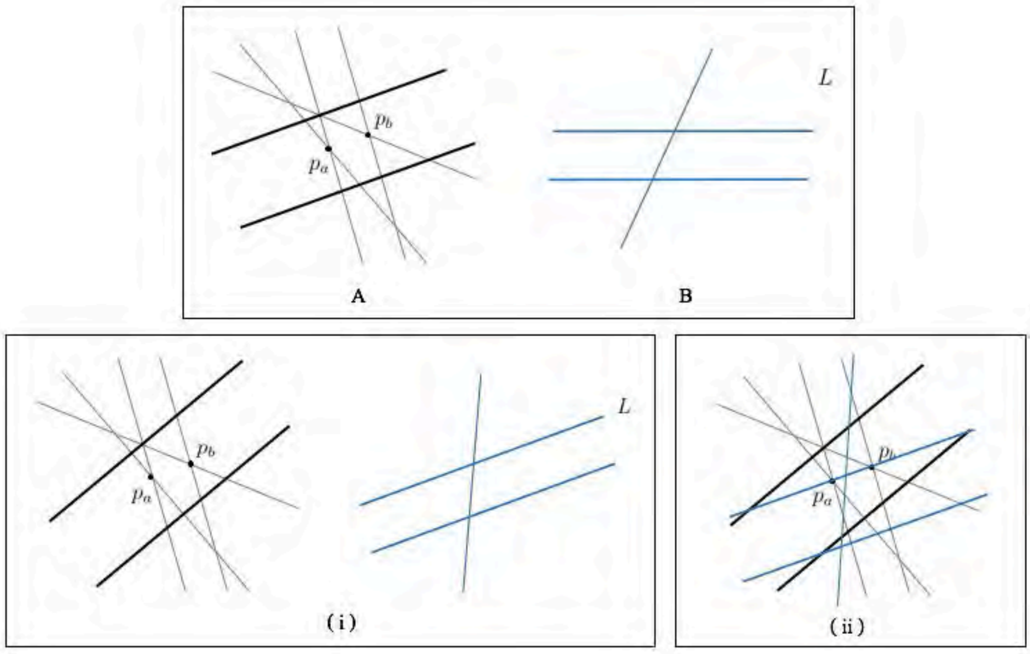

Figure 4.7: A combination of (1)a.

(2) A combination in (2) has the same process as the combination in Proposition 4.14(2) with considering p_a and p_b as p_{j_0}.

The codes of the combinations in (1)a, (1)b, and (2) are similar to the codes of the combinations in Proposition 4.14(1)a, (1)b, and (2) respectively but they have the same little bit difference. Since L passes both p_a and p_b, we consider p_a and p_b in the same way of p_{j_0}. Hence we get the codes as in (1)a, (1)b, and (2). □

Proposition 4.17. *Let A be a picture corresponding to the code $\langle\langle\lambda_1, \lambda_2, ..., \lambda_{M_1}|\mu_1, \mu_2, ..., \mu_{P_1}\rangle\rangle$ and p_a and p_b points in A with degree λ_a and λ_b respectively such that no any other point in A is collinear with p_a and p_b. Let B be a picture corresponding to the code $\langle\langle\lambda'_1, \lambda'_2, ..., \lambda'_{M_2}|\mu'_1, \mu'_2, ..., \mu'_{P_2}\rangle\rangle$ and p'_c and p'_d points in B with degree λ'_c and λ'_d respectively such that no any other point in B is collinear with p'_c and p'_d. Let fam_k and fam_l be the parallel families belonging to the lines in A parallel to segment $\overline{p_a p_b}$ and the parallel families belonging to the lines in B parallel to segment $\overline{p'_c p'_d}$ respectively.*

(1) Suppose there is no line in A passing both p_a and p_b and there is no line in B passing both p'_c and p'_d. Then we can combine A and B so that p'_c lies on p_a and p'_d lies on p_b, no line in one picture passes points in the other except for coinidence of p'_c and p_a and coinidence of p'_d and p_b, and

a. the lines in fam_k and the lines in fam_l are parallel but no other pair of lines from A and B are parallel, so the code of combination is

$$\langle\langle\tilde{\lambda}_1, ..., \tilde{\lambda}_{M_1+M_2-2}, 2^{\sum_{i=1}^{P_1}\mu_i\sum_{j=1}^{P_2}\mu'_j-\mu_k\mu'_l-(\lambda_a\lambda'_c+\lambda_b\lambda'_d)}|\tilde{\mu}_1, ..., \tilde{\mu}_{P_1+P_2-1}\rangle\rangle,$$

where
1. $\tilde{\lambda}_i$'s are nonincreasing list sorted from λ_q's, $q \in \{1, 2, ..., M_1\}\backslash\{a, b\}$, λ'_j's, $j \in \{1, 2, ..., M_2\}\backslash\{c, d\}$, and $\lambda_a + \lambda'_c$, $\lambda_b + \lambda'_d$,
2. $\tilde{\mu}_i$'s are nonincreasing list sorted from μ_q's, $q \in \{1, 2, ..., P_1\}\backslash\{k\}$, μ'_j's, $j \in \{1, 2, ..., P_2\}\backslash\{l\}$, and $\mu_k + \mu'_l$,

b. in the case that every line in fam_k or fam_l contain at most one multiple point, no pair of lines from A and B are parallel, so the code of combination is

$$\langle\langle\tilde{\lambda}_1, \tilde{\lambda}_2, ..., \tilde{\lambda}_{M_1+M_2-2}, 2^{\sum_{i=1}^{P_1}\mu_i \sum_{j=1}^{P_2}\mu'_j-(\lambda_a\lambda'_c+\lambda_b\lambda'_d)}|\tilde{\mu}_1, \tilde{\mu}_2, ..., \tilde{\mu}_{P_1+P_2}\rangle\rangle,$$

where

1. $\tilde{\lambda}_i$*'s are nonincreasing list sorted from* λ_q*'s,* $q \in \{1, 2, ..., M_1\}\backslash\{a, b\}$, λ'_j*'s,* $j \in \{1, 2, ..., M_2\}\backslash\{c, d\}$*, and* $\lambda_a + \lambda'_c$*,* $\lambda_b + \lambda'_d$*,*

2. $\tilde{\mu}_i$*'s are nonincreasing list sorted from* μ_q*'s,* $q \in \{1, 2, ..., P_1\}$*, and* μ'_j*'s,* $j \in \{1, 2, ..., P_2\}$*,*

(2) Suppose there is a line L'_0 *in* B *passing through* p'_c *and* p'_d *but there is no line in* A *passing both* p_a *and* p_b*. Then we can combine* A *and* B *so that* p'_c *lies on* p_a *and* p'_d *lies on* p_b*, no line in one picture passes points in the other except for coinidence of* p'_c *and* p_a *and coinidence of* p'_d *and* p_b*, and*

a. *the lines in* fam_k *and the lines in* fam_l *are parallel but no other pair of lines from* A *and* B *are parallel, so the code of combination is*

$$\langle\langle\tilde{\lambda}_1, ..., \tilde{\lambda}_{M_1+M_2-2}, 2^{\sum_{i=1}^{P_1}\mu_i \sum_{j=1}^{P_2}\mu'_j-\mu_k\mu'_l-(\lambda_a\lambda'_c+\lambda_b\lambda'_d)}|\tilde{\mu}_1, ..., \tilde{\mu}_{P_1+P_2-1}\rangle\rangle,$$

where

1. $\tilde{\lambda}_i$*'s are nonincreasing list sorted from* λ_q*'s,* $q \in \{1, 2, ..., M_1\}\backslash\{a, b\}$, λ'_j*'s,* $j \in \{1, 2, ..., M_2\}\backslash\{c, d\}$*, and* $\lambda_a + \lambda'_c$*,* $\lambda_b + \lambda'_d$*,*

2. $\tilde{\mu}_i$*'s are nonincreasing list sorted from* μ_q*'s,* $q \in \{1, 2, ..., P_1\}\backslash\{k\}$*,* μ'_j*'s,* $j \in \{1, 2, ..., P_2\}\backslash\{l\}$*, and* $\mu_k + \mu'_l$*,*

b. *in the case that every line in* fam_k *contain at most one multiple point, no pair of lines from* A *and* B *are parallel, so the code of combination is*

$$\langle\langle\tilde{\lambda}_1, \tilde{\lambda}_2, ..., \tilde{\lambda}_{M_1+M_2-2}, 2^{\sum_{i=1}^{P_1}\mu_i \sum_{j=1}^{P_2}\mu'_j-(\lambda_a\lambda'_c+\lambda_b\lambda'_d)}|\tilde{\mu}_1, \tilde{\mu}_2, ..., \tilde{\mu}_{P_1+P_2}\rangle\rangle,$$

where

1. $\tilde{\lambda}_i$*'s are nonincreasing list sorted from* λ_q*'s,* $q \in \{1, 2, ..., M_1\}\backslash\{a, b\}$, λ'_j*'s,* $j \in \{1, 2, ..., M_2\}\backslash\{c, d\}$*, and* $\lambda_a + \lambda'_c$*,* $\lambda_b + \lambda'_d$*,*

2. $\tilde{\mu}_i$*'s are nonincreasing list sorted from* μ_q*'s,* $q \in \{1, 2, ..., P_1\}$*, and* μ'_j*'s,* $j \in \{1, 2, ..., P_2\}$*,* *(3) Suppose there is a line L_0 in A passing through p_a and p_b and there is a line L_0' in B passing through p_c' and p_d'. Then we can combine A and B so that p_c' lies on p_a and p_d' lies on p_b, no line in one picture passes points in the other except for coinidence of p_c' and p_a and coinidence of p_d' and p_b, and the lines in fam_k and the lines in fam_l are parallel but no other pair of lines from A and B are parallel, so the code of combination is*

$$\langle\langle\tilde{\lambda}_1,...,\tilde{\lambda}_{M_1+M_2-2},2^{\sum_{i=1}^{P_1}\mu_i\sum_{j=1}^{P_2}\mu_j'-\sum_{i=1}^{P_1}\mu_i-\sum_{j=1}^{P_2}\mu_j'-\mu_l'\mu_{k'}+\mu_i+\mu_{k'}}|\tilde{\mu}_1,...,\tilde{\mu}_{P_1+P_2-1}\rangle\rangle,$$

where

1. $\tilde{\lambda}_i$'s are nonincreasing list sorted from λ_q's, $q\in\{1,2,...,M_1\}\backslash\{a,b\}$, λ_j''s, $j\in\{1,2,...,M_2\}\backslash\{c,d\}$, and $\lambda_a+\lambda_c'-1$, $\lambda_b+\lambda_d'-1$,

2. $\tilde{\mu}_i$'s are nonincreasing list sorted from μ_q's, $q\in\{1,2,...,P_1\}\backslash\{k\}$, μ_j''s, $j\in\{1,2,...,P_2\}\backslash\{l\}$, and $\mu_k+\mu_l'$.

Proof.

(1)a. We rotate B so that segment $\overline{p_c'p_d'}$ is parallel to segment $\overline{p_ap_b}$ and directional scale B using a line orthogonal to $\overline{p_c'p_d'}$ as a scaling axis so that $\overline{p_c'p_d'}$ have the same length as $\overline{p_ap_b}$ as Figure 4.8(i). Then we translate B by the vector from p_c' to p_a. We have that p_c' lies on p_a and p_d' lies on p_b. Note that there is no point in A and B collinear with $\overline{p_ap_b}$ but there may be some lines in fam_l coinciding with some lines in fam_k so we can avoid this by directional scaling B using $\overleftrightarrow{p_c'p_d'}$ as a scaling axis. Then we use a shear map with fixing L to map B so that no lines from A excluding fam_k and B excluding fam_l are parallel and no line in one picture passes points of the other except for coinidence of p_c' and p_a and coinidence of p_d' and p_b as Figure 4.8(ii)-(iii). Thus we get the desired combination.

(1)b. First, we perturb the lines in fam_k so that they are not parallel to $\overline{p_ap_b}$. Then we follow the process of the combination in (1)a.

The combinations in (2)a and (3) have the same process as the combination in (1)a and a combination in (2)b has the same process as the combination in (1)b.

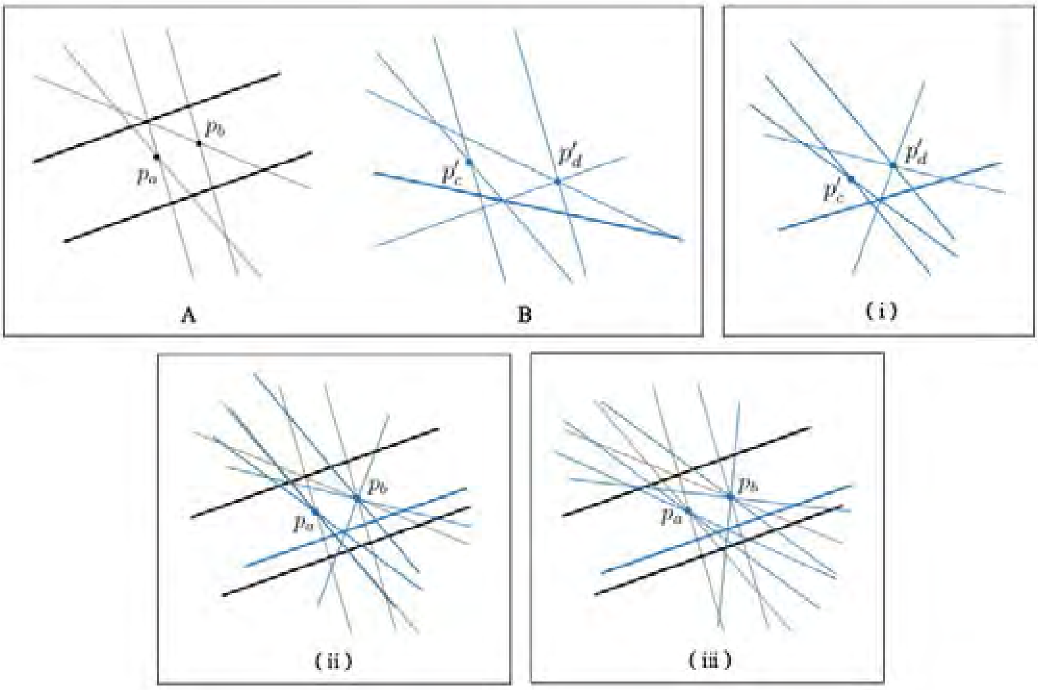

Figure 4.8: A combination of (1)a.

The codes of the combinations in (1)a and (2)a are similar to the code of the combination in Proposition 4.15(1)a and the codes of the combinations in (1)b and (2)b are similar to the code of the combination in Proposition 4.15(1)b but they all have the same little bit differences. Since p'_c lies on p_a, they are counted to be one point with degree $\lambda_a + \lambda'_c$ and the meetings between the lines containing p_a and the lines containing p'_c occur at a point p_a. This is also true for p'_d lying on p_b. Hence we get the codes as in (1)a, (1)b, (2)a, and (2)b. A codes of the combinations in (3) is similar to the code of the combination in Proposition 4.15(2) but p_a, p_b, p'_c and p'_d are counted to be two points with degree $\lambda_a + \lambda'_c - 1$ and $\lambda_b + \lambda'_d - 1$. □

In Proposition 4.17, we can see that there is a unique scaling map to scale one picture along a segment formed by two certain points to have distance between them equal to distance between the certain two points of the other picture. So we have to assume that there is no point collinear with these two segments. If not, the points collinear with those segments may inevitably coincide. Now consider the following combination of two pictures with each of three non-collinear points in one picture lying with each of three non-collinear points in other picture. Let A and B be pictures, p_{a_1}, p_{a_2}, p_{a_3} such non-collinear points points in A, and p'_{b_1}, p'_{b_2}, p'_{b_3} such non-collinear points in B. By Theorem 2.10, there is a unique affine transformation f mapping p'_{b_1}, p'_{b_2} and p'_{b_3} to p_{a_1}, p_{a_2} and p_{a_3} respectively. This show that we can combine A and B so that p'_{b_1}, p'_{b_2} and p'_{b_3} lie on p_{a_1}, p_{a_2} and p_{a_3} respectively but we can not use any other affine map to manipulate the pictures to have a certain kind of arrangement since there is a unique affine transformation to map those three points to those other three points except that we could use perturbations to manipulate them. So a code of this kind of combination can not be identified without additional information on other parts of the two picture.

Example 4.18. All 924 codes of 17 lines forming 101 points [4] can be divided into 3 cases as in Figure 4.9

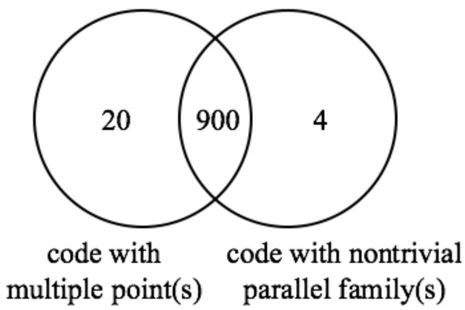

Figure 4.9: The number of codes for Fourier's 17 line problem satisfying the necessary conditions (1.1), (1.2) and (1.3) are divided into 3 cases

We will show examples for 3 codes from each case which can be directly drawn.

i) The code $\langle\langle 2^{101}|8, 4, 2, 1^3\rangle\rangle$ is from the case of nontrivial parallel family with no multiple point. We may add each line using Proposition 4.2. Then we get a pattern as Figure 4.10.

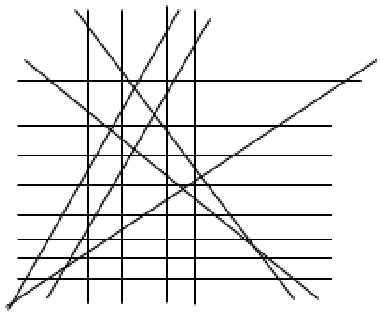

Figure 4.10: A patterns corresponding to $\langle\langle 2^{101}|8,4,2,1^3\rangle\rangle$

ii) The code $\langle\langle 8,3^4,2^{96}|1^{17}\rangle\rangle$ is from the case of trivial parallel family with multiple point. We may use 8 lines to form the point with degree 8 using Proposition 4.2 and use other 9 lines to form 4 points with degree 3 using Lemma 4.1(2) and Lemma 4.1(3). We then get two pictures as Figure 4.11.

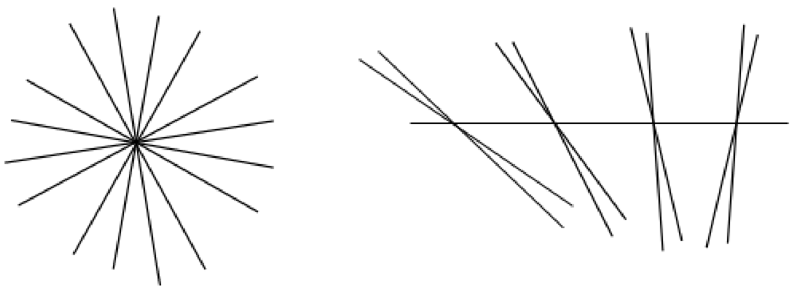

Figure 4.11: A patterns corresponding to $\langle\langle 8,3^4,2^{96}|1^{17}\rangle\rangle$

Finally, we may make a simple combination of these pictures using Proposition 4.12.

iii) The code $\langle\langle 4,3,2^{99}|8,1^9\rangle\rangle$ is from the case of nontrivial parallel family with multiple point. Since there are 9 single lines (no other line in its parallel family) and there are 2 multiple points with degree 4 and 3, we can use 4 and 3 single lines to form each multiple point. Then use other 10 lines to form a picture with 8 parallel lines, 2 free lines, and no multiple point. We get three pictures as Figure 4.12.

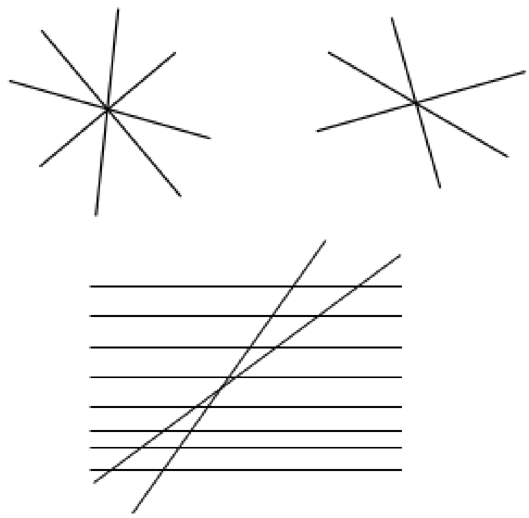

Figure 4.12: A patterns corresponding to $\langle\langle 4, 3, 2^{99}|8, 1^9\rangle\rangle$

Finally, we may make a simple combination of these pictures using Proposition 4.12.

4.3 Conclusion and Suggestion

The Sections 4.1 and 4.2 give some tools for directly drawing some patterns. In particular, some codes have the complicated line and point data which may not be drawable by using our lemmas and propositions straightforwardly. However, for some complicated codes, each can be divided into a lot of codes with smaller data in many possible cases so that they can be drawn using appropriate tools in this chapter.

CHAPTER V
REALIZATION OF AN ARRANGEMENT BY ANALYTIC METHOD

In the previous chapter, we have a method to check whether a code is drawable by directly drawing n lines having relation according to that code. For this section, one equivalence to that drawing is having a solution to a system of equations with some additional conditions.

For a code of n lines forming M points with $m = 1$ and $p = 1$, it can be easily drawn by first forming a multiple point by λ_1 lines, then drawing $\mu_1 - 1$ parallel lines to one of those λ_1 lines, and finally drawing the remaining lines not passing any point and not parallel to any line. We call this code a **trivial code**.

Example 5.1. The system of equations satisfying $\langle\langle 3, 2, 2|2, 1, 1\rangle\rangle$.

We can see that the code $\langle\langle 3, 2, 2|2, 1, 1\rangle\rangle$ is a trivial code so it is drawable as in Figure 3.1.

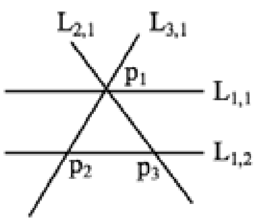

Figure 5.1: A picture for the code $\langle\langle 3, 2, 2|2, 1, 1\rangle\rangle$

To find a relation between each mentioned line and each mentioned point with each $p_j = (x_j, y_j)$ and each L_{ik} is the line $y = m_i x + c_{i,k}$, as in Figure 5.1, we get the relation between each line and each point on the plane equivalent to an equation in the following system

$$\begin{aligned} y_1 &= m_1x_1 + c_{1,1} & y_2 &= m_1x_2 + c_{1,2} \\ y_3 &= m_1x_3 + c_{1,2} & y_1 &= m_2x_1 + c_{2,1} \\ y_2 &= m_2x_2 + c_{2,1} & y_1 &= m_3x_1 + c_{3,1} \\ y_3 &= m_3x_3 + c_{3,1} \end{aligned} \tag{5.1}$$

Each equation represents each line passing through each point in the form of linear equation but here $m_i, c_{i,k}, x_j, y_j$'s are all variables.

Since $\langle\langle 3,2,2|2,1,1\rangle\rangle$ is drawable, the system of equations (5.1) has solutions $(m_1, m_2, m_3, c_{1,1}, c_{1,2}, c_{2,1}, c_{3,1}, x_1, y_1, x_2, y_2, x_3, y_3)$ corresponding to Figure 5.1.

In general, let $\mathcal{A} = \langle\langle \lambda_1, \lambda_2, ..., \lambda_M | \mu_1, \mu_2, ..., \mu_P \rangle\rangle$ be a code of n lines forming M points. The systems of equations satisfying to $\mathcal{A}$ and each equation in the system is in the form

$$y_j = m_i x_j + c_{i,k}, \tag{5.2}$$

where

1. $i = 1, 2, ..., P$ and for each i, $k = 1, 2, ..., \mu_i$,
2. $j = 1, 2, ..., M$,
3. For each $j = 1, 2, ..., M$, x_j and y_j appear in λ_j equations.

We call each equations $y = m_i x + c_{i,k}$ a **pseudo-line**, denoted by L_{ik}, and we say that L_{ik} contains p_j or p_j lies on L_{ik} if there is an equation $y_j = m_i x_j + c_{i,k}$ in the system of equations.

Suppose that (5.2) has solution $(m_1, m_2, ..., m_P, c_{1,1}, c_{1,2}, ..., c_{1,\mu_1}, c_{2,1}, ..., c_{P,\mu_P}, x_1, y_1, ..., x_M, y_M) = (\bar{m}_1, \bar{m}_2, ..., \bar{m}_P, \bar{c}_{1,1}, \bar{c}_{1,2}, ..., \bar{c}_{P,\mu_P}, \bar{x}_1, \bar{y}_1, ..., \bar{x}_M, \bar{y}_M)$. For each $i = 1, 2, ..., P$, for each $k = 1, 2, ..., \mu_i$, let $\bar{L}_{ik}$ be a line $y = \bar{m}_i x + \bar{c}_{ik}$. We consider the lines $\bar{L}_{ik}$'s and their points of intersection. They may not form a picture corresponding to $\mathcal{A}$ so we have to observe the conditions for the solution of (5.2) such that all $y = m_i x + c_{ik}$'s can form a picture corresponding to $\mathcal{A}$.

Theorem 5.2. *Let $\mathcal{A}$ be a code of an arrangement of n lines forming M points. Let S be a system of equation satisfying $\mathcal{A}$. Assume there is a solution of S satisfying the conditions*

C1: $m_{i_1} \neq m_{i_2}$ for each $i_1, i_2 = 1, 2, ..., P$ with $i_1 \neq i_2$,

C2: $c_{ik_1} \neq c_{ik_2}$ for each $i = 1, 2, ..., P$ and for all $k_1, k_2 \in \{1, 2, ..., \mu_i\}$ with $k_1 \neq k_2$,

C3: $(x_{j_1}, y_{j_1}) \neq (x_{j_2}, y_{j_2})$ for each $j_1, j_2 = 1, 2, ..., M$, and

C4: $y_j \neq m_l x_j + c_{lh}$ for each $j = 1, 2, ..., M$ and for each pseudo-line L_{lh} in S such that $L_{lh} \notin E_j$ where E_j is the set of pseudo-line L_{ik} in S such that L_{ik} containing p_j.

Then $\mathcal{A}$ is drawable.

Proof. Let $\mathcal{A} = \langle\langle \lambda_1, \lambda_2, ..., \lambda_M | \mu_1, \mu_2, ..., \mu_P \rangle\rangle$ and $y_j = m_i x_j + c_{ik}$'s equations in S. Let $(m_1, m_2, ..., m_P, c_{1,1}, c_{1,2}, ..., c_{P,\mu_P}, x_1, y_1, ..., x_M, y_M) = (\bar{m}_1, \bar{m}_2, ..., \bar{m}_P, , \bar{c}_{1,1}, \bar{c}_{1,2}, ..., \bar{c}_{P,\mu_P}, \bar{x}_1, \bar{y}_1, ..., \bar{x}_M, \bar{y}_M)$ be a solution of S satisfying C1, C2, C3, and C4. For each $i = 1, 2, ..., P$, for each $k = 1, 2, ..., \mu_i$, let $\bar{L}_{ik}$ be a line $y = m_i x + c_{ik}$.

First, we will show that the lines $\bar{L}_{ik}$'s are n distinct lines forming M points and have the arrangement satisfying $\mathcal{A}$. By C1 and C2, $\bar{L}_{ik}$'s are n distinct lines with P parallel families and the i^{th} parallel family consists of μ_i lines. Thus $\bar{L}_{ik}$'s have the arrangement satifying $\mu_1, \mu_2, ..., \mu_P \rangle\rangle$.

To prove that the arrangement of $\bar{L}_{ik}$'s has M points of intersection satisfying $\langle\langle \lambda_1, \lambda_2, ..., \lambda_M$, first we will show that $(\bar{x}_j, \bar{y}_j)$, $j = 1, 2, ..., M$ are distinct M points with degree λ_j, $j = 1, 2, ..., M$ respectively. Let R be the set of all points of intersection of $\bar{L}_{ik}$'s. For each $j = 1, 2, ..., M$, let $\bar{S}_j$ be the system of equation containing all equations $\bar{L}_{ik}$'s such that $L_{ik}(p_j) \in S$. Since $\bar{p}_j = (\bar{x}_j, \bar{y}_j) \in \bar{L}_{ik}$ for all $\bar{L}_{ik} \in S$, $\bar{p}_j$ is a solution of $\bar{S}_j$. Note that $\bar{L}_{ik}$'s $\in \bar{S}_j$ are λ_j distinct lines with distinct slopes and a solution of $\bar{S}_j$ exists. Then $\bar{S}_j$ has a unique solution which is $\bar{p}_j$. For the degree of $\bar{p}_j$, we have that $\bar{p}_j$ has degree at least λ_j since it lies on λ_j lines in $\bar{S}_j$. Suppose $\bar{p}_j$ has degree greater than λ_j. Then there is some line $\bar{L}_{lh}$ such that $\bar{p}_j \in \bar{L}_{lh}$, $\bar{L}_{lh} \notin \bar{S}_j$ and not parallel to the lines in $\bar{S}_j$. So $L_{lh}(p_j) \notin S$. By C4, $\bar{p}_j \notin \bar{L}_{lh}$, a contradiction. Thus $\bar{p}_j$ has degree λ_j. Hence $\bar{p}_1, \bar{p}_2, ..., \bar{p}_M$ are points of intersection of the lines $\bar{L}_{ik}$'s with degree $\lambda_1, \lambda_2, ..., \lambda_M$ respectively and they are all distinct by C3. Next we will show that R contains only $\bar{p}_1, \bar{p}_2, ..., \bar{p}_M$. Suppose there are points of intersection of $\bar{L}_{ik}$'s other than $\bar{p}_1, \bar{p}_2, ..., \bar{p}_M$, say $\bar{p}_{M+1}, \bar{p}_{M+2}, ..., \bar{p}_{M'}$, $M' \geq M+1$ having degree $\lambda_{M+1}, \lambda_{M+2}, ..., \lambda_{M'}$ respectively. Then the arrangement of $\bar{L}_{ik}$'s is represented by the code $\langle\langle \lambda_1, \lambda_2, ..., \lambda_M, \lambda_{M+1}, ..., \lambda_{M'} | \mu_1, \mu_2, ..., \mu_P \rangle\rangle$ which is drawable. Thus this code satisfies (3.1), we get

$$\sum_{j=1}^{M'} \left[\binom{\lambda_j}{2} - 1 \right] + \sum_{i=1}^{P} \binom{\mu_i}{2} = \binom{n}{2} - M',$$

so

$$\sum_{j=1}^{M} \left[\binom{\lambda_j}{2} - 1 \right] + \sum_{j=M+1}^{M'} \left[\binom{\lambda_j}{2} - 1 \right] + \sum_{i=1}^{P} \binom{\mu_i}{2} = \binom{n}{2} - M + M - M'. \quad (5.3)$$

Since $\mathcal{A}$ satisfies (3.1),

$$\sum_{j=1}^{M} \left[\binom{\lambda_j}{2} - 1 \right] + \sum_{i=1}^{P} \binom{\mu_i}{2} = \binom{n}{2} - M. \quad (5.4)$$

From (5.3) and (5.4), we get

$$\sum_{j=M+1}^{M'} \left[\binom{\lambda_j}{2} - 1 \right] = M - M'.$$

This is impossible because $\binom{\lambda_j}{2} - 1 \geq 0$ for all j but $M - M' < 0$. Thus $R = \{(\bar{x}_j, \bar{y}_j) | j = 1, 2, ..., M\}$. Hence $\mathcal{A}$ is drawable. □

In particular, to realize a code, we have to find possible relations between each line and each point regarding the geometric property of lines on the plane to eliminate some systems of equations with no solution satisfying the conditions C1, C2, C3, and C4. We will find the possible relations between each line and each point using incident matrix.

Definition 5.3. Let $\mathcal{A} = \langle\langle \lambda_1, \lambda_2, ..., \lambda_M | \mu_1, \mu_2, ..., \mu_P \rangle\rangle$ be a code of n lines forming M points. Let $D = [d_{ij}]$ be an $n \times M$ matrix. Assume that the first μ_1 rows of D is called the first row family, the next μ_2 rows of D is called the second row family, and so on. D is an **incident matrix** of A if D satisfies

1. for each $i = 1, 2, ..., n$ and $j = 1, 2, ..., M$, d_{ij} is 0 or 1,
2. for each row family and for each column $j = 1, 2, ..., M$, there is at most one entry equal to 1.
3. for each row $i, l = 1, 2, ..., n$ such that the i^{th} row and the l^{th} row are in distinct row families, there is a unique $j \in \{1, 2, ..., M\}$ such that $d_{ij} = d_{lj} = 1$.
4. for each column $j = 1, 2, ..., M$, $\sum_{i=1}^{n} d_{ij} = \lambda_j$.

Remark 5.4. Let $D = [d_{ij}]$ be a matrix in Definition 5.3. Let $\vec{u}_i$ be the i^{th} row vector of D and $\vec{v}_j$ the j^{th} column vector of D. We have that D is an incident matrix if D satisfies

1. for each $i = 1, 2, ..., n$, $\vec{u}_i$ has each entry equal to 0 or 1,
2. for each $i, l = 1, 2, ..., n$ such that the i^{th} row and the l^{th} row are in the same row family, $\vec{u}_i \cdot \vec{u}_l = 0$.
3. for each $i, l = 1, 2, ..., n$ such that the i^{th} row and the l^{th} row are in distinct row families, $\vec{u}_i \cdot \vec{u}_l = 1$.
4. for each $j = 1, 2, ..., M$, $\vec{v}_j \cdot [1, 1, ..., 1]^{\text{T}} = \lambda_j$.

Remark 5.5.

1. If a code does not have an incident matrix, then it is not realizable.

2. A code can have many different incident matrices up to reordering of the lines in the same parallel family, reordering of the parallel family with the same size, and reordering of the points with the same degree.

3. For a code $\mathcal{A}$ that is not assumed to satisfying the necessary conditions (3.1), (3.2), and (3.3), if $\mathcal{A}$ have an incident matrix then $\mathcal{A}$ satisfies such necessary conditions. This can be proved directly by the definition of an incident matrix.

Example 5.6. An incident matrix of $\langle\langle 3,3,3,3,2|2,2,1,1\rangle\rangle$.

We let L_1 and L_2 be lines in the first parallel family, L_3 and L_4 lines in the second parallel family, and L_5 and L_6 single lines each in the last two parallel families. We found that

$$\begin{array}{c} \\ L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \\ L_6 \end{array}\begin{array}{c} \begin{array}{ccccc} p_1 & p_2 & p_3 & p_4 & p_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

is an incident matrix of $\langle\langle 3,3,3,3,2|2,2,1,1\rangle\rangle$.

Let D be the incident matrix shown in Example 5.6. Then D represents relations between each line and each point. For each entry d_{lj} such that $d_{lj} = 1$, we have a pseudo-line L_{ik} containg p_j or $y_j = m_i x_j + c_{i,k}$ where i is the parallel family number of L_l, k is the line number of L_l in its paralell family. Thus the system of equations

$$\begin{array}{lll} y_1 = m_1x_1 + c_{1,1} & y_2 = m_1x_2 + c_{1,1} & y_3 = m_1x_3 + c_{1,2} \\ y_4 = m_1x_4 + c_{1,2} & y_1 = m_2x_1 + c_{2,1} & y_4 = m_2x_4 + c_{2,1} \\ y_2 = m_2x_2 + c_{2,2} & y_3 = m_2x_3 + c_{2,2} & y_1 = m_3x_1 + c_{3,1} \\ y_3 = m_3x_3 + c_{3,1} & y_5 = m_3x_5 + c_{3,1} & y_2 = m_4x_2 + c_{4,1} \\ y_4 = m_4x_4 + c_{4,1} & y_5 = m_4x_5 + c_{4,1}. & \end{array}$$

is equivalent to D.

Definition 5.7. Let $\mathcal{A}$ be a code. We said that S is an **incident system of equations** if it is equivalent to some incident matrix of $\mathcal{A}$.

Remark 5.8. Let S be an incident system of equations. Then S satisfies

1. for each two pseudo-lines L_{ik_1}, L_{ik_2} with $k_1 \neq k_2$, every p_j lying on L_{ik_1} does not lie on L_{ik_2}, and vice versa, and
2. for each two pseudo-lines L_{ik}, L_{lh} with $i \neq l$, there is only one $j \in \{1, 2, ..., M\}$ such that p_j lies on L_{ik}, L_{lh}.

This follows from the definition of incident matrix.

Next, we will observe the conditions for the solution of an incident system of equation satisfying a code $\mathcal{A}$ that make $\mathcal{A}$ drawable.

Theorem 5.9 and 5.10 will be used for a code with $n \geq 3$ and $M \geq 3$. On the other hand, for a code with $n < 3$ or $M < 3$, it is drawable by the following explanation. For a code with $n < 3$, there are only three different possible drawable codes, namely $\langle\langle\ |1\rangle\rangle$, $\langle\langle\ |2\rangle\rangle$, and $\langle\langle 2|1, 1\rangle\rangle$. For a code with $M = 1$, it can be drawn as a picture with a point degree n. For a code with $M = 2$, if such two points lie on distinct lines then those lines will form other points, so the two points must lie on the same line. Since the other lines passing the two points cannot form the other points, they can be only two parallel lines each one passing each point, hence a drawable code with $M = 2$ can be only $\langle\langle 2, 2|2, 1\rangle\rangle$.

Theorem 5.9. *Let $\mathcal{A}$ be a code. Let S be an incident system of equations satisfying $\mathcal{A}$ with $n \geq 3$ and $M \geq 3$. Assume that S has a solution satisfying C4. Then $\mathcal{A}$ is drawable.*

Proof. Let $\bar{d} = (\bar{m}_1, ..., \bar{m}_P, \bar{c}_{1,1}, ..., \bar{c}_{P,\mu_P}, \bar{x}_1, \bar{y}_1, ..., \bar{x}_M, \bar{y}_M)$ be a solution of the system of equations S satisfying C4. We will prove that $\bar{d}$ satisfies C1, C2, and C3, and then the Theorem 5.2 implies that $\mathcal{A}$ is drawable.

To prove that $\bar{d}$ satisfies C3, suppose that there are $j_1, j_2 \in \{1, 2, ..., M\}$ with $j_1 \neq j_2$ such that $\bar{p}_{j_1} = (\bar{x}_{j_1}, \bar{y}_{j_1}) = (\bar{x}_{j_2}, \bar{y}_{j_2}) = \bar{p}_{j_2}$. Since S is an incident system of equations and $j_1 \neq j_2$, every two pseudo-lines cannot contain both p_{j_1} and p_{j_2}. Then there is a pseudo-lines L_{ik} for some i, k, such that L_{ik} contains p_{j_1} but does not contain p_{j_2}. Thus the equation $y_{j_2} = m_i x_{j_2} + c_{i,k}$ is not in E_{j_2} so $\bar{y}_{j_2} \neq \bar{m}_i x_{j_2} + \bar{c}_{i,k}$ but $\bar{y}_{j_2} = \bar{y}_{j_1} = \bar{m}_i \bar{x}_{j_1} + \bar{c}_{i,k} = \bar{m}_i \bar{x}_{j_2} + \bar{c}_{i,k}$, a contradiction. Hence $\bar{d}$ satisfies C3.

To proof that $\bar{d}$ satisfies C1, suppose that there are $i_1, i_2 \in \{1, 2, ..., P\}$ with $i_1 \neq i_2$ such that $\bar{L}_{i_1 k}$ and $\bar{L}_{i_2 h}$ are parallel.

If they are the same lines, then for every $\bar{p}_j$, we have that $\bar{y}_j = \bar{m}_{i_1} \bar{x}_j + \bar{c}_{i_1,k}$ if and only if $\bar{y}_j = \bar{m}_{i_2} \bar{x}_j + \bar{c}_{i_2,h}$. Since S is an incident system of equations and $i_1 \neq i_2$, there is only one $j_0 \in \{1, 2, ..., M\}$ such that p_{j_0} lies on $L_{i_1 k}$ and $L_{i_2 h}$. Let p_r with $r \neq j_0$ be a point lying on L_{ik}. Then p_r does not lie on $L_{i_2 h}$. Thus the equation $y_r = m_{i_2} x_r + c_{i_2,h}$ is not in E_r so $\bar{y}_r \neq \bar{m}_{i_2} x_r + \bar{c}_{i_2,h}$. Since $\bar{y}_r = \bar{m}_{i_1} x_r + \bar{c}_{i_1,k}$, $\bar{y}_r = \bar{m}_{i_2} x_r + \bar{c}_{i_2,h}$, a contradiction.

If they are the distinct lines, then for every $\bar{p}_j$, we have that if $\bar{y}_j = \bar{m}_{i_1} \bar{x}_j + \bar{c}_{i_1,k}$, then $\bar{y}_j \neq \bar{m}_{i_2} \bar{x}_j + \bar{c}_{i_2,h}$, and vice versa. Since S is an incident system of equations and $i_1 \neq i_2$, there is only one $j_0 \in \{1, 2, ..., M\}$ such that p_{j_0} lies on $L_{i_1 k}$ and $L_{i_2 h}$. Since $\bar{d}$ is a solution, $\bar{y}_{j_0} = \bar{m}_{i_1} \bar{x}_{j_0} + \bar{c}_{i_1,k}$ and $\bar{y}_{j_0} = \bar{m}_{i_2} \bar{x}_{j_0} + \bar{c}_{i_2,h}$, a contradiction. Hence $\bar{d}$ satisfies C1.

To proof that $\bar{d}$ satisfies C2, let $i \in \{1, 2, ..., P\}$ such that $P \geq 2$ and suppose that there are $k_1, k_2 \in \{1, 2, ..., \mu_i\}$ with $k_1 \neq k_2$ such that $\bar{c}_{i,k_1} = \bar{c}_{i,k_2}$. Since S is an incident system of equations and $k_1 \neq k_2$, every p_j lying on L_{ik_1} does not lie on L_{ik_2}, and vice versa. Let p_{j_0} be a point such that p_{j_0} lies on L_{ik_1}. Then the equation $y_{j_0} = m_i x_{j_0} + c_{i,k_2}$ is not in E_{j_0} so $\bar{y}_{j_0} \neq \bar{m}_i \bar{x}_{j_0} + \bar{c}_{i,k_2}$ but $\bar{y}_{j_0} = \bar{m}_i \bar{x}_{j_0} + \bar{c}_{i,k_1} = \bar{m}_i \bar{x}_{j_0} + \bar{c}_{i,k_2}$, a contradiction. Hence $\bar{d}$ satisfies C2.

Since $\bar{d}$ satisfies C1, C2, C3, and C4, by Theorem 5.2, $\mathcal{A}$ is drawable. □

Theorem 5.10. *Let S be an incident system of equations satisfying a code $\mathcal{A}$ with $n \geq 3$ and $M \geq 3$. Assume that S has a solution satisfying C1 and C3. Then $\mathcal{A}$ is drawable.*

Proof. Let $\bar{d} = (\bar{m}_1, ..., \bar{m}_P, \bar{c}_{1,1}, ..., \bar{c}_{P,\mu_P}, \bar{x}_1, \bar{y}_1, ..., \bar{x}_M, \bar{y}_M)$ be a solution of the system of equations S satisfying C1 and C3. We will prove that $\bar{d}$ satisfies C4 and then the Theorem 5.9 implies that $\mathcal{A}$ is drawable.

Suppose there are $L_{i_0k_0}, p_j$ such that $L_{i_0k_0}$ does not contain p_j but $\bar{p}_j$ lies on $\bar{L}_{i_0k_0}$. Since S is an incident system of equations, there are $L_{i_1k_1}$ and $L_{i_2k_2}$ both containing p_j where $i_1 \neq i_2$. Suppose $i_1 = i_0$. Then $\bar{L}_{i_0k_0}$ and $\bar{L}_{i_1k_1}$ are parallel since $\bar{d}$ satisfies C1. Since $\bar{L}_{i_0k_0}$ and $\bar{L}_{i_1k_1}$ both contain p_j, they are the same line. Since S is an incident system of equations, there is p_s lying on $L_{i_0k_0}$ and $L_{i_2k_2}$ where $s \neq j$. By C3, $\bar{p}_s \neq \bar{p}_j$. Then $\bar{L}_{i_0k_0}$ and $\bar{L}_{i_2k_2}$ are distinct lines both containg two points $\bar{p}_s$ and $\bar{p}_j$ which is impossible. Thus $i_1 \neq i_0$. With the same reason, $i_2 \neq i_0$. Note that each pair of $\bar{L}_{i_0k_0}, \bar{L}_{i_1k_1}, \bar{L}_{i_2k_2}$ are not parallel since $\bar{d}$ satisfies C1. Since S is an incident system of equations, there is r such that p_r lies on $L_{i_0k_0}$ and $L_{i_1k_1}$. We have $\bar{p}_j$ is a point of intersection of $\bar{L}_{i_1k_1}$ and $\bar{L}_{i_2k_2}$, and $\bar{p}_r$ is a point of intersection of $\bar{L}_{i_0k_0}$ and $\bar{L}_{i_1k_1}$. Since $\bar{p}_j$ lies on $\bar{L}_{i_0k_0}$, $\bar{p}_j$ is also a point of intersection of $\bar{L}_{i_1k_1}$ and $\bar{L}_{i_0k_0}$. Note that $r \neq j$ since $L_{i_0k_0}$ does not contains p_j. By C3, $\bar{p}_r \neq \bar{p}_j$. So $\bar{L}_{i_1k_1}$ and $\bar{L}_{i_0k_0}$ have two points of intersection at $\bar{p}_r$ and $\bar{p}_j$ which is impossible. Hence $\bar{d}$ satisfies C4.

By Theorem 5.9, $\mathcal{A}$ is drawable. $\square$

From the above theorem, if there is an incident system of equations for some code such that the system of equations has solutions satisfying only C1 and C3, then the code is realizable. For the solution not satisfying C1 or C3, we will suggest a way to use the implicit function theorem to possibly adjust some variables.

Let S be an incident system of equations satisfying some code. Let E be the number of equations, V the number of variables. Then $E = \sum_{j=1}^{M} \lambda_j$ and $V = 2M + P + n$.

For the system of equations such that $V > E$, let $F : \mathbb{R}^{(V-E)+E} \to \mathbb{R}^E$ be a function

$$F(w, z) = (f_1(w, z), f_2(w, z), ..., f_E(w, z)),$$

where

1. $(w, z) = (w_1, w_2, ..., w_{V-E}, z_1, z_2, ..., z_E) \in \mathbb{R}^{V-E} \times \mathbb{R}^E$ is some ordering of

the variables m_i's, x_j's, y_j's, $c_{i,k}$'s, and

2. $f_q(w,z) = m_{i_q}x_{j_q} + c_{ik_q} - y_{j_q} = 0$, the q th equation of the system of equations, $q = 1, 2, ..., E$.

Then the Jacobian matrix of F, which is the matrix of the partial derivatives of F, is

$$J = DF = \begin{bmatrix} \frac{\partial f_1}{\partial w_1} & \cdots & \frac{\partial f_1}{\partial w_{V-E}} & \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_E} \\ \frac{\partial f_2}{\partial w_1} & \cdots & \frac{\partial f_2}{\partial w_{V-E}} & \frac{\partial f_2}{\partial z_1} & \cdots & \frac{\partial f_2}{\partial z_E} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \\ \frac{\partial f_E}{\partial w_1} & \cdots & \frac{\partial f_E}{\partial w_{V-E}} & \frac{\partial f_E}{\partial z_1} & \cdots & \frac{\partial f_E}{\partial z_E} \end{bmatrix}_{E \times V}$$

Assume there is $(\bar{w}, \bar{z}) = (\bar{w}_1, \bar{w}_2, ..., \bar{w}_{V-E}, \bar{z}_1, \bar{z}_2, ..., \bar{z}_E) \in \mathbb{R}^{V-E} \times \mathbb{R}^E$ such that $F(\bar{w}, \bar{z}) = 0$, where $0 \in \mathbb{R}^E$ and assume that

$$J_z(\bar{w}, \bar{z}) = \begin{bmatrix} \frac{\partial f_1}{\partial z_1}(\bar{w}, \bar{z}) & \cdots & \frac{\partial f_1}{\partial z_E}(\bar{w}, \bar{z}) \\ \frac{\partial f_2}{\partial z_1}(\bar{w}, \bar{z}) & \cdots & \frac{\partial f_2}{\partial z_E}(\bar{w}, \bar{z}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_E}{\partial z_1}(\bar{w}, \bar{z}) & \cdots & \frac{\partial f_E}{\partial z_E}(\bar{w}, \bar{z}) \end{bmatrix}_{E \times E}$$

is invertible. Then by Theorem 2.11, there exists an open set U of $\mathbb{R}^{V-E}$ containing $\bar{w}$, and such that there exists a unique continuously differentiable function $G : U \to \mathbb{R}^E$ such that $G(\bar{w}) = \bar{z}$ and $F(w, G(w)) = 0$ for all $w \in U$, i.e. $z_i = g_i(w_1, ..., w_{V-E})$, $i = 1, 2, ..., E$ where $G(w) = (g_1(w), g_2(w), ..., g_E(w))$.

Hence for a solution $(\bar{w}, \bar{z})$ of the incident system of equations with $V > E$ that does not satisfy conditions C1 or C3, assume that $(w_1, w_2, ..., w_{V-E}, z_1, z_2, ..., z_E)$ is the ordering of m_i's, x_j's, y_j's, $c_{i,k}$'s such that some variables making $(\bar{w}, \bar{z})$ fail such conditions are of $w_1, w_2, ..., w_{V-E}$ and $J_z(\bar{w}, \bar{z})$ is invertible, then there are many other solutions (w, z) around $(\bar{w}, \bar{z})$. Hence we may specify the variables w to possibly satisfy C1 and C3.

Next, we will show an example of a simple code realized by an analytic method using the implicit function theorem.

Example 5.11. Let $\mathcal{A} = \langle\langle 2^5|2, 1^2\rangle\rangle$ be a code for 4 lines forming 5 points. By finding an incident matrix satifying $\mathcal{A}$, we get an incident system S of equations

$$\begin{array}{lll} y_1 = m_1x_1 + c_{1,1} & y_2 = m_1x_2 + c_{1,1} & y_4 = m_1x_4 + c_{1,2} \\ y_5 = m_1x_5 + c_{1,2} & y_1 = m_2x_1 + c_{2,1} & y_3 = m_2x_3 + c_{2,1} \\ y_5 = m_2x_5 + c_{2,1} & y_2 = m_3x_2 + c_{3,1} & y_3 = m_3x_3 + c_{3,1} \\ y_4 = m_3x_4 + c_{3,1}. & & \end{array}$$

satisfying $\mathcal{A}$.

We found a solution $m_1 = \frac{1}{4}$, $m_2 = -\frac{3}{2}$, $m_3 = 2$, $c_{1,1} = 5$, $c_{1,2} = \frac{3}{2}$, $c_{2,1} = 12$, $c_{3,1} = -2$, $x_1 = x_2 = x_3 = 4$, $x_4 = 2$, $x_5 = 6$, $y_1 = y_2 = y_3 = 6$, $y_4 = 2$, and $y_5 = 3$. This solution fails conditions C3 and C4, and the lines $y = m_ix + c_{i,k}$'s form a picture as Figure 5.2 not satisfying $\mathcal{A}$.

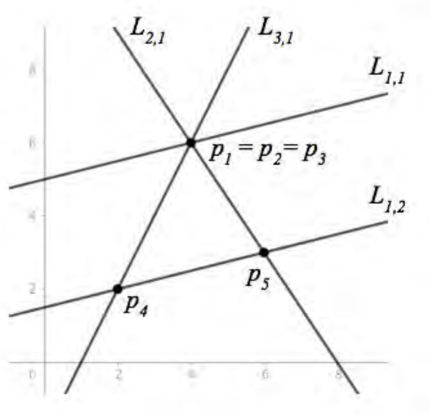

Figure 5.2: A picture formed by the lines $y = m_ix + c_{i,k}$'s

Note that $V = 17$ and $E = 10$. Let $f : \mathbb{R}^{17} \to \mathbb{R}^{10}$, for $d := (m_1, m_2, m_3, c_{1,1}, c_{1,2}, c_{2,1}, c_{3,1}, c_{4,1}, x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, x_5, y_5) \in \mathbb{R}^{17}$,

$$f(d) = (f_1(d), f_2(d), ..., f_E(d)),$$

where

$$\begin{array}{lll} f_1(d) = m_1x_1 + c_{1,1} - y_1, & f_2(d) = m_1x_2 + c_{1,1} - y_2, & f_3(d) = m_1x_4 + c_{1,2} - y_4, \\ f_4(d) = m_1x_5 + c_{1,2} - y_5, & f_5(d) = m_2x_1 + c_{2,1} - y_1, & f_6(d) = m_2x_3 + c_{2,1} - y_3, \\ f_7(d) = m_2x_5 + c_{2,1} - y_5, & f_8(d) = m_3x_2 + c_{3,1} - y_2, & f_9(d) = m_3x_3 + c_{3,1} - y_3, \\ f_{10}(d) = m_3x_4 + c_{3,1} - y_4. & & \end{array}$$

Let $\bar{d} = (\frac{1}{4}, -\frac{3}{2}, 2, 5, \frac{3}{2}, 12, -2, 4, 6, 4, 6, 4, 6, 2, 2, 6, 3)$. Then $f(\bar{d}) = 0$ since $\bar{d}$ is a solution of S.

Consider the Jacobian matrix J of f, we get

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial m_1} & \frac{\partial f_1}{\partial m_2} & \frac{\partial f_1}{\partial m_3} & \frac{\partial f_1}{\partial c_{1,1}} & \cdots & \frac{\partial f_1}{\partial c_{3,1}} & \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial x_5} & \frac{\partial f_1}{\partial y_5} \\ \frac{\partial f_2}{\partial m_1} & \frac{\partial f_2}{\partial m_2} & \frac{\partial f_2}{\partial m_3} & \frac{\partial f_2}{\partial c_{1,1}} & \cdots & \frac{\partial f_2}{\partial c_{3,1}} & \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial y_1} & \cdots & \frac{\partial f_2}{\partial x_5} & \frac{\partial f_2}{\partial y_5} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial f_{10}}{\partial m_1} & \frac{\partial f_{10}}{\partial m_2} & \frac{\partial f_{10}}{\partial m_3} & \frac{\partial f_10}{\partial c_{1,1}} & \cdots & \frac{\partial f_10}{\partial c_{3,1}} & \frac{\partial f_{10}}{\partial x_1} & \frac{\partial f_{10}}{\partial y_1} & \cdots & \frac{\partial f_{10}}{\partial x_5} & \frac{\partial f_{10}}{\partial y_5} \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & 0 & 0 & 1 & 0 & 0 & 0 & m_1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & m_1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_1 & -1 & 0 & 0 \\ x_5 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_1 & -1 \\ 0 & x_1 & 0 & 0 & 0 & 1 & 0 & m_2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & m_2 & -1 & 0 & 0 & 0 & 0 \\ 0 & x_5 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_2 & -1 \\ 0 & 0 & x_2 & 0 & 0 & 0 & 1 & 0 & 0 & m_3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & m_3 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & m_3 & -1 & 0 & 0 \end{bmatrix}.$$

Thus

$$J(\bar{d}) = \begin{bmatrix} 4 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{4} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & -1 & 0 & 0 \\ 6 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & -1 \\ 0 & 4 & 0 & 0 & 0 & 1 & 0 & -\frac{3}{2} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} & -1 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} & -1 \\ 0 & 0 & 4 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \end{bmatrix}.$$

Next, we will find the ordering $(w_1, w_2, ..., w_{V-E}, z_1, z_2, ..., z_E)$ of m_i's, x_j's, y_j's,

$c_{i,k}$'s such that $J_z(\bar{w}, \bar{z})$ is invertible. We found that a submatrix

$$\begin{bmatrix} 1 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & -1 \\ 0 & 1 & -\frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{3}{2} & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} & -1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \end{bmatrix}$$

of $J(\bar{d})$ has the i^{th} column vector, $i = 1, 2, ..., E$, from the column vectors partial derivatives of f by $c_{1,1}, c_{2,1}, x_1, x_2, x_3, y_3, x_4, y_4, x_5, y_5$ respectively at a point $\bar{d}$ and it is invertible. Hence we let $w := (m_1, m_2, m_3, c_{1,2}, c_{3,1}, y_1, y_2)$ and $z := (c_{1,1}, c_{2,1}, x_1, x_2, x_3, y_3, x_4, y_4, x_5, y_5)$. Then

$$J_z(\bar{w}, \bar{z}) = \begin{bmatrix} 1 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & -1 \\ 0 & 1 & -\frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{3}{2} & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} & -1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \end{bmatrix}$$

is invertible.

By Theorem 2.11, then there exist an open set U of $\mathbb{R}^7$ containing $\bar{w}$, and a unique continuously differentiable function $g : U \to \mathbb{R}^{10}$ such that

$$g(\frac{1}{4}, -\frac{3}{2}, 2, \frac{3}{2}, -2, 6, 6) = (5, 12, 4, 4, 4, 6, 2, 2, 6, 3)$$

and for all $(m_1, m_2, m_3, c_{1,2}, c_{3,1}, y_1, y_2) \in U$,

$$f(m_1, m_2, m_3, c_{1,2}, c_{3,1}, y_1, y_2, g(m_1, m_2, m_3, c_{1,2}, c_{3,1}, y_1, y_2)) = 0.$$

There are $\epsilon_1, \epsilon_2 \in \mathbb{R}\backslash\{0\}$ with $\epsilon_1 \neq \epsilon_2$ such that $(m'_1, m'_2, m'_3, c'_{1,2}, c'_{3,1}, y'_1, y'_2) = (\frac{1}{4}, -\frac{3}{2}, 2, \frac{3}{2}, -2, 6+\epsilon_1, 6+\epsilon_2) \in U$. Then we get $c'_{1,1} = 5+\frac{7}{8}\epsilon_2$, $c'_{2,1} = 24+7\epsilon_1-\frac{21}{4}\epsilon_2$, $x'_1 = 4 + 4\epsilon_1 - \frac{7}{2}\epsilon_2$, $x'_2 = 4 + \frac{1}{2}\epsilon_2$, $x'_3 = 4 + 2\epsilon_1 - \frac{3}{2}\epsilon_2$, $y'_3 = 6 + 4\epsilon_1 - 3\epsilon_2$, $x'_4 = 2$, $y'_4 = 2$, $x'_5 = 6 + 4\epsilon_1 - 3\epsilon_2$, and $y'_5 = 3 + \epsilon_1 - \frac{3}{4}\epsilon_2$.
We can observe that $m'_{i_1} \neq m'_{i_2}$, for all $i_1 \neq i_2$ and $(x'_{j_1}, y'_{j_1}) \neq (x'_{j_2}, y'_{j_2})$ for all $j_1 \neq j_2$. That is $(m_1, m_2, m_3, c_{1,2}, c_{3,1}, y_1, y_2, c_{1,1}, c_{2,1}, x_1, x_2, x_3, y_3, x_4, y_4, x_5, y_5) = (\frac{1}{4}, -\frac{3}{2}, 2, \frac{3}{2}, -2, 6+\epsilon_1, 6+\epsilon_2, 5+\frac{7}{8}\epsilon_2, 24+7\epsilon_1-\frac{21}{4}\epsilon_2, 4+4\epsilon_1-\frac{7}{2}\epsilon_2, 4+\frac{1}{2}\epsilon_2, 4+2\epsilon_1-\frac{3}{2}\epsilon_2, 6+4\epsilon_1-3\epsilon_2, 2, 2, 6+4\epsilon_1-3\epsilon_2, 3+\epsilon_1-\frac{3}{4}\epsilon_2)$ is another solution of S and it satisfies C1 and C3. So the code $\mathcal{A}$ is drawable.

One example of the other solution of S satisfying all condition C1, C2, C3, and C4 is that when we choose $m_1 = \frac{1}{4}$, $m_2 = -\frac{3}{2}$, $m_3 = 2$, $c_{1,2} = \frac{3}{2}$, $c_{3,1} = -2$, $y_1 = 8$, and $y_2 = \frac{26}{3}$ without considering U, we get $c_{1,1} = \frac{22}{3}$, $c_{2,1} = 12$, $x_1 = \frac{8}{3}$, $x_2 = \frac{16}{3}$, $x_3 = 4$, $y_3 = 6$, $x_4 = 2$, $y_4 = 2$, $x_5 = 6$, and $y_5 = 3$. So $(m_1, m_2, m_3, c_{1,2}, c_{3,1}, y_1, y_2, c_{1,1}, c_{2,1}, x_1, x_2, x_3, y_3, x_4, y_4, x_5, y_5) = (\frac{1}{4}, -\frac{3}{2}, 2, \frac{22}{3}, \frac{3}{2}, 12, -2, \frac{8}{3}, 8, \frac{16}{3}, \frac{26}{3}, 4, 6, 2, 2, 6, 3)$ is a solution of S, and the lines $y = m_i x + c_{i,k}$'s now form a picture as in Figure 5.3 satisfying $\mathcal{A}$.

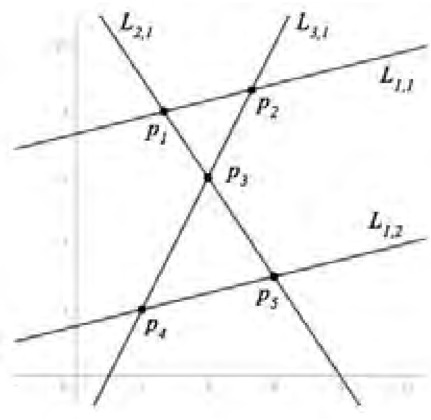

Figure 5.3: A picture formed by the lines $y = m_i x + c_{i,k}$'s

Unfortunately, the implicit function theorem is valid for some system of equations with some solution $\bar{d}$ since it has many assumptions. By the other ways, we may find the existence of solutions of the system satisfying all conditions. This is interesting to be investigated further more.

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