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Clifford Algebra-valued Segal-Bargmann Transform

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SORAWIT EAKNIPITSARI : CLIFFORD ALGEBRA-VALUED SEGAL-BARGMANN TRANSFORM
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A classical Segal-Bargmann transform maps square-integrable functions on $\mathbb{R}$ to holomorphic square-integrable functions on $\mathbb{C}$ with respect to some Gaussian measure. In this work, we extend the classical Segal-Bargmann transform to functions taking values in a Clifford algebra. We establish that in this setting, the generalized Segal-Bargmann transform is a unitary isomorphism mapping Clifford algebra-valued square-integrable functions on $\mathbb{R}^n$ with respect to some Gaussian measure to monogenic square-integrable functions on $\mathbb{R}^{n+1}$ with respect to another Gaussian measure. We also discuss about the differential and multiplication operators in the monogenic functions space as well as certain properties of their domains.

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CONTENTS

CHAPTER I
INTRODUCTION

For $x \in \mathbb{R}^n$, let $\rho(x) = (2\pi)^{-n/2}\, e^{-x^2/2}$. The classical Segal–Bargman transform, see e.g. [1], [9], [10], is a unitary isomorphism $V\colon L^2(\mathbb{R}^n, dx) \to \mathcal{H}L^2(\mathbb{C}^n, \nu\, dx\, dy)$ defined by

$$\begin{aligned} Vf(z) &= \int_{\mathbb{R}^n} \rho(z-x) f(x)\, dx \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-\frac{(z-x)^2}{2}} f(x)\, dx, \end{aligned}$$

where $\mathcal{H}L^2(\mathbb{C}^n, \nu\, dx\, dy)$ is the space of holomorphic square-integrable functions on $\mathbb{C}^n$ with respect to measure $\nu(y)\, dx\, dy = \pi^{-n/2}\, e^{-y^2} dx\, dy$. The map V can be regarded as the heat operator $e^{\frac{\Delta}{2}} f = \rho * f$, followed by the analytic continuation from $\mathbb{R}^n$ to $\mathbb{C}^n$, as in the following commutative diagram:

$$\begin{array}{ccc} & & \mathcal{H}L^2(\mathbb{C}^n, \nu\, dx\, dy) \\ & \overset{V}{\nearrow} & \uparrow \mathcal{C} \\ L^2(\mathbb{R}^n, dx) & \xrightarrow[e^{\frac{\Delta}{2}}]{} & \widetilde{\mathcal{A}}(\mathbb{R}^n) \end{array}$$

Here $\mathcal{C}$ denotes the analytic continuation from $\mathbb{R}^n$ to $\mathbb{C}^n$ and $\widetilde{\mathcal{A}}(\mathbb{R}^n)$ is the image of $L^2(\mathbb{R}^n, dx)$ by the operator $e^{\frac{\Delta}{2}}$.

Moreover, the creation and annihilation operators are intertwined by the Segal-Bargmann transform with differential and multiplication operators, respectively, in the holomorphic function space side.

Another variant of the Segal-Bargmann transform is considered by looking at a different pair of measures $U\colon L^2\left(\mathbb{R}^n, \rho\, dx\right) \to \mathcal{H}L^2(\mathbb{C}^n, \mu\, dz)$ defined by

$$\begin{aligned} Uf\,(z) &= \int_{\mathbb{R}^n} \rho\,(z-x)\, f(x)\, dx \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-\frac{(z-x)^2}{2}} f(x)\, dx, \end{aligned}$$

where $\mu(z) = \pi^{-n}\, e^{-z^2}$ and dz is Lebesgue measure on $\mathbb{C}^n$. The map U is a unitary isomorphism from $L^2\left(\mathbb{R}^n, \rho\, dx\right)$ onto $\mathcal{H}L^2(\mathbb{C}^n, \mu\, dz)$ and split as follows

$$\begin{array}{ccc} & \overset{U}{\nearrow} & \mathcal{H}L^2(\mathbb{C}^n, \mu\, dz) \\ & & \uparrow \mathcal{C} \\ L^2(\mathbb{R}^n, \rho\, dx) & \xrightarrow[e^{\frac{\Delta}{2}}]{} & \widetilde{\mathcal{A}}(\mathbb{R}^n) \end{array}$$

Note that the formula for V and U are exactly the same; the differences lie in their domains and ranges.

As the theory of complex analysis is generalized to Clifford analysis, a holomorphic function is generalized to a **monogenic** function. It is natural to consider a Clifford-valued function in place of a complex-valued function. In 2017, Mourão, Nunes and Qian [8] generalized the Segal-Bargmann transform to Clifford algebra-valued functions analogous to V. Denote by $\mathbb{C}_n$ the complex Clifford algebra on $\mathbb{C}^n$.

Theorem 1.1 ([8])**.** *The map* $\tilde{V}\colon L^2(\mathbb{R}^n, d\underline{x}) \otimes \mathbb{C}_n \to \mathcal{M}L^2(\mathbb{R}^{n+1}, \tilde{\nu}\, dx_0\, d\underline{x})$ *given by*

$$\tilde{V}(f)(x_0, \underline{x}) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{-\frac{\underline{p}^2}{2}} e^{i(\langle \underline{p}, \underline{x} - \underline{y}\rangle - x_0 \underline{p})}\, d\underline{p} \right) f(\underline{y}) d\underline{y},$$

is a unitary isomorphism. Here $\mathcal{M}L^2(\mathbb{R}^{n+1}, \tilde{\nu}\, dx_0\, d\underline{x})$ *is the Hilbert space of monogenic functions on* $\mathbb{R}^{n+1}$ *which are square-integrable with respect to measure* $\tilde{\nu}\, dx_0\, d\underline{x}$ *where* $\tilde{\nu}(x_0) = \dfrac{1}{\sqrt{\pi}} e^{-x_0^2}$.

The idea used here is to substitute the analytic continuation $\mathcal{C}$ with the CK extension, $e^{-x_0\underline{D}}$, as in the following diagram:

$$\begin{array}{ccc} & & \mathcal{M}L^2(\mathbb{R}^{n+1}, \tilde{\nu}\, dx_0\, d\underline{x}) \\ & \overset{\tilde{V}}{\nearrow} & \uparrow e^{-x_0\underline{D}} \\ L^2(\mathbb{R}^n, d\underline{x}) \otimes \mathbb{C}_n & \xrightarrow[e^{\frac{\Delta}{2}}]{} & \widetilde{\mathcal{A}}(\mathbb{R}^n) \otimes \mathbb{C}_n \end{array}$$

The main result of this work is to generalize the map U to Clifford algebra-valued functions. We obtain the following theorem:

Theorem 1.2. *The map $\tilde{U}$ given by*

$$\tilde{U}(f)(x_0, \underline{x}) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{-\frac{\underline{p}^2}{2}} e^{i((\underline{p}, \underline{x}-\underline{y})-x_0\underline{p})} d\underline{p} \right) f(\underline{y}) d\underline{y},$$

is a unitary isomorphism from $L^2(\mathbb{R}^n, \rho\, d\underline{x}) \otimes \mathbb{C}_n$ onto $\mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu})$, the Hilbert space of monogenic functions on $\mathbb{R}^{n+1}$ that are square-integrable with respect to the measure

$$d\tilde{\mu} = \frac{1}{\pi^{(n+1)/2}} e^{-x_0^2 - |\underline{x}|^2}\, dx_0\, d\underline{x}.$$

The map $\tilde{U}$ can also be factorized as in the following diagram:

$$\begin{array}{ccc} & & \mathcal{M}L^2(\mathbb{R}^{n+1},\, d\tilde{\mu}) \\ & \overset{\tilde{U}}{\nearrow} & \uparrow e^{-x_0\underline{D}} \\ L^2(\mathbb{R}^n, \rho\, d\underline{x}) \otimes \mathbb{C}_n & \xrightarrow[e^{\frac{\Delta}{2}}]{} & \widetilde{\mathcal{A}}(\mathbb{R}^n) \otimes \mathbb{C}_n \end{array}$$

Here is an outline of this thesis. In Chapter II, we review some preliminary definitions and theorems that will be used in this work. In Chapter III, we give a proof of Theorem 3.3 extending the Segal-Bargmann transform U to Clifford algebra-valued functions. In Chapter IV, we investigate the intertwining properties of the annihilation and creation operators under the generalized Segal-Bargamnn transform $\tilde{U}$.

CHAPTER II
PRELIMINARIES

In this chapter, we introduce some basic knowledge that will be used throughout this work.

2.1 Clifford analysis

2.1.1 Clifford algebra

We recall the definition of a Clifford algebra. See, e.g. [2], [3], [7], and [8]. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $B : V \times V \to \mathbb{K}$ be a symmetric bilinear form on a finite-dimensional vector space V over $\mathbb{K}$. Let $Q : V \to \mathbb{K}$ be the corresponding quadratic form defined by

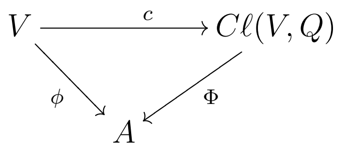
$$Q(x) = B(x, x) \quad \text{for any } x \in V.$$

We call the pair (V, Q) a **quadratic vector space**. A $\mathbb{K}$-linear map $\phi : V \to A$ from a quadratic vector space (V, Q) into a unital associative algebra A is called a **Clifford map** if for all $x \in V$,

$$\phi(x)^2 = -Q(x)1_A,$$

where 1_A is the unit of A.

Definition 2.1. A **Clifford algebra** is a unital associative algebra $C\ell(V, Q)$, together with a Clifford map $c : V \to C\ell(V, Q)$, such that for every Clifford map $\phi : V \to A$ into a unital associative algebra A there is a unique algebra homomorphism $\Phi : C\ell(V, Q) \to A$ such that $\Phi \circ c = \phi$, i.e., making the following diagram commute:



Proposition 2.2. *The Clifford algebra $C\ell(V,Q)$, if exists, is unique up to isomorphism.*

2.1.2 Construction of Clifford algebra

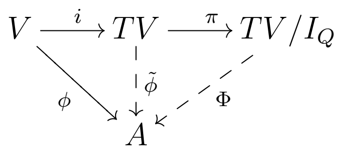
Let (V,Q) be a quadratic vector space and $TV = \oplus_{p\geq 0} V^{\otimes p}$ denote the tensor algebra of V, where $V^{\otimes 0} = \mathbb{K}$, $V^{\otimes 1} = V$ and $V^{\otimes p} = \otimes_{i=1}^{p} V$. The multiplication $V^{\otimes p} \times V^{\otimes q} \to V^{\otimes (p+q)}$ obtained by extending bilinearly the concatenation of monomials

$$(x_1 \otimes \cdots \otimes x_p)(y_1 \otimes \cdots \otimes y_q) = x_1 \otimes \cdots \otimes x_p \otimes y_1 \otimes \cdots \otimes y_q.$$

Let I_Q be a 2-sided ideal of TV generated by elements of the form

$$x \otimes x + Q(x) \in V^{\otimes 2} \oplus V^{\otimes 0}.$$

Consider the following diagram:



Let A be a unital associative algebra and $\phi : V \to A$ a Clifford map. By the universal property of the tensor algebra, there exists a unique algebra homomorphism $\tilde{\phi} : TV \to A$ such that $\tilde{\phi} \circ i = \phi$. Since ϕ is a Clifford map, it follows that

$I_Q \subseteq \ker \tilde{\phi}$:

$$\tilde{\phi}(x \otimes x + Q(x)) = \phi(x)^2 + Q(x) = 0$$

By the universal property of the quotient space, there exists a unique algebra homomorphism $\Phi : TV/I_Q \to A$ such that $\Phi \circ \pi = \tilde{\phi}$. Let $c = \pi \circ i$. Then $(TV/I_Q\,, c)$ satisfies the definition 2.1 of the Clifford algebra.

2.1.3 Real and Complex Clifford algebras

For computational purpose, let $V = \mathbb{K}^n$, equipped with a symmetric bilinear form B. Let $\{x_1, x_2, \ldots, x_n\}$ be a $\mathbb{K}-$basis, relative to which $B_{ij} := B(x_i, x_j) = \delta_{ij}$. For $j = 1, \ldots, n$, let e_j denote the image of x_j of the Clifford map c. Then we have

$$\begin{aligned} c(x_i + x_j)^2 &= -B(x_i + x_j, x_i + x_j) \\ c(x_i)^2 + c(x_i)c(x_j) + c(x_j)c(x_i) + c(x_j)^2 &= -(B_{ii} + B_{ij} + B_{ji} + B_{jj}) \\ e_i^2 + e_ie_j + e_je_i + e_j^2 &= -(B_{ii} + B_{ij} + B_{ji} + B_{jj}) \\ e_ie_j + e_je_i &= -2\delta_{ij}. \end{aligned}$$

Equivalently,

$$\begin{aligned} &e_ie_j = -e_je_i, \quad i \neq j, \quad i, j = 1, 2, \ldots, n \\ &e_i^2 = -1, \quad i, j = 1, 2, \ldots, n. \end{aligned}$$

We denote the Clifford algebra over $\mathbb{K}$ by $\mathbb{K}_n$. If $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we call it a real Clifford algebra or complex Clifford algebra, respectively. For example, when $n = 1, 2, 3$,

$$\mathbb{R}_1 \cong \mathbb{C}, \quad \mathbb{R}_2 \cong \mathbb{H} \text{ (the quaternions)}, \quad \mathbb{R}_3 \cong \mathbb{H}^2.$$

While,

$$\mathbb{C}_{2n+1} \cong M_{2^n}(\mathbb{C}) \oplus M_{2^n}(\mathbb{C}), \quad \mathbb{C}_{2n} \cong M_{2^n}(\mathbb{C}).$$

Note that $\{e_A \mid A \subset \{1, 2, \ldots, n\} = N\}$ is basis for $\mathbb{K}_n$ where $e_A = e_{i_1} e_{i_2} \cdots e_{i_k}$ if $A = \{i_1, i_2, \ldots, i_k\}$, with $1 \le i_1 < i_2 < \cdots < i_k \le n$, and $e_\varnothing = 1$. Moreover, for any $\lambda \in C\ell(V, Q)$, it can be written as

$$\lambda = \sum_{A \subset N} \lambda_A \, e_A,$$

where $\lambda_A \in \mathbb{K}$. Define the so-called **k-vector part** of λ, for $k = 0, 1, \ldots, n$, by

$$[\lambda]_k = \sum_{|A|=k} \lambda_A \, e_A.$$

Now, we focus at $\mathbb{C}_n$. One important operator of $\mathbb{C}_n$, the **Hermitian conjugation**, is defined by

$$\begin{aligned} &\overline{e_i} = -e_i, \quad i = 1, 2, \ldots, n, \\ &\overline{(\lambda_A e_A)} = \lambda_A^c \, \overline{e_A}, \quad \lambda_A \in \mathbb{C}, A \subset N, \\ &\overline{(\lambda \mu)} = \overline{\mu} \, \overline{\lambda}, \quad \lambda, \mu \in \mathbb{C}_n, \end{aligned}$$

where λ_A^c denotes the complex conjugate of the complex number λ_A. This contributes to a Hermitian inner product and its associated norm on $\mathbb{C}_n$, defined, respectively, by

$$(\lambda, \mu) = [\overline{\lambda}\mu]_0 \quad \text{and} \quad |\lambda|^2 = [\overline{\lambda}\lambda]_0 = \sum_A |\lambda_A|^2.$$

2.1.4 Clifford analysis

As we say earlier that the Cliford analysis generalize the Complex analysis. We first begin with defining the generalized Cauchy-Riemann operator by by

$$D = \partial_{x_0} + \underline{D},$$

where

$$\underline{D} = \sum_{j=1}^{n} \partial_{x_j} e_j.$$

To make things easier, we also identify the subspace of $\mathbb{R}_n$ of 1-vectors

$$\{\underline{x} = \sum_{j=1}^{n} x_j e_j : x = (x_1, \ldots, x_n) \in \mathbb{R}^n\},$$

which is identified with $\mathbb{R}^n$ and $-|\underline{x}|^2 = -(\underline{x}, \underline{x}) = \underline{x}^2$. Now, a generalized concept of holomorphic function is given. A continuously differentiable function f on an open domain $\mathcal{O} \subset \mathbb{R}^{n+1}$, taking values in $\mathbb{C}_n$, is called **monogenic** on $\mathcal{O}$ if it satisfies the generalized Cauchy-Riemann equation:

$$Df(x_0, \underline{x}) = 0.$$

Next, we introduce a space of Clifford algebra-valued square-integrable functions

$$L^2(\mathbb{R}^{n+1}, \tilde{\mu}\, d\underline{x}\,;\, \mathbb{C}_n) = \{f : \mathbb{R}^{n+1} \to \mathbb{C}_n \mid \int_{\mathbb{R}^{n+1}} |f(\underline{x})|^2 \tilde{\mu} d\underline{x} < \infty\},$$

which can be identified with the Hilbert space tensor product $L^2(\mathbb{R}^{n+1}, \tilde{\mu}\, d\underline{x}) \otimes \mathbb{C}_n$. It is equipped with the inner product:

$$\langle f, g\rangle = \int_{\mathbb{R}^{n+1}} (f(\underline{x}), g(\underline{x}))\, \tilde{\mu}\, d\underline{x} = \int_{\mathbb{R}^{n+1}} [\, \overline{f(\underline{x})} g(\underline{x})\,]_0 \tilde{\mu}\, d\underline{x}.$$

2.2 Schwartz space $\mathcal{S}(\mathbb{R}^n)$

We recall the Schwartz space on $\mathbb{R}^n$. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, write

$$\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^{\alpha_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x^{\alpha_n}} \quad \text{and} \quad x^\alpha = x^{\alpha_1} \cdots x^{\alpha_n}.$$

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is the space of smooth functions f such that f, together with all its derivatives, decrease rapidly at infinity, i.e. a smooth funtion f lie in

$\mathcal{S}(\mathbb{R}^n)$ if

$$\lim_{|x|\to\infty} x^\beta \frac{\partial^\alpha}{\partial x^\alpha} f(x) = 0 \text{ for all multi-indices } \alpha \text{ and } \beta.$$

Since $\mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$, we define the Fourier transform on $\mathcal{S}(\mathbb{R}^n)$ by

$$\hat{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(p) e^{-i\langle p,x\rangle} dp.$$

Let $C_c^\infty(\mathbb{R}^n)$ denote the space of smooth functions with compact support. We list some of the well-known results regarding Schwartz space and Fourier transform. See e.g. [6].

Theorem 2.3. *The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$.*

Theorem 2.4. *$C_c^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n, \rho\, dx)$.*

Proof. Let S be the class of measurable simple functions.
Step 1. We show that S is a dense subset of $L^2(\mathbb{R}^n, \rho\, dx)$.
If $\phi \in S$ then

$$\int |\phi|^2 \rho\, dx \leq \sup(|\phi|^2) \int \rho\, dx < +\infty,$$

and thus $\phi \in L^2(\mathbb{R}^n, \rho\, dx)$. Let $f \in L^2(\mathbb{R}^n, \rho\, dx)$. First assume that $f \geq 0$. Then there exist an increasing sequence of simple functions ϕ_m such that $0 \leq \phi_m \leq f$ and $\phi_m \nearrow f$. Also, $|\phi_m - f|^2 \leq 4|f|^2$ and since $|f|^2 \in L^1(\mathbb{R}^n, \rho\, dx)$, by DCT, $\phi_m \to f$ in $L^2(\mathbb{R}^n, \rho\, dx)$. The extension to the general case follows a routine procedure.

Step 2. We show that a simple function in S can be approxmiated by a continuous function with compact support. Let $\phi \in S$. Given $\epsilon > 0$, by Lusin's theorem, there exists a continuous function g with compact support such that $g = \phi$ except possibly on a set of measure ϵ, with respect to $\rho\, dx$, and

$$|g| \leq \sup(|\phi|).$$

Hence, for $E = \{x \mid (g-\phi)(x) \neq 0\}$,

$$\begin{aligned}\|g-\phi\|^2 &= \int_{\mathbb{R}^n} |g-\phi|^2 \rho\, dx \\ &\leq \int_E (|g|+|\phi|)^2 \rho\, dx \\ &\leq 4\epsilon \sup(|\phi|^2).\end{aligned}$$

That is we can estimate, in $L^2(\mathbb{R}^n, \rho\, dx)$, a simple function by continuous functions with compact support.

Step 3. Let $f \in C_c(\mathbb{R}^n)$. There exists a sequence (ϕ_n) in $C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp}(\phi_n)$, $\operatorname{supp}(f) \subset K$ for some fixed compact set K, and such that $\phi_n \to f$ uniformly on K. Thus

$$\int_{\mathbb{R}^n} |f-\phi_n|^2 \rho\, dx = \int_K |f-\phi_n|^2 \rho\, dx \leq \sup_K(|f-\phi_n|^2) \int_K \rho\, dx \to 0,$$

which completes the proof. □

Corollary 2.5. *$\mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, \rho\, dx)$ for $1 \leq p < \infty$.*

Proof. It follows directly from the fact that $C_c^\infty(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n)$. □

Theorem 2.6. *The Fourier transform is a unitary isomorphism on $\mathcal{S}(\mathbb{R}^n)$. Moreover, for any $f \in \mathcal{S}(\mathbb{R}^n)$,*

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(p) e^{i\langle p,x\rangle}\, dp.$$

Hence the Fourier transform can be extended to a unitary isomorphism on $L^2(\mathbb{R}^n)$.

CHAPTER III
CLIFFORD ALGEBRA-VALUED SEGAL-BARGMANN TRANSFORM

We briefly recall some useful facts in [8] regarding the Cauchy-Kowalevski extension. The Schwartz space of $\mathbb{C}_n$-valued functions is identified with the tensor product $\mathcal{S}(\mathbb{R}^n)\otimes\mathbb{C}_n$. Any $f\in\mathcal{S}(\mathbb{R}^n)\otimes\mathbb{C}_n$ can be written as $f=\sum_{A\subset N} f_A e_A$, where $f_A\in\mathcal{S}(\mathbb{R}^n)$. In addition, the Fourier transform of f is given by $\hat{f}=\sum_{A\subset N}\hat{f}_A e_A$.

Proposition 3.1. *Let $f\in\mathcal{S}(\mathbb{R}^n)\otimes\mathbb{C}_n$ be such that for all $x_0\in\mathbb{R}$*

$$e^{-ix_0\underline{p}}\hat{f}\in\mathcal{S}(\mathbb{R}^n)\otimes\mathbb{C}_n.$$

Then

$$F(x_0,\underline{x})=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(\langle\underline{p},\underline{x}\rangle-x_0\underline{p})}\hat{f}(\underline{p})\,d\underline{p}$$

defines a monogenic function satisfying the Cauchy problem

$$\begin{cases}\dfrac{\partial F}{\partial x_0}=-\underline{D}F\\ F(0,\underline{x})=f(\underline{x}).\end{cases}$$

Proposition 3.2 ([4])**.** *Let F be a monogenic function on $\mathbb{R}^{n+1}$ with $F(0,\underline{x})=f(\underline{x})$. Then*

$$F(x_0,\underline{x})=e^{-x_0\underline{D}}f(x_0,\underline{x}):=\left(\sum_{k=0}^{\infty}(-1)^k\frac{x_0^k}{k!}\underline{D}^k f\right)(\underline{x})$$

where the series converges uniformly on compact subsets.

Our main result is to extend map U to Clifford algebra-valued functions as the following theorem.

Theorem 3.3. *The map $\tilde{U}$ given by*

$$\tilde{U}(f)(x_0, \underline{x}) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{-\frac{p^2}{2}} e^{i((\underline{p}, \underline{x}-\underline{y}) - x_0 \underline{p})} d\underline{p} \right) f(\underline{y}) d\underline{y},$$

is a unitary isomorphism from $L^2(\mathbb{R}^n, \rho\, d\underline{x}) \otimes \mathbb{C}_n$ onto $\mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu})$, the Hilbert space of monogenic functions on $\mathbb{R}^{n+1}$ that are square-integrable with respect to the measure

$$d\tilde{\mu} = \frac{1}{\pi^{(n+1)/2}} e^{-x_0^2 - |\underline{x}|^2}\, dx_0\, d\underline{x}.$$

From Proposition 3.1, the Cauchy-Kowalevski extension of ρ is

$$e^{-x_0 \underline{D}} \rho(\underline{x}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i((\underline{p}, \underline{x}) - x_0 \underline{p})} \rho(\underline{p})\, d\underline{p}$$

Then

$$\tilde{U} f(x_0, \underline{x}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-x_0 \underline{D}} \rho(\underline{x} - \underline{y}) f(\underline{y})\, d\underline{y}$$

Lemma 3.4. *For $f \in L^2(\mathbb{R}^n, \rho\, d\underline{x}) \otimes \mathbb{C}_n$, $\tilde{U} f$ is a monogenic function on $\mathbb{R}^{n+1}$.*

Proof. From the above calculation and Leibniz rule,

$$\partial_{x_0} \tilde{U} f, \text{ and } \underline{D} \tilde{U} f$$

can be taken under the integral symbol and calculated only at $e^{-x_0 \underline{D}} \rho$. Hence, $\tilde{U} f$ is a monogenic function on $\mathbb{R}^{n+1}$. □

Since, for $f \in \mathcal{S}(\mathbb{R}^n) \otimes \mathbb{C}_n$, $\tilde{V} f = \tilde{U} f$, we have the following results from [8].

Remark 3.5. *The map $\tilde{U}$ can be shortly written as*

$$\tilde{U}(f) = e^{-x_0 \underline{D}} \circ e^{\frac{\triangle}{2}}(f).$$

Lemma 3.6. *Let $f \in \mathcal{S}(\mathbb{R}^n) \otimes \mathbb{C}_n$, with the Fourier transform $\hat{f}$. Then*

$$\tilde{U}(f)(x_0, \underline{x}) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|\underline{p}|^2}{2}} e^{i((\underline{p}, \underline{x}) - x_0 \underline{p})} \hat{f}(\underline{p})\, d\underline{p}.$$

Remark 3.7. $e^{-ix_0\underline{p}} = \cosh(x_0|\underline{p}|) - i\sinh(x_0|\underline{p}|)\dfrac{\underline{p}}{|\underline{p}|}$.

Let $\{He_k, k \in \mathbb{N}_0^n\}$ denote an orthogonal basis of $L^2(\mathbb{R}^n, \rho\, dx)$ consisting of Hermite polynomials on $\mathbb{R}^n$ with $\|He_k\|^2 = k!$, where the multi-index notation is used: for any $k \in \mathbb{N}_0^n$,

$$He_k(x) = He_{k_1}(x_1)\dots He_{k_n}(x_n),$$

$k! = k_1!\dots k_n!$ and $x^k = x_1^{k_1}\dots x_n^{k_n}$.

Lemma 3.8. $e^{\frac{\Delta}{2}} He_k = x^k$.

Proof. From the identity

$$e^{\langle x,y\rangle - \frac{(y)^2}{2}} = \sum_{k\in\mathbb{N}_0^n} He_k(x)\frac{y^k}{k!}, \quad x, y \in \mathbb{R}^n$$

and

$$\langle He_k, He_l\rangle_{L^2(\mathbb{R}^n,\rho\, dx)} = k!\delta_{kl},$$

we have

$$\begin{aligned} e^{\frac{\Delta}{2}} He_k &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{(x-y)^2}{2}} He_k(y)dy \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{y^2}{2}+yx-\frac{x^2}{2}} He_k(y)dy \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{y^2}{2}} \left(\sum_{k\in\mathbb{N}_0^n} He_k(y)\frac{x^k}{k!}\right) He_k(y)dy \\ &= x^k. \end{aligned}$$

□

Lemma 3.9. *For $f, h \in \mathcal{S}(\mathbb{R}^n)\otimes\mathbb{C}_n$, we have $\langle \tilde{U}(f), \tilde{U}(h)\rangle = \langle f, h\rangle$.*

Proof. Note that for any 1-vector $\underline{p} \in \mathbb{R}_n^1$, we have

$$(e^{i\underline{p}}u, v) = (u, e^{i\underline{p}}v), \quad \text{for any } u, v \in \mathbb{C}_n.$$

Using the Fourier transform, we have the following identity:

$$e^{-\frac{|\underline{p}-\underline{q}|^2}{4}} = \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{i(\underline{p}-\underline{q},\underline{x})} e^{-|\underline{x}|^2} d\underline{x}.$$

Let $A = \left(e^{-ix_0(\underline{p}+\underline{q})} \hat{f}(\underline{p}), \hat{h}(\underline{q})\right)$. By Lemma 3.6 and the above identity, we have

$$\langle \tilde{U}(f), \tilde{U}(h) \rangle = \frac{1}{\sqrt{\pi}(2\pi)^n} \int_{\mathbb{R}\times\mathbb{R}^{2n}} e^{-\frac{|\underline{p}-\underline{q}|^2}{4}} e^{-\frac{|\underline{p}|^2+|\underline{q}|^2}{2}} A e^{-x_0^2} dx_0\, d\underline{p}\, d\underline{q}$$

Also, from Remark 3.7 and the fact that sinh is an odd function, we have

$$\begin{aligned}\int_{\mathbb{R}} A e^{-x_0^2}\, dx_0 &= \int_{\mathbb{R}} \cosh(x_0|\underline{p}+\underline{q}|) \left(\hat{f}(\underline{p}), \hat{h}(\underline{q})\right) e^{-x_0^2}\, dx_0 \\ &= \sqrt{\pi} e^{\frac{|\underline{p}+\underline{q}|^2}{4}} \left(\hat{f}(\underline{p}), \hat{h}(\underline{q})\right).\end{aligned}$$

By changing variables in the above identity and using the Fubini's theorem, we obtain

$$\begin{aligned}&\langle \tilde{U}(f), \tilde{U}(h) \rangle \\ =&\frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{-\frac{|\underline{q}-\underline{p}|^2}{2}} \left(\hat{f}(\underline{p}), \hat{h}(\underline{q})\right) d\underline{p}\, d\underline{q} \\ =&\frac{1}{(2\pi)^{3n/2}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^n} e^{i(\underline{q}-\underline{p},\underline{x})} \left(\hat{f}(\underline{p}), \hat{h}(\underline{q})\right) e^{-\frac{|\underline{x}|^2}{2}}\, d\underline{x}\, d\underline{p}\, d\underline{q} \\ =&\frac{1}{(2\pi)^{3n/2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^{2n}} \overline{\left(e^{i(\underline{p},\underline{x})} \hat{f}(\underline{p})\right)} e^{i(\underline{q},\underline{x})} \hat{h}(\underline{q}) e^{-\frac{|\underline{x}|^2}{2}}\, d\underline{p}\, d\underline{q}\, d\underline{x} \\ =&\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} (f(\underline{x}), h(\underline{x}))\, e^{-\frac{|\underline{x}|^2}{2}} d\underline{x} \\ =&\langle f, h \rangle. \qquad \square\end{aligned}$$

Now, we are ready to prove Theorem 3.3.

Proof of Theorem 3.3. Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n, \rho\, d\underline{x})$, it follows that $\mathcal{S}(\mathbb{R}^n) \otimes \mathbb{C}_n$ is dense in $L^2(\mathbb{R}^n, \rho\, d\underline{x}) \otimes \mathbb{C}_n$. The isometry part follows from Lemma 3.9. It remains to show that $\tilde{U}$ is surjective.

Let $f \in \mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu})$. Note that $f_0(\underline{x}) = f(0, \underline{x})$ uniquely determines f by

Cauchy-Kowalevski extension. Since an entire monogenic function has a Taylor series with infinite radius of convergence, we can write f_0 uniquely as

$$f_0 = \sum_A \sum_{k \in \mathbb{N}_0^n} \alpha_{k,A} x^k e_A.$$

From Proposition 3.2, we have

$$\begin{aligned} f(x_0, \underline{x}) &= \sum_A \sum_{j=0}^{\infty} \frac{(-x_0)^j}{j!} \underline{D}^j \left(\sum_{k \in \mathbb{N}_0^n} \alpha_{k,A} x^k e_A \right) \\ &= \sum_A \sum_{j=0}^{\infty} \sum_{k \in \mathbb{N}_0^n} \frac{(-x_0)^j}{j!} \alpha_{k,A} \underline{D}^j (x^k) e_A, \end{aligned}$$

which follows from the fact that we can differentiate term-by-term in a power series. Since it has infinite radius of convergence, it also converges absolutely in $\mathbb{R}^{n+1}$ and thus we can interchange the summations resulting in

$$\begin{aligned} f(x_0, \underline{x}) &= \sum_A \sum_{k \in \mathbb{N}_0^n} \sum_{j=0}^{\infty} \frac{(-x_0)^j}{j!} \alpha_{k,A} \underline{D}^j (\underline{x}^k) e_A \\ &= \sum_A \sum_{k \in \mathbb{N}_0^n} \alpha_{k,A} e^{-x_0 \underline{D}} (\underline{x}^k) e_A. \end{aligned}$$

Take

$$g = \sum_A \sum_{k \in \mathbb{N}_0^n} \alpha_{k,A} (He_k) e_A$$

Then $g \in L^2(\mathbb{R}^n, \rho\, dx) \otimes \mathbb{C}_n$. By Lemma 3.8 and Remark 3.5,

$$\begin{aligned} \tilde{U}(g) &= \sum_A \sum_{k \in \mathbb{N}_0^n} \alpha_{k,A} e^{-x_0 \underline{D}} \left(e^{\frac{\triangle}{2}} (He_k) \right) e_A \\ &= \sum_A \sum_{k \in \mathbb{N}_0^n} \alpha_{k,A} e^{-x_0 \underline{D}} (\underline{x}^k) e_A = f. \end{aligned}$$

Hence $\tilde{U}$ is a unitary map from $L^2(\mathbb{R}^n, \rho\, dx) \otimes \mathbb{C}_n$ onto $\mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu})$. $\square$

For $k \subseteq \mathbb{N}^n$, we have

$$e^{-x_0 \underline{D}}\, \underline{x}^k = V_k(x_0, \underline{x}) = \frac{k!}{|k|!} Z^k,$$

where $|k| = \sum_{i=1}^{n} k_n$ and $Z^k = z_1 Z^{k-e^1} + \cdots + z_n Z^{k-e^n}$ with $Z^{(0,0,\ldots,0)} = 1$ and $Z^k = 0$ if one of k_i is negative, e^i is the vector in $\mathbb{R}^n$ having i^{th} coordinate being 1, and 0 otherwise and $z_i(x_0 + \underline{x}) = x_i - x_0 e_i$. Then we immediately have the following result.

Corollary 3.10. $\{V_k : k \subseteq \mathbb{N}_0^n\}$ *is an orthogonal basis for* $\mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu})$.

Proof. From Theorem 3.3, we have that the map $\tilde{U}$ is a unitary map. Since $V_k = e^{-x_0 \underline{D}}\, \underline{x}^k = \tilde{U}(He_k)$, we are immediately done. □

According to [2] and [4], a monogenic function admits the Taylor series of the form:

$$f(x_0, \underline{x}) = \sum_{k \in \mathbb{N}_0^n} \alpha_k V_k(x_0, \underline{x}).$$

Theorem 3.11. *For each* $f \in \mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu})$ *such that*

$$f(x_0, \underline{x}) = \sum_{k \in \mathbb{N}_0^n} \alpha_k V_k(x_0, \underline{x}),$$

we have $\|f\|^2 = \sum_{k \in \mathbb{N}_0^n} |\alpha_k|^2 k!$ *and this sum converges in* $L^2(\mathbb{R}^{n+1}, d\tilde{\mu}) \otimes \mathbb{C}_n$ *sense.*

Proof. It follows from Theorem 3.3 that

$$\begin{aligned}
\langle V_k, V_l \rangle &= \langle \tilde{U}(He_k), \tilde{U}(He_l) \rangle \\
&= \langle He_k, He_l \rangle \\
&= k! \delta_{kl}.
\end{aligned}$$

Since $\sum_{k\in\mathbb{N}_0^n} |\alpha_k|^2 k!$ is finite, $f_N(x_0,\underline{x}) = \sum_{\substack{k\in\mathbb{N}_0^n\\ |k|\le N}} \alpha_k V_k(x_0,\underline{x})$ converges in $L^2(\mathbb{R}^{n+1}, d\tilde{\mu})\otimes \mathbb{C}_n$ sense to some function g. Then there is a subsequence converges a.e. to g point wise and hence $f = g$ a.e.. $\square$

CHAPTER IV

CREATION AND ANNIHILATION OPERATORS

In this chapter, we investigate properties of creation and annihilation operators among different spaces. First, we recall the creation and annihilation operators on $L^2(\mathbb{R}^n, \rho\, dx)$, respectively,

$$\tilde{a}_k = \partial_{x_k}$$
$$\tilde{a}_k^* = x_k - \partial_{x_k}.$$

It is well-known, see e.g. [5], that these creation and annihilation operators are intertwined with diferrentiation operators and multiplication operators, respectively, on $\mathcal{H}L^2(\mathbb{C}^n, \mu\, dz)$ by the Segal-Bargmann transform U:

Theorem 4.1. *For all* $k = 1, \dots, n$

$$U\tilde{a}_k U^{-1} = \frac{\partial}{\partial z_k}$$
$$U\tilde{a}_k^* U^{-1} = z_k.$$

Now, we give an analogous result for the generalized Segal-Bargmann transform $\tilde{U}$:

Theorem 4.2. *For all* $k = 1, \dots, n$*,*

$$\tilde{U}\tilde{a}_k f(\underline{x}) = e^{-x_0 \underline{D}}\, \partial_{x_k}(\tilde{U}f)(0, \underline{x}) \quad \textit{and}$$
$$\tilde{U}\tilde{a}_k^* f(\underline{x}) = e^{-x_0 \underline{D}}\, x_k(\tilde{U}f)(0, \underline{x}).$$

Proof. Let $f = \sum_A f_A e_A \in \mathcal{S}(\mathbb{R}^n) \otimes \mathbb{C}_n$. By Theorem 4.1 and holomorphicity, we have

$$
\begin{aligned}
e^{\frac{\Delta}{2}}\partial_{x_k} f_A &= [U\partial_{x_k} f_A(x+iy)]_{y=0} \\
&= [\partial_{z_k} U f_A(x+iy)]_{y=0} \\
&= [\partial_{x_k} U f_A(x+iy)]_{y=0} \\
&= \partial_{x_k} e^{\frac{\Delta}{2}} f_A.
\end{aligned}
$$

Then

$$
\begin{aligned}
\tilde{U}\tilde{a}_k(f) &= e^{-x_0\underline{D}} \circ e^{\frac{\Delta}{2}}\partial_{x_k} f \\
&= e^{-x_0\underline{D}}\partial_{x_k} e^{\frac{\Delta}{2}} f \\
&= e^{-x_0\underline{D}}\partial_{x_k}(\tilde{U}f)(0,\underline{x}).
\end{aligned}
$$

The second equation follows by a similar argument. □

For $1 \leq j \leq n$, the operators P_j and X_j are defined by

$$
P_j f(x_0,\underline{x}) = \partial_{x_j} f(x_0,\underline{x}) \quad \text{and} \quad X_j f(x_0,\underline{x}) = e^{-x_0\underline{D}} x_j f(0,\underline{x}).
$$

Lemma 4.3. $P_j(V_k) = k_j V_{k-e_j} = e^{-x_0\underline{D}} \circ \partial_{x_j} V_k(0,\underline{x})$ *and* $X_j(V_k) = V_{k+e^j}$.

Proof. We will show only the first equality because the second is similar. First, we show that $P_j(Z^k) = |k| Z^{k-e_j}$ for all $k \subseteq \mathbb{N}^n$. It clearly holds for $|k| = 1$. Assume that it holds for $|k| < n$. For $|k| = n$, we have

$$
\begin{aligned}
\partial_{x_j} Z^k &= \sum_{i=1}^{n} \partial_{x_j} z_i Z^{k-e^i} \\
&= \sum_{i=1}^{n} \Big((\partial_{x_j} z_i) Z^{k-e^i} + z_i(\partial_{x_j} Z^{k-e^i}) \Big) \\
&= Z^{k-e^j} + (|k|-1)\sum_{i=1}^{n} z_i Z^{k-e^j-e^i} \\
&= Z^{k-e^j} + (|k|-1) Z^{k-e^j} = |k| Z^{k-e^j}.
\end{aligned}
$$

Then

$$\begin{aligned}\partial_{x_j} V_k &= \frac{k!}{|k|!}\partial_{x_j} Z^k \\ &= \frac{k!}{(|k|-1)!} Z^{k-e^j} \\ &= k_j \frac{(k-e^j)!}{|k-e^j|!} Z^{k-e^j} \\ &= k_j V_{k-e^j}.\end{aligned}$$

□

For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, write

$$P^\alpha = P_1^{\alpha_1} \ldots P_n^{\alpha_n} \quad \text{and} \quad X^\alpha = X^{\alpha_1} \cdots X^{\alpha_n}.$$

Define

$$\mathcal{D}_{P^\alpha} = \{f \in \mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu}) : P^\alpha f \in \mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu})\}$$

and

$$\mathcal{D}_{X^\alpha} = \{f \in \mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu}) : X^\alpha f \in \mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu})\}.$$

We will see that $\mathcal{D}_{P^\alpha} = \mathcal{D}_{X^\alpha}$. In order to prove this we need the following lemmas.

Lemma 4.4. $\mathcal{D}_{P^\alpha} \subseteq \mathcal{D}_{P^\beta}$ *if α and β are multi-indices such that $\beta \leq \alpha$.*

Proof. It suffices to show this for $\alpha = \beta + e^i$ where $1 \leq i \leq n$. Let $f(x_0, \underline{x}) = \sum_k a_k V_k(x_0, \underline{x})$ be a function in $\mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu})$. Then

$$P_i^{\beta_i} f = \sum_{\substack{k_i=\beta_i \\ k\subset\mathbb{N}^n}}^{\infty} \left(\frac{k_i!}{(k_i-\beta_i)!}\right) a_k V_{k-\beta_i e^i}$$

and

$$P_i^{\beta_i+1} f = \sum_{\substack{k_i=\beta_i+1 \\ k\subset\mathbb{N}^n}}^{\infty} \left(\frac{k_i!}{(k_i-\beta_i-1)!}\right) a_k V_{k-(\beta_i+1)e^i}\,.$$

By Theorem 3.11, we have

$$\begin{aligned}
\|P_i^{\beta_i} f\|^2 &= \sum_{\substack{k_i=\beta_i \\ k\subset\mathbb{N}^n}}^{\infty} \left(\frac{k_i!}{(k_i-\beta_i)!}\right)^2 |a_k|^2 (k-\beta_i e^i)! \\
&= \sum_{\substack{k_i=\beta_i \\ k\subset\mathbb{N}^n}}^{\infty} \left(\frac{k_i!}{(k_i-\beta_i)!}\right)^2 |a_k|^2 (k-k_i e^i)!(k_i-\beta_i)! \\
&= \sum_{\substack{k_i=\beta_i \\ k\subset\mathbb{N}^n}}^{\infty} \frac{(k_i!)^2}{(k_i-\beta_i)!} |a_k|^2 (k-k_i e^i)!
\end{aligned}$$

and, similarly,

$$\|P_i^{\beta_i+1} f\|^2 = \sum_{\substack{k_i=\beta_i+1 \\ k\subset\mathbb{N}^n}}^{\infty} \frac{(k_i!)^2}{(k_i-\beta_i-1)!} |a_k|^2 (k-k_i e^i)!\,.$$

Thus $\mathcal{D}_{P_i^{\beta_i+1}} \subseteq \mathcal{D}_{P_i^{\beta_i}}$. Now, for $f \in \mathcal{D}_{P^\alpha}$, consider

$$P^\alpha f = P_i^{\beta_i+1}(P^{\beta-\beta_i e_i} f) \in \mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu}).$$

That is $P^{\beta-\beta_i e_i} f \in \mathcal{D}_{P_i^{\beta_i+1}} \subseteq \mathcal{D}_{P_i^{\beta_i}}$. Thus $P^\beta f \in \mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu})$. □

Lemma 4.5. *$\mathcal{D}_{X^\alpha} \subseteq \mathcal{D}_{X^\beta}$ if α and β are multi-indices such that $\beta \le \alpha$.*

Proof. Let $f \in \mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu})$. Then $f(x_0, \underline{x}) = \sum_k a_k V_k(x_0, \underline{x})$. We have

$$X^\beta f = \sum_{k\subset\mathbb{N}^n} a_k V_{k+\beta}.$$

Then, by Theorem 3.11,

$$\begin{aligned}
\|X^\beta f\|^2 &= \sum_{k\subset\mathbb{N}^n} |a_k|^2 (|k+\beta|)! \\
&\le \sum_{k\subset\mathbb{N}^n} |a_k|^2 (|k+\alpha|)! = \|X^\alpha f\|^2
\end{aligned}$$ □

Now, we are ready to prove the main resulf of this section.

Theorem 4.6. *$\mathcal{D}_{P^\alpha} = \mathcal{D}_{X^\alpha}$ for any multi-indices α.*

Proof. We first show that $\mathcal{D}_{P_i} = \mathcal{D}_{X_i}$ for any $1 \leq i \leq n$. Let $f \in \mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu})$ with $f(x_0, \underline{x}) = \sum_k a_k V_k(x_0, \underline{x})$. Then

$$P_i f = \sum_k a_k k_i V_{k-e^i}$$

and

$$X_i f = \sum_k a_k V_{k+e^i}$$

Thus, by Theorem 3.11,

$$\begin{aligned} \|X_i f\|^2 - \|f\|^2 &= \sum_k |a_k|^2((k+e^i)! - k!) \\ &= \sum_k |a_k|^2 (k - k_i e^i)!((k_i+1)! - k_i!) \\ &= \sum_k |a_k|^2 k_i^2 (k - k_i e^i)!(k_i - 1)!) \\ &= \|P_i f\|^2. \end{aligned}$$

Hence $\mathcal{D}_{P_i} = \mathcal{D}_{X_i}$.

Next, assume that $\mathcal{D}_{P^\beta} = \mathcal{D}_{X^\beta}$ for some multi-index β. Let $\alpha = \beta + e^i$ for some $i \in \{1, 2, \ldots, n\}$. We will show that $\mathcal{D}_{P^\alpha} = \mathcal{D}_{X^\alpha}$. If $\beta_i = 0$, then we get

$$P_i(X^\beta f) = X^\beta (P_i f).$$

That is

$$\begin{aligned} f \in \mathcal{D}_{P^\alpha} &\iff P^\alpha f \in \mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu}) \\ &\iff P_i f \in \mathcal{D}_{P^\beta} = \mathcal{D}_{X^\beta} \\ &\iff X^\beta(P_i f) \in \mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu}) \\ &\iff P_i(X^\beta f) \in \mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu}) \\ &\iff X^\beta f \in \mathcal{D}_{P_i} = \mathcal{D}_{X_i} \\ &\iff f \in \mathcal{D}_{X^\alpha}. \end{aligned}$$

For the case $\beta_i \geq 1$, we will use the identity, for $f \in \mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu})$,

$$P_i(X^\beta f) = \beta_i(X^{\beta - e^i} f) + X^\beta(P_i f).$$

Let $f \in \mathcal{D}_{P^\alpha}$. Then $P_i f \in \mathcal{D}_{P^\beta} = \mathcal{D}_{X^\beta}$ so $X^\beta(P_i f) \in \mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu})$. Moreover, since $\mathcal{D}_{P^\alpha} \subseteq \mathcal{D}_{P^\beta} = \mathcal{D}_{X^\beta} \subseteq \mathcal{D}_{X^{\beta - e^i}}$, we have $X^{\beta - e^i} f \in \mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu})$. Thus $P_i(X^\beta f) \in \mathcal{M}L^2(\mathbb{R}^{n+1}, d\tilde{\mu})$. That is $X^\beta f \in \mathcal{D}_{P_i} = \mathcal{D}_{X_i}$. Hence $\mathcal{D}_{P^\alpha} = \mathcal{D}_{X^\alpha}$. $\square$

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