เลขชี้กำลังวิกฤตสำหรับสมการไม่เฉพาะที่ไม่เชิงเส้น

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CRITICAL EXPONENT FOR NONLINEAR NONLOCAL EQUATIONS

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วิทยานิพนธ์ฉบับนี้ศึกษาสมการไม่เฉพาะที่ไม่เชิงเส้น $\partial_t u = J * u - u + u^{1+p}$ เมื่อ $p > 0$ และ J เป็นฟังก์ชันที่ไม่เป็นลบมีขอบเขตมีสมมาตรเชิงรัศมีมีปริพันธ์เท่ากับ 1 การแปลงฟูเรียร์ของ J สอดคล้องกับเงื่อนไข $\hat{J}(\xi) = 1 - A|\xi|^\beta \left(\ln \frac{1}{|\xi|}\right)^\mu + o\left(|\xi|^\beta \left(\ln \frac{1}{|\xi|}\right)^\mu\right)$ เมื่อ $\xi \to 0$ สำหรับ $0 < \beta \leq 2$, $\mu \in \mathbb{R}$, และ $A > 0$ ในการศึกษานี้ได้พิสูจน์ การมีผลเฉลยเฉพาะที่ การมีผลเฉลยเพียงหนึ่งเดียว และได้หลักการเปรียบเทียบของผลเฉลย นอกจากนี้เราได้แสดงว่าเลขชี้กำลังวิกฤต Fujita ของสมการ คือ $\frac{\beta}{n}$ เมื่อ $\mu < 0$ ส่วนในกรณี $\mu > 1$ เราค้นพบว่าเลขชี้กำลังวิกฤต คือ $\frac{\beta}{n}$ และผลเฉลยสามารถอยู่ในช่วงการมีผลเฉลยวงกว้าง เมื่อค่าเริ่มต้นมีค่าน้อย

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In this thesis, we study the nonlinear nonlocal equation $\partial_t u = J * u - u + u^{1+p}$ where $p > 0$ and J is nonnegative, bounded, and radially symmetric with unit integral. The Fourier tranform of J satisfies $\hat{J}(\xi) = 1 - A|\xi|^\beta \left(\ln \frac{1}{|\xi|}\right)^\mu + o\left(|\xi|^\beta \left(\ln \frac{1}{|\xi|}\right)^\mu\right)$ as $\xi \to 0$, for $0 < \beta \leq 2$, $\mu \in \mathbb{R}$, and $A > 0$. In this study, we establish the local existence, uniqueness, and the comparison principle of solutions. Furthermore, we show that the Fujita critical exponent for this equation is $\frac{\beta}{n}$ when $\mu < 0$. For the case $\mu > 1$, we discover that the critical exponent is $\frac{\beta}{n}$, and the solutions belong to the global small initial data regime.

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CHAPTER I

INTRODUCTION

In 1966, Fujita [8] considered positive solutions of the nonlinear heat equation

$$\partial_t u = \Delta u + u^{1+p} \quad \text{in } (0, \infty) \times \mathbb{R}^n,$$

with a non-negative, nontrivial initial data. It was shown that the solutions blow up in finite time if $0 < p < \frac{2}{n}$, whereas the equation admits a global solution if $p > \frac{2}{n}$ provided that the initial condition is sufficiently small. Later on, the case $p = \frac{2}{n}$ was shown to belong to the finite time blow-up case (see [15]). Consequently, the number $p = \frac{2}{n}$ has been called the Fujita critical exponent for the nonlinear heat equation.

The Fujita phenomenon has become interested ever since. The investigation of such critical exponent has permeated other nonlinear equations.

In 1997, Bates, Fife, Ren, and Wang [4] considered the nonlinear nonlocal equation

$$\partial_t u = J * u - u - f(u) \quad \text{in } (0, \infty) \times \mathbb{R},$$

where the kernel J is nonnegative and radially symmetric with unit integral, and $f \in C^2(\mathbb{R})$ has only one zero in $(-1, 1)$ and no zero outside $[-1, 1]$ satisfying $f(\pm 1) = 0 < f'(\pm 1)$. They established the existence and the uniqueness of weak solutions. Moreover, they proved the regularities of weak solutions, and compared the asymptotic behavior to stationary-wave solutions, see [3] for more details on higher dimension. This leads to the study of the nonlocal equations of the form

$$\partial_t u = J * u - u, \tag{1.1}$$

where the kernel J is a nonnegative, radially symmetric, continuous function with unit integral. The nonlocal equations have been applied to various disciplines, e.g., biology, image processing, mathematical finances, etc., see [2] for more details.

In 2006, Chasseigne, Chaves, and Rossi [5] studied the nonlocal equation of the form (1.1) when the kernel J is a nonnegative, radially symmetric, continuous function with unit integral. They considered (1.1) on a bounded smooth domain, and showed that the Dirichlet or Neumann type boundary value problem has a unique solution, and such solution is also global. Furthermore, they studied (1.1) on the whole space with the Fourier transform of the kernel J satifying

$$\hat{J}(\xi) = 1 - A|\xi|^{\beta} + o(|\xi|^{\beta}) \quad \text{as } \xi \to 0, \tag{1.2}$$

for $0 < \beta \leq 2$ and $A > 0$. As a result, they found that the solutions is global, and have the same behavior as the solutions of $\partial_t u = -A(-\Delta)^{\frac{\beta}{2}} u$. Moreover, they also investigated the the case that the Fourier transform of the kernel J has the asymptotic expansion with logarithmic perturbation

$$\hat{J}(\xi) = 1 - A|\xi|^2 \left(\ln \frac{1}{|\xi|}\right) + o\left(|\xi|^2 \left(\ln \frac{1}{|\xi|}\right)\right) \quad \text{as } \xi \to 0, \tag{1.3}$$

where $A > 0$. They showed that the solutions have the same behavior as the solutions of the heat equation $\partial_t u = \frac{A}{2} \Delta u$ with a certain time scaling.

In 2009, Llanos and Rossi [16] studied the nonlinear nonlocal equation

$$\partial_t u = \int_{\Omega} J(x-y)(u(t,y) - u(t,x))dy + u^{1+p} \quad \text{in } (0,T) \times \Omega,$$

where the kernel $J \in C(\mathbb{R}^n)$ is nonnegative and radially symmetric with unit integral, and Ω is a bounded, connected, smooth domain. The local existence was establihed by using the contraction mapping theorem. They also showed that the solutions blow up when $p > 0$. On the other hand, the solutions are global when $p \leq 0$.

In 2010, García-Melián and Quirós [9] considered both (1.1) and the nonlinear nonlocal equation

$$\begin{cases} \partial_t u = J * u - u + u^{1+p} & \text{in } (0, \infty) \times \mathbb{R}^n, \\ u(0, x) = u_0(x), \end{cases} \tag{1.4}$$

where the kernel $J \in C(\mathbb{R}^n)$ is nonneagative, compactly supported, radially symmetric and radially decreasing with unit integral. Note that (1.4) is used to model the dispersal of a species by taking into some account of long-range effects, see [4] for more details. First, they used the idea in [16] to prove the local existence and the uniqueness of solution to (1.1) and (1.4). The comparison principle for positive solutions to (1.4) is also established by using the Gronwall's inequality. Furthermore, they showed that $p = \frac{2}{n}$ is the Fujita critical exponent to (1.4).

In 2017, Alfaro [1] considered positive solutions to (1.4) where $p > 0$ with a nonnegative, nontrivial initial data. He also assumed that the kernel J is nonnegative, bounded, and radially symmetric with unit integral, and the Fourier transform of the kernel J satisfies the condition (1.2). Consequently, Alfaro showed that the Fujita critical exponent for (1.4) is $\frac{\beta}{n}$.

In 2018, Khomrutai [13] studied the asymptotic behavior for (1.1) with a initial condition. He proved that there is a constant C depending on dimension n and the kernel J such that for large enough t,

$$||u(t, \cdot)||_{L^q} \leq C(||u_0||_{L^1} + ||\hat{u}_0||_{L^1}) t^{-\frac{n}{\beta}(1-\frac{1}{q})} \tag{1.5}$$

where $1 \leq q < \infty$, or q is ∞, and the kernel J is nonnegative and radially symmetric with unit integral satisfying (1.2). Moreover, he also considered the asymptotic behavior for (1.1) in the case that the Fourier tranform of the kernel J satisfies

$$\hat{J}(\xi) = 1 - A|\xi|^\beta \left(\ln \frac{1}{|\xi|}\right)^\mu + o\left(|\xi|^\beta \left(\ln \frac{1}{|\xi|}\right)^\mu\right) \quad \text{as } \xi \to 0, \tag{1.6}$$

for $0 < \beta \leq 2$, $\mu \in \mathbb{R}$, and $A > 0$. He obtained the same result as (1.5).

In this work, we consider positive solutions to (1.4) with a nonnegative, nontrivial initial data where the kernel J is nonnegative, bounded, and radially symmetric with unit integral. However, in our case, we assume that the Fourier tranform of the kernel J satisfies (1.6). First, we consider positive mild solutions of (1.4) with a nonnegative, nontrivial, smooth initial data. Provided that the kernel J is nonnegative bounded and radially symmetric with unit integral, we establish the local existence and the uniqueness of mild solutions by using the idea from [16]. We also investigate some qualitative behaviors, e.g., blow-up or global existence, of the solutions. In Chapter II, we review some background knowledges and some estimates which will be used thoughtout this thesis. In Chapter III, we use the idea from [16] and [9] to establish the local existence, the uniqueness and the comparison principle of positive mild solutions of (1.4). In Chapter IV, we apply the idea in [1] to prove some estimates, and show that the solutions blow up when $p > \frac{\beta}{n}$. When $p = \frac{\beta}{n}$, we show that the solutions blow up if $\mu < 0$, and the solutions can be global if $\mu > 1$, and the initial condition is sufficientlly small. Consequently, we prove that the solution can be global or blow-up depending upon the initial conditions when $p > \frac{\beta}{n}$.

CHAPTER II
PRELIMINARIES

In this chapter, we introduce basic definitions of some background knowledges needed in this thesis. We review the theorm of integration on Lebesgue measure, the properties of the Fourier transform, facts regarding the behavior of solutions, some basics of topology about the contraction mappings, and some useful inequalities.

Let $(\mathbb{R}^n, \mathfrak{L}, \mu)$ be a Lebesgue mesure space. We write $\displaystyle\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x$ for $\displaystyle\int_{\mathbb{R}^n} f\,\mathrm{d}\mu$.

Definition 2.1 ([7])**.** A statment is true **almost everywhere**, if a statement is true for all $x \in \mathbb{R}^n$ except for x in some null set, a set measures zero.

Definition 2.2 ([7])**.** Let $E \subset \mathbb{R}^n$. The **characteristic function** χ_E **of** E is defined by

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases}$$

Definition 2.3 ([7])**.** Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$. The **positive part of** f_+ **of** f is defined by

$$f_+(x) = \max(f(x), 0).$$

Theorem 2.4 ([7], The Monotone convergence theorem)**.** *Let f_n be a sequence of nonnegative integrable function such that $f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$. Then,*

$$\int_{\mathbb{R}^n} \lim_{n\to\infty} f_n(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}^n} f_n(x)\,\mathrm{d}x.$$

Theorem 2.5 ([7], The Dominated convergence theorem)**.** *Let f_n be a sequence of integrable function such that there exist a nonnegative, integrable function g with*

$|f_n| \leq g$ almost everywhere for all $n \in \mathbb{N}$. Then,

$$\int_{\mathbb{R}^n} \lim_{n\to\infty} f_n(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}^n} f_n(x)\,\mathrm{d}x.$$

Theorem 2.6 ([7]). *Let T be a bijection linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. If $f \geq 0$ or $f \in L^1(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x = |\det T| \int_{\mathbb{R}^n} foT(x)\,\mathrm{d}x.$$

For $x = (x_1, x_2, x_3, ..., x_n)$ and $y = (y_1, y_2, y_3, ..., y_n)$ in $\mathbb{R}^n$, we use the notation

$$x \cdot y = \sum_{k=1}^{n} x_k y_k.$$

Definition 2.7 ([1]). Let $f \in L^1(\mathbb{R}^n)$. The **Fourier tranform** $\hat{f}$ and **inverse Fourier transform** $\check{f}$, respectively, of f are the functions defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\,\mathrm{d}x \quad \text{and} \quad \check{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{ix\cdot\xi}\,\mathrm{d}x.$$

Lemma 2.8 ([1]). *If $f,\ \check{f} \in L^1(\mathbb{R}^n)$, then $(2\pi)^n f = \check{\hat{f}}$.*

Lemma 2.9 ([1], Plancherel formula). *Let $f,\ g \in L^2(\mathbb{R}^n)$. Then,*

$$\int_{\mathbb{R}^n} f(x)g(x)\,\mathrm{d}x = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\xi)\,\mathrm{d}\xi.$$

Lemma 2.10 ([10]). *If $f \in L^1(\mathbb{R}^n)$, then $\hat{f}$ and $\check{f}$ are bounded and continuous. Moreover,*

$$\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0 \quad \textit{and} \quad \lim_{|x|\to\infty} \check{f}(x) = 0.$$

Definition 2.11 ([10]). A function f on $\mathbb{R}^n$ is **radially symmetric**, if $f(x) = f(y)$ when $|x| = |y|$.

Note: By [10], the Fourier transform of a radially symmetric function f is a real-valued function when $n \geq 2$. For the case $n = 1$, a radially symmetric function f

is even. Therefore,

$$\int_{\mathbb{R}} \sin(x) f(x)\,\mathrm{d}x = 0.$$

Thus, for a radially symmetric function f,

$$\hat{f}(\xi) = \int_{\mathbb{R}} \cos(-x\cdot\xi) f(x)\,\mathrm{d}x \in \mathbb{R}.$$

Lemma 2.12. *Let f be a nonnegative, radially symmetric function on $\mathbb{R}^n$ with unit integral. Then, for $\xi \in \mathbb{R}^n$, $|\hat{f}(\xi)| \le 1$, and the equality holds when $\xi = 0$.*

Proof. Since $(1 - \cos(-x\cdot\xi))f(x)$ is a nonnegative, continuous function in x variable, and there is value that more than 0 when $\xi \neq 0$,

$$1 - \hat{f}(\xi) = \int_{\mathbb{R}^n} (1 - \cos(-x\cdot\xi)) f(x)\,\mathrm{d}x > 0.$$

For the case $\xi = 0$, we get $\hat{f}(0) = \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x = 1$. □

We define $||\cdot||_q := ||\cdot||_{L^q(\mathbb{R}^n)}$ for $q \in \mathbb{R}$ with $1 \le q < \infty$ and $||\cdot||_\infty := ||\cdot||_{L^\infty(\mathbb{R}^n)}$.

Definition 2.13 ([1])**.** We assume that u is a **solution** to (1.4) almost everywhere with

$$u \in C^1((0,T), L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)) \cap C^0([0,T), L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)),$$

for some $T > 0$.

1. u is **blow-up in finite time**, if the maximal of T is finite ($T_{max} < \infty$), i.e., for each $t < T_{max}$, $||u(t,\cdot)||_\infty < \infty$ and $\lim_{t\to T_{max}^-} ||u(t,\cdot)||_\infty = \infty$.
2. u is **global**, if the maximal of T is infinity, i.e., $||u(t,\cdot)||_\infty < \infty$ for all t.

Definition 2.14 ([12])**.** Let (X, d) be a metric space. A mapping $T : X \to X$ is a **contraction**, if there exists a constant C with $0 \le C < 1$ such that

$$d(T(x), T(y)) \le C d(x, y) \quad \text{for all } x, y \in X.$$

Theorem 2.15 ([12], The contraction mapping theorem)**.** *If $T : X \to X$ is a contraction on a complete metric space (X, d), then it has excactly one fixed point in X.*

Definition 2.16 ([7])**.** For any $f, g \in L^1(\mathbb{R}^n)$, the **convolution** of f and g is defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,\mathrm{d}y.$$

Remark 2.17. By [7], the convolution operator has commutative and associative properties.

Lemma 2.18 ([1])**.** *If $f,\ g \in L^1(\mathbb{R}^n)$, then $\widehat{f * g} = \hat{f}\hat{g}$.*

Definition 2.19 ([14])**.** The **Green operator** $\mathcal{G}(t)$ associated with (1.1) is defined by

$$\mathcal{G}(t)f = e^{-t}\sum_{k=0}^{\infty}\frac{t^k}{k!}J^{*(k)} * f,$$

where J^0 is the identity, and $J^{*(k)} = J * J^{*(k-1)}$ for $k \geq 1$.

Definition 2.20 ([14])**.** A function u is called a **mild solution** of (1.4) provided $u \in C([0,T], L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n))$, and satisfies

$$u(t,x) = \mathcal{G}(t)u_0(x) + \int_0^t \mathcal{G}(t-\tau)\{u(\tau,x)^{p+1}\}\,\mathrm{d}\tau \quad \text{in } (0,\infty) \times \mathbb{R}^n. \tag{2.1}$$

A function u is called a **mild subsolution** of (1.4) provided $u \in C([0,T], L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n))$, and satisfies

$$u(t,x) \leq \mathcal{G}(t)u_0(x) + \int_0^t \mathcal{G}(t-\tau)\{u(\tau,x)^{p+1}\}\,\mathrm{d}\tau \quad \text{in } (0,\infty) \times \mathbb{R}^n.$$

A function u is called a **mild supersolution** of (1.4) provided $u \in C([0,T], L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n))$, and satisfies

$$u(t,x) \geq \mathcal{G}(t)u_0(x) + \int_0^t \mathcal{G}(t-\tau)\{u(\tau,x)^{p+1}\}\,\mathrm{d}\tau \quad \text{in } (0,\infty) \times \mathbb{R}^n.$$

Theorem 2.21 ([18], Weight AM-GM). *Let p_1, p_2, ..., p_n be nonnegative rational numbers such that $\sum_{i=1}^{n} p_i = 1$. Then, for a_1, a_2, ..., $a_n \geq 0$,*

$$a_1^{p_1} a_1^{p_2} ... a_n^{p_n} \leq p_1 a_1 + p_2 a_2 + ... p_n a_n.$$

Theorem 2.22 ([7], The Hölder's inequality). *Let p and q be real numners such that $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are measurable functions, then*

$$||fg||_1 \leq ||f||_p ||g||_q.$$

Theorem 2.23 ([7], The Young's convolution inequality). *Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ where p, q are real numbers satisfying $1 \leq p, q < \infty$. Then, the following statements hold.*

1. *If $\frac{1}{p} + \frac{1}{q} = 1$, then $\|f * g\|_\infty \leq \|f\|_p \|g\|_q$.*
2. *If there exists $r \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, then $\|f * g\|_r \leq \|f\|_p \|g\|_q$.*

Moreover, if $f \in L^1(\mathbb{R}^n)$ and $g \in L^\infty(\mathbb{R}^n)$, then $\|f * g\|_\infty \leq \|f\|_1 \|g\|_\infty$.

Theorem 2.24 ([17], The Miscellaneous inequality). *Let I denote an interval of the form $[a, b]$, and continuous functions $f, A, B : I \to \mathbb{R}$. If B is nonnegative, and f satisfies the following condition*

$$f(t) \leq A(t) + \int_a^t B(s) f(s) ds \quad \text{in } I,$$

then

$$f(t) \leq A(t) + \int_a^t A(s) B(s) e^{\int_s^t B(r) dr} ds \quad \text{in } I.$$

Lemma 2.25 ([13]). *There is a constant $C > 0$ depending on the dimension n and the kernel J such that the solution to (1.1) under the assumption (1.6) with the initial condition v_0 satisfies, for large t,*

$$||v(t, \cdot)||_\infty \leq C(||v_0||_1 + ||\hat{v}_0||_1)(t(\ln t)^\mu)^{-\frac{n}{\beta}}.$$

Let us present the assumptions on J as presented in Chapter I and its properties.

Hypothesis 2.26. Let $J : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative, bounded, radially symmetric function with unit integral, and the Fourier transform of J satisfies

$$\hat{J}(\xi) = 1 - A|\xi|^\beta \left(\ln \frac{1}{|\xi|}\right)^\mu + o\left(|\xi|^\beta \left(\ln \frac{1}{|\xi|}\right)^\mu\right) \qquad \text{as } \xi \to 0,$$

for $0 < \beta \leq 2$, $\mu \in \mathbb{R}$, and $A > 0$.

Note: The kernel J satifies the following :

1. $\lim\limits_{|\xi| \to \infty} \hat{J}(\xi) = 0$;
2. $|\hat{J}(\xi)| \leq 1$ and the equality holds when $\xi = 0$;
3. The Hypothesis (2.26) implies that

$$\lim_{|\xi| \to 0} \frac{\hat{J}(\xi) - 1 + A|\xi|^\beta \left(\ln \frac{1}{|\xi|}\right)^\mu}{|\xi|^\beta \left(\ln \frac{1}{|\xi|}\right)^\mu} = 0.$$

CHAPTER III

LOCAL EXISTENCE, COMPARISON PRINCIPLE AND UNIQUENESS

In this chapter, we cosider the mild solutions $u(t,x)$ of the form (2.1). We will estimate the Green operator, and prove the local existence of positive mild solutions by using Theorem 2.15. This idea is inspired by [16] to apply to (1.4). The comparison principle is shown by considering mild supersolution and mild subsolution applying Theorem 2.24 to prove it. The uniqueness of mild solutions is the consequence of the comparison principle and the local existence.

Lemma 3.1. *For $f \in L^\infty(\mathbb{R}^n)$ and $t > 0$, then*

$$||\mathcal{G}(t)f||_\infty \leq ||f||_\infty.$$

Proof. Since $||J||_{L^1} = 1$, by Theorem 2.23, we have for each $k \in \mathbb{N}$ that

$$||J^{*(k+1)} * f||_\infty \leq ||J||_1||J^{*(k)} * f||_\infty = ||J^{*(k)} * f||_\infty.$$

Continuing the process, we have $||J^{*(k+1)} * f||_\infty \leq ||f||_\infty$, so that

$$||\mathcal{G}(t)f||_\infty \leq \left\| e^{-t}\sum_{k=0}^{\infty}\frac{t^k}{k!}J^{*(k)} * f\right\|_\infty \leq e^{-t}\sum_{k=0}^{\infty}\frac{t^k}{k!}||J^{*(k)} * f||_\infty = ||f||_\infty. \qquad \square$$

Lemma 3.2. *Let T be a positive constant. Let $\Omega = C([0,T]; L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n))$. For each $\varphi \in \Omega$, we denote*

$$||\varphi||_{L^\infty([0,T];L^\infty(\mathbb{R}^n))} = \sup_{t\in[0,T]} ||\varphi(t,\cdot)||_\infty.$$

Let ε be a positive constant. Let

$$\Omega_\varepsilon = \left\{\varphi \in \Omega : \varphi \geq 0, ||\varphi||_{L^\infty([0,T];L^\infty(\mathbb{R}^n))} < \varepsilon\right\}.$$

Then, Ω_ε is a complete metric space with norm $||\cdot||_{L^\infty([0,T];L^\infty(\mathbb{R}^n))}$.

Proof. Since $||\cdot||_\infty$ is a norm and definition of supremum, we have the following results:

For any $\alpha \in \mathbb{R}$ and $\varphi_1, \varphi_2 \in \Omega_\varepsilon$,

1. $\sup\limits_{t\in[0,T]} ||\varphi_1 + \varphi_2||_\infty \leq \sup\limits_{t\in[0,T]} (||\varphi_1||_\infty + ||\varphi_2||_\infty) \leq \sup\limits_{t\in[0,T]} ||\varphi_1||_\infty + \sup\limits_{t\in[0,T]} ||\varphi_2||_\infty$,
2. $\sup\limits_{t\in[0,T]} ||\alpha\varphi_1||_\infty \leq \sup\limits_{t\in[0,T]} |\alpha|||\varphi_1||_\infty$,
3. $\sup\limits_{t\in[0,T]} ||\varphi_1||_\infty \geq 0$.

Let $\varphi \in \Omega_\varepsilon$ be such that $||\varphi||_{L^\infty([0,T];L^\infty(\mathbb{R}^n))} = 0$. Then, for each $t \in [0,T]$, $||\varphi(t,\cdot)||_\infty = 0$, i.e., $\varphi(t,\cdot) = 0$ almost everywhere. Since $\varphi(t,\cdot)$ is continuous for all $t \in [0,T]$, we have $\varphi = 0$. Thus, $||\cdot||_{L^\infty([0,T];L^\infty(\mathbb{R}^n))}$ is a norm on Ω_ε.

Let (f_n) be a Cauchy sequence in Ω_ε, i.e.,

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \quad \text{with} \quad \forall m, n \geq N, ||f_m - f_n||_{L^\infty([0,T];L^\infty(\mathbb{R}^n))} < \epsilon.$$

Fixed $t \in [0,T]$, we have $(f_n(t,\cdot))$ is a Cauchy sequence in $L^\infty(\mathbb{R}^n)$. Since $L^\infty(\mathbb{R}^n)$ is a complete metric space, a limit of $(f_n(t,\cdot))$ converges to a function g_t in $L^\infty(\mathbb{R}^n)$. We define the function f by

$$f(t,x) = g_t(x) \quad \text{on } [0,T] \times \mathbb{R}^n.$$

Note that, for any $t \in [0,T]$, $f(t,\cdot) \in L^\infty(\mathbb{R}^n)$.

Fixed $t \in [0,T]$, we claim that $f(t,\cdot) \in C(\mathbb{R}^n)$. Let $x_0 \in \mathbb{R}^n$ and $\epsilon > 0$. By construction of $f(t,\cdot)$, there exists $N \in \mathbb{N}$ with $\forall n \geq N$,

$$||f_n(t,\cdot) - f(t,\cdot)||_\infty < \frac{\epsilon}{3}.$$

Since $f_N(t,\cdot) \in C(\mathbb{R}^n)$, there exists $\delta > 0$ with $\forall x, |x - x_0| < \delta$,

$$|f_N(t,x) - f_N(t,x_0)| < \frac{\epsilon}{3}.$$

Thus,

$$\begin{aligned}
&|f(t,x) - f(t,x_0)| \\
&\quad \leq |f(t,x) - f_N(t,x)| + |f_N(t,x) - f_N(t,x_0)| + |f_N(t,x_0) - f(t,x_0)|, \\
&\quad < \epsilon.
\end{aligned}$$

Next, we are going to show that $f \in \Omega$. Let $t_0 \in [0,T]$ and $\epsilon > 0$. By construction of $f(t_0,\cdot)$ and definition of Cauchy sequence of (f_n), there exists $N \in \mathbb{N}$ with $\forall m,n \geq N$,

$$||f_n(t_0,\cdot) - f(t_0,\cdot)||_\infty < \frac{\epsilon}{4} \quad \text{and} \quad ||f_m - f_n||_{L^\infty([0,T];L^\infty(\mathbb{R}^n))} < \frac{\epsilon}{4}.$$

Since $f_N \in \Omega$, there exists $\delta > 0$ with $\forall t$, $|t - t_0| < \delta$,

$$||f_N(t,\cdot) - f_N(t_0,\cdot)||_\infty < \frac{\epsilon}{4}.$$

Let t be such that $|t - t_0| < \delta$. By construction of $f(t,\cdot)$, there exists $M \in \mathbb{N}$ with $\forall n \geq M$,

$$||f_n(t,\cdot) - f(t,\cdot)||_\infty < \frac{\epsilon}{4}.$$

Then,

$$\begin{aligned}
||f(t,\cdot) - f(t_0,\cdot)||_\infty &\leq ||f(t_0,\cdot) - f_N(t_0,\cdot)||_\infty + ||f_N(t,\cdot) - f_N(t_0,\cdot)||_\infty \\
&\quad + ||f_N(t,\cdot) - f_{M+N}(t,\cdot)||_\infty + ||f_{M+N}(t,\cdot) - f(t,\cdot)||_\infty, \\
&< \epsilon.
\end{aligned}$$

Lastly, we will show that $f \in \Omega_\varepsilon$. Fixed $t \in [0,T]$, we have

$$\lim_{n\to\infty} ||f_n(t,\cdot) - f(t,\cdot)||_\infty = 0.$$

By triangle inequality and since $f_n \in \Omega_\varepsilon$,

$$||f(t,\cdot)||_\infty < ||f_n(t,\cdot) - f(t,\cdot)||_\infty + ||f_n(t,\cdot)||_\infty < \epsilon + ||f_n(t,\cdot) - f(t,\cdot)||_\infty.$$

Letting $n \to \infty$, then we are done. □

Lemma 3.3. *Let a, b and p be nonnegative real numbers. Then,*

$$|a^{p+1} - b^{p+1}| \le (p+1)|a-b|(\max(a,b))^p.$$

Proof. WLOG, let $a \ge b$. We define a function f by

$$f(p) = a^{p+1} - b^{p+1} - (p+1)(a-b)a^p \quad \text{on } [0,\infty).$$

Consider f when $p \in \mathbb{Q}_0^+$, i.e., $p = \frac{m}{n}$ where $m, n \in \mathbb{N}$. Let $x = \sqrt[n]{a}$ and $y = \sqrt[n]{b}$. Then, the inequality will be

$$n(x^{m+n} - y^{m+n}) \le (m+n)(x^n - y^n)x^m.$$

It is equivalent to

$$mx^{m+n} + ny^{m+n} \ge (m+n)x^m y^n,$$

which is implied from Theorem 2.21. Thus, $f \le 0$ on $\mathbb{Q}_0^+$. Since f is continuous and $\mathbb{Q}$ is dense in $\mathbb{R}$, we have $f \le 0$. Hence, we are done. □

Theorem 3.4. *Assume that $u_0 \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ is nonnegative. Then, there exists $T > 0$ such that (1.4) admits a unique mild solution.*

Proof. Let ε and T be positive constants. By Lemma 3.2, we have Ω_ε is a complete metric space with norm $||\cdot||_{L^\infty([0,T];L^\infty(\mathbb{R}^n))}$. Define the mapping $\mathcal{M}$ as follows

$$\mathcal{M}(v)(t,x) = \mathcal{G}(t)u_0(x) + \int_0^t \mathcal{G}(t-\tau)\{v(\tau,x)^{p+1}\}\,\mathrm{d}\tau \quad \text{on } \Omega_\varepsilon.$$

To use contraction mapping theorem, it is enough to show that $\mathcal{M}$ is a contraction mapping on Ω_ε. Since J is nonnegative, for nonnegative function f,

$$\mathcal{G}(t)f = e^{-t}\sum_{k=0}^\infty \frac{t^k}{k!}J^{*(k)} * f \geq 0.$$

Thus, $\mathcal{M}(v)(t,x)$ is nonnegative. Then, by Lemma 3.1, for each $v \in \Omega_\varepsilon$,

$$\begin{aligned}||\mathcal{M}(v)(t,\cdot)||_\infty &\leq ||\mathcal{G}(t)u_0||_\infty + \int_0^t \left\|\mathcal{G}(t-\tau)\{v(\tau,\cdot)^{p+1}\}\right\|_\infty \mathrm{d}\tau,\\ &\leq ||u_0||_\infty + \int_0^t \left\|v(\tau,\cdot)^{p+1}\right\|_\infty \mathrm{d}\tau.\end{aligned}$$

If we take

$$\varepsilon > 2||u_0||_\infty \quad \text{and} \quad 0 < T < \frac{1}{2(p+1)\varepsilon^p}, \tag{3.1}$$

then

$$||\mathcal{M}(v)||_{L^\infty([0,T];L^\infty(\mathbb{R}^n))} \leq ||u_0||_\infty + T\varepsilon^{p+1} < \varepsilon.$$

Consequently, $\mathcal{M}(v) \in \Omega_\varepsilon$. For $v_1, v_2 \in \Omega_\varepsilon$, we have by Lemma 3.1 and 3.3 that

$$\begin{aligned}&||\mathcal{M}(v_1)(t,\cdot) - \mathcal{M}(v_2)(t,\cdot)||_\infty\\ &\leq \int_0^t \left\|\mathcal{G}(t-\tau)\{v_1(\tau,\cdot)^{p+1} - v_2(\tau,\cdot)^{p+1}\}\right\|_\infty \mathrm{d}\tau,\\ &\leq \int_0^t \left\|v_1(\tau,\cdot)^{p+1} - v_2(\tau,\cdot)^{p+1}\right\|_\infty \mathrm{d}\tau,\\ &\leq \int_0^t \left\|(p+1)(v_1(\tau,\cdot) - v_2(\tau,\cdot))(\max_{x\in\mathbb{R}^n}\{v_1(\tau,\cdot), v_2(\tau,\cdot)\})^p\right\|_\infty \mathrm{d}\tau,\\ &\leq (p+1)\varepsilon^p \int_0^t \|v_1(\tau,\cdot) - v_2(\tau,\cdot)\|_\infty \mathrm{d}\tau\\ &\leq (p+1)\varepsilon^p T\,\|v_1 - v_2\|_{L^\infty([0,T];L^\infty(\mathbb{R}^n))}.\end{aligned}$$

By the condition (3.1), we have

$$||\mathcal{M}(v_1) - \mathcal{M}(v_2)||_{L^\infty([0,T];L^\infty(\mathbb{R}^n))} \leq \frac{1}{2} \|v_1 - v_2\|_{L^\infty([0,T];L^\infty(\mathbb{R}^n))}.$$

Then, $\mathcal{M}$ is a contraction mapping on Ω_ε. Therefore, by applying Theorem 2.15, we get the result that there exists a unique fixed point of $\mathcal{M}$ on Ω_ε, i.e., there exist $u \in \Omega_\varepsilon$ such that

$$u(t,x) = \mathcal{M}(u)(t,x) = \mathcal{G}(t)u_0(x) + \int_0^t \mathcal{G}(t-\tau)\{u(\tau,x)^{p+1}\}d\tau.$$

Hence, $u \in \Omega_\varepsilon \subset \Omega$ is a mild solution of (1.4) as desired. □

Theorem 3.5. *Let $u, v \in C([0,T]; L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n))$ be positive mild supersolution and subsolution of (1.4), respectively, with*

$$0 \leq v_0(x) \leq u_0(x) \quad \text{on } \mathbb{R}^n.$$

Then, $0 \leq v \leq u$ on $[0,T] \times \mathbb{R}^n$.

Proof. Since u is a mild supersolution of (1.4), we have

$$u(t,x) \geq \mathcal{G}(t)u_0(x) + \int_0^t \mathcal{G}(t-\tau)\{u(\tau,x)^{p+1}\}\, \mathrm{d}\tau.$$

For a mild subsolution v, we have

$$v(t,x) \leq \mathcal{G}(t)v_0(x) + \int_0^t \mathcal{G}(t-\tau)\{v(\tau,x)^{p+1}\}\, \mathrm{d}\tau.$$

Since $u_0 \geq v_0$ and J is nonnegative,

$$\mathcal{G}(t)(v_0 - u_0) = e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} J^{*(k)} * (v_0 - u_0) \leq 0.$$

Thus, we have

$$\begin{aligned} v(t,x)-u(t,x) &\le \mathcal{G}(t)(v_0(x)-u_0(x)) + \int_0^t \mathcal{G}(t-\tau)\{v(\tau,x)^{p+1}-u(\tau,x)^{p+1}\}\,\mathrm{d}\tau, \\ &\le \int_0^t \mathcal{G}(t-\tau)\{v(\tau,x)^{p+1}-u(\tau,x)^{p+1}\}\,\mathrm{d}\tau, \\ &= \int_0^t \mathcal{G}(t-\tau)\{(v(\tau,x)-u(\tau,x))H(\tau,x)\}\,\mathrm{d}\tau, \end{aligned}$$

where

$$H(t,x) = (p+1)\int_0^1 (\nu v(t,x) + (1-\nu)u(t,x))^p\,\mathrm{d}\nu \quad \text{on } [0,T]\times\mathbb{R}^n.$$

Since u and v are positve, H is nonnegative. Then,

$$(v-u)_+(t,x) \le \int_0^t \mathcal{G}(t-\tau)\{(v-u)_+(\tau,x)H(\tau,x)\}\,\mathrm{d}\tau.$$

By Lemma 3.1, we have

$$\|(v-u)_+(t,\cdot)\|_\infty \le \int_0^t \|(v-u)_+(\tau,\cdot)\|_\infty \|H(\tau,x)\|_\infty\,\mathrm{d}\tau.$$

By Theorem 2.24, by letting $f(t) = \|(v-u)_+(t,\cdot)\|_\infty$, $B(t) = \|H(t,x)\|_\infty$ and $A \equiv 0$, we get $f \equiv 0$, that is $u \ge v$. $\square$

This idea also can apply to the solution when $u \in C([0,T), L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$

Corollary 3.6. *Assume that $u_0 \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ is nonnegative. Then, there exists a unique positive mild solution of (1.4).*

Proof. By Theorem 3.4, we have the existence of mild solutions of (1.4). Let v and u be mild solutions with initial condition u_0. Then

$$v \in C([0,T_1]; L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)) \quad \text{and} \quad u \in C([0,T_2]; L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)),$$

for some $T_1, T_2 \ge 0$. Let $T' = \min(T_1, T_2)$. By Theorem 3.5, for $t \in [0, T']$, $u = v$ on $[0, T']$. $\square$

CHAPTER IV

CRITICAL EXPONENT

In this chapter, we discuss the behavior of solutions to (1.4) in the case $\mu \neq 0$. For the case $\mu = 0$, the kernel J is a dispersal kernel, and the result was discussed by Alfaro in [1]. We will divide the investigation into 3 cases:

1. When $0 < p < \frac{\beta}{n}$, we show that the solutions blow up in finite time;
2. When $p = \frac{\beta}{n}$, we show that the solutions blow up in finite time for $\mu < 0$ and solutions can be global provided the initial condition is sufficiently small for $\mu \geq 1$;
3. When $p > \frac{\beta}{n}$, we show that the solutions can be global or blow-up provided the initial condition is sufficiently small or sufficiently large, respectively.

4.1 Systematic blow up: $0 < p < \frac{\beta}{n}$

In this section, we prove that the positive solution to (1.4) is blow-up in finite time by contradiction. Thus we assume that the solution is global. We define for any $t > 0$, the quantity

$$f(t) = \int_{\mathbb{R}^n} e^{t(\hat{J}(\xi)-1)} \hat{u}_0(\xi) \, \mathrm{d}\xi.$$

This idea is originated by Kaplan [11]. We are going to estimate $f(t)$ from above and below as $t \to \infty$.

4.1.1 Estimate from below and above

We assume in this subsection that the initial condition u_0 satisfies

1. u_0 is nonnegative, nontrivial, radially symmetric, continuous and bounded.

2. u_0 and $\hat{u}_0$ are integrable function.

For the esitimate from below, we divide the proof into 2 cases: $\mu \in \mathbb{R}^+$ and $\mu \in \mathbb{R}^-$.

Theorem 4.1. *There is a positive constant G depending on the dimension n and the kernel J such that for large t, we have the following results.*

(i). If $\mu \in \mathbb{R}^-$, then

$$f(t) \geq \frac{||u_0||_1 G}{t^{\frac{n}{\beta}}}. \tag{4.1}$$

(ii). If $\mu \in \mathbb{R}^+$, then

$$f(t) \geq \frac{||u_0||_1 G}{(t(\ln t)^{\mu})^{\frac{n}{\beta}}}. \tag{4.2}$$

Proof of Theorem 4.1 (i). By the Hypothesis (2.26), there exists $\xi_0 > 0$ such that for any $|\xi| < \xi_0$,

$$\hat{J}(\xi) - 1 \geq -2A|\xi|^{\beta}\left(\ln \frac{1}{|\xi|}\right)^{\mu} \geq -2A|\xi|^{\beta}. \tag{4.3}$$

Since u_0 is nontrivial and non-negative, $\hat{u}_0(0) = \int_{\mathbb{R}^n} u_0(x)\,\mathrm{d}x > 0$. Therefore there exists $\xi_1 > 0$ such that if $|\xi| < \xi_1$, then $\hat{u}_0(\xi) \geq 0$. Thus, we set

$$\xi' = \min(\xi_0, \xi_1).$$

By Lemma 2.10, there exists $N \in \mathbb{N}$ such that for any $|\xi| > N$, $\hat{J}(\xi) < \frac{1}{2}$. By Lemma 2.12, we know that $\hat{J}$ achieves a maximum on $[\xi', N]$ with the value less than 1. Thus, there exists $\delta > 0$ such that for any $|\xi| \geq \xi'$,

$$\hat{J}(\xi) - 1 \leq -\delta. \tag{4.4}$$

Next, we are going to approximate the value of $t^{\frac{n}{\beta}} f(t)$ by dividing into 2 parts,

$$t^{\frac{n}{\beta}} f(t) = g_1(t) + g_2(t),$$

where $g_1(t) = t^{\frac{n}{\beta}} \int_{|\xi|\leq\xi'} e^{t(\hat{J}(\xi)-1)}\hat{u}_0(\xi)\,\mathrm{d}\xi$ and $g_2(t) = t^{\frac{n}{\beta}} \int_{|\xi|\geq\xi'} e^{t(\hat{J}(\xi)-1)}\hat{u}_0(\xi)\,\mathrm{d}\xi$.

By (4.4), we have

$$|g_2(t)| \le t^{\frac{n}{\beta}} e^{-\delta t} ||\hat{u}_0(\xi)||_1 \tag{4.5}$$

and the RHS of (4.5) converges to 0 as $t \to \infty$. On the other hand, from (4.3),

$$g_1(t) \ge t^{\frac{n}{\beta}} \int_{|\xi| \le \xi'} e^{-2At|\xi|^\beta} \hat{u}_0(\xi)\,\mathrm{d}\xi = t^{\frac{n}{\beta}} \int_{\mathbb{R}^n} e^{-2At|\xi|^\beta} \hat{u}_0(\xi) \chi_{(0,\xi')}(\xi)\,\mathrm{d}\xi.$$

By Theorem 2.6 with the map : $\xi \longmapsto t^{-\frac{1}{\beta}} z$, we have

$$\begin{aligned} g_1(t) &\ge t^{\frac{n}{\beta}} \int_{\mathbb{R}^n} e^{-2At|t^{-\frac{1}{\beta}} z|^\beta} \hat{u}_0(t^{-\frac{1}{\beta}} z) \chi_{(0,\xi')}(t^{-\frac{1}{\beta}} z) t^{-\frac{n}{\beta}}\,\mathrm{d}z, \\ &= \int_{\mathbb{R}^n} e^{-2A|z|^\beta} \hat{u}_0(t^{-\frac{1}{\beta}} z) \chi_{(0,\xi')}(t^{-\frac{1}{\beta}} z)\,\mathrm{d}z. \end{aligned} \tag{4.6}$$

By Lemma 2.10 and $u_0 \in L^1(\mathbb{R}^n)$, $\hat{u}_0$ is bounded. Thus,

$$e^{-2A|z|^\beta} \hat{u}_0(t^{-\frac{1}{\beta}} z) \le M e^{-2A|z|^\beta},$$

for some constant M. By [7], we know that $e^{-2A|z|^\beta}$ is an integrable function. Thus, we can apply Theorem 2.5 to (4.6) to get the integral converges, as $t \to \infty$, to

$$\int_{\mathbb{R}^n} e^{-2A|z|^\beta} \hat{u}_0(0)\,\mathrm{d}z.$$

Since $\hat{u}_0(0) = ||u_0||_1$, we can conclude that

$$f(t) \ge \frac{||u_0||_1 G}{t^{\frac{n}{\beta}}},$$

where $G = \frac{1}{2} \int_{\mathbb{R}^n} e^{-2A|z|^\beta}\,\mathrm{d}z$. □

Proof of Theorem 4.1 (ii). By the Hypothesis (2.26), there exists $\xi_0 > 0$ such that for any $|\xi| < \xi_0$,

$$\hat{J}(\xi) - 1 \ge -2A|\xi|^\beta \left(\ln \frac{1}{|\xi|}\right)^\mu. \tag{4.7}$$

Since u_0 is nontrivial and non-negative, $\hat{u}_0(0) = \int_{\mathbb{R}^n} u_0(x)\,\mathrm{d}x > 0$. Therefore there exists $\xi_1 > 0$ such that if $|\xi| < \xi_1$, then

$$\hat{u}_0(\xi) \geq \frac{\hat{u}_0(0)}{2} > 0. \tag{4.8}$$

Then, we set

$$\xi' = \min(\xi_0, \xi_1, 1).$$

By using the same idea as in the proof of Theorem 4.1(i), it is enough to show that there exists a positive constant G depending only on the dimension n and the kernel J such that

$$g(t) \geq ||u_0||_1 G,$$

where $g(t) = (t(\ln t)^{\mu})^{\frac{n}{\beta}} \int_{|\xi|\leq\xi'} e^{t(\hat{J}(\xi)-1)} \hat{u}_0(\xi)\, \mathrm{d}\xi$. From (4.7) and (4.8), we have

$$\begin{aligned} g(t) &\geq (t(\ln t)^{\mu})^{\frac{n}{\beta}} \int_{|\xi|\leq\xi'} e^{-2At|\xi|^{\beta}\left(\ln \frac{1}{|\xi|}\right)^{\mu}} \hat{u}_0(\xi)\, \mathrm{d}\xi, \\ &= (t(\ln t)^{\mu})^{\frac{n}{\beta}} \int_{\mathbb{R}^n} e^{-2At|\xi|^{\beta}\left(\ln \frac{1}{|\xi|}\right)^{\mu}} \left(\frac{\hat{u}_0(0)}{2}\right) \chi_{(0,\xi')}(\xi)\, \mathrm{d}\xi. \end{aligned}$$

By Theorem 2.6 with the map : $\xi \longmapsto (t(\ln t)^{\mu})^{-\frac{1}{\beta}} z$, we have

$$g(t) \geq \left(\frac{\hat{u}_0(0)}{2}\right) \int_{\mathbb{R}^n} e^{-2A\frac{|z|^{\beta}}{(\ln t)^{\mu}}\left(\ln \frac{(t(\ln t)^{\mu})^{\frac{1}{\beta}}}{|z|}\right)^{\mu}} \chi_{(0,\xi')}((t(\ln t)^{\mu})^{-\frac{1}{\beta}} z)\, \mathrm{d}z.$$

There exists t_0 such that for $t \geq t_0$,

$$0 < \frac{\ln \left(\ln t\right)^{\mu}}{\beta} - \frac{\mu}{\beta} < \ln[t^{\frac{1}{\beta}} (\ln t)^{\frac{\mu}{\beta}} (\xi')].$$

Then, for $t \geq t_0$,

$$g(t) \geq \left(\frac{\hat{u}_0(0)}{2}\right) \int_{\mathbb{R}^n} e^{-2A\frac{|z|^{\beta}}{(\ln t)^{\mu}}\left(\ln \frac{(t(\ln t)^{\mu})^{\frac{1}{\beta}}}{|z|}\right)^{\mu}} \chi_{(0,(\ln t)^{\frac{\mu}{\beta}} e^{-\frac{\mu}{\beta}})}(z)\, \mathrm{d}z. \tag{4.9}$$

Consider $h(t) = \frac{1}{\beta} + \frac{\mu \ln\ln t}{\beta \ln t} - \frac{\ln|z|}{\ln t}$. Then,

$$h'(t) = \frac{1}{(\ln t)^2}\left(\frac{\mu + \beta\ln|z| - \mu\ln\ln t}{\beta t}\right),$$

so h is decreasing function on (t_0, ∞). We can apply Theorem 2.4 in (4.9) to get that, as $t \to \infty$, the integral converges to

$$\int_{\mathbb{R}^n} e^{-2\frac{A}{\beta}|z|^\beta}\hat{u}_0(0)\,\mathrm{d}z = ||u_0||_1 G.$$

Hence we are done. □

For the esatimate from above, we use the result in [1]. This estimate requires the condition of the kernel J that J is nonnegative, bounded, and radially symmetric with unit integral.

Theorem 4.2 ([1]). *If u is a global solution to (1.4) with the kernel J satisfying that J is a nonnegative, radially symmetric function with unit integral, then, for any $t > 0$,*

$$f(t) \leq (2\pi)^n\left(\left(\frac{p+1}{p}\right)^{\frac{1}{p}}\frac{1}{t^{\frac{1}{p}}} + e^{-t}u_0(0)\right). \tag{4.10}$$

4.1.2 Systematic blow up

In this subsection, we show that, when $p < \frac{\beta}{n}$, the positive solution to (1.4) is blow-up in finite time by using estimate from below and above of $f(t)$. We assume the initial condition u_0 satisfies $u_0 \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ is nonnegative and nontrivial. For the comparison principle of solution, we use the result in [9].

Theorem 4.3 ([9]). *Let $u, v \in C^1((0,T), L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)) \cap C^0([0,T), L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n))$ be nonnegative functions satisfitying*

$$u_t \geq J * u - u + u^{1+p} \quad \textit{and} \quad v_t \leq J * v - v + v^{1+p} \quad \textit{in } (0,T) \times \mathbb{R}^n$$

for some $T > 0$, with $v_0 \leq u_0$ in $\mathbb{R}^n$. Then, $v \leq u$ in $[0,T) \times \mathbb{R}^n$.

Lemma 4.4. *For any $\alpha > 0$ and $\mu \in \mathbb{R}$,*

$$\lim_{|\xi|\to 0} |\xi|^{\alpha}\left(\ln\frac{1}{|\xi|}\right)^{\mu} = 0.$$

Theorem 4.5. *Assume that $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is nonnegative and nontrivial satisfying the condition that there exist $x_0 \in \mathbb{R}^n$, $\varepsilon > 0$, and $r > 0$ such that $u_0 \geq \varepsilon$ in $B(x_0, r)$. The solutions to (1.4) are blow-up in finite time when $p < \frac{\beta}{n}$.*

Proof. Suppose that there exists a global solution u to (1.4). Since there exist $x_0 \in \mathbb{R}^n$, $\varepsilon > 0$, and $r > 0$ such that $u_0 \geq \varepsilon$ in $B(x_0, r)$, there exists a nonnegative, nontrivial, radially symmetric, continuous function v with compact support that is smaller than u_0. By Theorem 4.3, the solution to (1.4) with v as the initial condition is global. Thus, we can assume, without loss of generality, that the initial condition is nonnegative, nontrivial, radially symmetric, and continuous with compact support. The result in Section 4.1.1 are available.

- $\mu \in \mathbb{R}^+$, $0 < p < \frac{\beta}{n}$: By (4.2) and (4.10), for a large t, we have

$$\frac{||u_0||_1 G}{(t(\ln t)^{\mu})^{\frac{n}{\beta}}} \leq (2\pi)^n \left(\left(\frac{p+1}{p}\right)^{\frac{1}{p}} \frac{1}{t^{\frac{1}{p}}} + e^{-t}u_0(0)\right).$$

  Letting $t \to \infty$,we have $||u_0||_1 \leq 0$. This is a contradiction since $||u_0||_1 > 0$.

- $\mu \in \mathbb{R}^-$, $0 < p < \frac{\beta}{n}$: By (4.1) and (4.10), letting $t \to \infty$, which gives a contradiction like the first case.

Hence the solutions are blow-up. □

4.2 Blow up vs extiction: $p = \frac{\beta}{n}$ and $\mu \in \mathbb{R} \setminus (0, 1]$

In this subsection, we show that, when $p = \frac{\beta}{n}$ and $\mu < 0$, the positive solution to (1.4) is blow-up in finite time by using estimate from below and above of $f(t)$. We assume the initial condition u_0 satisfies $u_0 \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ is nonnegative and nontrivial. For the comparison principle of solution, we use the result in [9].

Theorem 4.6. *Let $p = \frac{\beta}{n}$. Assume that $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is nonnegative and nontrivial satisfying the condition that there exist $x_0 \in \mathbb{R}^n$, $\varepsilon > 0$, and $r > 0$ such that $u_0 \geq \varepsilon$ in $B(x_0, r)$.*

i. If $\mu < 0$, then the solutions to (1.4) is blow-up in finite time.

ii. If $\mu > 1$, the solution to (1.4) is global if $||u||_1 + ||\hat{u}_0||_1 < \delta$ for some $\delta > 0$.

Proof of Theorem 4.6 (i). Suppose that there exists a global solution u to (1.4). Since there exist $x_0 \in \mathbb{R}^n$, $\varepsilon > 0$, and $r > 0$ such that $u_0 \geq \varepsilon$ in $B(x_0, r)$, there exists a nonnegative, nontrivial, radially symmetric, continuous function v with compact support that is smaller than u_0. By Theorem 4.3, the solution to (1.4) with v as the initial condition is global. Thus, we can assume, without loss of generality, that the initial condition is nonnegative, nontrivial, radially symmetric, and continuous with compact support. The result in Section 4.1.1 are available. By (4.1) and (4.10) and letting $t \to \infty$ as above, we have $||u_0||_1 < C$ where C is a constant that depends on the dimension n and the kernel J. Thus, by regarding $u(t, \cdot)$ as an initial value and apply the preceeding argrument, we derive that

$$m(t) := ||u(t, \cdot)||_{L_1} \leq C$$

for any $t \geq 0$. By Theorem **??**, we have

$$\frac{\mathrm{d}m(t)}{\mathrm{d}t} = \int_{\mathbb{R}^n} \partial_t u \,\mathrm{d}x = \int_{\mathbb{R}^n} (J * u - u) \,\mathrm{d}x + \int_{\mathbb{R}^n} u^{1+p}(t, x) \,\mathrm{d}x = \int_{\mathbb{R}^n} u^{1+p}(t, x) \,\mathrm{d}x.$$

Then, $m(t) - m(0) = \int_0^t \int_{\mathbb{R}^n} u^{1+p}(t, x) \,\mathrm{d}x \,\mathrm{d}t$. Thus,

$$\int_0^\infty \int_{\mathbb{R}^n} u^{1+p}(t, x) \,\mathrm{d}x \,\mathrm{d}t \leq C. \tag{4.11}$$

Let $\rho \in C_c^\infty(\mathbb{R})$ be such that $\rho \equiv 1$ on $(-1, 1)$, $0 \leq \rho \leq 1$ and $supp(\rho) = [-2, 2]$. For any $\varepsilon > 0$, $R > 0$ and $T > 0$, we define

$$\psi_R(t) = \rho\left(\frac{t - T}{R^\beta}\right) \quad \text{and} \quad \theta_R(x) = \rho\left(\varepsilon \frac{|x|}{R}\right).$$

Consider

- By Theorem **??**, we have

$$\int_{\mathbb{R}^n} (J * u - u)(t,x)\theta_R(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} (J * \theta_R - \theta_R)(x)u(t,x)\,\mathrm{d}x. \tag{4.12}$$

- By integration by parts with respect to time, we get

$$\begin{aligned}
&\int_T^\infty \int_{\mathbb{R}^n} u_t(t,x)\psi_R(t)\theta_R(x)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_{\mathbb{R}^n} u(t,x)\psi_R(t)\theta_R(x)\,\mathrm{d}x\Big|_T^\infty - \int_T^\infty \int_{\mathbb{R}^n} u(t,x)\psi_R'(t)\theta_R(x)\,\mathrm{d}x\,\mathrm{d}t, \\
&\quad = -\int_{\mathbb{R}^n} u(T,x)\theta_R(x)\,\mathrm{d}x - \int_T^\infty \int_{\mathbb{R}^n} u(t,x)\psi_R'(t)\theta_R(x)\,\mathrm{d}x\,\mathrm{d}t, \\
&\quad \le -\int_T^\infty \int_{\mathbb{R}^n} u(t,x)\psi_R'(t)\theta_R(x)\,\mathrm{d}x\,\mathrm{d}t. \qquad (4.13)
\end{aligned}$$

Multiplying (1.4) with $\psi_R(t)\theta_R(x)$ and integrating it over $(T,\infty)\times\mathbb{R}^n$. By (4.12) and (4.13), we obtain

$$\begin{aligned}
&\int_T^\infty \int_{\mathbb{R}^n} u^{1+p}(t,x)\psi_R(t)\theta_R(x)\,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_T^\infty \int_{\mathbb{R}^n} (J * u - u)(t,x)\psi_R(t)\theta_R(x)\,\mathrm{d}x\,\mathrm{d}t + \int_T^\infty \int_{\mathbb{R}^n} u_t(t,x)\psi_R(t)\theta_R(x)\,\mathrm{d}x\,\mathrm{d}t, \\
&\le -\int_T^\infty \int_{\mathbb{R}^n} (J * \theta_R - \theta_R)(x)u(t,x)\psi_R(t)\,\mathrm{d}x\,\mathrm{d}t - \int_T^\infty \int_{\mathbb{R}^n} u(t,x)\psi_R'(t)\theta_R(x)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Since $\theta_R(x) = 0$ on $|x| > 2\frac{R}{\varepsilon}$, $J * \theta_R \ge 0$. Thus,

$$\begin{aligned}
&\int_T^\infty \int_{\mathbb{R}^n} u^{1+p}(t,x)\psi_R(t)\theta_R(x)\,\mathrm{d}x\,\mathrm{d}t \\
&\le -\int_T^{T+2R^\beta} \int_{|x|<2\frac{R}{\varepsilon}} (J * \theta_R - \theta_R)(x)u(t,x)\,\mathrm{d}x\,\mathrm{d}t - \int_T^\infty \int_{|x|<2\frac{R}{\varepsilon}} u(t,x)\psi_R'(t)\,\mathrm{d}x\,\mathrm{d}t, \\
&= -I_1 - I_2,
\end{aligned}$$

where

$$I_1 = \int_T^{T+2R^\beta} \int_{|x|<2\frac{R}{\varepsilon}} (J * \theta_R - \theta_R)(x) u(t,x) \, \mathrm{d}x \, \mathrm{d}t \quad \text{and}$$
$$I_2 = \int_T^{\infty} \int_{|x|<2\frac{R}{\varepsilon}} u(t,x) \psi_R'(t) \, \mathrm{d}x \, \mathrm{d}t.$$

For any $t > T$, we have

$$\psi_R'(t)| = \left| \frac{1}{R^\beta} \rho' \left(\frac{t-T}{R^\beta} \right) \right| \leq \frac{C}{R^\beta} \chi_{(T+R^\beta, T+2R^\beta)}. \tag{4.14}$$

Then, by Theorem 2.22 and (4.14),

$$\begin{aligned}
|I_2| &\leq \frac{C}{R^\beta} \int_{T+R^\beta}^{T+2R^\beta} \int_{|x|<2\frac{R}{\varepsilon}} u(t,x) \, \mathrm{d}x \, \mathrm{d}t, \\
&\leq \frac{C}{R^\beta} \left(\int_{T+R^\beta}^{T+2R^\beta} \int_{|x|<2\frac{R}{\varepsilon}} 1 \, \mathrm{d}x \, \mathrm{d}t \right)^{\frac{p}{p+1}} \left(\int_{T+R^\beta}^{T+2R^\beta} \int_{|x|<2\frac{R}{\varepsilon}} u^{1+p}(t,x) \, \mathrm{d}x \, \mathrm{d}t \right)^{\frac{1}{p+1}}, \\
&= \frac{C}{\varepsilon^{\frac{\beta}{p+1}}} \left(\int_{T+R^\beta}^{T+2R^\beta} \int_{|x|<2\frac{R}{\varepsilon}} u^{1+p}(t,x) \, \mathrm{d}x \, \mathrm{d}t \right)^{\frac{1}{p+1}}.
\end{aligned} \tag{4.15}$$

Deriving from (4.11) by taking $R \to \infty$ in (4.15), we get that I_2 tends to zero. On the other hand, let

$$B(x) = (J * \theta_R - \theta_R)(x).$$

By Lemma 2.9, we have

$$\begin{aligned}
(2\pi)^n B(x) &= (2\pi)^n \int_{\mathbb{R}^n} \theta_R(z-x) J(z) \, \mathrm{d}z - (2\pi)^n \theta_R(x), \\
&= \int_{\mathbb{R}^n} \hat{J}(\xi) e^{-ix\cdot\xi} \hat{\theta_R}(\xi) \, \mathrm{d}\xi - (2\pi)^n \theta_R(x), \\
&= \int_{\mathbb{R}^n \setminus \{0\}} \left(1 - \mathcal{A}(\xi) |\xi|^\beta \left(\ln \frac{1}{|\xi|} \right)^\mu \right) e^{-ix\cdot\xi} \hat{\theta_R}(\xi) \, \mathrm{d}\xi - (2\pi)^n \theta_R(x),
\end{aligned}$$

where $\mathcal{A}$ is a bounded function. Since θ_R is radial, $(2\pi)^n \theta_R(x) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \hat{\theta_R}(\xi) \, \mathrm{d}\xi$.

Thus, $\hat{\theta_R}(\xi) = (\frac{R}{\varepsilon})^n \hat{\rho_n}(\frac{R}{\varepsilon}\xi)$ where $\rho_n(x) = \rho(|x|)$, we have

$$(2\pi)^n B(x) = -\left(\frac{\varepsilon}{R}\right)^{\beta} \int_{\mathbb{R}^n \setminus \{0\}} \left(\mathcal{A}\left(\frac{\varepsilon}{R}\xi'\right) |\xi'|^{\beta} \left(\ln \frac{R}{|\varepsilon \xi'|}\right)^{\mu}\right) e^{-i\frac{\varepsilon}{R}x\cdot\xi'} \hat{\rho_n}(\xi')\, \mathrm{d}\xi'.$$

Then,

$$|B(x)| \leq (2\pi)^{-n} \left(\frac{\varepsilon}{R}\right)^{\beta} ||\mathcal{A}||_{\infty} \int_{\mathbb{R}^n \setminus \{0\}} |\xi'|^{\beta} \left(\ln \frac{R}{|\varepsilon \xi'|}\right)^{\mu} |\hat{\rho_n}(\xi')|\, \mathrm{d}\xi'.$$

By Lemma 4.4 and $\hat{\rho_n} \in \mathcal{S}(\mathbb{R}^n)$, we have $|B(x)| \leq C\left(\frac{\varepsilon}{R}\right)^{\beta}$. Thus,

$$|I_1| < C\varepsilon^{\frac{\beta p}{p+1}} \left(\int_T^{T+2R^{\beta}} \int_{|x|<2\frac{R}{\varepsilon}} u^{1+p}(t,x)\, \mathrm{d}x\, \mathrm{d}t\right)^{\frac{1}{p+1}}. \tag{4.16}$$

Using (4.16) and (4.15) by letting $R \to \infty$, we have

$$\int_T^{\infty} \int_{\mathbb{R}^n} u^{1+p}(t,x)\, \mathrm{d}x\, \mathrm{d}t \leq C\varepsilon^{\frac{\beta p}{p+1}} \left(\int_T^{\infty} \int_{\mathbb{R}^n} u^{1+p}(t,x)\, \mathrm{d}x\, \mathrm{d}t\right)^{\frac{1}{p+1}}.$$

Since this holds for any $\varepsilon > 0$ and $T > 0$, $u \equiv 0$ on $(0,\infty) \times \mathbb{R}^n$. □

Proof of Theorem 4.6 (ii). Let v be according to Lemma 2.25 for a positive smooth function v_0 to be specified. Then there exists $t_0 > 0$ such that

$$||v(t,\cdot)||_{\infty} \leq C(||v_0||_1 + ||\hat{v}_0||_1)(t(\ln t)^{\mu})^{-\frac{n}{\beta}} \qquad \text{for any } t \geq t_0.$$

Since $\mu > 1$, there exists $t_1 > t_0$ such that $(\ln t)^{\mu} \geq 1$ for any $t > t_1$. Define g on $[t_1, \infty)$ by

$$g(t) = \left(\frac{1}{\frac{1}{(\ln t_1)^{\mu-1}} - \frac{pC^p(||v||_{L^1} + ||\hat{v}_0||_{L^1})^p}{\mu-1}\left(\frac{1}{(\ln t_1)^{\mu-1}} - \frac{1}{(\ln t)^{\mu-1}}\right)}\right)^{\frac{1}{p}},$$

where v_0 is sufficiently small so that

$$||v_0||_1 + ||\hat{v}_0||_1 \leq \frac{1}{C}\left(\frac{\mu-1}{p}\right)^{\frac{1}{p}}.$$

Then

$$\frac{g'(t)}{g^{1+p}(t)} = \frac{C^p(||v_0||_1 + ||\hat{v}_0||_1)^p}{t(\ln t)^\mu}.$$

For $t > t_1$, we have

$$\frac{g'(t)}{g^{1+p}(t)} \geq \frac{C^p(||v||_1 + ||\hat{v}_0||_1)^p}{(t(\ln t)^\mu)} \geq ||v(t,\cdot)||_\infty^p. \tag{4.17}$$

Since $||v(t,\cdot)||_\infty^p$ is continuous, $||v(t,\cdot)||_\infty^p$ attains a maximum on $[0, t_1]$, denoted by M. Given $M' = -Mt_1 - 1$, we define g on $[0, t_1)$ by

$$g = \left(-\frac{1}{p(Mt + M')}\right)^{\frac{1}{p}}.$$

Then $g > 0$ on $[0, \infty)$ and it satisfies (4.17). A straighforward computation shows that $g(t)v(t,x)$ is a supersolution to (1.4). If we make

$$u_0 \leq \left(\frac{1}{p(Mt_1 + 1)}\right)^{\frac{1}{p}} v_0,$$

then we can apply Theorem 4.3. So, we have

$$u(t,x) \leq g(t)v(t,x).$$

Hence, $u(t,x)$ is global with $\delta := \left(\frac{1}{p(Mt_0+1)}\right)^{\frac{1}{p}} \frac{1}{C} \left(\frac{\mu-1}{p}\right)^{\frac{1}{p}}$. □

4.3 Blow up vs extinction: $p > \frac{\beta}{n}$

In this subchapter, we prove that the solutions to (1.4) can be global or blow-up in finite time depending upon the initial condition.

Theorem 4.7. *There is $\delta > 0$ such that if $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $||u||_1 + ||\hat{u}_0||_1 < \delta$, then the solution to (1.4) is global and satisfies, for large t,*

$$||u(t,\cdot)||_{L^\infty} \leq C(||u||_{L^1} + ||\hat{u}_0||_{L^1})(t(\ln t)^\mu)^{-\frac{n}{\beta}},$$

for some constant $C > 0$.

Proof. Let v be according to Lemma 2.25 for a positive smooth function v_0 to be specified. Then, there exists $t_0 > 0$ such that

$$||v(t,\cdot)||_\infty \le C(||v_0||_1 + ||\hat{v}_0||_1)(t(\ln t)^\mu)^{-\frac{n}{\beta}} \quad \text{for any } t \ge t_0.$$

Since $p > \frac{\beta}{n}$, there exists $t_1 > t_0$ such that $(\ln t)^{\frac{n\mu p}{\beta}} t^{\frac{np}{2\beta}-\frac{1}{2}} \ge 1$ for any $t > t_1$. Define g on $[t_1, \infty)$ by

$$g(t) = \left(\frac{1}{\frac{1}{t_1^{\alpha-1}} - \frac{pC^p(||v||_1+||\hat{v}_0||_1)^p}{\alpha-1}\left(\frac{1}{t_1^{\alpha-1}} - \frac{1}{t^{\alpha-1}}\right)} \right)^{\frac{1}{p}},$$

where $\alpha = \frac{1+\frac{np}{\beta}}{2}$ and v_0 is sufficiently small so that

$$||v_0||_1 + ||\hat{v}_0||_1 \le \frac{1}{C}\left(\frac{\alpha-1}{p}\right)^{\frac{1}{p}}.$$

Then,

$$\frac{g'(t)}{g^{1+p}(t)} = \frac{C^p(||v_0||_1 + ||\hat{v}_0||_1)^p}{t^\alpha}.$$

For $t > t_1$, we have

$$\frac{g'(t)}{g^{1+p}(t)} \ge \frac{C^p(||v||_1 + ||\hat{v}_0||_1)^p}{(t(\ln t)^\mu)^{\frac{np}{\beta}}} \ge ||v(t,\cdot)||_\infty^p. \tag{4.18}$$

Since $||v(t,\cdot)||_\infty^p$ is continuous, $||v(t,\cdot)||_\infty^p$ attains a maximum on $[0, t_1]$, denoted by M. Given $M' = -Mt_1 - 1$, we define g on $[0, t_1)$ by

$$g = \left(-\frac{1}{p(Mt + M')}\right)^{\frac{1}{p}}.$$

Then, $g > 0$ on $(0, \infty)$ and it satisfies (4.18). A straighforward computation shows that $g(t)v(t,x)$ is a supersolution to (1.4). If we make

$$u_0 \leq \left(\frac{1}{p(Mt_1+1)}\right)^{\frac{1}{p}} v_0,$$

then we can apply Theorem (4.3). Thus, we have

$$u(t,x) \leq g(t)v(t,x).$$

Hence, $u(t,x)$ is global with $\delta := \left(\frac{1}{p(Mt_0+1)}\right)^{\frac{1}{p}} \frac{1}{C} \left(\frac{\alpha-1}{p}\right)^{\frac{1}{p}}$. □

Finally, we state without prove the following blow-up result which was establish in [1].

Theorem 4.8 ([1]). *Assume $\lambda > 0$ and $R > 0$ are such that*

$$\lambda > \left(1 - C_n \int_{|z|\leq R} J(z)dz\right)^{\frac{1}{p}},$$

where $0 < C_n < 1$ is a constant that depends only on the dimension n. Then, the solution to (1.4) with $\lambda\chi_{|x|\leq R}$ as an initial data blows up in finite time.

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