CHAPTER II

MATHEMATICAL MODEL

2.1 Flow through a pipe

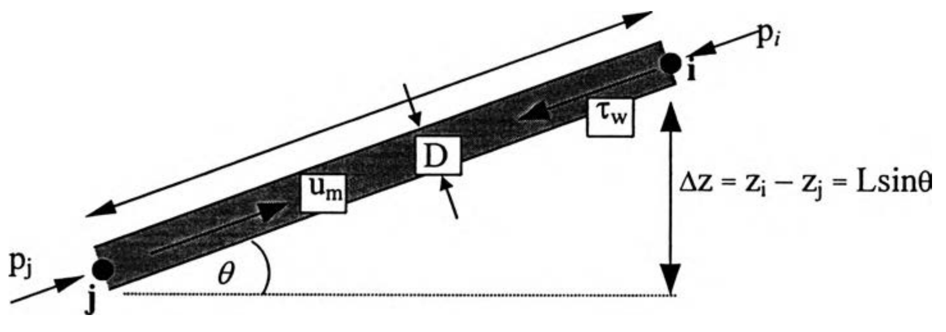

Figure 2.1 Steady flow from node j to node i in an inclined pipeline

A steady-state momentum balance in the direction of flow on the fluid from node j to node i in the pipe gives:

$$(p_j - p_i)\frac{\pi D^2}{4} - \tau_w \pi D L - \frac{\pi D^2}{4}\rho L g \sin\theta = 0 . \tag{2.1}$$

Equation (2.1) can be rewritten as:

$$-\Delta p = p_j - p_i = 4\tau_w \frac{L}{D} + \rho g \Delta z . \tag{2.2}$$

But the Fanning friction factor is defined as:

$$f_F = \frac{\tau_w}{\frac{1}{2}\rho u_m^2} . \tag{2.3}$$

So, equations (2.2) and (2.3) give:

$$-\Delta p = p_j - p_i = 2f_F \rho u_m^2 \frac{L}{D} + \rho g \Delta z\,. \tag{2.4}$$

Since:

$$u_m^2 = \frac{16Q^2}{\pi^2 D^2}\,, \tag{2.5}$$

equation (2.4) gives:

$$-\Delta p = \frac{32 f_F \rho Q^2 L}{\pi^2 D^5} + \rho g \Delta z \quad . \tag{2.6}$$

Subscripts are inserted to emphasize that the flow is from node j to node i, so equation (2.6) becomes:

$$p_j - p_i + \rho g(z_j - z_i) = \frac{32 f_{F_{ji}} \rho Q_{ji}^2 L_{ji}}{\pi^2 D_{ji}^5}\,. \tag{2.7}$$

In the same way, for flow from node i to node j :

$$p_j - p_i + \rho g(z_j - z_i) = \frac{32 f_{F_{ji}} \rho Q_{ij}^2 L_{ji}}{\pi^2 D_{ji}^5}\,. \tag{2.8}$$

Here, the second form is in term of the volumetric flow rate Q instead of the mean velocity. So, equation (2.7) can be written as:

$$p_j - p_i + \beta(z_j - z_i) = \alpha_{ji} f_{F_{ji}} Q_{ji}^2 \tag{2.9}$$

where:

$$\alpha_{ij} = \frac{32\rho L_{ji}}{\pi^2 D_{ji}^5} \qquad \text{and} \qquad \beta = \rho g . \tag{2.10}$$

If y is set as:

$$y = p_j - p_i + \beta(z_j - z_i) . \tag{2.11}$$

Then, from equation (2.9), the volumetric flow rate from node j to node i is given by:

$$Q_{ji} = \sqrt{\frac{y}{\alpha_{ji} f_{F_{ji}}}} \qquad \text{for } y > 0 . \tag{2.12}$$

and the flow from node i to node j, the volumetric flow rate is given by:

$$Q_{ij} = -\sqrt{\frac{-y}{\alpha_{ji} f_{F_{ji}}}} \qquad \text{for } y < 0 . \tag{2.13}$$

2.2 Flow across a pump

A pump is assumed not to involve any significant change of elevation between node j to node i, as shown in Figure 2.2.

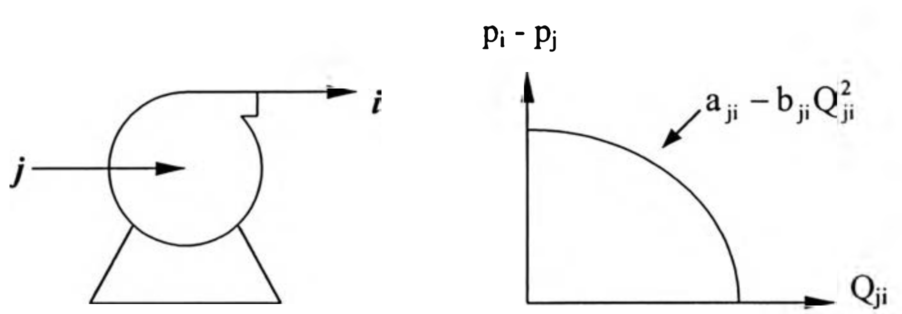

Figure 2.2 Centrifugal pump and performance curve

The pressure change across the pump (from node j to node i) is given by:

$$p_i - p_j = a_{ji} - b_{ji}Q_{ji}^2. \qquad (2.14)$$

If we consider the elevation of each node, equation (2.14) becomes:

$$p_i - p_j = a_{ji} - b_{ji}Q_{ji}^2 + \beta(z_j - z_i). \qquad (2.15)$$

2.3 Steady flow in an open channel

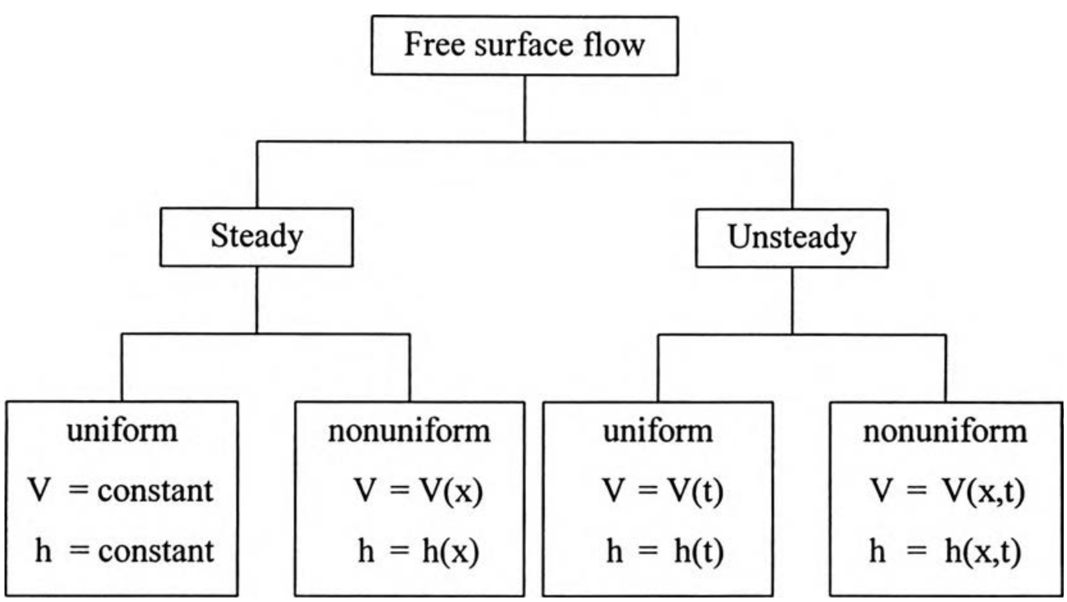

Figure 2.3 Tree diagram of types of flow

Consider the short reach of length L between node 1 and node 2 of a channel in steady and uniform flow with a free surface with cross-sectional area A (Figure 2.3). There is no change of depth h between the nodes. The only force in the direction of motion is gravity, because the free surface is uniformly exposed to the atmosphere, and this must be resisted by the boundary shear stress τ_w, acting over the area PL, where P is the wetted perimeter of the section.

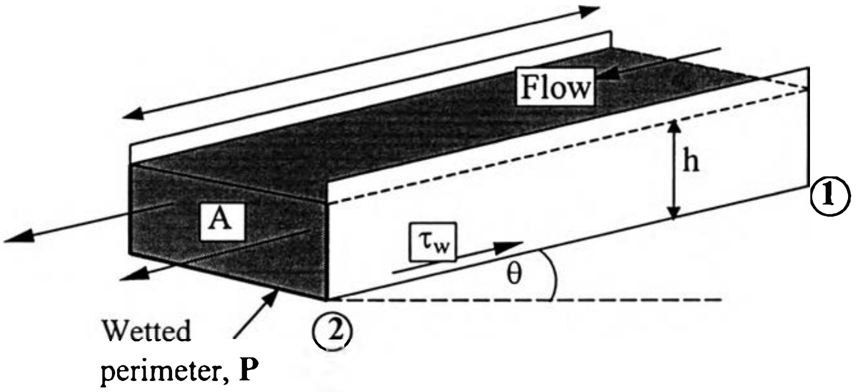

Figure 2.4 Flow in an open channel

A steady-state momentum balance in the direction of flow gives:

$$\rho A L g \sin\theta - \tau_w P L = 0\,. \tag{2.16}$$

Noting that:

$$L \sin\theta = z_1 - z_2\,, \tag{2.17}$$

division of equation (2.16) by $-\rho A$ gives:

$$g(z_2 - z_1) + \frac{\tau_w P L}{\rho A} = 0\,, \tag{2.18}$$

in which the second term can be rearranged as:

$$2\underbrace{\frac{\tau_w}{\frac{1}{2}\rho u_m^2}}_{f_F} u_m^2 \frac{L}{4A/P} = 2f_F u_m^2 \frac{L}{4A/P}. \qquad (2.19)$$

Since: $Q = u_m A$ and $D_e = 4A/P$ equation (2.19) can be rewritten as:

$$g(z_2 - z_1) + 2f_F \frac{Q^2}{A^2}\frac{L}{D_e} = 0, \qquad (2.20)$$

so, the volumetric flow rate from node 1 to node 2 is given by:

$$Q = \sqrt{\frac{g(z_1 - z_2)A^2 D_e}{2f_F L}} \qquad (2.21)$$

For the rectangular cross section shown in Fig. 2.5

$A = w \times h$ and $P = 2h + w$

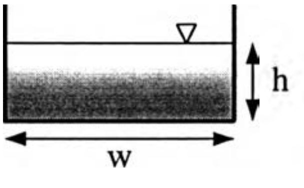

Figure 2.5 Rectangular cross section channel

therefore $D_e = \frac{4A}{P} = \frac{4wh}{2h + w}$

equation (2.21) becomes:

$$Q = \sqrt{\frac{g(z_1 - z_2)4(wh)^3}{2f_F L(2h + w)}}. \qquad (2.22)$$