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นายอภิวัฒน์ พิริยะพัฒน์

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สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2561
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CYCLIC CLIQUE DECOMPOSITIONS OF POWER OF CYCLES

Mr. Apiwat Peereeyaphat

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Mathematics

Department of Mathematics and Computer Science

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การแยกคลีควัฏจักร \mathcal{P} ของกราฟ G คือ หมู่ของคลีควัฏจักรของกราฟ G ซึ่งทำให้เส้นเชื่อมแต่ละเส้นบนกราฟ G ปรากฏอยู่บนคลีควัฏจักรเพียงเท่านั้นในหมู่ดังกล่าว โดยเราจะกล่าวว่า \mathcal{P} เป็น วัฏจักร ถ้ามีฟังก์ชันสมสัณฐาน $\alpha : V(G) \rightarrow V(G)$ ที่ทำให้ $K_k\{\alpha(v_1), \alpha(v_2), \alpha(v_3), \dots, \alpha(v_k)\}$ เป็นคลีควัฏจักรใน \mathcal{P} ก็ต่อเมื่อ $K_k\{v_1, v_2, v_3, \dots, v_k\}$ เป็นคลีควัฏจักรใน \mathcal{P} และ กำลังที่ k ของวงขนาด n เขียนแทนด้วยสัญลักษณ์ C_n^k คือกราฟที่มีเซตของจุดยอดเท่ากับเซตของจุดยอดของกราฟ C_n โดย uv เป็นเส้นเชื่อมของกราฟ C_n^k ก็ต่อเมื่อ $d_{C_n}(u, v) \leq k$ เราจะแนะนำการสร้างการแยกคลีควัฏจักรของกำลังที่ k ของวง n จุดซึ่งได้รับแรงบันดาลใจจากผลเฉลยของข้อปัญหาผลต่างของเฮฟเตอร์ สุดท้ายนี้เราได้สร้างการแยกคลีควัฏจักรของกำลังของวงออกเป็นคลีควัฏจักรอันดับไม่เกินสี่แบบเหมาะสมที่สุด สำหรับ $3 \leq k \leq 26$ และทุกจำนวนนับ $n > 3k$

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สาขาวิชา	คณิตศาสตร์
ปีการศึกษา	2561
	ลายมือชื่อ อ.ที่ปรึกษาหลัก
	ลายมือชื่อ อ.ที่ปรึกษาร่วม

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APIWAT PEEREYAPHAT : CYCLIC CLIQUE DECOMPOSITIONS OF
POWER OF CYCLES.

ADVISOR : ASSOCIATE PROFESSOR CHARIYA UIYYASATHIAN, Ph.D.,

CO-ADVISOR : ASSOCIATE PROFESSOR NATAPHAN KITISIN, Ph.D.,

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A *clique decomposition* \mathcal{P} of a graph G is a collection of cliques of G such that each edge of G belongs to exactly one clique in the collection. We say that \mathcal{P} is *cyclic* if there is an isomorphism $\alpha : V(G) \rightarrow V(G)$ such that $K_k\{\alpha(v_1), \alpha(v_2), \alpha(v_3), \dots, \alpha(v_k)\}$ is a clique in \mathcal{P} whenever $K_k\{v_1, v_2, v_3, \dots, v_k\}$ is. The *k-power of an n-cycle*, C_n^k , is the graph having the same vertex set as C_n and uv is an edge in C_n^k if and only if $d_{C_n}(u, v) \leq k$. Partly inspired by the solution of Heffter's difference problem, we introduce a certain construction of cyclic clique decompositions of the *k-power of an n-cycle*. Finally, we establish an optimal cyclic clique decomposition into cliques of order at most 4 for each $3 \leq k \leq 26$ and all natural numbers $n > 3k$.

Department: Mathematics and Computer Science Student's Signature:

Field of Study: Mathematics Advisor's Signature:

Academic Year 2018 Co-advisor's Signature:

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CHAPTER I

INTRODUCTION

1.1 History and Overview

For a *graph* G , we will use the notations $V(G)$ and $E(G)$ to denote the *vertex set* and the *edge set* of the graph G . Note that a graph in this work is always a *simple graph*.

A *clique* is a complete subgraph of G . A *clique decomposition* (or a *clique partition*) of G is a collection of cliques of G such that each edge of G belongs to exactly one clique in the collection. Note that a clique decomposition of a graph always exists since its edge set is certainly a clique decomposition. However, it is more interesting to find a clique decomposition with smaller number of cliques. A *minimum clique decomposition* of G is a clique decomposition of G with minimum number of elements among all possible clique decompositions of G . A *clique partition number* of G , denoted by $cp(G)$, is the number of elements in a minimum clique decomposition of G .

Clique decompositions of graph have been extensively studied for a long time. The first related paper came out around 1941. Hall [5] showed that an edge set of any graph can be covered by at most $\left\lfloor \frac{n^2}{4} \right\rfloor$ cliques whose order are at most 3. Later on, in 1966, Erdős, Goodman and Pósa [3] showed that the number $\left\lfloor \frac{n^2}{4} \right\rfloor$ happened to be the bound of clique partition number.

Various studies about graph decomposition are related to the decompositions of complete graphs. Furthermore, some of them investigate cyclic decompositions of complete graphs into certain subgraphs such as bipartite graphs, almost-bipartite graphs, cycles etc. See more details in [1], [2] and [4].

Note that when $k \geq \left\lfloor \frac{n}{2} \right\rfloor$, the complete graph K_n is a k -power of an n -cycle

which will be denoted by C_n^k . In 2007, Wichianpaisarn [9] studied clique decompositions of square (2-power) of cycles in order to determine the clique partition number.

Our aim is to investigate minimum cyclic clique decompositions of C_n^k , the k -power of n -cycles for $k \geq 3$ and all natural numbers n . Note that there is a well-known construction of a cyclic K_3 -decomposition of K_n for $n \equiv 1, 3 \pmod{6}$ which is equivalent to a cyclic Steiner triple system of order n . Such a construction relies on the solution of Heffter's difference problem [6]. In this work, we focus on C_n^k when $k < \lfloor \frac{n}{2} \rfloor$ and we introduce a certain method to construct their cyclic clique decompositions. Our method is a generalization of the construction of cyclic Steiner triple systems using Heffter's difference problem. The objective is to investigate a minimum cyclic clique decomposition of C_n^k into cliques of order at most 4.

This thesis is divided into 4 chapters. The first chapter includes most of the definitions and notation needed for our work. In Chapter II, we give details regarding Steiner triple system and Heffter's difference problem. In Chapter III, we talk about the k -power of an n -cycle and explain our idea to construct cyclic clique decomposition of C_n^k . Furthermore, we will introduce a classification of certain 2-cliques, 3-cliques and 4-cliques. Our main results will also be given in this chapter.

Finally, we conclude the results of our work in Chapter IV including some interesting open problems.

1.2 Definitions and Notation

A *path with n vertices*, denoted by P_n , is an n -vertex graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. The *length* of a path is the number of edges in it. A *u, v -path* is a path whose endpoints are u and v . The *distance* between vertices u and v in a graph G , denoted by $d_G(u, v)$ or $d(u, v)$, is the length of a shortest u, v -path in G . The

diameter of G , denoted by $\text{diam}(G)$, is the maximum distance $d(u, v)$ over vertex pairs u, v in $V(G)$.

A *cycle* is a closed path. The n -*cycle*, denoted by C_n , is a cycle with n vertices. Let \mathbb{Z}_n be the group of integers modulo n . In our work, we use \mathbb{Z}_n to represent $V(C_n)$ and $E(C_n) = \{(i)(i+1) \mid i \in \mathbb{Z}_n\}$ as shown in Figure 1.1. We denote elements $0, 1, 2, \dots, n-1$ in \mathbb{Z}_n as $(0), (1), (2), \dots, (n-1)$ in the figure when we label vertices of a graph.

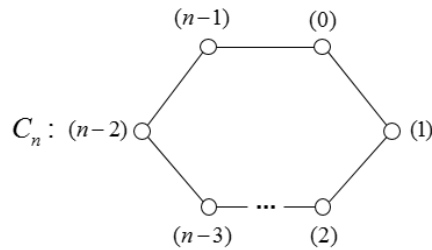


Figure 1.1: The n -cycle C_n

The k -*power* of a graph G , denoted by G^k , is the graph with the same vertex set as G , and for any u, v in $V(G^k)$, there exists an edge uv in $E(G^k)$ if and only if $d_G(u, v) \leq k$.

Example 1.1. Figure 1.2 illustrates the k -power of P_5 .

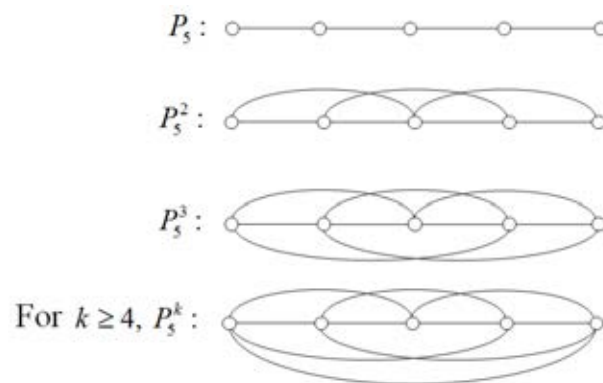


Figure 1.2: The k -power of P_5

□

Given a k -power of n -cycle C_n^k , we define the length function of edges as following.

Definition 1.2. Let $\ell : E(C_n^k) \rightarrow \mathbb{N}$ be a function of edges in C_n^k defined by

$$\ell(ij) = \min\{|i - j|, n - |i - j|\} \text{ for all } ij \in E(C_n^k).$$

For each edge ij in C_n^k , note that $\ell(ij)$ is the distance between vertices i and j in C_n . Thus, we call the function $\ell(ij)$ the *length* of the edge ij .

We write $K_n\{v_1, v_2, \dots, v_n\}$ to denote the clique of order n (or n -clique) on the vertex set $\{v_1, v_2, \dots, v_n\}$.

Example 1.3. Given a graph G such that $V(G) = \{u, v, x, y, z\}$ and $E(G) = \{uv, ux, uy, vx, vy, vw, wy, xy\}$ as in Figure 1.3, let $\mathcal{C}_1, \mathcal{C}_2$ be two collections of cliques of G defined as follows:

$$\begin{aligned} \mathcal{C}_1 &= \{K_4\{u, v, x, y\}, K_2\{v, w\}, K_2\{w, y\}\} \text{ and} \\ \mathcal{C}_2 &= \{K_3\{u, v, x\}, K_3\{v, w, y\}, K_2\{u, y\}, K_2\{x, y\}\}. \end{aligned}$$

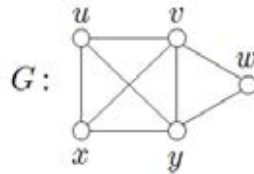


Figure 1.3: The graph G

Note that \mathcal{C}_1 and \mathcal{C}_2 are both clique decompositions of G . Consequently, clique decompositions of G might not be unique. Observe that in this example, two cliques are not enough to partition the graph G . Therefore, any clique decomposition of G has to contain at least 3 cliques. Hence, \mathcal{C}_1 is an example of a minimum clique decomposition of G . \square

Next, we mention a special type of decompositions of a graph. If all cliques in a clique decomposition \mathcal{C} have the same order k , then we call \mathcal{C} a K_k -decomposition.

A clique decomposition \mathcal{P} is *cyclic* if there is an isomorphism $\alpha : V(G) \rightarrow V(G)$, which is not an identity, such that $K_k\{\alpha(v_1), \alpha(v_2), \alpha(v_3), \dots, \alpha(v_k)\}$ is a clique in \mathcal{P} whenever $K_k\{v_1, v_2, v_3, \dots, v_k\}$ is. As in [2], we recall the notion of *clicking*. Given a graph G with $V(G) = \mathbb{Z}_n$, clicking G means applying the isomorphism $i \rightarrow i + 1$ on $V(G)$. Therefore, if clicking each element in \mathcal{P} yields another element in \mathcal{P} , then \mathcal{P} is cyclic. Figure 1.4 illustrates an example of clicking the clique $K_3\{x, y, z\}$. If edges xy, yz and xz are of length ℓ_1, ℓ_2 and ℓ_3 , respectively, then edges $(x+1)(y+1), (y+1)(z+1)$ and $(x+1)(z+1)$ of $K_3\{x+1, y+1, z+1\}$ are of the length ℓ_1, ℓ_2 and ℓ_3 , respectively. Therefore, clicking preserves the length of edges of 3-clique.

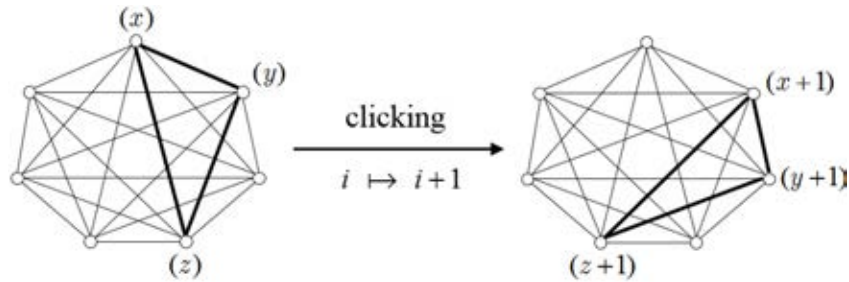


Figure 1.4: Clicking $K_3\{x, y, z\}$

Example 1.4. We illustrate three different clique decompositions of the complete graph K_7 with $V(K_7) = \mathbb{Z}_7$. Let $\mathcal{C}_1, \mathcal{C}_2$ and \mathcal{C}_3 be three clique decompositions of K_7 as follows:

$$\mathcal{C}_1 = \{K_7\{0, 1, 2, 3, 4, 5, 6\}\},$$

$$\mathcal{C}_2 = \{K_6\{1, 2, 3, 4, 5, 6\}, K_2\{0, 1\}, K_2\{0, 2\},$$

$$K_2\{0, 3\}, K_2\{0, 4\}, K_2\{0, 5\}, K_2\{0, 6\}\} \text{ and}$$

$$\mathcal{C}_3 = \{K_3\{i, i+1, i+3\} : 0 \leq i \leq 6\}.$$

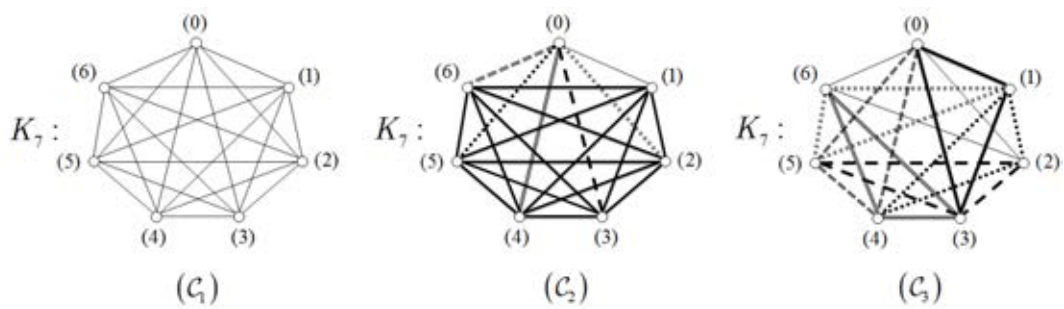


Figure 1.5: Clique decompositions of the complete graph K_7

Clearly, \mathcal{C}_1 is a minimum clique decomposition of K_7 .

Furthermore, \mathcal{C}_3 is a cyclic K_3 -decomposition of K_7 but \mathcal{C}_2 is certainly not. \square

CHAPTER II

PRELIMINARIES

Before we discuss a method to construct cyclic clique decompositions of C_n^k , we would like to review a Steiner triple system. Note that $C_n^k = C_n^{\lfloor \frac{n}{2} \rfloor} = K_n$ if $k \geq \lfloor \frac{n}{2} \rfloor$. Essentially, the well-known cyclic Steiner triple system is equivalent to a cyclic K_3 -decomposition of the complete graph. The idea behind the construction of a cyclic Steiner triple system motivates us to make certain generalization and then use it in this work to construct cyclic clique decompositions of C_n^k into cliques of order at most 4.

A *Steiner triple system of order n* , $\text{STS}(n)$, is an ordered pair (S, \mathcal{T}) , where S is a finite set of points such that $|S| = n$ and \mathcal{T} is a set of 3-element subsets of S called *triples*, such that each pair of distinct elements of S occurs together in exactly one triple of \mathcal{T} .

Graphically, if we represent each point in S by a vertex and represent each triple $\{x, y, z\}$ by a triangle whose vertices are x, y and z , then the Steiner triple system (S, \mathcal{T}) is equivalent to a decomposition of a complete graph $K_{|S|}$ into 3-cliques.

Next, we give some examples of Steiner triple systems.

Example 2.1. We consider the following Steiner triple systems.

$$(i) S_1 = \{0, 1, 2, 3, 4, 5, 6\},$$

$$\mathcal{T}_1 = \{\{0, 1, 3\}, \{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 0\}, \{5, 6, 1\}, \{6, 0, 2\}\}.$$

Note that (S_1, \mathcal{T}_1) is a $\text{STS}(7)$ which is equivalent to a K_3 -decomposition of K_7 shown in Figure 1.5 (\mathcal{C}_3).

$$(ii) S_2 = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$$

$$\begin{aligned} \mathcal{T}_2 = \{ & \{0, 1, 2\}, \{3, 4, 5\}, \{6, 7, 8\}, \{0, 3, 6\}, \{1, 4, 7\}, \{2, 5, 8\} \\ & \{0, 4, 8\}, \{1, 5, 6\}, \{2, 3, 7\}, \{0, 5, 7\}, \{1, 3, 8\}, \{2, 4, 6\}\}. \end{aligned}$$

Note that (S_2, \mathcal{T}_2) is a $\text{STS}(9)$.

In 1847, Kirkman solved the problem regarding to a positive integers n such that a Steiner triple system of order n exists. The reader may see more details in [7].

Theorem 2.2. [7] *An STS(n) exists if and only if $n \equiv 1, 3 \pmod{6}$.*

More studies have been conducted on STS. Some of them are to determine whether STS is cyclic. Recall that an *automorphism* of an STS (S, \mathcal{T}) is a bijection $\alpha : S \rightarrow S$ such that $t = \{x, y, z\} \in \mathcal{T}$ if and only if $\alpha(t) = \{\alpha(x), \alpha(y), \alpha(z)\}$. An STS(n) is *cyclic* if there is an automorphism which is a permutation consisting of a single cycle of length n . Note that $(\mathbb{Z}_n, \mathcal{T})$ is a cyclic STS(n) if clicking gives a permutation of \mathcal{T} .

Note that in Example 2.1, (S_1, \mathcal{T}_1) is a cyclic STS(7) because clicking is a permutation on \mathcal{T}_1 . On the other hand, (S_2, \mathcal{T}_2) is a STS(9) but it is not cyclic.

Next, we would like to introduce *Heffter's difference problems* [6] which will be a key to construct a cyclic Steiner triple system. In particular, the Heffter's problems concern the partition of a given set into *difference triples*. In details, for each integer n , a *difference triple* is a subset of 3 distinct elements of $\{1, 2, 3, \dots, k-1\}$ such that either

- (i) their sum is $0 \pmod{k}$, or
- (ii) one element is the sum of the other two \pmod{k} .

In 1896, Heffter [6] posed the following problems which are known as **Heffter's difference problems** which compose of two difference problems:

(1) Let $n = 6m + 1$. Is it possible to partition the set $\left\{1, 2, 3, \dots, \frac{n-1}{2} = 3m\right\}$ into difference triples?

(2) Let $n = 6m + 3$. Is it possible to partition the set $\left\{1, 2, 3, \dots, \frac{n-1}{2} = 3m + 1\right\} \setminus \left\{\frac{n}{3} = 2m + 1\right\}$ into difference triples?

For more details regarding these problems, see [7]. We mention here some results regarding to the above problems in Example 2.3.

Example 2.3. Here are some certain solutions to the Heffter's first difference problem.

(i) For $n = 7$, the solution is $\{\{1, 2, 3\}\}$.

(ii) For $n = 13$, the solution is $\{\{1, 3, 4\}, \{2, 5, 6\}\}$.

The following are some solutions to the Heffter's second difference problem.

(iii) For $n = 9$, the set we have to partition is $\{1, 2, 4\}$ which is not a difference triple. Therefore, there is no solution in this case.

(iv) For $n = 15$, the only solution is $\{\{1, 3, 4\}, \{2, 6, 7\}\}$. \square

In 1939, Peltesohn [8] completely solved the Heffter's difference problems except when $n = 9$, which is shown that the solution does not exist. Peltesohn's solutions can be used to prove the existence of a cyclic STS(n) which is precisely stated in Theorem 2.4.

Theorem 2.4. [7] *For all $n \equiv 1, 3 \pmod{6}$ and $n \neq 9$, there exists a cyclic STS(n).*

In order to construct a cyclic STS(n), we introduce the notion of a base block. Given a difference triple $\{x, y, z\}$, we define the corresponding *base block* to be the triple $\{0, x, x + y\}$. Graphically, observe that the base block $\{0, x, x + y\}$ is a 3-clique $K_3\{0, x, x + y\}$ with the set of edge lengths $\{x, y, z\}$ as illustrated in Figure 2.1. The base blocks corresponding to difference triples in the solution to Heffter's difference problems are then used to construct a cyclic STS(n).

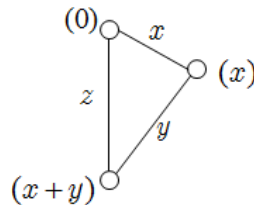


Figure 2.1: Base block $K_3\{0, x, x + y\}$ corresponding to a difference triple $\{x, y, z\}$

According to a solution of Heffter's difference problems, Examples 2.5, 2.6 and 2.7 illustrate the construction of a cyclic K_3 -decomposition of K_n , which is equivalent to cyclic $\text{STS}(n)$. Recall that for $K_n = C_n^k$ when $k \geq \lfloor \frac{n}{2} \rfloor$, we define the length function of edges in K_n as in Definition 1.2.

Example 2.5. We consider the complete graph K_7 with $V(K_7) = \mathbb{Z}_7$.

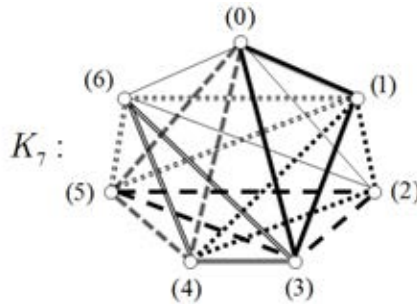


Figure 2.2: The complete graph K_7

The set $\{1, 2, 3\}$ is the set of all edge lengths of K_7 , which is itself a difference triple. Therefore, the solution to Heffter's difference problem when $n = 7$ is $\{\{1, 2, 3\}\}$. We obtain the corresponding base block $K_3\{0, 1, 3\}$ with the set of edge lengths $\{1, 2, 3\}$. As clicking preserves the length of edges and K_7 contains 7 edges of each length 1, 2 and 3, we have a cyclic K_3 -decomposition \mathcal{C}_1 of K_7 where

$$\mathcal{C}_1 = \{K_3\{i, i+1, i+3\} : 0 \leq i \leq 6\}$$

after clicking this base block 6 times.

Equivalently, $(V(K_7), \mathcal{T}_1)$ is a cyclic $\text{STS}(7)$ where

$$\mathcal{T}_1 = \{\{i, i+1, i+3\} : 0 \leq i \leq 6\}.$$

□

Example 2.6. We consider the complete graph K_{13} with $V(K_{13}) = \mathbb{Z}_{13}$.

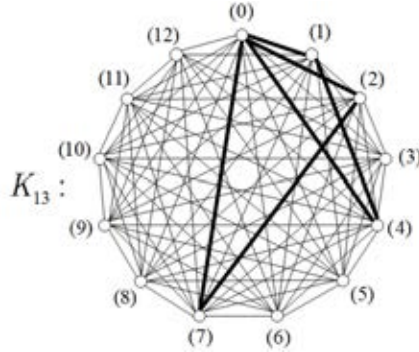


Figure 2.3: The complete graph K_{13}

The set $\{1, 2, 3, 4, 5, 6\}$ is the set of edge lengths of K_{13} . A solution to Heffter's difference problem when $n = 13$ is $\{\{1, 3, 4\}, \{2, 5, 6\}\}$. Consequently, we obtain the base blocks $K_3\{0, 1, 4\}$ and $K_3\{0, 2, 7\}$ with the set of edge lengths $\{1, 3, 4\}$ and $\{2, 5, 6\}$, respectively. Those two base blocks are shown in Figure 2.3. Note that edges of those two 3-cliques contain all edge lengths that occur in K_{13} . Also, observe that K_{13} contains 13 edges of each length $i \in \{1, 2, 3, 4, 5, 6\}$. By clicking each base block 12 times, we obtain that

$$\mathcal{C}_2 = \{K_3\{i, i+1, i+4\}, K_3\{i, i+2, i+7\} : 0 \leq i \leq 12\}$$

which is a cyclic K_3 -decomposition of K_{13} .

Equivalently, $(V(K_{13}), \mathcal{T}_2)$ is a cyclic STS(13), where

$$\mathcal{T}_2 = \{\{i, i+1, i+4\}, \{i, i+2, i+7\} : 0 \leq i \leq 12\}.$$

□

Example 2.7. We consider the complete graph K_{15} with $V(K_{15}) = \mathbb{Z}_{15}$.

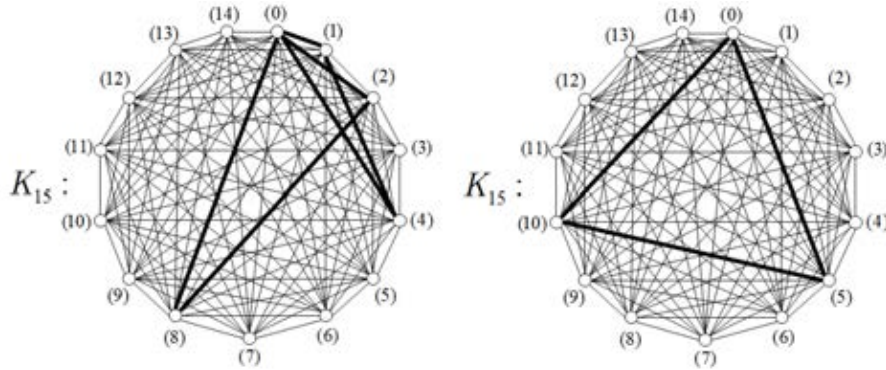


Figure 2.4: The complete graph K_{15}

The set of edge lengths of K_{15} is $\{1, 2, 3, 4, 5, 6, 7\}$. A solution to Heffter's second difference problem when $n = 15$ is $\{\{1, 3, 4\}, \{2, 6, 7\}\}$. Thus the corresponding base blocks are $K_3\{0, 1, 4\}$ and $K_3\{0, 2, 8\}$ which are shown in Figure 2.4 on the left.

Note that edges of length 5 are not included in any base block. However, we can form 3-cliques using these edges, consequently, there must be one more base block, namely $K_3\{0, 5, 10\}$, which is shown on the right of Figure 2.4.

Thus, all the base blocks in this case are $K_3\{0, 1, 4\}, K_3\{0, 2, 8\}$ and $K_3\{0, 5, 10\}$.

Therefore,

$$\mathcal{C}_3 = \{K_3\{i, i+1, i+4\}, K_3\{i, i+2, i+8\}, K_3\{j, j+5, j+10\} : 0 \leq i \leq 14, 0 \leq j \leq 4\}$$

which is a cyclic K_3 -decomposition of K_{15} that we are looking for.

Equivalently, $(V(K_{15}), \mathcal{T}_3)$ is a cyclic STS(15) where

$$\mathcal{T}_3 = \{\{i, i+1, i+4\}, \{i, i+2, i+8\}, \{j, j+5, j+10\} : 0 \leq i \leq 14, 0 \leq j \leq 4\}.$$

□

CHAPTER III

CYCLIC CLIQUE DECOMPOSITIONS OF C_n^k

In this chapter, we explain the method to construct cyclic clique decompositions of C_n^k . First, we classify certain cliques of order at most 4. Finally, we can construct cyclic clique decompositions of C_n^k .

Recall that the k -power of an n -cycle, denoted by C_n^k , is an n -cycle with additional edges for each pair of vertices u and v such that $d_{C_n}(u, v) \leq k$. It is easy to see that $C_n^k = K_n$ for $k \geq \lfloor \frac{n}{2} \rfloor$ since $\text{diam}(C_n^k) = \lfloor \frac{n}{2} \rfloor$. In this case, the complete graph K_n itself can be considered as the n -clique and therefore, we have $cp(K_n) = 1$ for any natural number n . In order to focus on the case that C_n^k is not the complete graph, from now on, we write C_n^k to denote the k -power of n -cycle where $k < \lfloor \frac{n}{2} \rfloor$. We let $V(C_n^k) = V(C_n) = \mathbb{Z}_n$ and $E(C_n^k) = E(C_n) \cup \{uv : u, v \in V(C_n) \text{ and } d_{C_n}(u, v) \leq k\}$.

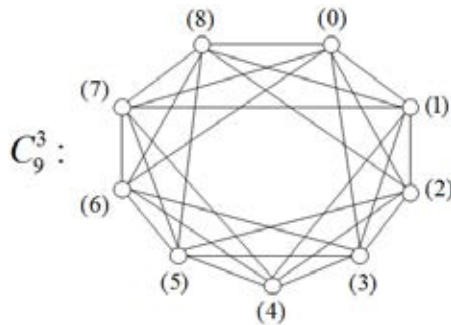


Figure 3.1: The 3-power of a 9-cycle C_9^3

Remark 3.1. Let C_n^k be a k -power of an n -cycle with $k < \lfloor \frac{n}{2} \rfloor$. We have the following observations:

- (i) There are kn edges in C_n^k .
- (ii) The set of all lengths of edges in C_n^k is $\{1, 2, 3, \dots, k\}$.
- (iii) For each $i \in \{1, 2, 3, \dots, k\}$, there are n edges of length i in C_n^k .

As in Chapter II, the Heffter's difference problems can be used to decompose K_n into 3-cliques. In this work, we generalize such an idea and establish the construction method for cyclic clique decompositions of C_n^k into cliques of order at most 4.

3.1 Pure Difference Tuple Problem

Given a natural number $k < \lfloor \frac{n}{2} \rfloor$, from Remark 3.1, there are n edges of length i in C_n^k for each $i \in \{1, 2, 3, \dots, k\}$. We would like to find base blocks containing cliques of order at most 4 such that there is only one edge of length i in these base blocks. Afterward, we can use them to construct a cyclic clique decomposition of C_n^k . In order to find such base blocks, we generalize the Heffter's difference problems by allowing a partition of the set $\{1, 2, 3, \dots, k\}$ of edge lengths of C_n^k to contain *pure difference 3-tuples*, *pure difference 6-tuples* and *singletons* which we define as follows.

Definition 3.2. Let k be a positive integer.

(i) A *pure difference 3-tuple* is an ordered triple (x, y, z) of 3 distinct elements of $\{1, 2, 3, \dots, k\}$ such that $z = x + y$. In particular, a pure difference 3-tuple is a triple that can be written in the form $(x, y, x + y)$.

(ii) A *pure difference 6-tuple* is an ordered 6-tuple of the form

$$(x, y, z, x + y, y + z, x + y + z)$$

containing 6 distinct elements of $\{1, 2, 3, \dots, k\}$.

(iii) A *singleton* is a 1-tuple containing x , denoted by $\{x\}$, where $x \in \{1, 2, 3, \dots, k\}$.

Our construction of a cyclic clique decomposition of C_n^k into cliques of order at most 4 will rely on the partition of the set of edge lengths of C_n^k into certain subsets. Inspired by the Heffter's difference problems, we pose the following problem in order to investigate certain partition of the set. This problem plays an important role in our work.

Pure Difference Tuple Problem of order k . *Let $k \geq 3$, is it possible to partition the set $\{1, 2, 3, \dots, k\}$ into pure difference 6-tuples, pure difference 3-tuples, or singletons where the number of elements in the partition is the minimum among all such partitions?*

We denote $\text{PDT}(k)$ as a solution to Pure Difference Tuple Problem of order k .

Example 3.3. Here, we give some examples of partitions of the set $\{1, 2, 3, \dots, k\}$ into pure difference 6-tuples or pure difference 3-tuples or singletons and therefore, solutions to Pure Difference Tuple Problem of order k .

$$(i) \text{ Let } A = \{\{i\} : 1 \leq i \leq 9\},$$

$$B = \{(1, 3, 2, 4, 5, 6), \{7\}, \{8\}, \{9\}\} \text{ and}$$

$$C = \{(3, 2, 4, 5, 6, 9), (1, 7, 8)\}$$

be ones of the partitions of the set $\{1, 2, 3, \dots, 9\}$.

Observe that C has smaller size than the sizes of A and B . In this case, C is a $\text{PDT}(9)$ because we need at least 2 of such tuples to form a partition of the set $\{1, 2, 3, \dots, 9\}$.

(ii) Given D_1, D_2 and D_3 as follows:

$$D_1 = \{(1, 16, 6, 17, 22, 23), (2, 12, 7, 14, 19, 21), (3, 5, 10, 8, 15, 18), (4, 9, 11, 13, 20, 24)\},$$

$$D_2 = \{(1, 6, 16, 7, 22, 23), (2, 12, 5, 14, 17, 19), (3, 10, 8, 13, 18, 21), (4, 11, 9, 15, 20, 24)\}$$

and

$$D_3 = \{(1, 16, 6, 17, 22, 23), (5, 2, 12, 7, 14, 19), (3, 10, 8, 13, 18, 21), (4, 11, 9, 15, 20, 24)\}.$$

Then, D_1, D_2 and D_3 are ones of possible partitions for $k = 24$. Note that they are $\text{PDT}(24)$. Therefore, a solution to Pure Difference Tuple Problem might not be unique.

(iii) One of the possible partition for $k = 18$ is

$$\{(1, 6, 5, 7, 11, 12), (2, 13, 3, 15, 16, 18), (4, 10, 14), (8, 9, 17)\}.$$

Surprisingly, it is a $\text{PDT}(18)$ which will be proved in Theorem 3.10. \square

Let D be a partition of the set $\{1, 2, 3, \dots, k\}$ into a pure difference 6-tuples, b pure difference 3-tuples and c singletons. Note that if $a = \left\lfloor \frac{k}{6} \right\rfloor$, $b = \left\lfloor \frac{k - 6a}{3} \right\rfloor$ and $c = k - 6a - 3b$, then D is certainly a PDT(k). However, the converse does not hold as we see in Example 3.3 (iii).

3.2 Classification of 2-cliques, 3-cliques and 4-cliques

In this section, we define the *base blocks* corresponding to a pure difference 6-tuple, a pure difference 3-tuple and a singleton. This set of base blocks obtained from a solution PDT(k) will be used to form our cyclic clique decomposition of C_n^k into cliques of order at most 4 in our construction.

We introduce a classification of 2-cliques, 3-cliques and 4-cliques according to the parity of their edge lengths. For convenience, we call an edge with odd length (and even length) an *odd edge* (and *even edge*, respectively).

3.2.1 2-cliques

For a singleton $\{x\}$, where $x \in \{1, 2, 3, \dots, k\}$, we define the corresponding base block to be the 2-clique $K_2\{0, x\}$ in C_n^k with length x . Such graph is shown in Figure 3.2.

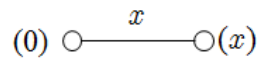


Figure 3.2: $K_2\{0, x\}$ corresponding to a singleton $\{x\}$

3.2.2 3-cliques

For a pure difference 3-tuple $(x, y, x + y)$, we define the corresponding base block to be the 3-clique $K_3\{0, x, x + y\}$ which contains edges of lengths x, y and $x + y$ as illustrated in Figure 3.3.

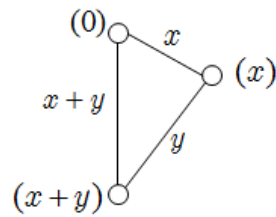


Figure 3.3: $K_3\{0, x, x + y\}$ corresponding to a pure difference 3-tuple $(x, y, x + y)$

By considering each case which depends on the parities of elements x and y in a pure difference triple $(x, y, x + y)$, there are four different cases of a 3-clique as shown in Table 3.1 where we represent odd edges and even edges as dash lines and bold lines, respectively.

Remark 3.4. There are only two types of base blocks corresponding to pure difference 3-tuples as follows:

- 1) 3-clique of type O^2E containing 2 odd edges and 1 even edge, and
- 2) 3-clique of type E^3 containing all 3 even edges.

x	y	$x + y$	Graph $K_3\{0, x, x + y\}$	Type
Even	Odd	Odd		O^2E
Odd	Even	Odd		O^2E
Odd	Odd	Even		O^2E
Even	Even	Even		E^3

Table 3.1: Types of $K_3\{0, x, x + y\}$

3.2.3 4-cliques

For a pure difference 6-tuple $(x, y, z, x + y, y + z, x + y + z)$, we define the corresponding base block to be the 4-clique $K_4\{0, x, x + y, x + y + z\}$ illustrated in Figure 3.4. The edges of length x, y, z and $x + y + z$ are called *border edges* and the edges of length $x + y$ and $y + z$ are called *diagonal edges*. Note that there are exactly four 3-cliques contained in the 4-clique $K_4\{0, x, x + y, x + y + z\}$ as its subgraphs.

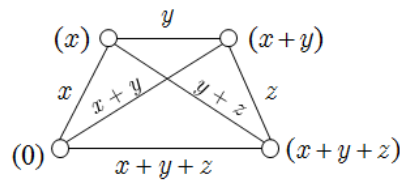


Figure 3.4: $K_4\{0, x, x + y, x + y + z\}$ corresponding to a pure difference 6-tuple $(x, y, z, x + y, y + z, x + y + z)$

By considering each case which depends on the parities of elements x, y and z in a pure difference 6-tuple $(x, y, z, x + y, y + z, x + y + z)$, we have eight different forms of a 4-clique as shown in Table 3.2.

Remark 3.5. There are only three types of base blocks corresponding to pure difference 6-tuples as follows:

- 1) 4-clique of type E^6 containing all 6 even edges,
- 2) 4-clique of type O^4E^2 containing 4 odd edges and 2 even edges, and
- 3) 4-clique of type O^3E^3 containing 3 odd edges and 3 even edges.

x	y	z	$x+y$	$y+z$	$x+y+z$	Graph $K_4\{0, x, x+y, x+y+z\}$	Type
Odd	Odd	Odd	Even	Even	Odd		$O^4 E^2$
Odd	Odd	Even	Even	Odd	Even		$O^3 E^3$
Odd	Even	Odd	Odd	Odd	Even		$O^4 E^2$
Odd	Even	Even	Odd	Even	Odd		$O^3 E^3$
Even	Odd	Odd	Odd	Even	Even		$O^3 E^3$
Even	Odd	Even	Odd	Odd	Odd		$O^4 E^2$
Even	Even	Odd	Even	Odd	Odd		$O^3 E^3$
Even	Even	Even	Even	Even	Even		E^6

Table 3.2: Types of $K_4\{0, x, x+y, x+y+z\}$

For Table 3.2, we obtain some properties of base blocks corresponding to pure difference 6-tuples concluded in the next proposition.

Proposition 3.6. *Let $K_4(V)$ be the base block corresponding to a pure difference 6-tuple.*

(i) *If $K_4(V)$ is of type O^3E^3 , then it always contains exactly one 3-clique subgraph of type E^3 .*

(ii) *If $K_4(V)$ is of type O^4E^2 , then two even edges must be both diagonal edges or two opposite border edges.*

3.3 Cyclic Clique Decompositions of C_n^k into Cliques of Order at Most 4 for $3 \leq k \leq 26$

From now on, in this work, B_k denotes the corresponding set of base blocks obtained from a $\text{PDT}(k)$. Note that if a $\text{PDT}(k)$ contains a, b and c pure difference 6-tuples, pure difference 3-tuples and singletons, respectively, then B_k contains a 4-cliques, b 3-cliques and c 2-cliques.

We say that a cyclic clique decomposition of C_n^k into cliques of order at most 4 is *optimal* if it contains the minimum number of cliques among all such possible clique decompositions obtained by clicking the base blocks. Recall that clicking means applying the isomorphism $i \rightarrow i + 1$ on $V(C_n^k)$.

Our study establishes the condition on integers k and n to ensure that a cyclic clique decomposition of C_n^k into cliques of order at most 4 obtained from $\text{PDT}(k)$ is optimal.

Theorem 3.7. *Let k and n be integers such that $k \geq 3$ and $k < \lfloor \frac{n}{2} \rfloor$. If $n > 3k$, then a solution $\text{PDT}(k)$ yields an optimal cyclic clique decomposition of C_n^k into cliques of order at most 4.*

Proof. In order to find a cyclic clique decomposition of C_n^k into cliques of order at most 4, we first partition $\{1, 2, 3, \dots, k\}$, the set of edge lengths of C_n^k , into pure difference 6-tuples, pure difference 3-tuples, or singletons. Afterward, we obtain the corresponding set of base blocks containing exactly one edge of length $i \in \{1, 2, 3, \dots, k\}$. Since clicking preserves edge lengths of a graph and C_n^k has

n edges of length i for each $i \in \{1, 2, 3, \dots, k\}$, it follows that clicking each base block $n - 1$ times yields the desired decomposition of C_n^k .

Our construction method can be summarized as the following steps:

Step 1 Obtain a solution, $\text{PDT}(k)$, of the Pure Difference Tuple Problem.

Step 2 Get the set of base blocks B_k corresponding to $\text{PDT}(k)$ in Step 1.

Step 3 Apply clicking $n - 1$ times to each base block in B_k and get the desired decomposition of C_n^k .

Therefore, a $\text{PDT}(k)$ yields a cyclic clique decomposition of C_n^k into cliques of order at most 4.

Next, we show that such a decomposition is optimal when $n > 3k$.

First, we claim that the union of base blocks must not contain repeated edge lengths. As an optimal cyclic clique decomposition is obtained from clicking base blocks $n - 1$ times, the only possible base blocks of C_n^k with repeated edge lengths must be as given in Figure 3.5.

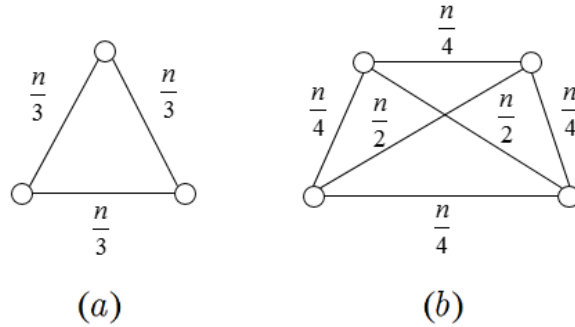


Figure 3.5: Base blocks with repeated edge lengths

However, since $n > 3k$ and $k < \lfloor \frac{n}{2} \rfloor$, C_n^k has neither edge of length $\frac{n}{3}$ nor $\frac{n}{2}$. Thus the union of all base blocks cannot contain repeated edge lengths. In other words, the set of edge lengths of each base block form a partition of $\{1, 2, 3, \dots, k\}$ which can be obtained from a $\text{PDT}(k)$.

It remains to make sure that some 2-cliques and 3-cliques obtained from $\text{PDT}(k)$ cannot form a new 3-clique nor a new 4-clique in C_n^k . Otherwise, we will obtain a smaller set of base blocks. In order to achieve that, we claim that the following two properties hold when $n > 3k$.

(i) C_n^k contains no 3-clique $K_3\{0, x, x + y\}$ with edges of the length x, y and z where $x + y + z = n$.

(ii) C_n^k contains no 4-clique $K_4\{0, x, x + y, x + y + z\}$ with border edges of the length x, y, z and w where $x + y + z + w = n$.

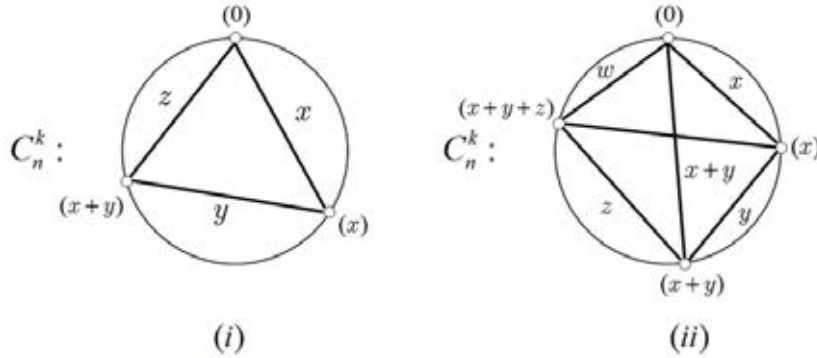


Figure 3.6: 3-clique and 4-clique mentioned in (i) and (ii), respectively

We prove (i) by using contrapositive arguments. Suppose that $K_3\{0, x, x + y\}$ is a clique of C_n^k with edges of the length x, y and z where $x + y + z = n$ as in Figure 3.6 (i). Since x, y and z are all at most k , we have $n = x + y + z \leq k + k + k \leq 3k$. Thus, the property (i) holds if $n > 3k$.

Next, we prove (ii), suppose that $K_4\{0, x, x + y, x + y + z\}$ is a clique of C_n^k with border edges of length x, y, z and w where $x + y + z + w = n$ as in Figure 3.6 (ii). Then, the diagonal edges joining vertices 0 and $x + y$ is of the length at most k . Since $z \leq k$ and $w \leq k$, we have $n = (x + y) + z + w \leq k + k + k \leq 3k$. Thus, the property (ii) holds if $n > 3k$.

Hence, a solution $\text{PDT}(k)$ yields an optimal cyclic clique decomposition of C_n^k into cliques of order at most 4. \square

Consequently, it is crucial to determine a solution $\text{PDT}(k)$. In this work, we have succeeded in determining $\text{PDT}(k)$ for $3 \leq k \leq 26$. By Theorem 3.7, this yields an optimal cyclic clique decomposition of C_n^k into cliques of order at most 4 for $3 \leq k \leq 26$ and all natural numbers $n > 3k$.

From now on, we let $\mathcal{C}_k(n)$ be such an optimal cyclic clique decomposition of C_n^k into cliques of order at most 4 when $3 \leq k \leq 26$ and $n > 3k$.

Next, we give a solution $\text{PDT}(k)$ and $\mathcal{C}_k(n)$ for $3 \leq k \leq 26$ and all natural numbers $n > 3k$.

3.3.1 $\text{PDT}(k)$ and $\mathcal{C}_k(n)$ for $3 \leq k \leq 11$

A $\text{PDT}(k)$ and an optimal cyclic clique decomposition $\mathcal{C}_k(n)$ of C_n^k into cliques of order at most 4 for $3 \leq k \leq 11$ and all natural numbers $n > 3k$ are given below.

Case $k = 3, 4, 5$

When $k = 3$, we have $\text{PDT}(3) = \{(1, 2, 3)\}$ and hence, the set of corresponding base block is $B_3 = \{K_3\{0, 1, 3\}\}$. The base block in B_3 is shown in Figure 3.7.

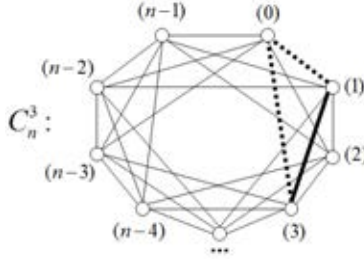


Figure 3.7: The base block $K_3\{0, 1, 3\}$ in C_n^3

Apply clicking $n - 1$ times to the base block in B_3 , we obtain that

$$\mathcal{C}_3(n) = \{K_3\{i, i + 1, i + 3\} : 0 \leq i \leq n - 1\}$$

is an optimal cyclic clique decomposition of C_n^3 into cliques of order at most 4.

Next, by adding the singleton $\{4\}$ to $\text{PDT}(3)$, we obtain $\text{PDT}(4)$. Similarly, $\text{PDT}(5)$ is obtained by adding the singleton $\{5\}$ to $\text{PDT}(4)$. Hence, we have

$$\text{PDT}(4) = \{(1, 2, 3), \{4\}\} \text{ and}$$

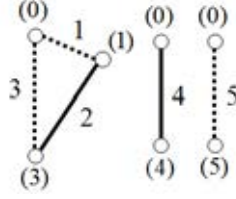
$$\text{PDT}(5) = \{(1, 2, 3), \{4\}, \{5\}\}.$$

Consequently, the corresponding sets of base blocks are

$$B_4 = \{K_3\{0, 1, 3\}, K_2\{0, 4\}\} \text{ and}$$

$$B_5 = \{K_3\{0, 1, 3\}, K_2\{0, 4\}, K_2\{0, 5\}\}.$$

The base blocks in B_5 are shown in Figure 3.8.

Figure 3.8: The base blocks in B_5

Therefore, we have

$$\mathcal{C}_4(n) = \{K_3\{i, i+1, i+3\}, K_2\{i, i+4\} : 0 \leq i \leq n-1\} \text{ and}$$

$$\mathcal{C}_5(n) = \{K_3\{i, i+1, i+3\}, K_2\{i, i+4\}, K_2\{i, i+5\} : 0 \leq i \leq n-1\}.$$

Since the set of base blocks B_k are determined by a solution $\text{PDT}(k)$, from now on, we only show a solution $\text{PDT}(k)$ and the decomposition $\mathcal{C}_k(n)$ of C_n^k obtained from B_k .

Case $k = 6, 7, 8$

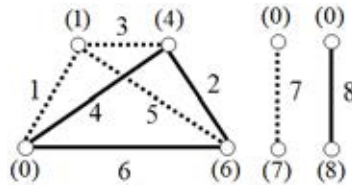
When $k = 6$, we have $\text{PDT}(6) = \{(1, 3, 2, 4, 5, 6)\}$ which yields the corresponding set of base blocks $B_6 = \{K_4\{0, 1, 4, 6\}\}$. Hence,

$$\mathcal{C}_6(n) = \{K_4\{i, i+1, i+4, i+6\} : 0 \leq i \leq n-1\}.$$

Using $\text{PDT}(6)$, we obtain

$$\text{PDT}(7) = \{(1, 3, 2, 4, 5, 6), \{7\}\} \text{ and}$$

$$\text{PDT}(8) = \{(1, 3, 2, 4, 5, 6), \{7\}, \{8\}\}.$$

Figure 3.9: The base blocks corresponding to elements in $\text{PDT}(8)$

Therefore, we have

$$\mathcal{C}_7(n) = \{K_4\{i, i+1, i+4, i+6\}, K_2\{i, i+7\} : 0 \leq i \leq n-1\} \text{ and}$$

$$\mathcal{C}_8(n) = \{K_4\{i, i+1, i+4, i+6\}, K_2\{i, i+7\}, K_2\{i, i+8\} : 0 \leq i \leq n-1\}.$$

Case $k = 9, 10, 11$

When $k = 9$, we have $\text{PDT}(9) = \{(3, 2, 4, 5, 6, 9), (1, 7, 8)\}$ which yields the corresponding set of base blocks $B_9 = \{K_4\{0, 3, 5, 9\}, K_3\{0, 1, 8\}\}$. Hence,

$$\mathcal{C}_9(n) = \{K_4\{i, i+3, i+5, i+9\}, K_3\{i, i+1, i+8\} : 0 \leq i \leq n-1\}.$$

Using $\text{PDT}(9)$, we obtain

$$\text{PDT}(10) = \{(3, 2, 4, 5, 6, 9), (1, 7, 8), \{10\}\} \text{ and}$$

$$\text{PDT}(11) = \{(3, 2, 4, 5, 6, 9), (1, 7, 8), \{10\}, \{11\}\}.$$

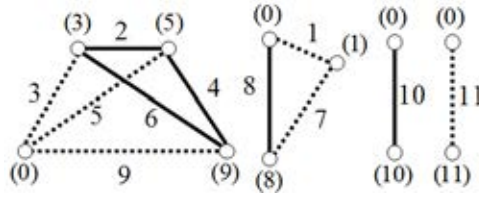


Figure 3.10: The base blocks corresponding to elements in $\text{PDT}(11)$

Therefore, we have

$$\mathcal{C}_{10}(n) = \{K_4\{i, i+3, i+5, i+9\}, K_3\{i, i+1, i+8\},$$

$$K_2\{i, i+10\} : 0 \leq i \leq n-1\} \text{ and}$$

$$\mathcal{C}_{11}(n) = \{K_4\{i, i+3, i+5, i+9\}, K_3\{i, i+1, i+8\},$$

$$K_2\{i, i+10\}, K_2\{i, i+11\} : 0 \leq i \leq n-1\}.$$

3.3.2 $\text{PDT}(k)$ and $\mathcal{C}_k(n)$ for $k = 12, 18$

When $k = 12, 18$, it is natural to try to partition the set $\{1, 2, 3, \dots, k\}$ into $\frac{k}{6}$ pure difference 6-tuples. However, the following two lemmas and Theorem 3.10 show that such a partition does not exist. To achieve that, we begin with the investigation of a partition of the set $\left\{1, 2, 3, \dots, \frac{k}{2}\right\}$ into pure difference 3-tuples in Lemma 3.8.

Lemma 3.8. *Let k be a positive integer. If $k \equiv 12, 18 \pmod{24}$, then there is no partition of the set $\left\{1, 2, 3, \dots, \frac{k}{2}\right\}$ into $\frac{k}{6}$ pure difference 3-tuples.*

Proof. For the sake of convenience, we write $k = 6a$ where $a \equiv 2, 3 \pmod{4}$.

Let $X = \left\{1, 2, 3, \dots, \frac{k}{2}\right\}$. Suppose that X can be partitioned into $\frac{k}{6}$ pure difference 3-tuples, namely $\left\{(x_i, x'_i, y_i) : i \in \left\{1, 2, 3, \dots, \frac{k}{6}\right\}\right\}$.

Thus, we have $y_i = x_i + x'_i$ for $1 \leq i \leq \frac{k}{6}$.

Now, we compute the sum of all elements in X as

$$\sum_{i=1}^{k/6} (x_i + x'_i + y_i) = \sum_{i=1}^{k/2} i = \frac{k}{4} \left(\frac{k}{2} + 1\right).$$

On the other hand, $\sum_{i=1}^{k/6} (x_i + x'_i + y_i) = \sum_{i=1}^{k/6} (y_i + y_i) = 2 \sum_{i=1}^{k/6} y_i$.

Therefore, we get $\sum_{i=1}^{k/6} y_i = \frac{k}{8} \left(\frac{k}{2} + 1\right) = \frac{3a(3a+1)}{4}$.

Next, we consider $\frac{3a(3a+1)}{4}$ where $a \equiv 2$ or $3 \pmod{4}$.

Case 1 $a \equiv 2 \pmod{4}$

We get $3a \equiv 2 \pmod{4}$ and $3a+1 \equiv 3 \pmod{4}$.

Therefore, $3a(3a+1) \equiv 2 \pmod{4}$.

Thus, we have $\sum_{i=1}^{k/6} y_i = \frac{3a(3a+1)}{4} \notin \mathbb{Z}$, which is a contradiction.

Hence, the partition of the set $\left\{1, 2, 3, \dots, \frac{k}{2}\right\}$ into $\frac{k}{6}$ pure difference 3-tuples does not exist when $k = 6a$ and $a \equiv 2 \pmod{4}$.

Case 2 $a \equiv 3 \pmod{4}$

Then, $3a \equiv 1 \pmod{4}$ and $3a+1 \equiv 2 \pmod{4}$.

That is, $3a(3a+1) \equiv 2 \pmod{4}$.

We have $\sum_{i=1}^{k/6} y_i = \frac{3a(3a+1)}{4} \notin \mathbb{Z}$, which is a contradiction.

Hence, the partition of the set $\left\{1, 2, 3, \dots, \frac{k}{2}\right\}$ into $\frac{k}{6}$ pure difference 3-tuples does not exist when $k = 6a$ and $a \equiv 3 \pmod{4}$. \square

Lemma 3.9. *Let k be a positive integer such that $k \equiv 12, 18 \pmod{24}$. The base blocks corresponding to pure difference 3-tuples and pure difference 6-tuples satisfy the followings.*

(i) *There is no $\frac{k}{6}$ base blocks which are 3-cliques of type E^3 whose sets of edge lengths form a partition of the set $\{2, 4, 6, \dots, k\}$.*

(ii) *There is no $\frac{k}{6}$ base blocks which are 4-cliques of type O^3E^3 whose sets of edge lengths form a partition of the set $\{1, 2, 3, \dots, k\}$.*

Proof. (i) Suppose the contrary, there is $\frac{k}{6}$ 3-cliques of type E^3 whose sets of edge lengths form a partition of the set $\{2, 4, 6, \dots, k\}$.

This yields a partition of $\left\{1, 2, 3, \dots, \frac{k}{2}\right\}$ into $\frac{k}{6}$ pure difference 3-tuples which contradicts the statement of Lemma 3.8.

(ii) Again, suppose the contrary, there is $\frac{k}{6}$ 4-cliques of type O^3E^3 whose sets of edge lengths form a partition of the set $\{1, 2, 3, \dots, k\}$. By Proposition 3.6 (i), each 4-clique of type O^3E^3 always contains exactly one 3-clique subgraphs of type E^3 . Hence, we obtain $\frac{k}{6}$ 3-cliques of type E^3 whose sets of edge lengths form a partition of the set $\{2, 4, 6, \dots, k\}$. However, this contradicts (i). \square

The next theorem is the key to solve the Pure Difference Tuple Problem of order $k = 12, 18$

Theorem 3.10. *If $k = 12, 18$, then there is no PDT(k) which contains $\frac{k}{6}$ pure difference 6-tuples.*

Proof. Suppose that the statement is false when $k = 12$. Then, there is a set of base blocks B_{12} containing two 4-cliques whose sets of edge lengths form a partition of $\{1, 2, 3, \dots, 12\}$.

According to our classification of 4-cliques described in Remark 3.5, there are only three types of 4-cliques, namely O^3E^3 , O^4E^2 and E^6 .

We assume that B_k contains p, q and r 4-cliques of type O^3E^3 , O^4E^2 and E^6 , respectively. Hence, $p + q + r = 2$.

Since the set $\{1, 2, 3, \dots, 12\}$ contains 6 odd integers and 6 even integers, we obtain the equation $3p + 4q = 6 = 3p + 2q + 6r$. Thus, $q = 3r$.

Therefore, we have $2 = p + q + r = p + 4r$ which forces $r = 0 = q$ and hence, $p = 2$. Consequently, we must have two 4-cliques of type O^3E^3 whose sets of edge lengths form a partition of $\{1, 2, 3, \dots, 12\}$, which contradicts Lemma 3.9 (ii).

We can obtain the result for the case $k = 18$ by the similar argument. \square

Theorem 3.10 implies that when $k = 12, 18$, any $\text{PDT}(k)$ contains at least $\frac{k}{6} + 1$ elements. Therefore, if we can find a partition of the set $\{1, 2, 3, \dots, k\}$ into $\frac{k}{6} - 1$ pure difference 6-tuples and 2 pure difference 3-tuples, then such a partition has to be a $\text{PDT}(k)$. Finally, we obtain a $\text{PDT}(12)$ and a $\text{PDT}(18)$ as shown below.

Case $k = 12$

We have

$$\text{PDT}(12) = \{(1, 8, 3, 9, 11, 12), (2, 5, 7), (4, 6, 10)\}$$

which yields the corresponding set of base blocks

$$B_{12} = \{K_4\{0, 1, 9, 12\}, K_3\{0, 2, 7\}, K_3\{0, 4, 10\}\}$$

as shown in Figure 3.11.

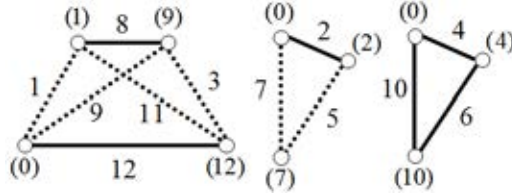


Figure 3.11: The base blocks in B_{12}

Therefore, we have

$$\begin{aligned} \mathcal{C}_{12}(n) = \{ & K_4\{i, i + 1, i + 9, i + 12\}, K_3\{i, i + 2, i + 7\}, \\ & K_3\{i, i + 4, i + 10\} : 0 \leq i \leq n - 1\}. \end{aligned}$$

Case $k = 18$

We have

$$\text{PDT}(18) = \{(1, 6, 5, 7, 11, 12), (2, 13, 3, 15, 16, 18), (4, 10, 14), (8, 9, 17)\}$$

which yields the corresponding set of base blocks

$$B_{18} = \{K_4\{0, 1, 7, 12\}, K_4\{0, 2, 15, 18\}, K_3\{0, 4, 14\}, K_3\{0, 8, 17\}\}$$

as shown in Figure 3.12.

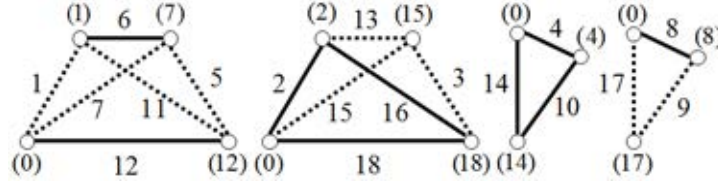


Figure 3.12: The base blocks in B_{18}

Therefore, we have

$$\begin{aligned} \mathcal{C}_{18}(n) = \{ & K_4\{i, i+1, i+7, i+12\}, K_4\{i, i+2, i+15, i+18\}, \\ & K_3\{i, i+4, i+14\}, K_3\{i, i+8, i+17\} : 0 \leq i \leq n-1\}. \end{aligned}$$

3.3.3 PDT(k) and $\mathcal{C}_k(n)$ for $k = 13, 14, 19, 20$

According to Subsection 3.3.1, a solution PDT(k) for each of the cases $k = 3, 6$ or 9 yields a solution PDT($k+1$) and a PDT($k+2$). However, a PDT(13) and PDT(14) cannot be obtained from a PDT(12). Similarly, PDT(19) and PDT(20) cannot be obtained from a PDT(18). On the other hand, a PDT(13) and a PDT(19) can yield a PDT(14) and a PDT(20), respectively.

Case $k = 13, 14$

We have

$$\text{PDT}(13) = \{(3, 1, 8, 4, 9, 12), (2, 5, 6, 7, 11, 13), \{10\}\}$$

which yields the corresponding set of base blocks

$$B_{13} = \{K_4\{0, 3, 4, 12\}, K_4\{0, 2, 7, 13\}, K_2\{0, 10\}\}.$$

Therefore, we have

$$\begin{aligned} \mathcal{C}_{13}(n) = \{ & K_4\{i, i+3, i+4, i+12\}, K_4\{i, i+2, i+7, i+13\}, \\ & K_2\{i, i+10\} \mid 0 \leq i \leq n-1\}. \end{aligned}$$

Using PDT(13), we obtain

$$\text{PDT}(14) = \{(3, 1, 8, 4, 9, 12), (2, 5, 6, 7, 11, 13), \{10\}, \{14\}\}.$$

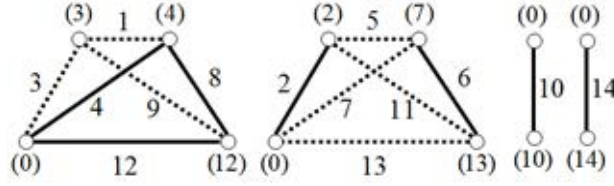


Figure 3.13: The base blocks corresponding to elements in PDT(14)

Therefore, we have

$$\mathcal{C}_{14}(n) = \{K_4\{i, i + 3, i + 4, i + 12\}, K_4\{i, i + 2, i + 7, i + 13\}, \\ K_2\{i, i + 10\}, K_2\{i, i + 14\} \mid 0 \leq i \leq n - 1\}.$$

Case $k = 19, 20$

We have

$$\text{PDT}(19) = \{(1, 7, 10, 8, 17, 18), (2, 4, 9, 6, 13, 15), (3, 11, 5, 14, 16, 19), \{12\}\}$$

which yields the corresponding set of base blocks

$$B_{19} = \{K_4\{0, 1, 8, 18\}, K_4\{0, 2, 6, 15\}, K_4\{0, 3, 14, 19\}, K_2\{0, 12\}\}.$$

Therefore, we have

$$\mathcal{C}_{19}(n) = \{K_4\{i, i + 1, i + 8, i + 18\}, K_4\{i, i + 2, i + 6, i + 15\}, \\ K_4\{i, i + 3, i + 14, i + 19\}, K_2\{i, i + 12\} : 0 \leq i \leq n - 1\}.$$

Using PDT(20), we obtain

$$\text{PDT}(20) = \{(1, 7, 10, 8, 17, 18), (2, 4, 9, 6, 13, 15), (3, 11, 5, 14, 16, 19), \{12\}, \{20\}\}.$$

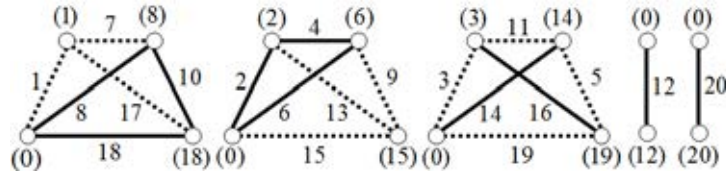


Figure 3.14: The base blocks corresponding to elements in PDT(20)

Therefore, we have

$$\mathcal{C}_{20}(n) = \{K_4\{i, i + 1, i + 8, i + 18\}, K_4\{i, i + 2, i + 6, i + 15\}, \\ K_4\{i, i + 3, i + 14, i + 19\}, K_2\{i, i + 12\}, K_2\{i, i + 20\} : 0 \leq i \leq n - 1\}.$$

3.3.4 PDT(k) and $\mathcal{C}_k(n)$ for $k = 15, 16, 17, 21, 22, 23$

Similarly, we determine a PDT(k) when $k = 15, 21$, which will consequently yield a PDT($k + 1$) and PDT($k + 2$). Note that a solution PDT(k) when $k = 15, 21$ contains $\left\lfloor \frac{k}{6} \right\rfloor$ pure difference 6-tuples and one pure difference 3-tuple.

Case $k = 15, 16, 17$

We have

$$\text{PDT}(15) = \{(5, 1, 8, 6, 9, 14), (2, 10, 3, 12, 13, 15), (4, 7, 11)\}$$

which yields the corresponding set of base blocks

$$B_{15} = \{K_4\{0, 5, 6, 14\}, K_4\{0, 2, 12, 15\}, K_3\{0, 4, 11\}\}.$$

Therefore, we have

$$\begin{aligned} \mathcal{C}_{15}(n) = \{ & K_4\{i, i + 5, i + 6, i + 14\}, K_4\{i, i + 2, i + 12, i + 15\}, \\ & K_3\{i, i + 4, i + 11\} : 0 \leq i \leq n - 1\}. \end{aligned}$$

Using PDT(15), we obtain

$$\text{PDT}(16) = \{(5, 1, 8, 6, 9, 14), (2, 10, 3, 12, 13, 15), (4, 7, 11), \{16\}\} \text{ and}$$

$$\text{PDT}(17) = \{(5, 1, 8, 6, 9, 14), (2, 10, 3, 12, 13, 15), (4, 7, 11), \{16\}, \{17\}\}.$$

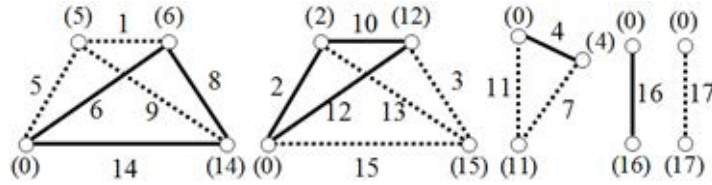


Figure 3.15: The base blocks corresponding to elements in PDT(17)

Therefore, we have

$$\mathcal{C}_{16}(n) = \{K_4\{i, i + 5, i + 6, i + 14\}, K_4\{i, i + 2, i + 12, i + 15\},$$

$$K_3\{i, i + 4, i + 11\}, K_2\{i, i + 16\} : 0 \leq i \leq n - 1\} \text{ and}$$

$$\mathcal{C}_{17}(n) = \{K_4\{i, i + 5, i + 6, i + 14\}, K_4\{i, i + 2, i + 12, i + 15\},$$

$$K_3\{i, i + 4, i + 11\}, K_2\{i, i + 16\}, K_2\{i, i + 17\} : 0 \leq i \leq n - 1\}.$$

Case $k = 21, 22, 23$

We have

$$\text{PDT}(21) = \{(1, 13, 6, 14, 19, 20), (3, 8, 10, 11, 18, 21), (4, 5, 7, 9, 12, 16), (2, 15, 17)\}$$

which yields the corresponding set of base blocks

$$B_{21} = \{K_4\{0, 1, 14, 20\}, K_4\{0, 3, 11, 21\}, K_4\{0, 4, 9, 16\}, K_3\{0, 2, 17\}\}.$$

Therefore, we have

$$\begin{aligned} \mathcal{C}_{21}(n) = \{ & K_4\{i, i+1, i+14, i+20\}, K_4\{i, i+3, i+11, i+21\}, \\ & K_4\{i, i+4, i+9, i+16\}, K_3\{i, i+2, i+17\} : 0 \leq i \leq n-1\}. \end{aligned}$$

Using PDT(21), we obtain

$$\text{PDT}(22) = \{(1, 13, 6, 14, 19, 20), (3, 8, 10, 11, 18, 21), (4, 5, 7, 9, 12, 16), (2, 15, 17), \{22\}\} \text{ and}$$

$$\text{PDT}(23) = \{(1, 13, 6, 14, 19, 20), (3, 8, 10, 11, 18, 21), (4, 5, 7, 9, 12, 16), (2, 15, 17), \{22\}, \{23\}\}.$$

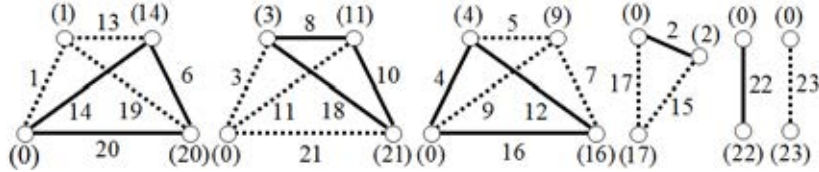


Figure 3.16: The base blocks corresponding to elements in PDT(23)

Therefore, we have

$$\begin{aligned} \mathcal{C}_{22}(n) = \{ & K_4\{i, i+1, i+14, i+20\}, K_4\{i, i+3, i+11, i+21\}, \\ & K_4\{i, i+4, i+9, i+16\}, K_3\{i, i+2, i+17\}, \\ & K_2\{i, i+22\} : 0 \leq i \leq n-1\} \text{ and} \\ \mathcal{C}_{23}(n) = \{ & K_4\{i, i+1, i+14, i+20\}, K_4\{i, i+3, i+11, i+21\}, \\ & K_4\{i, i+4, i+9, i+16\}, K_3\{i, i+2, i+17\}, \\ & K_2\{i, i+22\}, K_2\{i, i+23\} : 0 \leq i \leq n-1\}. \end{aligned}$$

3.3.5 PDT(k) and $\mathcal{C}_k(n)$ for $k = 24, 25, 26$

In this subsection, we determine a PDT(24). In contrast to the cases when $k = 12, 18$, we show that it is possible to find a PDT(24) containing 4 pure difference 6-tuples. Consequently, a PDT(25) and a PDT(26) immediately follow.

Case $k = 24, 25, 26$

We have

$$\text{PDT}(24) = \{(1, 16, 6, 17, 22, 23), (2, 12, 7, 14, 19, 21), \\ (3, 5, 10, 8, 15, 18), (4, 9, 11, 13, 20, 24)\}$$

which yields the corresponding set of base blocks

$$B_{24} = \{K_4\{0, 1, 17, 23\}, K_4\{0, 2, 14, 21\}, K_4\{0, 3, 8, 18\}, K_4\{0, 4, 13, 24\}\}.$$

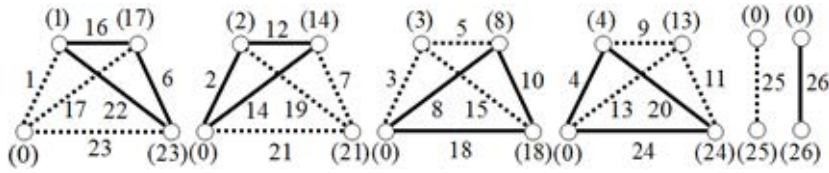


Figure 3.17: The base blocks corresponding to elements in PDT(26)

Therefore, we have

$$\mathcal{C}_{24}(n) = \{K_4\{i, i+1, i+17, i+23\}, K_4\{i, i+2, i+14, i+21\}, \\ K_4\{i, i+3, i+8, i+18\}, K_4\{i, i+4, i+13, i+24\} : 0 \leq i \leq n-1\}.$$

Using PDT(24), we obtain

$$\text{PDT}(25) = \{(1, 16, 6, 17, 22, 23), (2, 12, 7, 14, 19, 21), \\ (3, 5, 10, 8, 15, 18), (4, 9, 11, 13, 20, 24), \{25\}\} \text{ and} \\ \text{PDT}(26) = \{(1, 16, 6, 17, 22, 23), (2, 12, 7, 14, 19, 21), \\ (3, 5, 10, 8, 15, 18), (4, 9, 11, 13, 20, 24), \{25\}, \{26\}\}.$$

Therefore, we have

$$\begin{aligned}
\mathcal{C}_{25}(n) = & \{K_4\{i, i + 1, i + 17, i + 23\}, K_4\{i, i + 2, i + 14, i + 21\}, \\
& K_4\{i, i + 3, i + 8, i + 18\}, K_4\{i, i + 4, i + 13, i + 24\}, \\
& K_2\{i, i + 25\} : 0 \leq i \leq n - 1\} \text{ and} \\
\mathcal{C}_{26}(n) = & \{K_4\{i, i + 1, i + 17, i + 23\}, K_4\{i, i + 2, i + 14, i + 21\}, \\
& K_4\{i, i + 3, i + 8, i + 18\}, K_4\{i, i + 4, i + 13, i + 24\}, \\
& K_2\{i, i + 25\}, K_2\{i, i + 26\} : 0 \leq i \leq n - 1\}.
\end{aligned}$$

CHAPTER IV

CONCLUSION AND OPEN PROBLEMS

4.1 Conclusion

1) Given natural numbers k and $n \geq 3$ such that $k < \lfloor \frac{n}{2} \rfloor$, we recount what we have done so far. First, we introduce Pure Difference Tuple Problem and use it to construct an optimal cyclic clique decomposition of k -power of an n -cycle C_n^k into cliques of order at most 4. Secondly, the desired decomposition can be obtained by the set of the base blocks B_k corresponding to a solution to Pure Difference Tuple Problem, $\text{PDT}(k)$. Therefore, it suffices to determine a $\text{PDT}(k)$.

For each natural number $3 \leq k \leq 26$, a solution $\text{PDT}(k)$ where $k \equiv 0 \pmod{3}$ and $k \neq 12, 18$ yields $\text{PDT}(k+1)$ and $\text{PDT}(k+2)$. In particular, $\text{PDT}(k+1)$ is obtained by adding a singleton $\{k+1\}$ to $\text{PDT}(k)$. Similarly, $\text{PDT}(k+2)$ is obtained by adding a singleton $\{k+2\}$ to $\text{PDT}(k+1)$. Hence, $|\text{PDT}(k+1)| = |\text{PDT}(k)| + 1$. On the other hand, we found that $|\text{PDT}(12)| = |\text{PDT}(13)|$ and $|\text{PDT}(18)| = |\text{PDT}(19)|$

Table 4.1 lists a solution $\text{PDT}(k)$ to Pure Difference Tuple Problem for $3 \leq k \leq 26$.

k	pure difference 6-tuple(s)	pure difference 3-tuple(s)	singleton(s)
3	-	(1, 2, 3)	-
4	-	(1, 2, 3)	{4}
5	-	(1, 2, 3)	{4}, {5}

k	pure difference 6-tuple(s)	pure difference 3-tuple(s)	singleton(s)
6	(1, 3, 2, 4, 5, 6)	-	-
7	(1, 3, 2, 4, 5, 6)	-	{7}
8	(1, 3, 2, 4, 5, 6)	-	{7}, {8}
9	(3, 2, 4, 5, 6, 9)	(1, 7, 8)	-
10	(3, 2, 4, 5, 6, 9)	(1, 7, 8)	{10}
11	(3, 2, 4, 5, 6, 9)	(1, 7, 8)	{10}, {11}
12	(1, 8, 3, 9, 11, 12)	(2, 5, 7), (4, 6, 10)	-
13	(3, 1, 8, 4, 9, 12), (2, 5, 6, 7, 11, 13)	-	{10}
14	(3, 1, 8, 4, 9, 12), (2, 5, 6, 7, 11, 13)	-	{10}, {14}
15	(5, 1, 8, 6, 9, 14), (2, 10, 3, 12, 13, 15)	(4, 7, 11)	-
16	(5, 1, 8, 6, 9, 14), (2, 10, 3, 12, 13, 15)	(4, 7, 11)	{16}
17	(5, 1, 8, 6, 9, 14), (2, 10, 3, 12, 13, 15)	(4, 7, 11)	{16}, {17}
18	(1, 6, 5, 7, 11, 12), (2, 13, 3, 15, 16, 18)	(4, 10, 14), (8, 9, 17)	-
19	(1, 7, 10, 8, 17, 18), (2, 4, 9, 6, 13, 15), (3, 11, 5, 14, 16, 19)	-	{12}
20	(1, 7, 10, 8, 17, 18), (2, 4, 9, 6, 13, 15), (3, 11, 5, 14, 16, 19)	-	{12}, {20}
21	(1, 13, 6, 14, 19, 20), (3, 8, 10, 11, 18, 21), (4, 5, 7, 9, 12, 16)	(2, 15, 17)	-
22	(1, 13, 6, 14, 19, 20), (3, 8, 10, 11, 18, 21), (4, 5, 7, 9, 12, 16)	(2, 15, 17)	{22}
23	(1, 13, 6, 14, 19, 20), (3, 8, 10, 11, 18, 21), (4, 5, 7, 9, 12, 16)	(2, 15, 17)	{22}, {23}

k	pure difference 6-tuple(s)	pure difference 3-tuple(s)	singleton(s)
24	(1, 16, 6, 17, 22, 23), (2, 12, 7, 14, 19, 21), (3, 5, 10, 8, 15, 18), (4, 9, 11, 13, 20, 24)	-	-
25	(1, 16, 6, 17, 22, 23), (2, 12, 7, 14, 19, 21), (3, 5, 10, 8, 15, 18), (4, 9, 11, 13, 20, 24)	-	{25}
26	(1, 16, 6, 17, 22, 23), (2, 12, 7, 14, 19, 21), (3, 5, 10, 8, 15, 18), (4, 9, 11, 13, 20, 24)	-	{25}, {26}

Table 4.1: A solution PDT(k) for $3 \leq k \leq 26$

Let $3 \leq k \leq 26$ and $k \neq 12, 18$. Note that $|\text{PDT}(k)|$ depends on the largest possible number of 6-tuples which can be obtained from $\{1, 2, 3, \dots, k\}$. Hence, we have that

$$|\text{PDT}(k)| = \left\lfloor \frac{k}{6} \right\rfloor + \left\lfloor \frac{k - 6 \lfloor \frac{k}{6} \rfloor}{3} \right\rfloor + \left(k - 6 \left\lfloor \frac{k}{6} \right\rfloor - 3 \left\lfloor \frac{k - 6 \lfloor \frac{k}{6} \rfloor}{3} \right\rfloor \right).$$

If we write $k = 3s + t$ where $0 \leq t \leq 2$, then

$$|\text{PDT}(3s + t)| = \left\lfloor \frac{s}{2} \right\rfloor + t \quad \text{when } 3s + t \neq 12, 18.$$

Therefore, when $n > 3k$, the size of an optimal cyclic clique decomposition $\mathcal{C}_k(n)$ of C_n^k into cliques of order at most 4 is

$$|\mathcal{C}_{3s+t}(n)| = n \left\lfloor \frac{s}{2} \right\rfloor + nt.$$

Note further that nt is the number of 2-cliques in $\mathcal{C}_{3s+t}(n)$ when $3s + t \neq 12, 18$. Thus if $t = 0$ (k is divisible by 3), then $\mathcal{C}_{3s+t}(n)$ will be composed of all 4-cliques and 3-cliques.

2) It is important to point out that an optimal cyclic clique decomposition does not necessarily give the minimum size among cyclic clique decomposition of C_n^k into cliques of order at most 4. Example 4.1 illustrates this situation.

Example 4.1. We give two different cyclic clique decompositions of C_{16}^5 into cliques of order at most 4.

(i) We use a PDT(5) to construct a cyclic clique decomposition of C_{16}^5 . First, we have $\text{PDT}(5) = \{(1, 2, 3), \{4\}, \{5\}\}$. Consequently, the set of corresponding base blocks is $B_5 = \{K_3\{0, 1, 3\}, K_2\{0, 4\}, K_2\{0, 5\}\}$. By clicking each of these base blocks for 15 times, we obtain that

$$\mathcal{C}_1 = \{K_3\{i, i+1, i+3\}, K_2\{i, i+4\}, K_2\{i, i+5\} : 0 \leq i \leq 15\}$$

is a cyclic clique decomposition of C_{16}^5 into cliques of order at most 4, where $|\mathcal{C}_1| = 48$. Observe that the above decomposition is optimal.

(ii) Consider the set of base block $B = \{K_4\{0, 1, 2, 5\}, K_3\{1, 3, 6\}, K_2\{0, 4\}\}$.

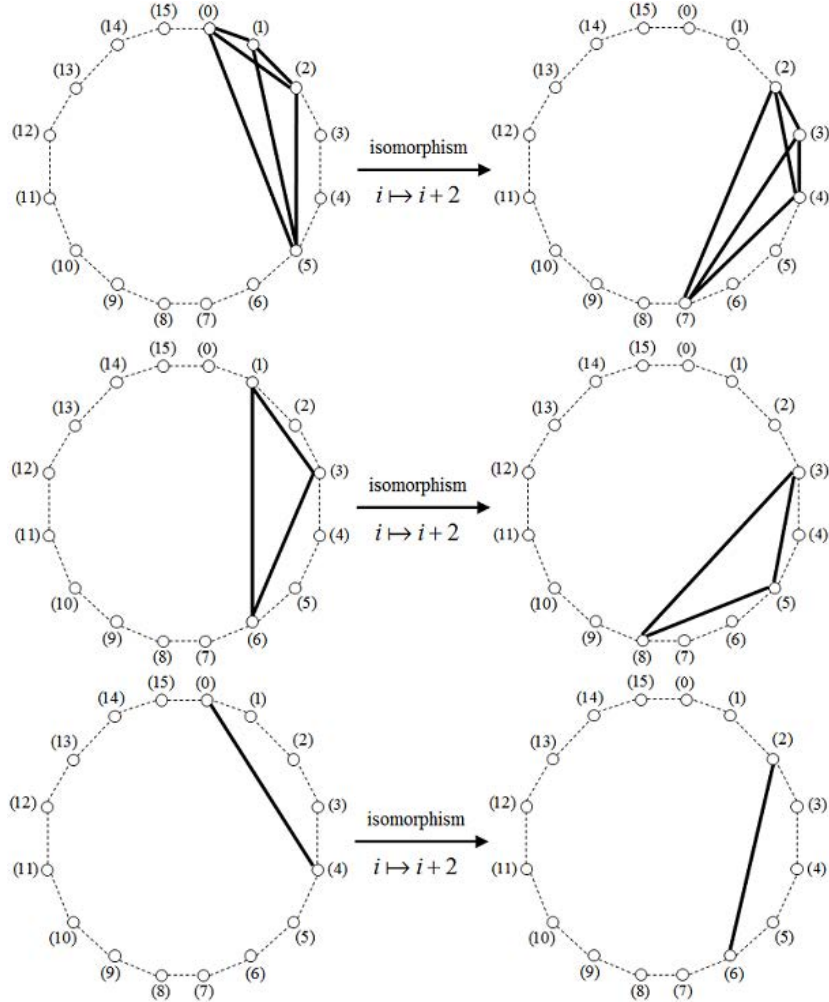


Figure 4.1: Applying the isomorphism $i \rightarrow i + 2$ on each base block in B

For each $i \in \{1, 2, 3, 4, 5\}$, there are totally 2 edges of length i occurring in base blocks in B . By applying the isomorphism $i \rightarrow i + 2$ on $V(C_{16}^5)$ to each base block

of B for 7 times as shown in Figure 4.1, we obtain that

$$\mathcal{C}_2 = \{K_4\{i, i+1, i+2, i+5\}, K_3\{i+1, i+3, i+6\}, \\ K_2\{i, i+4\} : i \in \{0, 2, 4, \dots, 14\}\}$$

is a cyclic clique decomposition of C_{16}^5 into cliques of order at most 4, where $|\mathcal{C}_2| = 24$.

Note that the construction in (ii) is not optimal but it yields a decomposition which has less number of cliques than the construction in (i). \square

3) We have investigated the construction of cyclic clique decomposition of C_n^k when $k < \lfloor \frac{n}{2} \rfloor$. However, we would discuss the case when $k \geq \lfloor \frac{n}{2} \rfloor$. Recall that the set of edge lengths of K_n is $\{1, 2, 3, \dots, \lfloor \frac{n}{2} \rfloor\}$ where $C_n^k = K_n$. We can then apply our construction to obtain a cyclic clique decomposition of K_n as follows.

When n is odd, Remark 3.1 is still valid for K_n where $k = \lfloor \frac{n}{2} \rfloor$. Therefore, we can construct a cyclic clique decomposition of K_n into cliques of order at most 4 by using $\text{PDT}\left(\lfloor \frac{n}{2} \rfloor\right) = \text{PDT}\left(\frac{n-1}{2}\right)$. However, this decomposition is not guaranteed to be optimal.

When n is even, K_n has n edges of each length $i \in \{1, 2, 3, \dots, \frac{n}{2} - 1\}$ and has $\frac{n}{2}$ edges joining antipodal vertices of length $\frac{n}{2}$. Thus,

$$\mathcal{C} \cup \{K_2\{i, i + \frac{n}{2}\} : 0 \leq i \leq \frac{n}{2} - 1\}$$

is a cyclic clique decomposition of K_n into cliques of order at most 4, where \mathcal{C} is a decomposition obtained from $\text{PDT}\left(\frac{n}{2} - 1\right)$. Again, this decomposition is not guaranteed to be optimal.

4.2 Open Problems

1) We mentioned that a $\text{PDT}(k)$ might not be unique, see Example 3.3 (ii). However, to determine a solution $\text{PDT}(k)$ can be very complicated especially when k is large. Therefore, the problem to find $\text{PDT}(k)$ still remains open for all natural numbers $k \geq 27$.

Although our work reveals that $|\text{PDT}(3s+t)| \neq \left\lceil \frac{s}{2} \right\rceil + t$ when $3s+t = 12, 18$, we believe that

$$|\text{PDT}(3s+t)| = \left\lceil \frac{s}{2} \right\rceil + t \quad \text{for all } 3s+t \geq 27 \text{ and } 0 \leq t \leq 2.$$

Furthermore, we think $\text{PDT}(3s+t)$ should contain $\left\lfloor \frac{k}{6} \right\rfloor$ pure difference 6-tuples, $\left\lfloor \frac{k-6\lfloor \frac{k}{6} \rfloor}{3} \right\rfloor$ pure difference 3-tuples and $k-6\left\lfloor \frac{k}{6} \right\rfloor - 3\left\lfloor \frac{k-6\lfloor \frac{k}{6} \rfloor}{3} \right\rfloor$ singletons.

2) If $n > 3k$, then our method provides an optimal cyclic clique decomposition of C_n^k into cliques of order at most 4. However, whether a cyclic clique decomposition of C_n^k into cliques of order at most 4 for natural numbers n for $2k+2 \leq n \leq 3k$ is optimal still remains open.

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VITA

Name	Mr. Apiwat Peereeyaphat
Date of Birth	June 23, 1993
Place of Birth	Bangkok, Thailand
Education	B.Sc. (Mathematics), Srinakharinwirot University, 2016
Scholarship	Science Achievement Scholarship of Thailand (SAST)
Conference	Presenter Some Constructions of Cyclic Clique Decompositions of Power of Cycles, at International Conference in Mathematics and Applications (ICMA-MU 2018), 16–18 December 2018 at Century Park Hotel, Bangkok, Thailand