CHAPTER III

TYPE I SEMIRINGS

It is easy to show that every field is congruence free. Suppose F is a field and $\sim$ is a congruence on F which is not Δ. Then there exist $x \neq y \in F$ such that $x \sim y$. Therefore $x + (-x) \sim y + (-x)$, so $0 \sim y - x \neq 0$. Thus $(y - x)^{-1} . 0 \sim (y - x)^{-1} . (y - x)$. Therefore $0 \sim 1$. Thus we have that for all $a \in F$, $a.0 \sim a.1$. Thus $0 \sim a$, so $\sim = F \times F$. Thus F is congruence free. Thus every field is a type I semiring. The purpose of this chapter is to prove the converse of this statement if the order of the type I semiring is greater than 2.

<u>3.1.1 Proposition</u> : Every type I semiring S is AC if $||S|| > 2$.

Proof : Since S is type I, S has an additive inverse 0. Clearly 0 is AC. Now apply Theorem 2.1.6. #

<u>3.1.2 Proposition</u> : Let S be a type I semiring such that $||S|| > 2$. Then S has at least two elements with additive inverses.

Proof : Since S is type I, S has an additive identity 0 which is its own inverse . Suppose that S has no non-zero element with an additive inverse. Claim that $S \setminus \{0\}$ is closed with respect to addition and multiplication. $S \setminus \{0\}$ is closed with respect to multiplication since S is congruence free and thus MC, i.e. if $x, y \neq 0$ and $xy = 0$ then $xy = x.0$. Therefore $y = 0$ which is a contradiction. $S \setminus \{0\}$ is closed with respect to addition by assumption. Thus the claim is true. But then $(S \setminus \{0\} \times S \setminus \{0\}) \cup (0,0)$ is a nontrivial congruence on S. (since $||S \setminus \{0\}|| > 1$). This contradicts the fact that S is congruence-free. Thus S has at least one nonzero element with an additive inverse. #

Using the two results above we are able to prove the following interesting result.

3.1.3 Theorem : Let S be a type I semiring such that $||S|| > 2$. Then S is a field.

Proof : Let $I = \{x \in S \mid x \text{ has an additive inverse}\}$. $||I|| \geq 2$ by Proposition 3.1.2. Suppose $x,y \in I$. Then there exist $-x,-y \in S$. Thus $(-x) + (-y) \in S$. Thus $x + y \in I$ so I is an additive subsemigroup of S. Suppose $x \in I$ and $s \in S$. Then $-x \in S$. But $(-x)s + xs = ((-x) + x)s = 0s = 0$ Therefore $xs \in I$. Thus I is a multiplicative ideal in S. Note also that $x \in I$ implies that $-x \in I$. Define a relation $\sim$ on S as follows. For $x,y \in S$ say that $x \sim y$ iff there exists $i \in I$ such that $x = y + i$. Now $x \sim x$ since $x = x + 0$ and $0 \in I$. Suppose $x \sim y$. Then there exists $i \in I$ such that $x = y + i$. But $i \in I$, so $-i \in I$. Thus $x + (-i) = y + i + (-i) = y$ and therefore $y \sim x$. Suppose $x \sim y$ and $y \sim z$. Then there exist $a,b \in I$ such that $x = y + a$ and $y = z + b$. Thus $x = (z + b) + a = z + (b + a)$. But by the argument above $a + b \in I$, so $x \sim z$. Therefore $\sim$ is an equivalence relation on S. Now suppose that $x \sim y$. Then there exists $i \in I$ such that $x = y + i$. Thus for all $a \in S$ $x + a = (y + a) + i$. Thus $x + a \sim y + a$. Also $ax = a(y + i) = ay + ai$. But by the argument above, $ai \in I$. Therefore $ax \sim ay$. So we get that $\sim$ is a congruence on S. Choose a nonzero $x \in I$. Then $x + (-x) = 0$, so $x \sim 0$. Thus $\sim \neq \Delta$ so $\sim = S \times S$. Thus $0 \sim 1$

Therefore there exists $k \in I$ such that $0 = 1 + k$ i.e. $-1 \in S$. Now let $a \in S$. $a + (-1)a = 1.a + (-1)a = (1 + (-1))a = 0a = 0$. Therefore $-a \in S$. Thus S is a ring. But by a previous result, every congruence free ring is a field. This is the desired result.

As shown by Example 2.1.1 if $||S|| = 2$ then S may be type I but not a ring. If $||S|| = 2$ and S is type I then S has two elements 1 and 0 and two possible algebraic structures. We know that $0.1 = 0.0 = 1.0 = 0$. Also $1.1 = 1$. Also $0 + 1 = 1 + 0 = 1$, and $0 + 0 = 0$ since 0 is an additive identity. But there are two possible choices for $1 + 1$ i.e. $1 + 1 = 1$ or $1 + 1 = 0$. The case where $1 + 1 = 1$ is the Boolean semifield. The case where $1 + 1 = 0$ is just the integers modulo 2. which is of course a field. Thus we have completely described type I semirings i.e. except for those type I semirings isomorphic to the Boolean 0-semifield, all type I semirings are fields. In fact, every type I semiring is a 0-semifield so every nonzero element has a multiplicative inverse.

The following two results are an application of the arguments above.

<u>3.1.4 Proposition</u> : Let S be a commutative semiring with 1 and a multiplicative zero α. Then if $||S|| > 1$, S has maximal proper congruences.

Proof : Let $\mathcal{Q}$ be the set of proper congruences on S, $\mathcal{Q} = \emptyset$ since $\Delta \in \mathcal{Q}$. Now let $\{A_i\}_{i \in I}$ be a nonempty chain in $\mathcal{Q}$. Let $A = \bigcup_{i \in I} A_i$. For all $i \in I$ $(1, \alpha) \notin A_i$ since if it were then $(a.1, a.\alpha) = (a, \alpha) \in A_i$ for all $a \in S$. Therefore A_i would not be proper $(1, \alpha) \notin A$ so if A is a congruence then A is a proper congruence. But for all $i \in I$ and for all $x \in S$, $(x,x) \in A_i$. Thus $(x,x) \in A$. Suppose $(x,y) \in A$. Then for some $i \in I$ $(x,y) \in A_i$

Thus $(y,x) \in A_i$ and so $(y,x) \in A_i$. Now suppose (x,y), $(y,z) \in A$. We can choose an $i \in I$ such that $(x,y) \in A_i$ and $(y,z) \in A_i$. Thus $(x,z) \in A_i$ so $(x,z) \in A$. Thus $\sim$ is an equivalence relation on S. Suppose $(x,y) \in A$ and $s \in S$. Choose $i \in I$ such that $(x,y) \in A_i$. Then $(x + s, y + s) \in A_i$ and $(sx, sy) \in A_i$. Therefore $(x + s, y + s)$ and $(sx, sy) \in A$. Thus A is a congruence on S. Thus each nonempty chain in $\mathcal{Q}$ has an upper bound. Thus by Zorn's Lemma A has maximal elements. # Note : Proposition 3.1.4 is also true if S has an additive identity and an additive zero.

3.1.5 Theorem. Let S be a commutative semiring with 1 and a multiplicative zero a. Let $\sim$ be a maximal proper congruence on S. Then $\frac{S}{\sim}$ is congruence-free

Proof : $[a]$ is clearly a multiplicative zero in $\frac{S}{\sim}$. Similarly. $[1]$ is a multiplicative identity in $\frac{S}{\sim}$. Claim that $\frac{S}{\sim}$ is congruence free. Well suppose $p \neq \Delta$ is a nontrivial congruence on $\frac{S}{\sim}$. Then there exist $x,y \in S$ such that $[x] \not p [y]$. Now define a congruence p^1 on S as follows. For $a,b \in S$ say that $a\, p^1\, b$ iff $[a]\, p\, [b]$. p^1 is clearly an equivalence relation on S. But suppose $a\, p^1\, b$ and $s \in S$. Then $[a]\, p\, [b]$. Therefore $[s] + [a]\, p\, [s] + [b]$ and $[s][a]\, p\, [s][b]$. Thus $[s + a]\, p\, [s + b]$ and $[sa]\, p\, [sb]$. Thus $s + a\, p^1\, s + b$ and $sa\, p^1\, sb$. Thus p^1 is a congruence relation on S. But $[x] \not p [y]$. Thus p^1 is a proper congruence relation on S. Clearly $p^1 \supseteq \sim$. But since $p \neq \Delta$, there exists $a, b \in S$ such that $[a] \neq [b]$ i.e. $a \not\sim b$ but $[a]\, p\, [b]$. Therefore $a\, p^1\, b$ so $p^1 \supset \sim$ which contradicts the maximality of $\sim$. Thus $\frac{S}{\sim}$ is congruence-free. #

Thus given any commutative semiring S with 1 and a multiplicative zero ∞ , we are able to construct a congruence-free semiring $\frac{S}{\sim}$ where $\sim$ is a maximal proper congruence. The next theorem uses the results of this chapter to prove a interesting fact about $\frac{S}{\sim}$.

<u>3.1.6 Theorem.</u> Let S be a commutative semiring with 1 and 0,and $\sim$ a maximal proper congruence on S. Assume also that $|S| > 1$. Then $\frac{S}{\sim}$ is a field or the Boolean 0-semifield.

Proof : $[0]$ is clearly the additive identity and multiplicative zero in $\frac{S}{\sim}$. By Theorem 3.1.5 $\frac{S}{\sim}$ is congruence free. Now apply Theorem 3.1.3 and the remarks following that theorem. #

This is an interesting result since the algebraic structure on S might be pathological (e.g.not MC or with zero divisors etc.) but nevertheless $\frac{S}{\sim}$ is a field or the Booleon 0-semifield. As an example consider Z_6 , the integers modulo 6. By inspection $\sim = (\{0,2,4\} \times \{0,2,4\}) \cup (\{1,3,5\} \times \{1,3,5\})$ is a maximal proper congruence on S. In this case $\frac{Z_6}{\sim} \cong Z_2$, the integers modulo 2.