CHAPTER II

THE GOVERNING EQUATIONS

The motions of the atmosphere are governed by the fundamental physical laws of mass, momentum, and energy and equation of state. In this chapter we will show how these principles can be applied to the atmosphere, in order to obtain the governing equations.

Momentum Equation

Newton's second law of motion states that the rate of change of momentum of an object referred to inertial fame in space equals the sum of all the forces acting. For atmospheric motions of meteorological interest, the forces which are of primary concern are the pressure gradient force, the gravitational force, and friction. If the motion is referred to a coordinate system rotating with the earth, Newton's second law may still be applied provided that certain apparent forces, the centrifugal force and the Coriolis force, are included among the forces acting. We can write Newton's second law in rotating coordinate as

$$\frac{d\vec{V}}{dt} = -\vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right) - 2\vec{\Omega} \times \vec{V} - \frac{1}{\rho}\nabla P + \vec{g}_a + \vec{F}r \qquad (2.1)$$

where

$\vec{V}$ ≡ The velocity relative to the rotating earth

$\vec{\Omega} \cong$ The angular speed of rotation of the earth

$\rho \cong$ The density of an air parcel on the rotating earth

$P \cong$ The air pressure

$\vec{g}_a \cong$ The gravitational force

$\vec{Fr} \cong$ The friction force

$\vec{r} \cong$ The position vector of the particle as measured from the origin at the earth's center.

and the right-hand side are the centrifugal force, the Coriolis force, the pressure gradient force, the gravitational force, and friction force respectively.

A. Component Equation in Spherical Coordinates

It is convenient to expand eq.(2.1) in spherical coordinates so that the (level) surface of the earth corresponds to a coordinate surface. The coordinate axes are then (λ, ϕ, z) where λ is longitude, ϕ is latitude, and z is the vertical distance above the surface of the earth. If the unit vector $\hat{i}, \hat{j}, \hat{k}$ are now taken to direct eastward, northward, and upward, respectively (see Fig. 2.1), the relative velocity becomes

$$\vec{V} = \hat{i}u + \hat{j}v + \hat{k}w$$

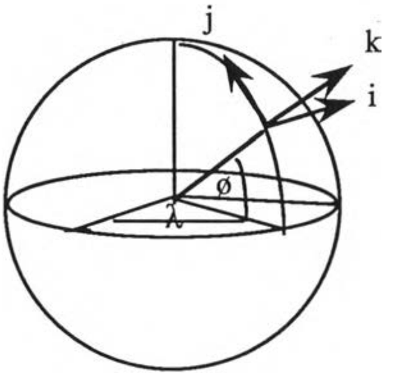

Figure 2.1 Spherical coordinates with Cartesian tangent plane.

Sphere	Cartesian tangent plane	
$r = a + z$	dz	$w = \frac{dz}{dt}$
λ	$dx = a \cos\phi \, d\lambda$	$u = a \cos\phi \frac{d\lambda}{dt}$
ϕ	$dy = a \, d\phi$	$v = a \frac{d\phi}{dt}$

If we define

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

and friction $\vec{Fr}$ is expanded in components as

$$\vec{Fr} = \hat{i}\,F_x + \hat{j}\,F_y + \hat{k}\,F_z$$

Then the eastward, northward, and vertical component momentum equations become

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + 2\,\Omega\,v\sin\phi - 2\,\Omega\,w\cos\phi + Fx \tag{2.2}$$

$$\frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} = -\frac{1}{\rho}\frac{\partial P}{\partial y} - 2\,\Omega\,u\sin\phi + Fy \tag{2.3}$$

$$\frac{dw}{dt} + \frac{u^2 + v^2}{a} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g + 2\,\Omega\,u\cos\phi + Fz \tag{2.4}$$

The terms proportional to $1/a$ on the left-hand sides in eq.(2.2)-eq.(2.4) are called *the curvature terms* because they arise due to the curvature of the earth.

B. Scale Analysis of the Equation of Motion.

In order to simplify eq.(2.2)-eq.(2.4) for synoptic scale motions we define the following characteristic scales of the field variables based on observed values for midlatitude synoptic systems.

U	~	10	m / s	horizontal velocity scale
W	~	1	cm / s	vertical velocity scale
L	~	10^6	m	length scale
D	~	10^4	m	depth scale
$\Delta P / \rho$	~	10^3	m^2/s^2	horizontal pressure fluctuation scale
L / U	~	10^5	s	time scale

It is convenient to consider disturbances centered at latitude $\phi_0 = 45^\circ$, and introduce the notation $f_0 = 2\,\Omega \sin\phi_0 \approx 10^{-4}\ s^{-1}$ then we can now estimate the characteristic magnitude of each term in eq.(2.2) and eq.(2.3) based on the scaling considerations as shown in Table 2.1.

Table 2.1 Scale analysis of the horizontal momentum equations

	A	B	C	D	E		F
x- Component	$\frac{du}{dt}$	$-2\,\Omega\, v \sin\phi$	$+2\,\Omega\, w \cos\phi$	$+\frac{uw}{a}$	$-\frac{uv \tan\phi}{a}$	=	$-\frac{1}{\rho}\frac{\partial P}{\partial x}$
y- Component	$\frac{dv}{dt}$	$+2\,\Omega\, u \sin\phi$		$+\frac{vw}{a}$	$+\frac{u^2 \tan\phi}{a}$	=	$-\frac{1}{\rho}\frac{\partial P}{\partial y}$
Scales	$\frac{U^2}{L}$	f_0U	f_0W	$\frac{UW}{a}$	$\frac{U^2}{a}$		$\frac{\Delta P}{\rho L}$
Magnitudes (cm / sec^2)	10^{-2}	10^{-1}	10^{-4}	10^{-6}	10^{-3}		10^{-1}

From Table 2.1, the Coriolis force (term B) and the pressure gradient force (term F) are in approximate balance. Therefore, retaining only these two terms, we obtain as a first approximation the geostrophic relationship.

$$-fv \cong -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad , \quad fu \cong -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{2.5}$$

Where $f = 2\,\Omega\sin\phi$ is called the Coriolis parameter, by analogy to the geostrophic approximation eq.(2.5) it is possible to define a horizontal velocity field, $\vec{V}_g \equiv \hat{i}\,u_g + \hat{j}\,v_g$, call the geostrophic wind, which satisfies eq.(2.5) identically. Thus in vectorial form

$$\vec{V}_g \equiv \hat{k} \times \frac{1}{\rho f}\nabla P \tag{2.6}$$

And, so as to obtain prediction equations it is necessary to retain the acceleration (term A), then the horizontal momentum equations are

$$\frac{du}{dt} - fv = -\frac{1}{\rho}\frac{\partial P}{\partial x} \tag{2.7}$$

$$\frac{dv}{dt} + fu = -\frac{1}{\rho}\frac{\partial P}{\partial y} \tag{2.8}$$

A similar analysis can be applied to the vertical component of the momentum eq.(2.4), as shown in Table 2.2.

Table 2.2 Scale analysis of the vertical momentum equation

	A	B	C	D	E
z- Component	$\frac{dw}{dt}$	$-2\,\Omega\, u \cos\,\phi$	$\frac{u^2 + v^2}{a}$	$= -\frac{1}{\rho}\frac{\partial P}{\partial z}$	$-g$
Scales	$\frac{UW}{L}$	$f_0 U$	$\frac{U^2}{a}$	$\frac{P_0}{\rho H}$	g
Magnitudes (cm / sec^2)	10^{-5}	10^{-1}	10^{-3}	10^{3}	10^{3}

The scaling indicates that to a high degree of accuracy the gravity force must be exactly balance by the vertical component of the pressure gradient force .

$$\frac{\partial P}{\partial z} = -\rho g \qquad (2.9)$$

This condition of hydrostatic balance provides an excellence approximation for the vertical dependence of the pressure field in the real atmosphere. It is often useful to express the hydrostatic equation in terms of the geopotential [$\Phi(z)$] which is defined as the work required to raise a unit mass to height z from mean sea level.

$$\Phi = \int_0^z g\, dz \qquad (2.10)$$

If g is assumed to be a constant, then we obtain

$$\Phi(z) = g\, z \qquad (2.11)$$

The Continuity Equation

The second fundamental law is the conservation of mass which may be expressed in mathematical form as follows.

$$\frac{\partial \rho}{\partial t} = - \nabla(\rho \vec{V})$$
$$= - \rho \nabla.\vec{V} - \vec{V}.\nabla\rho \qquad (2.12)$$

Where ρ is the density , and $\vec{V}$ is the velocity. Alternate forms are obtainable by combining local derivatives and using specific volume ($\alpha = \frac{1}{\rho}$)

$$\frac{1}{\rho}\frac{d\rho}{dt} = \frac{1}{\alpha}\frac{d\alpha}{dt} = \nabla.\vec{V} \qquad (2.13)$$

Equation of State

The thermodynamic state of the atmosphere at any point is determined by the values of pressure, temperature, and density (or specific volume) at that point. These field variables are related to each other by the equation of state for an ideal gas. Letting P, T, ρ, and α denote pressure, temperature, density, and specific volume, respectively, we can express the equation of state for dry air as

$$P\alpha = RT \quad \text{or} \quad P = \rho R T \qquad (2.14)$$

Where R is the gas constant for dry air ($R = 287\ J\ kg^{-1}\ K^{-1}$) .

Equation of Energy

The third fundamental conservation principle is the conservation of thermodynamic energy. For such a system the first law, states that the change in internal energy of the system is equal to the difference between the heat added to the system and the work done by the system. Which can be written as

$$\frac{d\left[\rho\left(e+\frac{1}{2}\,\vec{V}\cdot\vec{V}\right)\delta V\right]}{dt} = -\nabla.\left(P\,\vec{V}\right)\,\delta V + \rho\, g.\,\vec{V}\,\delta V + \rho\,\dot{q}\,\delta V \qquad (2.15)$$

Where

$-\nabla.\left(P\,\vec{V}\right)\,\delta V$ ≅ The total rate of working by the pressure force.

$\rho\, g.\,\vec{V}\,\delta V$ ≅ The rate at which body forces do work on the mass element.

$\dot{q}$ ≅ The rate of heating per unit mass

g ≅ The gravity.

e ≅ The internal energy per unit mass

δV ≅ The volume element.

With the aid of the chain rule of differentiation, we can rewrite eq.(2.15) as.

$$\rho\frac{de}{dt} + \rho\,\frac{d\left(\frac{1}{2}\,\vec{V}\cdot\vec{V}\right)}{dt} = -\,\vec{V}.\nabla P - P\nabla.\,\vec{V} - \rho\, g\, w + \rho\,\dot{q} \qquad (2.16)$$

This equation can be further simplified by noting that if we take the dot product of $\vec{V}$ with the momentum eq.(2.1) we obtain (neglecting friction)

$$\rho\,\frac{d\left(\frac{1}{2}\,\vec{V}\cdot\vec{V}\right)}{dt} = -\,\vec{V}.\nabla P - \rho\, g\, w \qquad (2.17)$$

Subtracting eq.(2.17) from eq.(2.16) we obtain

$$\rho\frac{de}{dt} = - P\nabla.\vec{V} + \rho\,\dot{q} \tag{2.18}$$

The term in eq.(2.16) which were eliminated by subtracting. Eq.(2.17) represents the balance of mechanical energy due to the motions of the fluid element; the remaining terms represent the thermal energy balance. Using the definition of geopotential we have

$$g\,w = g\frac{dz}{dt} = \frac{d\Phi}{dt}$$

So that eq.(2.17) can be rewritten as

$$\rho\frac{d\left(\frac{1}{2}\vec{V}\cdot\vec{V} + \Phi\right)}{dt} = -\vec{V}.\nabla P \tag{2.19}$$

Which is referred to as the mechanical energy equation. Thus eq.(2.19) states that following the motion, the rate of change of mechanical energy per unit volume equals the rate at which work is done by the pressure gradient force. The thermal energy eq.(2.18) can be written in more familiar form as

$$\frac{1}{\rho}\nabla.\vec{V} = -\frac{1}{\rho^2}\frac{d\rho}{dt} = \frac{d\alpha}{dt}$$

And that for dry air the internal energy per unit mass is given by $e = C_v T$, where C_v [=717 J kg^{-1} K^{-1}] is the specific heat at constant volume. We then obtain

$$C_v\frac{dT}{dt} + P\frac{d\alpha}{dt} = \dot{q} \tag{2.20}$$

Which is the usual form of the energy equation.

A. The Thermodynamic Energy Equation

The first law of thermodynamics expresses the principle of conservation of energy, which may be written in the simple form as eq.(2.20). After taking the total derivative of the equation of state eq.(2.14), substituting for $p\frac{d\alpha}{dt}$ in eq.(2.20) and using $C_p = C_v + R$, where C_p [= 1004 J kg^{-1}K^{-1}] is the specific heat at constant pressure, we can rewrite the first law of thermodynamics as.

$$C_p \frac{dT}{dt} - \alpha \frac{dP}{dt} = \dot{q} \tag{2.21}$$

Dividing through by T and again using the equation of state we obtain the entropy form of the first law of thermodynamics:

$$C_p \frac{d\ln T}{dt} - R \frac{d\ln P}{dt} = \frac{\dot{q}}{T} = \frac{ds}{dt} \tag{2.22}$$

Eq.(2.22) gives the rate of change of entropy per unit mass following the motion for a thermodynamically reversible process.

B. Potential Temperature

For an idea gas undergoing an adiabatic process the first law of thermodynamics can be written in the form

$$Cp \, d\ln T - R \, d\ln P = 0$$

Integrating this expression from a state at pressure P and temperature T to a state in which the pressure is P_S [=1000 hPa] and the temperature is θ, we obtain

$$\theta = T\left(\frac{P_s}{P}\right)^{R/C_p} \tag{2.23}$$

This relationship is referred to as Poisson's equation, and the temperature defined by eq.(2.23) is called the potential temperature. Taking the logarithm of eq.(2.23) and differentiating, we find that

$$C_p\frac{d\ln\theta}{dt} = C_p\frac{d\ln T}{dt} - R\frac{d\ln P}{dt} \tag{2.24}$$

comparing eq.(2.22) and eq.(2.24), we obtain

$$C_p\frac{d\ln\theta}{dt} = \frac{ds}{dt} \tag{2.25}$$

Thus, for reversible dry adiabatic processes, fractional potential temperature changes are indeed proportional to entropy changes.

The Complete System of Equations

For dry air, eq.(2.1), eq.(2.13), eq.(2.14), and eq.(2.21) comprise a complete system of six scalar equations and six unknowns P, α, T, u, v, and w. The friction force Fr and diabatic heating $\dot{q}$ are assumed to be either known functions or expressible in terms of the other variable; hence, in principle, all future states can be determined by solution of this system.

When moisture is included, modifications are necessary in the equation of state and the first law of thermodynamics; in addition, an equation is needed to express the conservation of the water substance. For the present, only dry air will be considered.

<u>The Vorticity Equation</u>

Vorticity is a vector field defined as the curl of velocity. The absolute vorticity $\vec{\omega}_a$ is given by the curl of the absolute velocity, while the relative vorticity $\vec{\omega}$ is given by the curl of the relative velocity.

$$\vec{\omega}_a = \nabla \times \overrightarrow{Va} \quad , \quad \vec{\omega} = \nabla \times \vec{V}$$

However, in dynamic meteorology we are in general concerned only with the vertical components of absolute and relative vorticity.

$$\eta = \hat{k} \cdot \left(\nabla \times \overrightarrow{Va}\right) \quad , \quad \zeta = \hat{k} \cdot \left(\nabla \times \vec{V}\right) \tag{2.26}$$

The difference between absolute and relative vorticity is given by the vertical component of the vorticity of the earth due to its rotation; or $\hat{k} \cdot \left(\nabla \times \overrightarrow{Ve}\right) = 2\,\Omega \sin \phi = f$. Thus we have $\eta = \zeta + f$, or using Cartesian coordinates.

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad , \quad \eta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f \tag{2.27}$$

For motions of synoptic scale, the vorticity equation can be derived using the approximate horizontal momentum eq.(2.7) and eq.(2.8). We differentiate the x component equation with respect to y and the y component equation with respect to x

$$\frac{\partial}{\partial y}\left\{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - f\,v = -\frac{1}{\rho}\frac{\partial P}{\partial x}\right\} \tag{2.28}$$

$$\frac{\partial}{\partial x}\left\{\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} - f\,u = -\frac{1}{\rho}\frac{\partial P}{\partial y}\right\} \tag{2.29}$$

Subtracting eq.(2.28) from eq.(2.29) and recall that $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$, we obtain the vorticity equation

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + w\frac{\partial \zeta}{\partial z} + (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$+\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + v\frac{df}{dy} = \frac{1}{\rho^2}\left[\frac{\partial \rho}{\partial x}\frac{\partial P}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial P}{\partial x}\right] \qquad (2.30)$$

Using the fact that the Coriolis parameter depends only on y so that $\frac{df}{dt} = v\frac{df}{dy}$ and eq.(2.30) may be rewritten in the form

$$\frac{d(\zeta + f)}{dt} = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)$$

$$+ \frac{1}{\rho^2}\left[\frac{\partial \rho}{\partial x}\frac{\partial P}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial P}{\partial x}\right] \qquad (2.31)$$

Eq.(2.31) states that the rate of change of the absolute vorticity following the motion is given by the sum of the three terms on the right, called the divergence term, the tilting or twisting term and the solenoidal term, respectively.