สมมาตรยวดยิ่งเชิงปรากฏการณ์

นายวิรินทร์ สนธิ์เศรษฐี

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาฟิสิกส์ ภาควิชาฟิสิกส์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2548

ISBN 974-53-2775-1

PHENOMENOLOGICAL SUPERSYMMETRY

Mr. Wirin Sonsrettee

A Thesis Submitted in Partial Fulfillment of the Requirements

for the Degree of Master of Science Program in Physics

Department of Physics

Faculty of Science

Chulalongkorn University

Academic Year 2005

ISBN 974-53-2775-1

Thesis Title	Phenomenological Supersymmetry
By	Mr. Wirin Sonsrettee
Field of Study	Physics
Thesis Advisor	Rujikorn Dhanawittayapol, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree.

....Dean of the Faculty of Science

(Professor Piamsak Menasveta, Ph.D.)

THESIS COMMITTEE

....Chairman

(Ahpisit Ungkitchanukit, Ph.D.)

....Rujikorn Dhanawittayapol........Thesis Advisor

(Rujikorn Dhanawittayapol, Ph.D.)

....Member

(Burin Asavapibhop, Ph.D.)

....Member

(Sathon Vijarnwannaluk, Ph.D.)

วิรินทร์ สนธิ์เศรษฐี : สมมาตรยวดยิ่งเชิงปรากฏการณ์. (PHENOMENOLOGICAL SUPERSYMMETRY) อ. ที่ปรึกษา : อ.ดร. รุจิกร ธนวิทยาพล, 122 หน้า. ISBN 974-53-2775-1.

วิทยานิพนธ์ฉบับนี้ปริทัศน์การขยายเชิงสมมาตรยวดยิ่งอย่างน้อยที่สุดของแบบจำลองมาตรฐาน (The Minimal Supersymmetric Standard Model) โดยเฉพาะอย่างยิ่งในส่วนของอนุภาคฮิกส์ จากการวิเคราะห์การสูญเสียสมมาตรของอิเล็กโทรวีก (electroweak symmetry breaking) พบว่ามีอนุภาคฮิกส์โบซอนห้าตัวในทฤษฎีซึ่งมวลของฮิกส์เหล่านี้ได้ถูกคำนวณทั้งในการประมาณในระดับคลาสสิค (classical level) และการประมาณในระดับหนึ่งลูป (one-loop correction) ในการประมาณที่ระดับหนึ่งลูปนั้นวิทยานิพนธ์ฉบับนี้ได้คำนวณด้วยวิธีการศักย์ยังผล (effective potential) ในการคำนวณศักย์ยังผลนี้ได้คำนึงถึงผลอันเนื่องมาจากอนุภาคท๊อปควาร์ก (top quark) และอนุภาคสท๊อปสควาร์ก (stop squark) เป็นสำคัญ จากการคำนวณนี้แสดงให้เห็นว่าการคำนวณมวลของอนุภาคฮิกส์ตัวที่เบาที่สุดในทฤษฎีจำเป็นต้องคำนวณถึงระดับหนึ่งลูปจึงจะได้ผลที่ถูกต้อง

ภาควิชา..ฟิสิกส์.......................................ลายมือชื่อนิสิต...วรินทร์ สนธิ์เศรษฐี...............
สาขาวิชา..ฟิสิกส์.......................................ลายมือชื่ออาจารย์ที่ปรึกษา..รุจิกร ธนวิทยาพล...........
ปีการศึกษา..2548.......................................

4572496023 : MAJOR PHYSICS

KEY WORDS : SUPERSYMMETRY/STANDARD MODEL/HIGGS BOSONS/QUANTUM FIELD THEORY/EFFECTIVE POTENTIAL/PARTICLE PHYSICS

WIRIN SONSRETTEE : PHENOMENOLOGICAL SUPERSYMMETRY. THESIS ADVISOR : RUJIKORN DHANAWITTAYAPOL, PH.D., 122 pp. ISBN 974-53-2775-1.

A review on the Minimal Supersymmetric Standard Model (MSSM) is given, putting a special emphasis on the Higgs sector. The electroweak symmetry breaking in the context of the MSSM is analyzed. It is found that there are five Higgs bosons in the model, and their tree-level masses are determined. However these masses receive quantum corrections, which can be large if the particles involved in the loop diagrams have large masses. In this thesis, the one-loop correction, due to the top quark and stop squark loops, to the Higgs effective potential is obtained. It is found that such a one-loop correction changes the upper bound on the mass of the lightest Higgs boson substantially with the known value of top quark mass and a sufficiently high supersymmetry breaking scale.

Department ...Physics................................ Student's signature Wirin Sonsrettee

Field of study ...Physics.............................. Advisor's signature Rujikorn Dhanawittayapol

Academic year ...2005................................

ACKNOWLEDGEMENTS

I would like to express my gratitude toward my advisor, Dr. Rujikorn Dhanawittayapol. His teaching inspired my appreciation in theoretical physics. His suggestions always led me to the heart of the problems and the way to investigate and solve them.

I would like to thank Dr. Ahpisit Ungkitchanukit for his kindness. He not only taught me physics but also guided me to the way of life.

I also would like to express my gratefulness to Dr. Auttakit Chattarabhuti for his suggestions and his helps.

I am very grateful to Dr. Burin Asavapibhop and Dr. Sathon Vijarnwannaluk for serving as my thesis committee and for their suggestions.

A special thank goes to Houang A. T. Kiet for discussions about supersymmetry and his helps. A lot of thanks also go to Pawin Ittisami and Pitayuth Wongjun for discussions and their helps, especially in lending me some of their very useful books.

I thank the XI Vietnam School of Physics for giving me the opportunity to attend the school with full support.

I also thank my true friend, Pairud Kamsing, for his friendship and encouragements.

I sincerely thank Rangsima Chanphana for her mental supports. There were many times that I confronted serious problems in life, and thought of ending my dream and my graduate study. I may not have passed those difficult times without her.

Lastly, I would like to thank my family and friends for their encouragements and supports throughout the whole period of my study.

CONTENTS

LIST OF TABLES

Table **Page**

LIST OF FIGURES

CHAPTER I

INTRODUCTION

Over a long period of time, physicists had started to discover the laws of Nature by observing facts about Nature. The patterns of Nature behaviors have been seen and the equations were written down. In this way, the electromagnetic (EM) theory was formulated. The EM theory, however, was seen in a deeper view. In the 19th century, physicists knew that the Maxwell's equations have the Lorentz symmetry. Einstein put this fact to use in developing his theory of Special Relativity. Today, physicists extract symmetries from the assumptions and formulate a law of Nature with regarding to the symmetries. Symmetries in modern physics have taken an even stronger role to such an extent that the laws of modern physics cannot even be formulated without the concept of symmetry [1]. In particle physics, the Standard Model (SM) describes elementary particles and their fundamental interactions based on symmetries. The SM action has the Lorentz symmetry and the symmetry under translations in spacetime. All fields in the SM therefore belong to some representations of the Poincaré group. The interactions in the SM are described by the gauge bosons which mediate *strong*, *weak* and *electromagnetic* interactions. The laws of interactions are constructed from a *gauge principle* which states that the Lagrangian is invariant under the local symmetry transformations. The electromagnetic and weak interactions can be unified into the electroweak (EW) interactions through the process of spontaneous symmetry breaking. The spontaneous symmetry breaking implies the existence of the scalar field called the Higgs field. The interactions of the Higgs field with any particle cause the particle to acquire mass after the symmetry breaking.

The results from decades of theoretical and experimental research have confirmed that the SM is extremely successful in that its predictions have agreed very well with the experimental results of high precision. The SM, however, is not the best theory for understanding the fundamental laws of Nature. There have been many unsolved problems in the SM. We now give a brief overview of some important problems:

1. The SM cannot explain a number of arbitrary parameters:

Parameter	Amount
Quark masses	6
Leptons masses	3
Mixing angles θ_i	3
QCD $\overline{\theta}$	1
Phase δ	1
Coupling constants	3
Higgs sector	2
Total	**19**

   These parameters appear in the equations, and they just have to be put in to make the theory fit observations. For example, if one asks "Why is the top quark, which is the heaviest known elementary particle, something like 300,000 times heavier than the electron?" The answer is "We don't know."

2. People believe that the SM is just an effective low energy theory. So one needs more fundamental theory which has a larger domain of validity extending to smaller distances, or equivalently, to higher energies that have not yet been explored by particle accelerators. A possible theory, so-called the *Grand Unified Theory* (GUT), has been proposed. The GUT unifies the strong, weak and electromagnetic interactions in the sense that they become a part of a larger gauge group with a single coupling constant. One motivation for the unification comes from renormalization group calculations which show that the strengths of three effective (running) coupling constants tend to the same value at the grand unification scale of about $M_{GUT} \sim 10^{16}$ GeV. This motivated some people to construct a GUT model

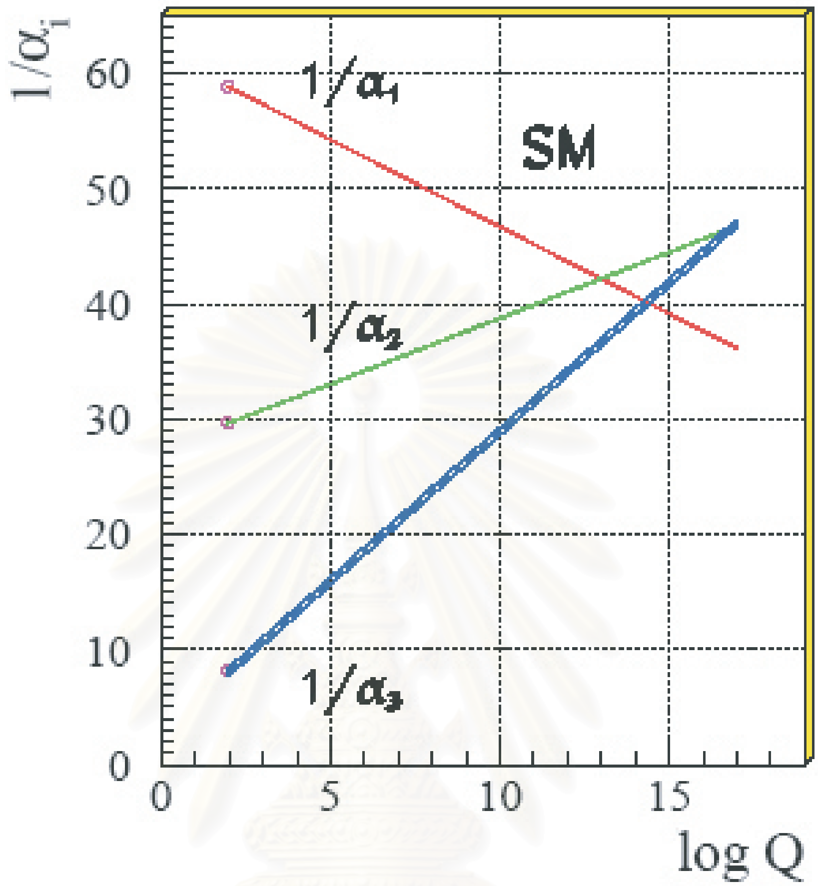

Figure 1.1: The plot of the inverse of the coupling constants of electromagnetic (α_1), weak nuclear (α_2), and strong nuclear (α_3) forces as functions of the energy scale according to the renormalization group calculations in the Standard Model.

with one gauge group with only one coupling constant above the M_{GUT} scale. Below this scale, the GUT gauge group should be broken to the Standard Model gauge group in the same way that the electroweak gauge group $SU(2)_L \times U(1)_Y$ is broken to the electromagnetic gauge group $U(1)_{EM}$ below the weak scale $M_W \sim 100$ GeV. However, using the data of high precision for the coupling constants at the weak scale, the SM calculation indicated that the coupling constants do not really tend to a single value at any high energy scale. the standard model miss each other. This is shown in Fig. 1.1.

3. The *hierarchy problem* cannot be solved by the SM itself. The problem arises from the fact that the difference between the EW scale and the GUT scale is extremely large. In the calculation of the Higgs mass in the SM, one obtains the result

$$m_H^2 = m_{H,bare}^2 + \mathcal{O}(\Lambda^2), \tag{1.1}$$

where m_H is the observed Higgs mass (physical Higgs mass), $m_{H,bare}$ is the bare Higgs mass which is the mass parameter that appears in the SM Lagrangian, and $\mathcal{O}(\Lambda^2)$ represents the quantum corrections from the effect of interactions between Higgs and other particles with "Λ" being the cutoff energy. One can view Λ as the energy scale above which the SM is not valid and the new physics occurs. One expects that Λ is the GUT scale ($\sim 10^{16}$ GeV). There are, however, bounds on the Higgs mass. By imposing the condition to preserve the unitarity of the $W^+W^- \rightarrow W^+W^-$ scattering amplitude [2], the physical Higgs mass needs to have an upper bound, $m_\phi \leq$ 1 TeV. Thus in order to get the acceptable value of Higgs mass, the bare mass must be fine-tuned to a very high precision in order to cancel the very large quantum correction terms. This has annoyed physicists so much that they called it a hierarchy problem.

4. The gravity cannot fundamentally be unified with the other interactions of the Standard Model. Although it is possible to study quantum field theories on a curved spacetime (in which gravity is treated as a classical background field, and particles and other fundamental forces are described by quantum fields), it is far from how to unify or connect a quantization of gravity with the SM. In this context, a second related problem is the cosmological constant, the energy of the vacuum. The energy density calculated by invoking spontaneous symmetry breaking in the SM is 50 orders of magnitude

higher than the observational limit. This necessitates excessive fine tuning between bare pieces, which have, a priori, no reason to be related to each other at all.

There are many theories proposed to solve these problems. Once again, physicists also expect the symmetry to play the central role in all of them. Which symmetries are available? Looking at the SM, it has both the spacetime symmetry via the Poincaré group and the internal symmetry in the field space via the gauge group separately. Are there any more general kinds of symmetry than these? In 1974, Haag, Lopuszanski and Sohnius showed that the supersymmetry constitutes the only possible non-trivial generalization involving the Poincaré and internal symmetries [3]. Thus supersymmetry is one of the candidates to extend the SM.

Supersymmetry (SUSY) is, by definition, a symmetry between fermions and bosons [4]. Hence SUSY implies that there are equal numbers of fermionic and bosonic degrees of freedom in Nature. SUSY assigns to each fermion a bosonic partner, and vice versa. They are called the superpartner of each other.

The problems of the SM above can be solved by supersymmetry. For example, because every fermion has a bosonic superpartner, the hierarchy problem can be solved in supersymmetry by reducing the quadratic divergences ($\mathcal{O}(\Lambda^2)$) to logarithmic divergences ($\mathcal{O}(\log \Lambda^2)$) with a cancellation of Feynman diagrams which separately correspond to fermions and bosons. However, in a truly supersymmetric theory the masses of a fermion and its superpartner have to be the same, but the superpartner of each SM particle has not yet been observed in Nature. Thus in a viable theory, supersymmetry has to be broken by introducing the mass difference between each fermion and its superpartner without causing the quadratic divergences to reappear.

One of the simplest model of the supersymmetric extension of the Standard Model is the Minimal Supersymmetric Standard Model (MSSM). In the

MSSM, all particles of SM are doubled with their superpartner except the Higgs particle. There are five Higgs bosons in the MSSM. The lightest supersymmetric Higgs which is denoted by h^0 and has the tree-level mass less than that of the Z boson, yet it has not been observed experimentally. Unfortunately (for experimentalists at least!) m_{h^0} can receive the radiative corrections which possibly cause it to become heavier than m_Z. Thus it is interesting to compute the radiative corrections to m_{h^0}.

The organization of this thesis is as follows. In Chapter 2, we review the construction of the Standard Model from the postulate that it obeys some practical symmetries. In detail, we briefly review some necessary Lie group theory which is the mathematical language of continuous symmetry, and use the formalism to construct the dynamics of the SM. In Chapter 3, the general supersymmetric field theory and the MSSM are constructed analogously, excepted that they are based the graded Lie groups instead of the ordinary Lie groups. In Chapter 4, the Higgs phenomenology will be reviewed. The one-loop effective potential for the supersymmetric Higgs fields is calculated to analyze the bound on the mass of the lightest supersymmetric Higgs particle. Finally the conclusions are made in Chapter 5.

CHAPTER II

THE STANDARD MODEL

In order to comprehend the extension of the Standard Model with supersymmetry, the construction of the Standard Model (SM) with symmetries should be understood before. To describe the symmetries of the physical theories, physicists use the idea of group theory. In this chapter, we start with a brief review of group theory paying a particular attention to the Poincaré group, and then go on to discuss particles in quantum field theory, as representations of the Poincaré group. To describe the interactions in the SM, we next discuss gauge symmetries which, together with the Poincaré symmetry, dictate the possible form of the dynamics of the theory. After that, the symmetry group $SU(3)_C \times SU(2)_L \times U(1)_Y$ of the SM is presented. We end this chapter with a discussion of the breaking of $SU(2)_L \times U(1)_Y$ (weak isospin $\times$ hypercharge electroweak symmetry) via the Higgs mechanism which gives masses to the particles.

2.1 Mathematical Descriptions of Symmetries

2.1.1 Some Group Theory

By studying the composition of symmetry transformations, one can conclude that they form a group. In this section, we briefly discuss basic ideas of group theory, paying a particular attention to Lie groups and Lie algebras. We begin with the definition of group.

Definition 1. *A* group $(G, \circ)$ *consists of a set* G *together with a composition law denoted by* $\circ$ *which associates an element* $x \circ y \in G$ *to each pair of elements* $(x, y) \in G \times G$ *such that the following properties are satisfied:*

1. *Associativity:* $x \circ (y \circ z) = (x \circ y) \circ z$ *for all* $x, y, z \in G$.

2. *There exists an identity element,* $e \in G$*, such that* $e \circ x = x \circ e = x$ *for all* $x \in G$*.*

3. *For each* $x \in G$*, there exists an inverse element,* $x^{-1} \in G$*, such that* $x \circ x^{-1} = x^{-1} \circ x = e$*.*

The group is said to be abelian *if the commutative law* $x \circ y = y \circ x$ *holds for all* $x, y \in G$*.*

If the elements of G *only satisfy the first two properties, then* $(G, \circ)$ *is called a* semigroup.

The kind of groups that plays most important roles in particle physics is *Lie group* [5]. A Lie group is the group whose elements are parametrized by a set of continuous parameters, which are normally treated as the coordinates of a manifold, called a *group manifold*, whose dimension is called the *dimension* of the Lie group. By convention, these parameters are set to zero for an identity element; this means that the identity element is associated with the origin of the coordinate system on the group manifold. A small deviation from the identity element (that is, the group element "nearby" the identity) is thus specified by a direction (or vector) from the origin, which is expressed as a linear combination of the basis vectors (with either real or complex coefficients which play the role of the continuous parameters) on the tangent space of the identity element. Such the basis vectors are called the *generators* of the Lie group.

Let A be a linear combination of the generators. Then it can be proved that any element g of a Lie group, which is connected to an identity element by a continuous curve on the group manifold, takes the form of the exponential map,

$$g = \exp(A).$$

Moreover, the group generators are required to form the basis of a *Lie algebra* [6] defined formally as follows.

Definition 2. *A* Lie algebra *consists of a vector space $\mathcal{L}$ over a field $\mathbb{F}$ (here $\mathbb{R}$ or $\mathbb{C}$) with a composition rule called product, written $\circ$, defined as follows:*

$$\circ \ : \ \mathcal{L} \times \mathcal{L} \longmapsto \mathcal{L}$$

If $v_1, v_2, v_3 \in \mathcal{L}$, then the following properties define the Lie algebra:

1. *Closure : $v_1 \circ v_2 \in \mathcal{L}$.*

2. *Linearity : $v_1 \circ (v_2 + v_3) = v_1 \circ v_2 + v_1 \circ v_3$.*

3. *antisymmetry : $v_1 \circ v_2 = -v_2 \circ v_1$.*

4. *Jacobi identity : $v_1 \circ (v_2 + v_3) + v_2 \circ (v_3 + v_1) + v_3 \circ (v_1 + v_2) = 0$.*

For a Lie group of which the generators are basis vectors, the product $\circ$ is just a Lie bracket of two vectors.

It can be shown that most Lie groups are matrix groups, in which the elements are square matrices and the product of two group elements is just an ordinary matrix multiplication. In such cases, the generators are simply the square matrices; the exponential map is just an ordinary exponential of a matrix, defined as a Taylor series $\exp A = \sum_{n=0}^{\infty} A^n/n!$; and the Lie algebra product $\circ$ is a commutator of two matrices.

A familiar example of matrix Lie groups is a group of 2×2 unitary matrices with unit determinant, known as the special unitary group $SU(2, \mathbb{C})$ with complex parameters. Its generators are complex, traceless, antihermitian 2×2 matrices with the Lie product

$$a \circ b \equiv [a, b] = ab - ba. \tag{2.1}$$

The associated Lie algebra $\mathcal{L}$ is called $su(2, \mathbb{C})$. The most popular basis consists of three matrices

$$\tau_i = \frac{i}{2}\sigma_i$$

where the σ_i are the three Pauli matrices.

2.1.2 Lorentz and Poincaré Groups

In the special theory of relativity, one demands that physics be invariant under Lorentz transformations. The group of Lorentz transformations, called the *Lorentz group*, is a Lie group. In detail, the Lorentz group is the group which leaves an interval $(x-y)^2 \equiv (x^0-y^0)^2 - |\vec{x}-\vec{y}|^2$ in the Minkowski space invariant, i.e., all linear coordinate transformations

$$x \rightarrow x' = \Lambda x$$

such that $(x-y)^2 = (x'-y')^2$. Thus it is a special orthogonal group $SO(1,3)$, and its element $\Lambda \in SO(1,3)$ can be written in the exponential form

$$\Lambda = \left[\exp(-\frac{i}{2}\omega^{\rho\sigma}M_{\rho\sigma})\right] \tag{2.2}$$

where the generators $M_{\rho\sigma}$ and the parameters $\omega^{\rho\sigma}$ are antisymmetric in ρ and σ, and the factor i appears so as to make $(M_{\rho\sigma})$ hermitian. $M_{\mu\nu}$ are related to the rotation generators M_1, M_2, M_3 and the Lorentz boost generators N_1, N_2, N_3 by

$$(M_{\mu\nu}) = \begin{pmatrix} 0 & -K_1 & -K_2 & -K_3 \\ K_1 & 0 & J_3 & -J_2 \\ K_2 & -J_3 & 0 & J_1 \\ K_3 & J_2 & -J_1 & 0 \end{pmatrix},$$

where $J_l = iM_l$, $K_l = iN_l$. The 4×4 matrices $(M_{\mu\nu})$ constitute a basis of the Lie algebra $o(1,3)$, with the commutation relations

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\mu\nu}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}) \tag{2.3}$$

where $\eta_{\mu\nu} = diag(1,-1,-1,-1)$ is the metric tensor.

Besides the Lorentz transformations, one can see that a translation $T_x(a)$ which changes the coordinates x by

$$x \rightarrow x' \equiv T_x(a)x = x + a$$

with a a constant four-vector, also leaves the Minkowski interval invariant. Certainly, one can write $T_x(a)$ in the exponential form

$$T_x(a) = \exp[-ia^\lambda P_\lambda] \tag{2.4}$$

where P_λ are the translation generators.

By combining Lorentz transformations and translations, one obtains the *Poincaré group* which transforms the spacetime coordinates as

$$x \to x' = \Lambda x + a. \tag{2.5}$$

It is the largest group that leaves the Minkowski interval invariant. Thus the Poincaré group has the generators of Lorentz transformations $M_{\mu\nu}$ and the generators of translations P_μ satisfying the Lie algebra

$$\begin{aligned} [P_\mu, P_\nu] &= 0 && (2.6)\\ [M_{\mu\nu}, P_\lambda] &= -i(\eta_{\nu\lambda} P_\mu - \eta_{\mu\lambda} P_\nu) && (2.7)\\ [M_{\mu\nu}, M_{\rho\sigma}] &= -i(\eta_{\mu\nu} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho}). && (2.8) \end{aligned}$$

In quantum field theory, an elementary particle may be viewed as the "quantum" of a classical relativistic field $A_i(x)$ which may be labeled by some set of indices, denoted collectively by i. To form a relativistically invariant action, the field $A_i(x)$ must be a representation of the Poincaré group (that is, it must have the consistent transformation properties with respect to the Poincaré transformations) otherwise one cannot construct a Poincaré invariant action using the field together with some other objects, such as spacetime derivatives [7]. Thus the collective index i may consist of some indices responsible for the Poincaré transformations together with some other indices which make the field transform as some representations of other symmetry groups of the theory.

2.1.3 Representations of Lorentz and Poincaré Groups

So far, we have considered a group element as being an abstract mathematical object, defined by its composition rules with other group members. To incorporate the symmetries into the theory, we have to construct a concrete form of the group elements in terms of the objects that we already knew (such as matrices or differential operators) together with a space on which the group elements act. Such a space may be a finite dimensional vector space or a space of functions, and is called a "representation space."

Starting with the Lorentz group, we are interested in the representations in the form of functions of spacetime possibly with indices. More specifically, a set of objects φ^i, with $i = 1, \ldots, n$, is said to transform as an n dimensional representation of the Lorentz group if it transforms as

$$\varphi^i \to \varphi'^i \equiv \left[\exp(-\frac{i}{2}\omega^{\rho\sigma} M_{\rho\sigma})\right]^i{}_j \varphi^j \tag{2.9}$$

where $\left[\exp(-\frac{i}{2}\omega^{\rho\sigma} M_{\rho\sigma})\right]^i{}_j$ is a matrix representation of dimension n of the Lorentz group. If φ_i are also spacetime functions $\varphi_i(x)$, then the Lorentz transformation generally affects x.

The simplest example of this is the case of a scalar field $\phi(x)$, which is invariant under the Lorentz transformation in the sense that

$$\phi(x) \to \phi'(x') \equiv \phi'(\Lambda x) = \phi(x). \tag{2.10}$$

A more complicated example is given by a vector field $V^\mu(x)$ with one spacetime index, which under $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$, transforms as

$$V^\mu(x) \to V'^\mu(x') = \Lambda^\mu{}_\nu V^\nu(x). \tag{2.11}$$

Here the matrix representation in (2.9) is just a transformation matrix for the spacetime coordinates.

In general, a tensor field of arbitrary rank $T^{\mu_1,\ldots,\mu_m}{}_{\nu_1,\ldots,\nu_n}(x)$ can be built out of a vector by adding more indices, and transforms multilinearly with the transformation matrices Λ:

$$T^{\mu_1,\ldots,\mu_m}{}_{\nu_1,\ldots,\nu_n}(x) = \Lambda^{\mu_1}{}_{\mu'_1}\ldots\Lambda^{\mu_m}{}_{\mu'_m}\Lambda_{\nu_1}{}^{\nu'_1}\ldots\Lambda_{\nu_n}{}^{\nu'_n}T^{\mu'_1,\ldots,\mu'_m}{}_{\nu'_1,\ldots,\nu'_n}(x'). \quad (2.12)$$

These are tensor representations which yield a large class of relativistic fields. Besides the tensor representation, however, there are also spinor representations of the Lorentz group, which are less obvious. Such representations can be found by a trick due to Dirac.

We start with defining the 4×4 matrices γ^μ which satisfy

$$\{\gamma^\mu,\gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}I_{4\times 4}. \quad (2.13)$$

Then we can construct a representation of Lorentz generators as

$$\frac{1}{2}\Sigma^{\mu\nu} \equiv \frac{i}{4}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) \quad (2.14)$$

which satisfy the Lorentz algebra (2.8). The representation space on which $\Sigma^{\mu\nu}$ acts is 4-dimensional complex representation space, called the Dirac spinor Ψ_D. In quantum field theory, this representation describes a spin-1/2 particle.

A particularly useful representation of the gamma matrices is the Weyl representation defined by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \qquad (\mu = 0,1,2,3) \quad (2.15)$$

where

$$\sigma^\mu \equiv (I_{2\times 2}, \overrightarrow{\sigma}) \quad (2.16)$$

$$\overline{\sigma}^\mu \equiv (I_{2\times 2}, -\overrightarrow{\sigma}) = \sigma_\mu. \quad (2.17)$$

That this representation is useful is due to the fact that the “chirality” operator γ^5 defined by

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -I_{2\times 2} & 0 \\ 0 & I_{2\times 2} \end{pmatrix} \quad (2.18)$$

is block diagonal. Moreover, by substituting the representation (2.15) into (2.14), the Lorentz generators in this representation read

$$\frac{1}{2}\Sigma^{0i} = \frac{i}{4}\left[\gamma^0, \gamma^i\right] = -\frac{i}{2}\begin{pmatrix}\sigma^i & 0\\ 0 & -\sigma^i\end{pmatrix}, \tag{2.19}$$

and

$$\frac{1}{2}\Sigma^{ij} = \frac{i}{4}\left[\gamma^i, \gamma^j\right] = -\frac{1}{2}\epsilon^{ijk}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix} \equiv \frac{1}{2}\epsilon^{ijk}\Sigma^k \tag{2.20}$$

which are also block diagonal. This means that the 4×4 representation above is reducible. To decompose it into irreducible parts, we express the Dirac spinor Ψ_D in the Weyl representation as

$$\Psi_D = \begin{pmatrix}\Psi_L\\ \Psi_R\end{pmatrix} \tag{2.21}$$

where

$$\Psi_L = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}\psi_A\end{pmatrix} \qquad (A = 1, 2) \tag{2.22}$$

$$\Psi_R = \begin{pmatrix}\overline{\chi}^{\dot{1}}\\ \overline{\chi}^{\dot{2}}\end{pmatrix} = \begin{pmatrix}\overline{\chi}^{\dot{A}}\end{pmatrix} \qquad (\dot{A} = 1, 2). \tag{2.23}$$

The two component objects Ψ_L and Ψ_R, called the left-handed and the right-handed Weyl spinors respectively, are irreducible representations of the Lorentz group. The reason behind the names left-handed and right-handed is as follows. In field theory, these spinors satisfy the Dirac equations. If the masses of these spinors are zero, then it can be shown that the particle's spin is always parallel (anti-parallel) to its momentum for the right-handed (left-handed) spinor. Thus the operators $P_L = (1-\gamma^5)/2$ and $P_R = (1+\gamma^5)/2$ are projection operators that project out the left-handed and right-handed parts of the Dirac spinor respectively, hence the name chirality operator for γ^5.

It is easily seen that, under an infinitesimal rotation with parameters θ^i (and Σ^{0i} as generators) and an infinitesimal boost with parameters β^i (and Σ^{ij}

as generators), these spinors change by

$$\Psi_L \rightarrow \left(1 - i\theta\cdot\frac{\sigma}{2} - \beta\cdot\frac{\sigma}{2}\right)\Psi_L = \Lambda_L\Psi_L \tag{2.24}$$

$$\Psi_R \rightarrow \left(1 - i\theta\cdot\frac{\sigma}{2} + \beta\cdot\frac{\sigma}{2}\right)\Psi_R = \Lambda_R\Psi_R. \tag{2.25}$$

From (2.24) and (2.25), we obtain the properties

$$\Lambda_L^\dagger\Lambda_R = \Lambda_R^\dagger\Lambda_L = 1, \tag{2.26}$$

where Λ_L and Λ_R are Lorentz transformation operators for left-handed Weyl and right-handed Weyl spinors respectively.

We now discuss some algebras of Weyl spinors. The indices of ψ_A are raised and the indices of $\overline{\chi}^{\dot{A}}$ are lowered using the matrices

$$\epsilon = \left(\epsilon_{AB}\right) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \left(\epsilon^{AB}\right)^{-1} \tag{2.27}$$

and

$$\overline{\epsilon} = \left(\epsilon^{\dot{A}\dot{B}}\right) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \left(\epsilon_{\dot{A}\dot{B}}\right)^{-1} \tag{2.28}$$

according to the rule

$$\psi_A = \epsilon^{AB}\psi_B \qquad \text{and} \qquad \overline{\chi}_{\dot{A}} = \epsilon_{\dot{A}\dot{B}}\overline{\chi}^{\dot{B}}. \tag{2.29}$$

Short-hand notations for summations over indices are defined differently for dotted and undotted indices. For undotted indices, sum over indices is defined according to the northwest-southeast rule, while the southwest-northeast rule is applied for the summation over dotted indices:

$$(\psi\chi) = \psi^A\chi_A = \epsilon^{AB}\psi_B\chi_A = \psi_2\chi_1 - \psi_1\chi_2 \tag{2.30}$$

$$(\overline{\psi}\overline{\chi}) = \overline{\psi}_{\dot{A}}\overline{\chi}^{\dot{A}} = \epsilon_{\dot{A}\dot{B}}\overline{\psi}^{\dot{B}}\overline{\chi}^{\dot{A}} = \overline{\psi}^{\dot{1}}\overline{\chi}^{\dot{2}} - \overline{\psi}^{\dot{2}}\overline{\chi}^{\dot{1}}. \tag{2.31}$$

For σ matrices, they have mixed indices as

$$\sigma^\mu = \left(\sigma^\mu_{A\dot{B}}\right), \qquad \overline{\sigma}^\mu = \left(\overline{\sigma}^{\mu\dot{A}B}\right) \tag{2.32}$$

since they provide a mixing of left-handed and right-handed Weyl spinors according to (2.15). It can be checked that σ^μ and $\overline{\sigma}^\mu$ are related as follows:

$$\sigma^\mu_{A\dot{A}} = \epsilon_{AB}\epsilon_{\dot{A}\dot{B}}\overline{\sigma}^{\mu\dot{B}B} \tag{2.33}$$

$$\overline{\sigma}^{\mu\dot{A}A} = \epsilon^{AB}\epsilon^{\dot{A}\dot{B}}\sigma^\mu_{B\dot{B}} \quad . \tag{2.34}$$

One can also show that $\sigma^2\psi^*$ transforms like $\overline{\chi}$ under the Lorentz transformation and that $\sigma^2\overline{\chi}^*$ transforms like ψ. Left-handed and right-handed Weyl spinors thus can transform into one another by complex conjugation:

$$(\psi_A)^* = \overline{\psi}^{\dot{A}}. \tag{2.35}$$

As a Dirac spinor is a direct sum of two irreducible representations of the Lorentz group, there is a standard way of reducing the number of degrees of freedom of the Dirac spinor so that the resulting spinor contains only one irreducible representation. One defines a Majorana spinor Ψ_M as a Dirac spinor in which ψ_A and $\overline{\chi}^{\dot{A}}$ are not independent, but are related by $\overline{\chi}^{\dot{A}} = i\overline{\sigma}^2\psi^*_A$. Thus

$$\Psi_M = \begin{pmatrix} \psi_A \\ i\overline{\sigma}^2\psi^*_A \end{pmatrix} = \begin{pmatrix} \psi_A \\ \overline{\psi}^{\dot{A}} \end{pmatrix}. \tag{2.36}$$

Thus, it has the same number of degrees of freedom as that of a Weyl spinor, although it is written in the form of a Dirac spinor. From this definition, it follows that a Majorana spinor is invariant under the charge conjugation defined by

$$\Psi^c_M \equiv C\overline{\Psi}^T_M = \Psi_M \tag{2.37}$$

where

$$C = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\overline{\sigma}^2 \end{pmatrix} \tag{2.38}$$

and

$$\overline{\Psi} = \Psi^\dagger\gamma^0. \tag{2.39}$$

We see that the Majorana spinor is invariant under "complex conjugation," and so it is sometimes called a real spinor. Note that the process of reducing the degrees of freedom above is equivalent to imposing the constraint $\Psi = \Psi^c$ on the Dirac spinor.

So far, all necessary field representations of the Lorentz group have been considered, but the representations of the Poincaré group have not been discussed yet. As mentioned earlier, the Poincaré group is the largest group that leaves the metric in the Minkowski space invariant. Thus, to have a theory whose physics does not depend on both the Lorentz frame and the origin of the coordinate system, all fields must be the representations of the Poincaré group. In particular, after the theory is quantized, the basis of the Hilbert space of a free particle is considered the representation of the Ponicaré group.

To label the physical states, we need the operators whose eigenvalues are invariant under the action of all elements of the Ponicaré group. Thus these operators must commute with all generators of the Ponicaré algebra, and they are called the Casimir operators. As the Casimir operators are normally constructed from the generators, they commute among themselves and so have simultaneous eigenvectors and eigenvalues. As a result, all eigenvectors with the same set of eigenvalues of the Casimir operators form an irreducible representation of the group.

There are two Casimir operators for the Poincaré group. The first one is quite obvious; we first observe that $P^2 = m^2$, or the mass squared, commutes with all generators and is therefore a Casimir operator. Under the Lorentz transformations, it transforms as a scalar and hence is invariant. Certainly, it is invariant under translations because all translations commute.

To find the other Casimir operator, one introduces

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\varrho\sigma} P_\nu M_{\rho\sigma} \tag{2.40}$$

which is called the Pauli-Lubanski tensor. Using the commutation relations of the Poincaré group, one can verify that the square of this tensor W^2 is a Casimir operator. Thus all physical states in quantum field theory can be labeled with the eigenvalues of these two Casimir operators. However, the physical significance of W^2 operator is not easy to understand.

To find the physical significance of W^2, let us consider the rest frame of a massive particle: $P_\mu = (m, \vec{0})$. Inserting this into (2.40), we find that

$$W^i = -\frac{1}{2} m \epsilon_{ijk0} M^{jk} \tag{2.41}$$

$$= -mJ_i \tag{2.42}$$

$$W^0 = 0 \tag{2.43}$$

where J_i is just the usual rotation matrix in three dimensions. Thus, in the rest frame of a massive particle, the Pauli-Lubanski tensor is just the spin generators. Its square is therefore the Casimir of $SO(3)$ (rotation group), which we know yields the spin of the particle:

$$W^2 = -m^2 s(s+1) \tag{2.44}$$

where s is the spin of the particle.

However, we have not discussed massless particles. Since the massless particle has $m^2 = 0$, then $P^2 = 0$. If moreover $W^2 = 0$, or

$$W \cdot W|p\rangle = 0, \qquad P \cdot P|p\rangle = 0 \tag{2.45}$$

where $|p\rangle$ is a particle state of momentum p which belongs to the subspace $P^2 = W^2 = 0$. From (2.40), it is clear that W^μ and P^μ are orthogonal:

$$W \cdot P = 0 \Rightarrow W \cdot P|p\rangle = 0. \tag{2.46}$$

From (2.45) and (2.46), we conclude that they must be proportional to each other,

$$\left(W^\mu - \lambda P^\mu\right)|p\rangle = 0 \tag{2.47}$$

with λ being a proportionate constant. Thus the massless states in this subspace can be characterized by one number λ, which is the ratio of W^μ and P^μ and so has the dimension of the angular momentum. It is called the *helicity*. If the parity is included, the helicity can have both plus and minus signs, $\pm|\lambda|$. To see what λ actually is, we use the frame in which $P^\mu = (P, 0, 0, P)$. In this frame, $P^0 = P = |\vec{P}|$ and $W^0 = \epsilon^{0ijk} P_i M_{jk}/2 = \vec{P} \cdot \vec{S}$ where $\vec{S}$ is the spin of the particle. This implies $\lambda = \vec{P} \cdot \vec{S}/|\vec{P}|$ is a projection of the particle's spin along its 3-momentum direction, hence the name helicity.

Now, we can label all one-particle states with the eigenvalues of these Casimir operators. A complete list of states is give in terms of the mass (m), spin (s) and helicity (λ):

$$P^2 > 0 : \qquad |m, s\rangle, s = 0, 1/2, 1, 3/2 \ldots \tag{2.48}$$

$$P^2 = 0 : \qquad | \pm \lambda\rangle. \tag{2.49}$$

2.2 Dynamics of Fields from Symmetries

In the previous section, we considered the fields as the representations of, but not limited to, the Poincaré group. In this section, we study the dynamics of fields using only the symmetry principles.

2.2.1 The Actions with Lorentz Symmetry

When you believe that the law of physics does not change upon some transformations, it is the law of symmetry. From the modern point of view, the laws of physics are described by the actions. Thus to say that a law of physics has a symmetry is equivalent to saying that the action is invariant under the transformations associated with that symmetry. Thus the symmetries dictate the possible forms of the action one would like to construct.

For a real scalar field, an action describing the non-trivial dynamics must contain $\partial_\mu\phi$. In order to be a Lorentz invariant action, the index μ must be contracted with another factor $\partial^\mu\phi$. Therefore the kinetic term must be proportional to two time derivatives $\partial_\mu\phi\partial^\mu\phi$ (because we don't know how to quantize actions with more than two time derivatives). The other terms involve the polynomials of ϕ. If we moreover demand that the action of the scalar field be invariant under the transformation $\phi \to -\phi$ so as to exclude terms of odd power in ϕ (which may render the potential unstable) form the action, we end up with the Lorentz invariant action of the form

$$S(\phi) \;=\; \int d^4x \left[\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4 + \cdots\right], \tag{2.50}$$

where various numerical factors have been put in for convenience. The terms without the spacetime derivatives form the negative of the potential $V(\phi)$, so we generally write

$$\begin{aligned} S(\phi) &= \int d^4x\, \mathcal{L}(\phi) && (2.51)\\ &= \int d^4x \left[\frac{1}{2}(\partial_\mu\phi)^2 - V(\phi)\right], && (2.52) \end{aligned}$$

where $\mathcal{L}(\phi)$ is the Lagrangian of the scalar field. The quadratic term of the potential is the mass term, while higher powers, like ϕ^4, ϕ^6, etc. give non-linear contributions to the equation of motion, and therefore correspond to self-interactions.

For spinor fields, each term in the Lagrangian is non-trivial. To construct the scalar terms from spinors, one notices that $\Psi_L^\dagger\Psi_R$ and $\Psi_R^\dagger\Psi_L$ are Lorentz scalars:

$$\begin{aligned} \Psi_L^\dagger\Psi_R &\to \Psi_L^\dagger\Lambda_L^\dagger\Lambda_R\Psi_R = \Psi_L^\dagger\Psi_R && (2.53)\\ \Psi_R^\dagger\Psi_L &\to \Psi_R^\dagger\Lambda_R^\dagger\Lambda_L\Psi_L = \Psi_R^\dagger\Psi_L, && (2.54) \end{aligned}$$

where the properties $\Lambda_L^\dagger\Lambda_R = \Lambda_R^\dagger\Lambda_L = 1$ have been used.

To construct vector quantities from spinors, more properties of γ matrices are needed. Using the antisymmetry of $\omega_{\mu\nu}$ and (2.14), we find

$$\left[\frac{i}{4}\omega_{\mu\nu}\Sigma^{\mu\nu}, \gamma^{\lambda}\right] = -\frac{1}{4}\omega_{\mu\nu}\left[\gamma^{\mu}\gamma^{\nu}, \gamma^{\lambda}\right] \tag{2.55}$$

$$= -\frac{1}{4}\omega_{\mu\nu}\left(\gamma^{\mu}\gamma^{\nu}\gamma^{\lambda} - \gamma^{\lambda}\gamma^{\mu}\gamma^{\nu}\right) \tag{2.56}$$

$$= \frac{1}{4}\omega_{\mu\nu}\left(\gamma^{\lambda}\gamma^{\mu}\gamma^{\nu} + \gamma^{\mu}\gamma^{\lambda}\gamma^{\nu} - 2\gamma^{\mu}g^{\lambda\nu}\right) \tag{2.57}$$

$$= \frac{1}{2}\omega_{\mu\nu}\left(g^{\lambda\mu}\gamma^{\nu} - \gamma^{\mu}g^{\lambda\nu}\right) \tag{2.58}$$

$$= \omega^{\lambda}{}_{\mu}\gamma^{\mu}, \tag{2.59}$$

or equivalently,

$$\left(1 + \frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right)\gamma^{\mu}\left(1 - \frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right) = \left(1 - \frac{i}{2}\omega^{\lambda}{}_{\mu}\right)\gamma^{\nu}. \tag{2.60}$$

This equation is just the infinitesimal form of

$$\Lambda_{\frac{1}{2}}^{-1}\gamma^{\mu}\Lambda_{\frac{1}{2}} = \Lambda^{\mu}{}_{\nu}\gamma^{\nu}, \tag{2.61}$$

where

$$\Lambda_{\frac{1}{2}} = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right). \tag{2.62}$$

By using (2.61) and the properties $\Lambda_L^{\dagger} = \Lambda_R^{-1}$ and $\Lambda_R^{\dagger} = \Lambda_L^{-1}$, it is clear that the quantities $\Psi_L^{\dagger}\overline{\sigma}^{\mu}\Psi_L$ and $\Psi_R^{\dagger}\sigma^{\mu}\Psi_R$ transform as four-vectors. Therefore it is possible to write the Lorentz invariant kinetic terms which are of first order in the derivatives,

$$\mathcal{L}(\Psi_L) = i\Psi_L^{\dagger}\overline{\sigma}^{\mu}\partial_{\mu}\Psi_L \tag{2.63}$$

and

$$\mathcal{L}(\Psi_R) = i\Psi_R^{\dagger}\sigma^{\mu}\partial_{\mu}\Psi_R. \tag{2.64}$$

Now, the spinor action is defined as

$$S(\psi) = \int d^4x\,\mathcal{L}_D \tag{2.65}$$

$$= \int d^4x\left[\Psi_L^{\dagger}\overline{\sigma}^{\mu}\partial_{\mu}\Psi_L + i\Psi_R^{\dagger}\overline{\sigma}^{\mu}\partial_{\mu}\Psi_R + m(\Psi_L^{\dagger}\Psi_R + \Psi_R^{\dagger}\Psi_L)\right], \tag{2.66}$$

where $\mathcal{L}_D$ denotes the Dirac Lagrangian. In terms of the Dirac spinor, the above action reads

$$S(\psi) = \left[\int d^4x \overline{\Psi}_D(i\gamma^\mu\partial_\mu - m)\Psi_D\right] \tag{2.67}$$

which is a compact form.

Finally, we consider the Lorentz invariant action of the **vector fields**. An electromagnetic field is a good example of the vector fields. It is described by a four-vector A_μ, the gauge potential. The field strength tensor is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{2.68}$$

which is related to the electric and magnetic fields by

$$F^{0i} = \partial^0 A^i - \partial^i A^0 = -E^i, \tag{2.69}$$

$$F^{ij} = -\epsilon^{ijk} B^k. \tag{2.70}$$

The Maxwell equations in the absence of sources can be derived from the action:

$$S_{EM} = \int d^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right] = \int d^4x \left[\frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2)\right]. \tag{2.71}$$

So far, we have considered the construction of the actions based on only the requirement of Lorentz invariance. The actions considered up to now only describe free fields and some self-interactions, and therefore do not give the real descriptions of Nature. In the real world, different fields interact with one another, such as the interactions of charged particles with the electromagnetic fields. We therefore need to consider more general forms of the action, possibly with some new invariance principles; this will be done in the following subsection.

2.2.2 Interactions from Gauge Symmetries

The Lorentz symmetry considered in the previous subsection is not enough for modeling all the particle interactions. As is well known nowadays, fundamental interactions among elementary particles are based on the existence of another kind of symmetry, known as gauge symmetry.

To discuss this kind of symmetry in detail, consider a set of N scalar fields forming a vector in an "internal space,"

$$\mathbf{\Phi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_N \end{pmatrix} \tag{2.72}$$

subject to the internal symmetry transformations

$$U \equiv \exp\left[-iT^a \alpha^a\right], \tag{2.73}$$

where $a = 1, 2, \ldots, n$ (n is the number of generators of the transformation), the generators T^a are $N \times N$ matrices and $\alpha^a \in \mathbb{R}$. If α^a are constant at every spacetime point, an internal symmetry transformation is called a *global phase transformation*. On the other hand, if α^a are spacetime dependent, the internal symmetry transformation is called a *local phase transformation* or *gauge transformation*. We will see that the latter case is responsible for most interactions in the SM as the theory is required to be invariant under the local gauge transformations. The Nœther's theorem to be discussed later says that the invariance of the equations of motion under a continuous symmetry implies the existence of a conserved charge. Thus there are conserved quantities in the SM associated with the local gauge symmetries.

The theory invariant under the local phase transformations is called gauge theory. The first version of such theory is the electromagnetic theory of Maxwell with the local phase transformations forming an Abelian group. Thus it is an Abelian gauge theory, in contrast to the non-Abelian Yang-Mills theories which are based on non-Abelian groups. Since the $SU(2)$ symmetry of the Standard Model is a non-Abelian gauge symmetry, we will first take a look at the *gauge principle* before continuing to the full Standard Model gauge group.

2.2.3 Gauge Principle

We start with the simplest gauge theory in the SM based on an Abelian Lie group of complex numbers of modulus one, called $U(1)$. As we shall see, the requirement that the theory be invariant under the Abelian $U(1)$ gauge symmetry implies the existence of a massless vector boson (photon) which mediates the electromagnetic interactions. By *local* $U(1)$ *symmetry*, we mean that the field actions are invariant under the following transformations:

$$\Psi_D(x) \;\; \rightarrow \;\; \Psi'_D(x) = \exp[-i\alpha(x)]\Psi_D(x) \tag{2.74}$$

$$\overline{\Psi}_D(x) \;\; \rightarrow \;\; \overline{\Psi}'_D(x) = \exp[i\alpha(x)]\overline{\Psi}_D(x). \tag{2.75}$$

Because the derivatives in the Dirac equation act on $\alpha(x)$ as well as on the field $\Psi_D(x)$, one can show that the free Dirac Lagrangian is not invariant under this transformation,

$$\mathcal{L}_0 \;\; = \;\; \overline{\Psi}_D(x)(i\gamma^\mu\partial_\mu - m)\Psi_D(x) \tag{2.76}$$

changes to

$$\begin{aligned} \mathcal{L}'_0 \;\; = \;\; & \overline{\Psi}_D(x)(i\gamma^\mu\partial_\mu - m)\Psi_D(x) \\ & + \overline{\Psi}_D(x)\gamma^\mu\Psi_D(x)\partial_\mu\alpha(x). \end{aligned} \tag{2.77}$$

To make the Dirac Lagrangian invariant under the $U(1)$ transformations, one introduces the *gauge field* A_μ through the *minimal coupling* with coupling constant e,

$$D_\mu \equiv \partial_\mu + ieA_\mu, \tag{2.78}$$

where D_μ is called a *covariant derivative*, and demands that $D_\mu\Psi_D$ transforms in the same way as the spinor field does,

$$D_\mu\Psi_D \rightarrow \exp[-i\alpha(x)](D_\mu\Psi_D). \tag{2.79}$$

This can be accomplished if A_μ transforms as

$$A_\mu \rightarrow A'_\mu = A_\mu + \frac{1}{e}\partial_\mu \alpha. \tag{2.80}$$

The gauge field A_μ is identified with the electromagnetic 4-dimensional potential in the previous subsection.

From the covariant property of D_μ, one can construct new covariant objects from the products of covariant derivatives. For example, consider the anti-symmetric product of two covariant derivatives acting on a spinor, it transforms covariantly, i.e.,

$$\begin{aligned}
[D_\mu, D_\nu]\,\Psi_D &\rightarrow D'_\mu\,(D'_\nu)\,\Psi'_D - D'_\mu\,(D'_\nu)\,\Psi'_D && (2.81)\\
&= D_\mu\left(e^{-i\alpha(x)} D_\nu \Psi\right) - D_\mu\left(e^{-i\alpha(x)} D_\nu \Psi\right) && (2.82)\\
&= e^{-i\alpha(x)}\,(D_\mu D_\nu \Psi) - e^{-i\alpha(x)}\,(D_\mu D_\nu \Psi) && (2.83)\\
&= e^{-i\alpha(x)}\,[D_\mu, D_\nu]\,\Psi. && (2.84)
\end{aligned}$$

If one write $[D_\mu, D_\nu]\,\Psi_D$ in terms of A_μ, one finds

$$\begin{aligned}
[D_\mu, D_\nu] &= ie\,[\partial_\mu A_\nu - \partial_\nu A_\mu] && (2.85)\\
&= ieF_{\mu\nu} && (2.86)
\end{aligned}$$

or

$$F_{\mu\nu} = \frac{-i}{e}\,[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.87}$$

Thus $F_{\mu\nu}$ is gauge invariant and is the field strength in (2.68).

It is now straightforward to couple a Dirac field of charge e to the electromagnetic field by replacing ∂_μ by D_μ in the Dirac Lagrangian as

$$\mathcal{L}_D = \overline{\Psi}_D(x)(i\gamma^\mu D_\mu - m)\Psi_D(x). \tag{2.88}$$

Then the Dirac Lagrangian is invariant under the $U(1)$ transformation:

$$
\begin{aligned}
\mathcal{L}_D \to \mathcal{L}'_D &= \overline{\Psi}'_D(x)(i\gamma^\mu D'_\mu - m)\Psi'_D(x) \\
&= \overline{\Psi}_D(x)\exp[+i\alpha(x)]\left\{i\gamma^\mu\partial_\mu - e\left(\gamma^\mu A_\mu + \frac{1}{e}i\gamma^\mu\partial_\mu\right) - m\right\} \\
&\quad \times \exp[-i\alpha(x)]\Psi_D(x) \\
&= \overline{\Psi}_D(x)(i\gamma^\mu D_\mu - m)\Psi_D(x) \\
&= \mathcal{L}_D.
\end{aligned}
$$

From (2.88), one can conclude that "the requirement that the theory be invariant under gauge transformations imposes the specific form of the interactions with the gauge fields." In other words, the symmetries imply dynamics.

Instead of an Abelian Lie group $U(1)$, one may consider a non-Abelian Lie group. This idea was implemented by Utiyama in 1956 for any Abelian group G with generators t_a satisfying the Lie algebra

$$[t_a, t_b] = iC^{abc}t_c \tag{2.89}$$

with C_{abc} being the structure constants of the group. Let the multiplet of (scalar or spinor) fields

$$\mathbf{\Phi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_N \end{pmatrix}.$$

transform according to

$$
\begin{aligned}
\mathbf{\Phi}(x) \to \mathbf{\Phi}'(x) &= \exp\left[-i\mathbf{T}\cdot\alpha(x)\right]\mathbf{\Phi}(x) && (2.90) \\
&\equiv U(\alpha)\mathbf{\Phi}(x), && (2.91)
\end{aligned}
$$

where T_a $(a = 1, \ldots, n)$ are $N \times N$ matrices representing the generator t_a and $\alpha_a(x)$ $(a = 1, \ldots, n)$ are arbitrary function of space-time. To make the Lagrangian of $\mathbf{\Phi}$ invariant under the non-Abelian transformations, one gauge field for each generator is introduced. The *covariant derivative* is defined using the gauge field as

$$D_\mu \equiv \partial_\mu + igT^a A^a_\mu. \tag{2.92}$$

To ensure that the Lagrangian is invariant under the local non-Abelian gauge transformation, the covariant derivative acting on the field must transform like the field itself, i.e., $D_\mu \mathbf{\Phi} \to U(D_\mu \mathbf{\Phi})$. To achieve this, one requires that

$$T^a A^a_\mu \to U\left(T^a A^a_\mu + \frac{i}{g}\partial_\mu\right) U^{-1}. \tag{2.93}$$

Finally, we obtain the field strength tensor for a non-Abelian Lie group in the same way as we did in the case of an Abelian symmetry. Thus

$$\begin{aligned} T^a F^a_{\mu\nu} &= \frac{-i}{g}\left[D_\mu, D_\nu\right] &(2.94)\\ &= \partial_\mu T^a A^a_\nu - \partial_\nu T^a A^a_\mu - ig\left[T^a A^a_\mu, T^a A^a_\nu\right] &(2.95)\end{aligned}$$

or

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gC^{abc} A^b_\mu A^c_\nu. \tag{2.96}$$

Thus $F^a_{\mu\nu}$ is a gauge covariant quantity and transforms as

$$igF^a_{\mu\nu}T^a \;\to\; U(\alpha) igF^a_{\mu\nu}T^a U^{-1}(\alpha). \tag{2.97}$$

Similar to the electromagnetic case, the gauge-invariant kinetic term of the non-Abelian gauge bosons takes the form

$$\mathcal{L}_{kin} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}. \tag{2.98}$$

2.2.4 Conservation Laws from Symmetries

In the last two subsections, we saw that the local symmetries of the action demand the existence of the gauge fields and also put the restrictions on the possible forms of the terms in the Lagrangian. There is another important property of the theory with local symmetries discovered by Emmy Nœther. The theorem of Nœther states that for every continuous symmetry of the action there results a conserved quantity. From this theorem, the symmetries and conservation laws are connected. To appreciate the theorem, we review its derivation.

Consider a field ϕ, which transforms as

$$\phi(x) \rightarrow \phi'(x) = \phi(x) + \alpha\Delta\phi(x), \tag{2.99}$$

where α is an infinitesimal continuous parameter and $\triangle\phi(x)$ is some deformation of the field configuration. The changed Lagrangian which is the result of the field change is

$$\begin{aligned} \alpha\Delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi}(\alpha\Delta\phi) + \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\partial_\mu(\alpha\Delta\phi) \\ &= \alpha\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) + \alpha\left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right]\Delta\phi. \end{aligned} \tag{2.100}$$

The second term vanishes by the Euler-Lagrange equations.

If this transformation is a symmetry transformation, the Lagrangian does not change or changes by a 4-divergence term:[1]

$$\mathcal{L}(x) \rightarrow \mathcal{L}(x) + \alpha\partial_\mu J^\mu(x), \tag{2.101}$$

for some $J^\mu(x)$.

To compare (2.100) with (2.101), one finds that

$$\begin{aligned} \alpha\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) &= \alpha\partial_\mu J^\mu(x) \\ \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - J^\mu\right) &= 0 \\ \partial_\mu j^\mu(x) &= 0, \end{aligned} \tag{2.102}$$

[1]Under this condition, the action is invariant or changes by a surface term respectively, and the Euler-Lagrange equations of motion are not affected.

where $j^\mu(x) = \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - J^\mu\right)$ and is called a current. (2.102) says that the current $j^\mu(x)$ is conserved. By integrating (2.102) over the spacelike hypersurface in the Minkowski space and demanding that the current vanishes at spatial infinity, one finds that the charge

$$Q \equiv \int j^0 d^3x \tag{2.103}$$

is constant in time, and hence conserved. An important property of this conserved charge is that, using the methods of either classical or quantum field theory, one can show that it generates the infinitesimal transformation associated with it. What this really means is that:

- In classical field theory, the infinitesimal change of the field (divided by the transformation parameter) is obtained by calculating the Poisson bracket of the conserved charge and the field.

- In quantum field theory, the charge is the generator of the unitary transformations on the field operator, and on the Hilbert space of the states.

In the case of rotations, for example, the infinitesimal transformation of the field under a rotation about the x_3 axis is

$$\phi \to \phi' = (1 - i\epsilon L_3)\phi, \tag{2.104}$$

where L_3 is the third component of the quantum-mechanical orbital angular momentum operator. In quantum field theory in which the field is an operator, then the field must transform according to the rule

$$\phi \to \phi' = U\phi U^{-1}, \tag{2.105}$$

under such rotation, where infinitesimally

$$U = \left(1 - i\epsilon M^3\right) \tag{2.106}$$

where M^3 is a rotation generator about x_3 axis. Thus (2.104) and (2.105) agree only if

$$\left[M^3, \phi\right] = -L_3\phi \tag{2.107}$$

or

$$\left[i\epsilon M^3, \phi\right] = \delta\phi. \tag{2.108}$$

If the action is invariant under this rotational symmetry, then the above consideration leads to the expression of the corresponding charge Q in terms of the field ϕ. Once ϕ becomes an operator with the usual commutation relations, one can check that $[Q, \phi] = -L_3\phi$ so that the charge Q is indeed the rotation generator M^3.

2.2.5 Gauge Groups of the Standard Model

The gauge group of the Standard Model is the direct product group $SU(3)_C \times SU(2)_L \times U(1)_Y$, which means that the three factors separated by the $\times$ sign commute. This gauge group is composed of the symmetry group of the strong interactions, $SU(3)_C$, and the symmetry group of the electroweak interactions, $SU(2)_L \times U(1)_Y$. The symmetry group of the electromagnetic interactions, $U(1)_{EM}$, is just a subgroup of $SU(2)_L \times U(1)_Y$. In this sense, the weak and electromagnetic interactions are said to be unified. The $SU(3)_C$ is believed to be an exactly symmetry while the $SU(2)_L \times U(1)_Y$ is spontaneous symmetry broken. The $SU(2)_L \times U(1)_Y$ symmetry is broken through the *Higgs mechanism.* This mechanism gives the masses of the $W^{\pm}$ and Z bosons as well as the mass splitting among the leptons through their interactions with the *Higgs field.* Because the Higgs field is the main topic of this thesis, we put emphasis on the electroweak theory and Higgs mechanism in the remain part of this chapter.

2.3 The Electroweak Theory without Spontaneous Symmetry Breaking

In the first step of constructing the electroweak theory, one identifies the appropriate representations of $SU(2) \times U(1)$. We begin with the lepton sector. For the $SU(2)$ part, the left-handed parts of the charged leptons and neutrinos form doublets of $SU(2)$ with a charged lepton on the top and the corresponding neutrino at the bottom:

$$L = \begin{pmatrix} \nu \\ l \end{pmatrix}_L \tag{2.109}$$

where the family index has been suppressed. The right-handed charged leptons, on the contrary, are $SU(2)$ singlets:

$$R = l_R. \tag{2.110}$$

Thus this $SU(2)$ is normally referred to as $SU(2)_L$ where L stands for "left-handed."

L transforms under the weak isospin transformation (a specific name for the 2-dimensional representation of the $SU(2)$ group) as

$$L(x) \quad \rightarrow \quad e^{-\frac{1}{2} i\alpha^a \tau^a} L(x), \tag{2.111}$$

$$\overline{L}(x) \quad \rightarrow \quad e^{\frac{1}{2} i\alpha^a \tau^a} \overline{L}(x), \tag{2.112}$$

where τ are the 2×2 Pauli matrices and α is spacetime independent, while $R(x)$ is invariant,

$$R(x) \quad \rightarrow \quad R(x), \tag{2.113}$$

$$\overline{R}(x) \quad \rightarrow \quad \overline{R}(x). \tag{2.114}$$

For the transformation under a global $U(1)$ group, each component of $L(x)$ are multiplied by the same phase factor, $e^{-i\alpha}$,

$$L \quad \rightarrow \quad \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} L, \tag{2.115}$$

while

$$R \;\rightarrow\; e^{i\alpha}R. \tag{2.116}$$

The kinetic terms of the lepton Lagrangian take the form

$$\mathcal{L}_{\text{kin}} = \overline{L}i\gamma^{\mu}\partial_{\mu}L + \overline{R}i\gamma^{\mu}\partial_{\mu}R. \tag{2.117}$$

and thus is invariant under both weak isospin and global $U(1)$ transformations.

Because one requires that the gauge group $U(1)_Q$ of electromagnetic interactions is a subgroup of $SU(2)\times U(1)$, then $L(x)$ and $R(x)$ should transform under the $U(1)_Q$ group. The $U(1)_Q$ is generated by the electric charge Q. From the fact that each charged lepton has charge -1 and the neutrino has no charge (all charges are given in units of the elementary charge e), Q acts on $L(x)$ and $R(x)$ by

$$QL(x) \;=\; \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} L(x) \tag{2.118}$$

$$QR(x) \;=\; -R(x). \tag{2.119}$$

Thus $U(1)_Q$ transforms L as

$$\begin{pmatrix} \nu \\ l \end{pmatrix}_L \rightarrow \begin{pmatrix} \nu' \\ l' \end{pmatrix}_L \;=\; e^{-iQ\alpha}\begin{pmatrix} \nu \\ l \end{pmatrix}_L \tag{2.120}$$

$$=\; \begin{pmatrix} e^{-i(0)\alpha}\nu \\ e^{-i(-1)\alpha}l \end{pmatrix}_L \tag{2.121}$$

$$=\; \begin{pmatrix} \nu \\ e^{i\alpha}l \end{pmatrix}_L. \tag{2.122}$$

Note that $U(1)_Q$ affects only the l_L component of L.

To see how $U(1)_Q$ is embedded in $SU(2)\times U(1)$, we observe that its generator Q, when action on L, can be expressed as

$$Q = \frac{1}{2}\tau_3 - \frac{1}{2}I. \tag{2.123}$$

Let the weak $SU(2)$ generators be denoted by T_i, $i = 1,\,2,\,3$, and satisfy

$$[T_i, T_j] = i\epsilon_{ijk}T_k, \tag{2.124}$$

and the generator of the $U(1)$ group be Y. Y is called the *weak hypercharge*, so this $U(1)$ is referred to as $U(1)_Y$. The explicit form of T^i acting on the weak isospin doublet L is $T^i = \frac{\tau^i}{2}$, while its representation acting on the singlet R is $T^i = 0$. The values of the weak hypercharge for leptons are assigned as follows: $Y_L = -\frac{1}{2}$ for $SU(2)$ doublets and $Y_R = -1$ for singlets, as implied by (2.115) and (2.116). With the above assignments, the generator Q in (2.123) takes the form of a linear sum of two generators, one for each group in the direct product,

$$Q = T_3 + Y. \tag{2.125}$$

Thus the eigenvalue of Q for each lepton is really its electric charge.

To construct the electroweak interactions, the global symmetry of L_{kin} has to be implemented to a local symmetry, and so the appropriate gauge fields have to be introduced:

$$SU(2)_L \;\Rightarrow\; W_\mu^1,\ W_\mu^2,\ W_\mu^3 \tag{2.126}$$
$$U(1)_Y \;\Rightarrow\; B_\mu. \tag{2.127}$$

With these gauge fields, the covariant derivatives for the lepton fields are

$$D_\mu L(x) = \left(\partial_\mu + i\frac{g}{2}\tau^i W_\mu^i + i\frac{g'}{2}Y^i B_\mu\right) L(x) \tag{2.128}$$
$$D_\mu R(x) = \left(\partial_\mu + i\frac{g'}{2}Y^i B_\mu\right) R(x), \tag{2.129}$$

where g and g' are the coupling constants associated with the groups $SU(2)_L$ and $U(1)_Y$ respectively. The field strengths are defined according to (2.87) and (2.96):

$$W_{\mu\nu}^i \equiv \partial_\mu W_\nu^i - \partial_\nu W_\mu^i + g\epsilon^{ijk} W_\mu^j W_\nu^k \tag{2.130}$$
$$B_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu. \tag{2.131}$$

With the above gauge fields, the kinetic terms for the lepton sector in (2.117) can now be modified to yield the local gauge invariant terms as

$$\mathcal{L}_{\text{lepton}}(x) = \overline{L} i\gamma^\mu D_\mu L + \overline{R} i\gamma^\mu D_\mu R. \tag{2.132}$$

Table 2.1: Particle contents of the Standard Model.

Name	Notations	Spin	$SU(3)_C$, $SU(2)_L$, $U(1)_Y$
quarks, Q (3 families)	(u_L, d_L) (c_L, s_L) (t_L, b_L) u_R c_R t_R d_R s_R b_R	1/2 1/2 1/2	$(\mathbf{3}, \mathbf{2}, \frac{1}{6})$ $(\bar{\mathbf{3}}, 1, \frac{2}{3})$ $(\bar{\mathbf{3}}, 1, -\frac{1}{3})$
leptons, L (3 families)	(ν_e, e_L) (ν_μ, μ_L) (ν_τ, τ) e_R μ_R τ_R	1/2 1/2	$(\mathbf{1}, \mathbf{2}, -\frac{1}{2})$ $(\bar{\mathbf{1}}, 1, -1)$
Higgs, ϕ	(ϕ_1, ϕ_2)	0	$(\mathbf{1}, \mathbf{2}, 1)$
gluons	g	1	$(\mathbf{8}, \mathbf{1}, 0)$
W bosons	$W^\pm$ W^0	1	$(\mathbf{1}, \mathbf{3}, 0)$
B boson	B^0	1	$(\mathbf{1}, \mathbf{1}, 0)$

The kinetic terms of the gauge fields are

$$\mathcal{L}_{\text{gauge}}(x) = -\frac{1}{4} W^i_{\mu\nu} W^{i\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}. \tag{2.133}$$

With the above results, we are now done with the lepton sector.

For the quark sector, things are almost the same. Left-handed quarks form $SU(2)$ doublets, while the right-handed ones are singlets. The hypercharge assignments for the first generation of fermions are given in Table 2.1. Covariant derivatives are defined in almost the same way, except that, since quarks are $SU(3)_C$ triplets, the $SU(3)_C$ generators together with the gauge fields describing the gluons have to be included in the covariant derivatives.

Thus one can write down the kinetic terms for any fermion field similar to (2.132) as

$$\mathcal{L}_F = \overline{\Psi}_L i\gamma^\mu D_\mu \Psi_L + \overline{\Psi}_R i\gamma^\mu D_\mu \Psi_R, \tag{2.134}$$

where $\Psi_L = L, Q_L$, and $\Psi_R = e_R, u_R, d_R$.

Note that the Lagrangian $\mathcal{L}_\psi(x)$ in (2.134) does not contain the mass terms, $m\overline{\Psi}\Psi = m(\overline{\Psi}_R\Psi_L + \overline{\Psi}_L\Psi_R)$, because the left-handed and right-handed fermions transform differently under $SU(2)_T$ and $U(1)_Y$ and so one cannot construct the gauge invariant mass terms in the usual way. Similarly, $\mathcal{L}_{\text{gauge}}$ cannot have the mass term $\frac{1}{2}m^2 A^\mu A_\mu$, because $A^\mu A_\mu$ is not gauge invariant. Thus in the unbroken electroweak theory, all fields have no masses.

To give the masses to some fields, the so-called Higgs mechanism will have to be employed. This will be considered in the next section.

2.4 Spontaneous Symmetry Breaking of the Electroweak Theory

According to many phenomenological and experimental results, the gauge bosons of weak interactions have masses while the photon of the electromagnetic interactions does not. Thus the electroweak theory needs some mechanism for giving masses to weak gauge bosons. The solution to this problem is the spontaneous breaking of $SU(2)\times U(1)$ down to $U(1)_Q$ through the Brout-Englert-Higgs mechanism, which is popularly abbreviated as the Higgs mechanism. In order to explain the Higgs mechanism, the spontaneous breaking of an abelian U(1) gauge symmetry is usually taken as an example before the full understanding of the non-Abelian cases.

2.4.1 Higgs Mechanism in an Abelian Theory

In this subsection, the Higgs mechanism is used to yield the "photon mass." Even though such a situation does not occur in the real world, it is the simplest way to study the Higgs mechanism.

The Higgs mechanism occurs via the process of spontaneous symmetry breaking. So before learning what it means by Higgs mechanism, we need to know what the spontaneous symmetry breaking is. Indeed, it is the situation in which the Lagrangian is invariant under some continuous symmetries but the vacuum is not. To see how one can actually achieve such a goal, consider a Lagrangian describing the interactions between a $U(1)$ gauge field and a complex scalar field which is also subject to self-interactions,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu\phi|^2 - V(\phi), \tag{2.135}$$

where

$$\begin{aligned} D_\mu &= \partial_\mu + ieA_\mu \\ V(\phi) &= \mu^2|\phi|^2 + \lambda\left(|\phi|^2\right)^2. \end{aligned} \tag{2.136}$$

This Lagrangian is invariant under $U(1)$ gauge transformations:

$$\phi(x) \rightarrow \phi'(x) = e^{-i\alpha(x)}\phi(x) \tag{2.137}$$

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x) \tag{2.138}$$

for arbitrary $\alpha(x)$.

To find the vacuum state (the state of lowest energy), the kinetic term is set to zero and the potential term $V(\phi)$ is minimized,

$$0 = \frac{dV}{d|\phi|} = 2\mu^2|\phi| + 4\lambda|\phi|^3. \tag{2.139}$$

λ must be positive for the potential to have a lower bound, but there are two possible choices for μ^2. If $\mu^2 > 0$, then the scalar field has the vacuum expectation value (VEV) $\langle\phi\rangle = 0$. Such a theory is quantum electrodynamics with massless photon and a charged scalar field of mass μ. If $\mu^2 < 0$, then the scalar field has non-trivial VEVs with the modulus

$$\langle|\phi|\rangle = \sqrt{\frac{-\mu^2}{2\lambda}} \equiv \frac{\nu}{\sqrt{2}}. \tag{2.140}$$

In this case, μ cannot be interpreted as the mass of the scalar field ϕ. As the Lagrangian is invariant under the global $U(1)$ symmetry, it doesn't matter which VEV is chosen. Nevertheless, once a specific VEV is chosen, this VEV is not invariant under $U(1)$ and the symmetry is spontaneously broken.

It is convenient to choose a real VEV $\langle\phi\rangle = \nu/\sqrt{2}$, so one may express the scalar field as

$$\phi \equiv \frac{1}{\sqrt{2}} e^{i\frac{\chi}{\nu}}(\nu + \xi), \tag{2.141}$$

where χ and ξ are real fields of zero VEVs. If one substitutes the field ϕ in (2.141) into the Lagrangian in (2.135), one obtains

$$\begin{aligned} \mathcal{L} \;=\; & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e\nu A_\mu \partial^\mu \chi + \frac{e^2\nu^2}{2}A_\mu A^\nu \\ & + \frac{1}{2}\left(\partial_\mu \xi \partial^\mu \xi + 2\mu^2\xi^2\right) + \frac{1}{2}\partial_\mu \chi \partial^\mu \chi \\ & + (\xi, \chi \text{ interaction terms}). \end{aligned} \tag{2.142}$$

This Lagrangian presents a theory with a photon of mass $M^A = e\nu$, a scalar field ξ of mass $\sqrt{-2\mu^2} > 0$, and a massless scalar field χ called the Goldstone boson. Because this Lagrangian is equivalent to the Lagrangian in (2.135), it is still gauge invariant.

Let us count the degrees of freedom (d.o.f) in this Lagrangian. There are five degrees of freedom of massive fields: three for A_μ (since the longitudinal mode is now allowed) and two for real scalar fields, ξ and χ. At first, this result seems to disagree with the number of d.o.f of (2.135) in which there are two d.o.f of a massless gauge field (corresponding to the two independent transverse modes) and two for a complex scalar field (corresponding two real components); hence one less d.o.f. than that of the previous counting.

This seeming appearance of an extra d.o.f can be eradicated if one notices that the above Lagrangian is still gauge invariant (even though such an invariance is "hidden") and therefore the gauge degree of freedom can be used to get rid of this extra degree of freedom. This can be done by choosing a particular gauge. A convenient way is that one chooses the gauge such that the function $\alpha(x)$ in (2.137) at each spacetime point equals to the phase of ϕ,

$$\phi \;\rightarrow\; \phi' = e^{-i\frac{\chi}{\nu}}\frac{1}{\sqrt{2}}e^{i\frac{\chi}{\nu}}(\nu+\xi) = \frac{1}{\sqrt{2}}(\nu+\xi) \tag{2.143}$$

$$A_\mu \;\rightarrow\; A'_\mu = A_\mu + \frac{1}{e\nu}\partial_\mu\chi. \tag{2.144}$$

This gauge is called a unitary gauge. Once the gauge choice is fixed, the Goldstone boson disappears and the Lagrangian becomes

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \frac{e^2\nu^2}{2}A'_\mu A'^\nu + \frac{1}{2}\partial_\mu\xi\partial^\mu\xi \\ &\quad -\frac{1}{2}(-2\mu^2)\xi^2 + \frac{1}{2}(e^2)(\xi+2\nu)\xi A'_\mu A'^\nu - \frac{\lambda}{4}\xi^3(\xi+4\nu).\end{aligned} \tag{2.145}$$

This Lagrangian describes the interactions between a massive vector boson A'_μ and the massive real scalar field ξ, called the *Higgs boson*. The Higgs boson has the mass square

$$m_\xi^2 = -2\mu^2 = 2\lambda\nu^2. \tag{2.146}$$

Now all massless fields disappear after choosing a particular gauge choice, and one can say that the massless field has been eaten to give a mass to the photon. This is called the Higgs mechanism.

It is instructive to check the consistency of the theory regarding to the number of degrees of freedom (d.o.f.) before and after the spontaneous symmetry breaking. Before the spontaneous symmetry breaking, there was a massless photon (which contributed two d.o.f. corresponding to the two independent transverse modes) and a complex scalar field (with two real d.o.f.), hence four total degrees of freedom. Since the massless fields disappeared after the spontaneous symmetry breaking, and the massive photon has three d.o.f. (two for transverse modes and one for a longitudinal mode), the total number of d.o.f. after spontaneous symmetry breaking is still four, just like the original Lagrangian (2.135).

In general, for N gauge vector fields to become massive via the Higgs mechanism, there must be at least $N + 1$ real scalar fields: N of them become unphysical and disappear, and the other one becomes the Higgs boson.

2.4.2 Electroweak Theory

Just as the Abelian case, the Higgs Mechanism can be applied to a theory with non-Abelian symmetry such as the electroweak theory.

In the electroweak symmetry breaking, one needs three massive gauge vector bosons of weak interactions and a massless photon of electromagnetic interactions. This implies that at least four real scalars are needed. As it is the $SU(2)_L \times U(1)_Y$ symmetry that we want to break, these scalar degrees of freedom must be arranged such that they form a representation of this group. A convenient choice is that they form two complex components of an $SU(2)$ doublet:

$$\mathbf{\Phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{2.147}$$

The Lagrangian for this doublet is

$$\mathcal{L}_{\mathbf{\Phi}} = (D_\mu \mathbf{\Phi})^\dagger (D^\mu \mathbf{\Phi}) - V(\mathbf{\Phi}), \tag{2.148}$$

where

$$D_\mu = \left(\partial_\mu + i\frac{g}{2}\tau^i W^i_\mu + i\frac{g'}{2} B_\mu Y \right) \tag{2.149}$$

and

$$V(\mathbf{\Phi}) = \mu^2 |\mathbf{\Phi}^\dagger \mathbf{\Phi}| + \lambda \left(|\mathbf{\Phi}^\dagger \mathbf{\Phi}| \right)^2, \ \lambda > 0. \tag{2.150}$$

Like the Abelian model, the vacuum state for $\mu^2 < 0$ is not at $\mathbf{\Phi} = 0$. Since the potential depends on $|\mathbf{\Phi}^\dagger \mathbf{\Phi}|$, then it can be checked that any $\mathbf{\Phi}$ that satisfies $|\mathbf{\Phi}^\dagger \mathbf{\Phi}| = \nu^2/2$ can be a VEV. For convenience, one chooses

$$\langle \mathbf{\Phi} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \nu \end{pmatrix}. \tag{2.151}$$

For this vacuum state to be invariant under the $U(1)_Q$ group of electromagnetic interactions, the above $\langle\mathbf{\Phi}\rangle$ has to be annihilated by the charge operator, $Q\langle\Phi\rangle = 0$. According to (2.123), one gets the condition

$$Q\langle\mathbf{\Phi}\rangle = (T_3 + Y)\,\langle\mathbf{\Phi}\rangle = 0. \tag{2.152}$$

Because $\mathbf{\Phi}$ is a doublet and Y is a $U(1)_Y$ generator, one can write Y as

$$Y = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} \tag{2.153}$$

and so

$$\begin{aligned} 0 &= Q\langle\mathbf{\Phi}\rangle & (2.154)\\ &= \left[\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}\right]\begin{pmatrix} 0 \\ \frac{\nu}{\sqrt{2}}. \end{pmatrix}. & (2.155) \end{aligned}$$

Thus the hypercharge of the doublet is $Y_{\mathbf{\Phi}} = \frac{1}{2}I$. Thus $Q\phi_1 = (+1)\phi_1$ and $Q\phi_2 = 0$, so it is convenient for $\mathbf{\Phi}$ to be written as

$$\mathbf{\Phi} = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}. \tag{2.156}$$

To consider the Higgs mechanism in this model, one expresses $\mathbf{\Phi}(x)$ as

$$\mathbf{\Phi}(x) = \frac{1}{\sqrt{2}} e^{i\frac{\tau^i}{2}\cdot\frac{\chi^i}{\nu}} \begin{pmatrix} 0 \\ \nu + H \end{pmatrix} \tag{2.157}$$

where the real scalar fields χ^i $(i = 1,\ 2,\ 3)$ and H have zero VEVs. The above expression is obtained from the vacuum state by first changing its magnitude along the same direction in the $SU(2)$ space (by adding $H(x)$ to ν) and then rotating it using an element of $SU(2)$ (this process preserves its magnitude). Choosing the unitary gauge,

$$\mathbf{\Phi}'(x) = e^{-i\frac{\tau^i}{2}\cdot\frac{\chi^i}{\nu}}\mathbf{\Phi} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \nu + H \end{pmatrix}, \tag{2.158}$$

the scalar field Lagrangian in (2.148) reads

$$\begin{aligned} \mathcal{L} &= \left|\left(\partial_\mu + i\frac{g}{2}\tau^i W^i_\mu + i\frac{g'}{2}B_\mu Y\right)\frac{(\nu + H)}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right|^2 \\ &\quad -\mu^2\frac{(\nu + H)^2}{2} - \lambda\frac{(\nu + H)^4}{4}. \end{aligned} \tag{2.159}$$

The gauge boson mass terms comes from

$$\begin{aligned}&\left|\left(\partial_\mu + i\frac{g}{2}\tau^i W^i_\mu + i\frac{g'}{2}B_\mu I\right)\frac{\nu}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}\right|^2 \\ = \;& \frac{\nu^2}{8}\left\{g^2\left[(W^1_\mu)^2+(W^2_\mu)^2\right]+\left(gW^3_\mu - g'B_\mu\right)^2\right\}.\end{aligned} \tag{2.160}$$

If one introduces the linear combinations

$$W^+_\mu = \frac{1}{\sqrt{2}}(W^1_\mu - iW^2_\mu) \tag{2.161}$$

and

$$W^-_\mu = (W^+_\mu)^\dagger = \frac{1}{\sqrt{2}}(W^1_\mu + iW^2_\mu), \tag{2.162}$$

the first term of (2.160) becomes

$$\frac{g^2\nu^2}{8}\left[(W^1_\mu)^2+(W^2_\mu)^2\right] = \frac{g^2\nu^2}{8}W^+_\mu W^{-\mu}. \tag{2.163}$$

Since W^+_μ is a complex field, its mass squared is

$$(M_W)^2 = \frac{g^2\nu^2}{4}. \tag{2.164}$$

Thus W^+_μ and W^-_μ are identified as the *charged bosons*.

For the second term in (2.160), we can rewrite it as

$$\frac{\nu^2}{8}\left(gW^3_\mu - g'B_\mu\right)^2 = \frac{\nu^2}{8}\begin{pmatrix}W^3_\mu & B_\mu\end{pmatrix}\begin{pmatrix}g^2 & -gg' \\ -gg' & g'^2\end{pmatrix}\begin{pmatrix}W^3_\mu \\ B_\mu\end{pmatrix}. \tag{2.165}$$

In order to diagonalize the mass matrix in (2.165), one uses the following orthogonal transformation:

$$\begin{pmatrix}A_\mu \\ Z_\mu\end{pmatrix} = \begin{pmatrix}\cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W\end{pmatrix}\begin{pmatrix}B_\mu \\ W^3_\mu\end{pmatrix}, \tag{2.166}$$

where θ_W is called the *Weinberg angle* defined by

$$\sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}} \tag{2.167}$$

and

$$\cos\theta_W \;=\; \frac{g}{\sqrt{g^2+g'^2}}. \tag{2.168}$$

Then (2.165) becomes

$$\frac{1}{2}\begin{pmatrix} Z_\mu & A_\mu \end{pmatrix}\begin{pmatrix} \frac{1}{4}\nu^2(g^2+g'^2) & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \frac{1}{2}\left[\frac{1}{4}\nu^2(g^2+g'^2)\right] Z_\mu Z^\mu. \tag{2.169}$$

It can be seen immediately that A_μ is massless and thus identified as a photon, while Z_μ is massive boson of mass squared

$$(M_Z)^2 = \frac{1}{4}\nu^2(g^2+g'^2). \tag{2.170}$$

It is instructive to count the degrees of freedom both before and after the spontaneous symmetry breaking. Before the symmetry breaking, one had a complex scalar $SU(2)_L$ doublet $\mathbf{\Phi}$ together with some gauge fields. The total number of degrees of freedom of the theory is 12: 4 d.o.f. for $\mathbf{\Phi}$, 6 d.o.f. for a massless $SU(2)_L$ gauge field, W_i, and 2 d.o.f. for a massless $U(1)_Y$ gauge field, B_μ. After the symmetry breaking, one was left with a physical real scalar field H (1 d.o.f.), three massive vector bosons, W and Z (9 d.o.f.), and a massless photon (2 d.o.f.). One can say that the three d.o.f. of the scalar doublet have been eaten by $W^\pm$ and Z to give the longitudinal components of $W^\pm$ and Z.

To find the values of coupling constants in the theory, one writes the kinetic terms for the lepton fields in (2.132) in terms of the physical gauge fields:

$$\begin{aligned}\mathcal{L}_{\text{lepton}} \;=\;& -g\sin\theta_W \bar{l}\gamma^\mu l A_\mu \\ & -\frac{g}{2\sqrt{2}}[\overline{\nu}_L\gamma^\mu l_L W_\mu^+ + \bar{l}_L\gamma^\mu\nu_L W_\mu^-] + \ldots.\end{aligned} \tag{2.171}$$

The electromagnetic coupling constant or electric charge is identified as

$$e = g\sin\theta_W = g'\cos\theta_W. \tag{2.172}$$

From the low-energy phenomenology, the weak interactions are described by

$$\mathcal{L}_{\text{weak}} = \left(\frac{M_W^2 G_F}{\sqrt{2}}\right)^{1/2}[\overline{\nu}_L\gamma^\mu l_L W_\mu^+ + \bar{l}_L\gamma^\mu\nu_L W_\mu^-], \tag{2.173}$$

where G_F is the Fermi coupling constant. By comparing (2.171) with the second term on the left hand side of (2.173), one identifies

$$\frac{g}{2\sqrt{2}} = \left(\frac{M_W^2 G_F}{\sqrt{2}}\right)^{1/2}. \tag{2.174}$$

Note that (2.164) gives

$$\nu = (\sqrt{2}G_F)^{-1/2} = 246 \text{ GeV}. \tag{2.175}$$

Even though the explicit fermion mass terms $m\overline{\Psi}\Psi = m(\overline{\Psi}_R\Psi_L + \overline{\Psi}_L\Psi_R)$ were prohibited by gauge symmetry as discussed before, the Higgs boson, H, can be used to give the fermion masses via the gauge invariant Yukawa interactions of the form

$$-\lambda^a\overline{\Psi}_L\mathbf{\Phi}\Psi_R, \tag{2.176}$$

where $\lambda^{(a)}, a = e, d, u$ are Yukawa couplings.

The Yukawa coupling of the Higgs boson to the up and down quarks is

$$-\lambda^{(d)}\overline{Q}_L\mathbf{\Phi}d_R + h.c., \tag{2.177}$$

or more explicitly

$$-\frac{\lambda^{(d)}}{\sqrt{2}}\begin{pmatrix}\overline{u}_L & \overline{d}_L\end{pmatrix}\begin{pmatrix}0 \\ \nu + H\end{pmatrix}d_R + h.c.. \tag{2.178}$$

This yields a mass term for the down quark if one identifies

$$\lambda^{(d)} = \frac{m_d\sqrt{2}}{\nu} \tag{2.179}$$

with m_d being the down quark mass. For the up quark mass term, one defines $\mathbf{\Phi}^c \equiv \widetilde{\mathbf{\Phi}} = -i\tau_2\mathbf{\Phi}^*$ and write the $SU(2)_L$ invariant coupling as

$$-\lambda^{(u)}\overline{Q}_L\mathbf{\Phi}^c u_R + h.c., \tag{2.180}$$

which generates an up quark mass term. Similar couplings can be used to generate mass terms for the charged leptons. Since the neutrinos have no right-handed components, they remain massless.

2.4.3 The Electroweak Action

To write the general action of the electroweak theory, all particle families have to be included. All the results obtained so far have been for the first generation of fermions only. For other generations, we merely use the previous results with the following substitutions:

$$\begin{array}{ccccc} e & \rightarrow & e_A & = & (e, \mu, \tau) \\ \nu_e & \rightarrow & \nu_e & = & (\nu_e, \nu_\mu, \nu_\tau) \\ u & \rightarrow & p_A & = & (u, c, t) \\ d & \rightarrow & n_A & = & (d, s, b) \end{array} \tag{2.181}$$

where the particles μ, ν_μ, c, and s belong to the second generation particles, and τ, ν_τ, t, and b belong to the third generation, and so all the $SU(2)$ doublets are of the generic form:

$$L_A = \begin{pmatrix} \nu_A \\ e_A \end{pmatrix}_L \tag{2.182}$$

$$Q_{AL} = \begin{pmatrix} p_A \\ n_A \end{pmatrix}_L . \tag{2.183}$$

Once the Lagrangian for each fermion generation has been obtained, one merely sums over all generations to obtain the electroweak action:

$$S = \int dx^4 \mathcal{L} \tag{2.184}$$

$$= \int dx^4 \left(\mathcal{L}_G + \mathcal{L}_F + \mathcal{L}_{\mathbf{\Phi}} + \mathcal{L}_Y \right). \tag{2.185}$$

It consists of four parts:

1. $\mathcal{L}_G$ is the kinetic terms for the vector gauge fields:

$$\mathcal{L}_G = -\frac{1}{4} W^i_{\mu\nu} W^{i\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}. \tag{2.186}$$

2. $\mathcal{L}_F$ is fermionic kinetic terms:

$$\mathcal{L}_F = \overline{\Psi}_L i\gamma^\mu D_\mu \Psi_L + \overline{\Psi}_R i\gamma^\mu D_\mu \Psi_R, \tag{2.187}$$

where Ψ_L and Ψ_R represents all left-handed and right-handed fermionic fields respectively, and the sum over the femionic species is understood.[2]

3. $\mathcal{L}_{\mathbf{\Phi}}$ is the Higgs boson Lagrangian:

$$\mathcal{L}_{\mathbf{\Phi}} = (D_\mu \mathbf{\Phi})^\dagger (D^\mu \mathbf{\Phi}) - V(\mathbf{\Phi}), \tag{2.188}$$

where

$$D_\mu \mathbf{\Phi} = \left(\partial_\mu + i\frac{g}{2}\tau^i W_\mu^i + i\frac{g'}{2} B_\mu Y \right) \frac{(\nu + H)}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.189}$$

and

$$V(\mathbf{\Phi}) = -\mu^2 \frac{(\nu + H)^2}{2} - \lambda \frac{(\nu + H)^4}{4}. \tag{2.190}$$

4. $\mathcal{L}_Y$ is the general Yukawa interactions between scalars and fermions:

$$\mathcal{L}_Y = \lambda_{AB}^{(e)} l_{AL} \mathbf{\Phi} e_{BR} + \lambda_{AB}^{(p)} \overline{Q}_{AL} \widetilde{\mathbf{\Phi}} p_{BR} + \lambda_{AB}^{(n)} \overline{Q}_{AL} \mathbf{\Phi} n_{BR} + h.c., \tag{2.191}$$

which contains family indices, A and B.

The fields $l_{AL}, e_{BR}, Q_{AL}, p_{BR}, n_{BR}$ are *gauge eigenfields*, i.e., they transform as singlets or doublets under $SU(2)$ gauge transformations. After the spontaneous symmetry breaking,

$$\mathbf{\Phi} \rightarrow \mathbf{\Phi}' = \frac{(\nu + H)}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$\mathcal{L}_Y$ takes the form

$$\begin{aligned} \mathcal{L}_Y \;\; = \;\; & \frac{H(x)}{\sqrt{2}} [\lambda_{AB}^{(e)} \overline{e}_{AL} e_{BR} + \lambda_{AB}^{(p)} \overline{p}_{AL} p_{BR} + \lambda_{AB}^{(n)} \overline{n}_{AL} \mathbf{\Phi} n_{BR}] \\ & + \frac{\nu}{\sqrt{2}} [\lambda_{AB}^{(e)} \overline{e}_{AL} e_{BR} + \lambda_{AB}^{(p)} \overline{p}_{AL} p_{BR} + \lambda_{AB}^{(n)} \overline{n}_{AL} \mathbf{\Phi} n_{BR}] \\ & + h.c. \end{aligned} \tag{2.192}$$

which gives the fermion mass terms with the mass matrix

$$M_{AB}^{(a)} = \frac{-\nu}{\sqrt{2}} \lambda_{AB}^{(a)} \;, \quad a = e, p, n. \tag{2.193}$$

[2] Note that the kinetic terms of the vector gauge fields and fermions remain unaltered after the symmetry breaking.

whose eigenvectors are called the *mass eigenfields*, which represent the particles that one observes in Nature.

Since the mass matrix in (2.193) is not diagonal, some of the gauge eigenfields may not represent particles that one observes in the experiments. To obtain the particle spectrum observed in experiments, the mass matrix has to be diagonalized with the result that all the fields in the Lagrangian are now the mass eigenfields, some of which are the linear combinations of the original gauge eigenfields. For the theory with only two fermion generations, one has the mass eigenfields

$$d_\theta = \cos\theta_c d + \sin\theta_c s \tag{2.194}$$

$$s_\theta = \cos\theta_c s - \sin\theta_c d \tag{2.195}$$

where the mixing angle θ_c is called the *Cabibbo angle*.

Let us finally note that such a difference between gauge and mass eigenfields has already been encountered before in the cases of the vector fields in (2.169), where the physical gauge bosons Z_μ and A_μ are linear combinations of the gauge eigenfields W_μ^3 and B_μ in (2.166).

CHAPTER III

SUPERSYMMETRIC FIELD THEORY

For decades, theoretical and experimental research has confirmed that the Standard Model is very successful in explaining and predicting lots of experimental results, yet there have been many unsolved problems. As discussed in Chapter 1, supersymmetry (SUSY) is one of important candidates for solving such problems. In this chapter, we review the construction of supersymmetry algebra (SUSY algebra) and supersymmetric field theories using the superspace technique.

3.1 Construction of the Supersymmetry Algebra

According to the first chapter, the hierarchy problem in the Standard Model can be solved if the theory has a symmetry which relates bosons and fermions. Such a kind of symmetry indeed exists and is known as supersymmetry or SUSY. Thus a SUSY transformation on a field changes its spin by one-half unit and turns a boson into a fermion and vice-versa:

$$\begin{aligned} Q\,|\text{boson}\rangle &= |\text{fermion}\rangle\,, \\ Q\,|\text{fermion}\rangle &= |\text{boson}\rangle\,, \end{aligned} \tag{3.1}$$

where Q is a generator of SUSY. Thus the operator Q should transforms under the Lorentz transformations as a spinor.

It is natural to ask how one can extend the Standard Model using supersymmetry. The answer of this question is non-trivial because the Standard Model already has Lorentz and gauge symmetries. So if one wants to add supersymmetry (whose generators are spinors) to the theory, then supersymmetry should combine with the Lorentz symmetry (whose generators have effects on spinors) in a non-trivial way.

Consider first the generators of gauge and Poincaré groups:

1. The internal symmetry generators: they do not affect the spacetime coordinates.

2. The 4-momentum operators P_μ: they are the vector quantities which affect the spacetime coordinates by translations.

3. The generator of the Lorentz group $M_{\mu\nu}$: they affect the spacetime coordinates by "rotations."

These generators form the Lie algebras via the commutation relations. As none of these generators changes the spin of the field they act on, they are said to be the "bosonic generators."

On the other hand, as the supersymmetry generator Q is a spinor quantity (and so it is called a "fermionic generator"), it must have non-trivial commutation relations with the rotation generators $M_{\mu\nu}$. Also in field theory, the spinors have anti-commutation relations among themselves, then so should the SUSY generators.

This above argument implies that the whole algebra should involve both commutators and anti-commutators. Such a kind of algebra is not a new idea in mathematics; it belongs to a class of algebra, known as the *graded Lie algebras* which have been investigated by mathematicians. The supersymmetry algebra we are interested in is actually known as the $\mathbb{Z}_2$ graded Lie algebra.

3.1.1 $\mathbb{Z}_2$ Graded Lie Algebras

Definition 3. *A $\mathbb{Z}_2$ Graded Lie algebra consists of the direct sum of two vector spaces $\mathcal{L}_0$ and $\mathcal{L}_1$:*

$$\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1 \tag{3.2}$$

together with a product

$$\circ : \mathcal{L} \times \mathcal{L} \to \mathcal{L} \tag{3.3}$$

If $\forall x_i \in \mathcal{L}_i, i = 0, 1$ *, then the following properties define the* $\mathbb{Z}_2$ *Lie algebra:*

1. *Grading:*

$$x_i \circ x_j \in \mathcal{L}_{i+j \ mod \ 2}$$

2. *Supersymmetrization:*

$$x_i \circ x_j = -(-1)^{ij} x_i \circ x_j$$

3. *Generalized Jacobi identities:*

$$x_i \circ (x_j \circ x_k)(-1)^{ik} + x_j \circ (x_k \circ x_i)(-1)^{ji} + x_k \circ (x_i \circ x_j)(-1)^{kj} = 0$$

It is important to note that $\mathcal{L}$ is not a Lie algebra, since, as defined in supersymmetrization, the product is in general not antisymmetric.

Example: Let $\mathcal{L} = \text{Span}\{X_\mu\}$ be the direct sum of $\mathcal{L}_0$ and $\mathcal{L}_1$, where

$$\begin{array}{ll} \mathcal{L}_0 = \text{Span}\{E_i\} & i = 1, \ldots, \text{dim}\mathcal{L}_0 \\ \mathcal{L}_1 = \text{Span}\{Q_a\} & i = 1, \ldots, \text{dim}\mathcal{L}_1. \end{array} \tag{3.4}$$

Let $g(E_i) = 0$ and $g(Q_a) = 1$. The product $\circ$ is defined by

$$\circ : (X_\mu, X_\nu) \to X_\mu \circ X_\nu = X_\mu X_\nu - (-1)^{g(X_\mu)g(X_\nu)} X_\mu X_\nu. \tag{3.5}$$

Consider this product separately on the subspaces $\mathcal{L}_0$ and $\mathcal{L}_1$:

i) $\circ : \mathcal{L}_0 \times \mathcal{L}_0 \to \mathcal{L}_o$

Let $E_i, E_j \in \mathcal{L}_0$. Then

$$E_i \circ E_j = E_i E_j - (-1)^{(0)(0)} E_j E_i = [E_i, E_j] \tag{3.6}$$

ii) $\circ : \mathcal{L}_0 \times \mathcal{L}_1 \rightarrow \mathcal{L}_1$

Let $E_i \in \mathcal{L}_0$ and $Q_a \in \mathcal{L}_1$. Then

$$E_i \circ Q_a = E_i Q_a - (-1)^{(0)(1)} Q_a E_i = [E_i, Q_a] \tag{3.7}$$

iii) $\circ : \mathcal{L}_1 \times \mathcal{L}_1 \rightarrow \mathcal{L}_0$

Let $Q_a, Q_b \in \mathcal{L}_1$. Then

$$Q_a \circ Q_b = Q_a Q_b - (-1)^{(1)(1)} Q_b Q_a = \{Q_a, Q_b\} \tag{3.8}$$

The above construction is easily seen to satisfy the grading and supersymmetrization properties of $\mathbb{Z}_2$ graded Lie algebra. It is not hard to verify that (3.5) obeys the generalized Jacobi identities.

3.1.2 Supersymmetry Algebra

In order to construct a supersymmetric version of the Standard Model, one starts with the construction the supersymmetric extension of the Poincaré algebra.

The supersymmetric extension of Poincaré algebra is a $\mathbb{Z}_2$ graded Lie algebra consisting of the direct sum of the Poincaré algebra as the subspace $\mathcal{L}_0$ and a vector space $\mathcal{L}_1 = \text{Span}\{Q_a\}$, $a = 1, 2, 3, 4$. The super Poincaré algebra consistent with the generalized Jacobi identities is given by

$$[P_\mu, P_\nu] = 0 \tag{3.9}$$

$$[M_{\mu\nu}, P_\lambda] = -i(\eta_{\nu\lambda} P_\mu - \eta_{\mu\lambda} P_\nu) \tag{3.10}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\mu\nu} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho}) \tag{3.11}$$

$$[P_\mu, Q_a] = 0 \tag{3.12}$$

$$[M_{\mu\nu}, Q_a] = -\frac{1}{2} (\Sigma_{\mu\nu})_{ab} Q_b \tag{3.13}$$

$$\{Q_a, \overline{Q}_b\} = -2(\gamma^\mu)_{ab} P_\mu \tag{3.14}$$

$$\{Q_a, Q_b\} = -2(\gamma^\mu C)_{ab} P_\mu \tag{3.15}$$

$$\{\overline{Q}_a, \overline{Q}_b\} = -2(C^{-1}\gamma^\mu)_{ab} P_\mu, \tag{3.16}$$

where the generators are

1. P_μ – the generators of translations;

2. $M_{\mu\nu}$ – the generators of Lorentz transformations and spatial rotations;

3. Q_a and $\overline{Q}_a$, $a = 1, 2, 3, 4$ – the spinor generators of supersymmetry transformations.

Above, Q is a Majorana spinor and $\overline{Q}$ is its Dirac conjugate, so they contain totally two independent complex spinor components. Also $\frac{1}{2}\Sigma_{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right]$ and C is the charge conjugation matrix defined in Chapter 2.

In the above set of (anti)commutation relations, the first three lines are the Poincaré algebra. For the rest of them, (3.12)-(3.16), they are derived from the general Jacobi identities of the graded Lie algebra. One can note, in particular, that

1. (3.12) implies that Q transforms trivially under translations;

2. (3.13) implies that Q transforms as a spinor under Lorentz transformations.

Also since the anticommutator of two Qs gives a momentum P, then people sometimes refer to Q as a "square root" of P.

As the supersymmetry generator Q is a Majorana spinor, it is more convenient to express the super Poincaré algebra in terms of its Weyl spinor components. Thus the anti-commutator part of the super Poincaré algebra

$$Q_a\overline{Q}_b + \overline{Q}_b Q_a = 2\gamma^\mu_{ab} P_\mu \tag{3.17}$$

can be expressed in terms of 2-component Weyl spinors as

$$\begin{aligned} \left\{Q_A, Q^B\right\} &= 0, & \left\{Q_A, \overline{Q}_{\dot{B}}\right\} &= 2\sigma^\mu_{A\dot{B}} P_\mu, \\ \left\{\overline{Q}^{\dot{A}}, Q^B\right\} &= 2\overline{\sigma}^{\mu\dot{A}B} P_\mu & \left\{\overline{Q}^{\dot{A}}, \overline{Q}_{\dot{B}}\right\} &= 0. \end{aligned} \tag{3.18}$$

For the commutator (3.13), it reads

$$\begin{aligned}[M_{\mu\nu}, Q_A] &= -(\sigma_{\mu\nu})_A{}^B Q_B && (3.19)\\ \left[M_{\mu\nu}, \overline{Q}^{\dot{A}}\right] &= -(\overline{\sigma}_{\mu\nu})^{\dot{A}}{}_{\dot{B}}\overline{Q}^{\dot{B}}, && (3.20)\end{aligned}$$

where

$$\frac{1}{2}\Sigma_{\mu\nu} = \begin{pmatrix} (\sigma_{\mu\nu})_A{}^B & 0 \\ 0 & (\overline{\sigma}_{\mu\nu})^{\dot{A}}{}_{\dot{B}} \end{pmatrix}. \tag{3.21}$$

This 2-component spinor formulation will be used throughout this chapter.

3.2 Superspace and Superfields

3.2.1 What are Superspace and Superfields?

To formulate a field theory having some continuous symmetries, one first constructs the Lie group of that symmetries, whose elements are obtained by exponentiating the Lie algebra elements. The fields then belong to some representations of this group.

How about a supersymmetric field theory, whose symmetries are described by a graded Lie group? Certainly, one cannot express the graded Lie group elements as the exponential of the fermionic generators with complex number coefficients. The reason for this is that, once this exponential factor is Taylor expanded, half of the terms in the expansion are bosonic while the other half contains fermionic terms; such a summation cannot be defined consistently. The way out of this problem is to use the anti-commuting or Grassmann numbers as the coefficients of the fermionic generators. In the super Poincaré algebra, the supersymmetry generators Q are Weyl spinors. So to make a product of Q and its Grassmann number coefficient Lorentz invariant, this Grassmann number must also be a Weyl spinor. Thus the Grassmann variables in the Weyl representation are written as

$$\{\theta_A\}_{A=1,2} \quad \text{and} \quad \{\bar{\theta}_{\dot{B}}\}_{\dot{B}=\dot{1},\dot{2}}$$

and have properties:

$$\{\theta_A, \theta_B\} = \{\bar{\theta}_{\dot{A}}, \bar{\theta}_{\dot{B}}\} = \{\theta_A, \bar{\theta}_{\dot{B}}\} = 0. \tag{3.22}$$

With the multiplication of Grassmann numbers on the left of the supersymmetry generators, all the anti-commutators become the commutators as

$$\begin{aligned}
\left[\theta^A Q_A, \bar{\theta}_{\dot{B}}\bar{Q}^{\dot{B}}\right] &= 2\theta^A \sigma^{\mu}_{A\dot{B}}\bar{\theta}^{\dot{B}} P_\mu && (3.23)\\
\left[\theta^A Q_A, \theta^B Q_B\right] &= 0 && (3.24)\\
\left[\bar{\theta}_{\dot{A}}\bar{Q}^{\dot{A}}, \bar{\theta}_{\dot{B}}\bar{Q}^{\dot{B}}\right] &= 0 && (3.25)\\
\left[P^\mu, \theta^A Q_A\right] &= 0 && (3.26)\\
\left[P^\mu, \bar{\theta}_{\dot{B}}\bar{Q}^{\dot{B}}\right] &= 0. && (3.27)
\end{aligned}$$

Note that the Grassmann variables are assumed constant, so they commute with the momentum generators. As the above subalgebra is closed, yet nontrivial, it motivates one to write the corresponding group elements as the exponentials

$$G\left(x^\mu, \theta, \overline{\theta}\right) = \exp\left[i\left(\theta Q + \bar{\theta}\bar{Q} - x^\mu P_\mu\right)\right] \tag{3.28}$$

which form a subgroup of the super Poincaré group. This transformation operator, $G\left(x^\mu, \theta, \overline{\theta}\right)$, is called a finite "supertranslation" [8].

The reason behind the name "supertranslation" can be understood as follows. One expands a spacetime with coordinates x^μ by including θ and $\overline{\theta}$ as additional Grassmann coordinates. The resulting "superspace" thus has a set of coordinates $(x^\mu, \theta, \overline{\theta})$ on it, and the group element in (3.28) has the superspace as its group "supermanifold." Thus there is a one-to-one map between group elements and points on the supermanifold.

The action of the group element $G(a^\mu, \xi, \overline{\xi})$ on the superspace coordinates $(x^\mu, \theta, \overline{\theta})$ is defined by the right-action of $G(a^\mu, \xi, \overline{\xi})$ on the group element

$G(x^\mu,\theta,\overline{\theta})$ which corresponds to coordinates $(x^\mu,\theta,\overline{\theta})$:

$$\begin{aligned}
&G\left(x^\mu,\theta,\overline{\theta}\right)G\left(a^\mu,\xi,\overline{\xi}\right)\\
&\quad=\exp\left[i\left(\theta Q+\bar{\theta}\bar{Q}-x^\mu P_\mu\right)\right]\exp\left[i\left(\xi Q+\bar{\xi}\bar{Q}-a^\mu P_\mu\right)\right]\\
&\quad=\exp\left[i\left(\left(\theta+\xi\right)Q+\left(\bar{\theta}+\bar{\xi}\right)\bar{Q}-\left(x^\mu+a^\mu-i\xi\sigma^\mu\bar{\theta}+i\theta\sigma^\mu\bar{\xi}\right)P_\mu\right)\right]\\
&\quad=G\left(x^\mu+a^\mu-i\xi\sigma^\mu\bar{\theta}+i\theta\sigma^\mu\bar{\xi},\theta+\xi,\bar{\theta}+\bar{\xi}\right),
\end{aligned}$$

where the Baker-Campbell-Hausdorff formula has been used to evaluate the product of exponentials. The parameters of the group element on the right-hand side are the transformed superspace coordinates, or mathematically speaking:

$$G\left(a^\mu,\xi,\overline{\xi}\right):\left(x^\mu,\theta,\overline{\theta}\right)\to\left(x^\mu+a^\mu-i\xi\sigma^\mu\bar{\theta}+i\theta\sigma^\mu\bar{\xi},\theta+\xi,\bar{\theta}+\bar{\xi}\right). \tag{3.29}$$

It is clear that the above result corresponds to a translation of superspace coordinates; hence the name supertranslation. Note that if one includes $M_{\mu\nu}$ terms in the exponents of the group elements and repeats the calculations, then it is not hard to verify that the superspace coordinates will get rotated in addition to the translations.

A function $\Phi\left(x^\mu,\theta,\overline{\theta}\right)$ on the superspace is called a "superfield." Under the above super Poincaré group action on the superspace coordinates, the superfield transforms as

$$\begin{aligned}
&\Phi\left(x^\mu+a^\mu-i\xi\sigma^\mu\bar{\theta}+i\theta\sigma^\mu\bar{\xi},\theta+\xi,\bar{\theta}+\bar{\xi}\right)\\
&\quad=\Phi\left(x^\mu,\theta,\overline{\theta}\right)+\left(a^\mu-i\xi\sigma^\mu\bar{\theta}+i\theta\sigma^\mu\bar{\xi}\right)\tfrac{\partial\Phi}{\partial x^\mu}+\xi^A\tfrac{\partial\Phi}{\partial\theta^A}+\bar{\xi}^{\dot{A}}\tfrac{\partial\Phi}{\partial\bar{\theta}^{\dot{A}}}+\ldots\\
&\quad=\left[1+\left(a^\mu-i\xi\sigma^\mu\bar{\theta}+i\theta\sigma^\mu\bar{\xi}\right)\tfrac{\partial}{\partial x^\mu}+\xi^A\tfrac{\partial}{\partial\theta^A}+\bar{\xi}^{\dot{A}}\tfrac{\partial}{\partial\bar{\theta}^{\dot{A}}}+\ldots\right]\Phi\left(x^\mu,\theta,\overline{\theta}\right)
\end{aligned}$$

and infinitesimally,

$$\begin{aligned}
\delta_G\left(a^\mu,\xi,\bar{\xi}\right)\Phi\left(x^\mu,\theta,\bar{\theta}\right) &= \left[\left(a^\mu-i\xi\sigma^\mu\bar{\theta}+i\theta\sigma^\mu\bar{\xi}\right)\frac{\partial}{\partial x^\mu}\right.\\
&\qquad\left.+\xi^A\frac{\partial}{\partial\theta^A}+\bar{\xi}^{\dot{A}}\frac{\partial}{\partial\bar{\theta}^{\dot{A}}}\right]\Phi\left(x^\mu,\theta,\bar{\theta}\right)
\end{aligned} \tag{3.30}$$

from which it follows that the action of supertranslation on the superfield is generated by

$$P_\mu = i\partial_\mu, \tag{3.31}$$

$$iQ_A = \frac{\partial}{\partial\theta^A}-i\sigma^\mu_{A\dot{A}}\bar{\theta}^{\dot{A}}\partial_\mu, \tag{3.32}$$

$$i\overline{Q}_{\dot{A}} = -\frac{\partial}{\partial\overline{\theta}^{\dot{A}}}+i\theta^A\sigma^\mu_{A\dot{A}}\partial_\mu \tag{3.33}$$

so that $\delta_G\left(a^\mu,\xi,\bar{\xi}\right)\Phi\left(x^\mu,\theta,\bar{\theta}\right) = \left(-ia^\mu P_\mu + i\xi^A Q_A + i\bar{\xi}_{\dot{A}}\overline{Q}^{\dot{A}}\right)\Phi\left(x^\mu,\theta,\bar{\theta}\right)$. One can verify that the above representation of the supertranslation operators satisfy the super Poincaré algebra.

Observe that this representation treats θ and $\overline{\theta}$ on equal footing. There are, however, other representations that treat θ and $\overline{\theta}$ slightly differently. Such representations are known as the *chiral representations*. They are defined in precisely the same way as above, but using different forms of the group elements (i.e., different from (3.28)). There are two of them defined as follows:

1. L-representation: The group elements take the form

$$G_L\left(x^\mu,\theta,\bar{\theta}\right) = \exp\left(-ix^\mu P_\mu + i\theta Q\right)\cdot\exp\left(i\bar{\theta}\bar{Q}\right) \tag{3.34}$$

which is equivalent to (3.28) if x^μ in (3.28) is changed to $x^\mu - i\theta\sigma^\mu\overline{\theta}$. Repeating the steps in the original case leads to the supertranslation rule on the superfield,

$$\delta_{(L)G}(\xi,\bar{\xi})\Phi_L\left(x^\mu,\theta,\bar{\theta}\right) = \left(\xi^A\frac{\partial}{\partial\theta^A} + \bar{\xi}^{\dot{A}}\frac{\partial}{\partial\bar{\theta}^{\dot{A}}} + 2i\theta\sigma^\mu\bar{\xi}\partial_\mu\right)\Phi_L\left(x^\mu,\theta,\bar{\theta}\right) \tag{3.35}$$

so that the generators in this representation read

$$P_\mu = i\partial_\mu \tag{3.36}$$

$$iQ_A = \frac{\partial}{\partial\theta^A} \tag{3.37}$$

$$i\overline{Q}_{\dot{A}} = -\frac{\partial}{\partial\overline{\theta}^{\dot{A}}} + 2i\theta^A\sigma^\mu_{A\dot{A}} \tag{3.38}$$

which satisfy the super Poincaré algebra as they should. If one defines the new bosonic coordinates

$$y^\mu = x^\mu + i\theta\sigma^\mu\overline{\theta}, \tag{3.39}$$

then the above representation takes the form (3.31)-(3.33) with x^μ being replaced by y^μ. Thus this L-representation is completely equivalent to the original representation, and so the superfield in this representation is conveniently written in the form $\Phi_L\left(x^\mu + i\theta\sigma^\mu\bar{\theta},\theta,\bar{\theta}\right)$.

2. R-representation: The group elements take the form

$$G_R\left(x^\mu,\theta,\bar\theta\right) = \exp\left(-ix^\mu P_\mu + i\bar\theta\bar Q\right)\cdot\exp\left(i\theta Q\right) \tag{3.40}$$

which is equivalent to (3.28) if x^μ in (3.28) is changed to $x^\mu + i\theta\sigma^\mu\overline{\theta}$. This leads to the transformation rule

$$\delta_{(R)G}(\xi,\bar\xi)\Phi_R\left(x^\mu,\theta,\bar\theta\right) = \left(\xi^A\frac{\partial}{\partial\theta^A} + \bar\xi^{\dot A}\frac{\partial}{\partial\bar\theta^{\dot A}} - 2i\xi\sigma^\mu\bar\theta\partial_\mu\right)\Phi_R\left(x^\mu,\theta,\bar\theta\right) \tag{3.41}$$

which implies the representation

$$P_\mu = i\partial_\mu \tag{3.42}$$

$$iQ_A = \frac{\partial}{\partial\theta^A} - 2i\sigma^\mu_{A\dot A}\overline{\theta}^{\dot A}\partial_\mu \tag{3.43}$$

$$i\overline{Q}_{\dot A} = -\frac{\partial}{\partial\overline{\theta}^{\dot A}} \tag{3.44}$$

satisfying the super Poincaré algebra as one can check. Similar to the L-representation case, one can check that with the new bosonic coordinates

$$z^\mu = x^\mu - i\theta\sigma^\mu\overline{\theta}, \tag{3.45}$$

the above representation takes the form (3.31)-(3.33) with x^μ being changed to z^μ. So the superfield in the R-representation is normally written in the form $\Phi_R\left(x^\mu - i\theta\sigma^\mu\bar\theta,\theta,\bar\theta\right)$.

It will be seen later on that the L- and R-representations play an important role in the representations of the super Poincaré group.

3.2.2 Component Fields and their Supersymmetry Transformations

To find a connection between a superfield, $\Phi\left(x^\mu,\theta,\bar\theta\right)$, and the ordinary fields (functions of spacetime coordinates only), one expands the superfields as a Taylor series with respect to θ and $\overline{\theta}$. The coefficients obtained in this way are the ordinary component fields.

Consider a general (Lorentz scalar or pseudoscalar) superfield $\Phi(x, \theta, \overline{\theta})$. Its Taylor expansion in θ and $\overline{\theta}$ terminates as

$$\begin{aligned}
\Phi\left(x^{\mu}, \theta, \bar{\theta}\right) &= f(x) + \theta^{A}\varphi_{A}(x) + \bar{\theta}_{\dot{A}}\bar{\chi}^{\dot{A}}(x) \\
&\quad + (\theta\theta)\, m(x) + \left(\bar{\theta}\bar{\theta}\right) n(x) + \left(\theta\sigma^{\mu}\bar{\theta}\right) V_{\mu}(x) \\
&\quad + (\theta\theta)\,\bar{\theta}_{\dot{A}}\bar{\lambda}^{\dot{A}}(x) + \left(\bar{\theta}\bar{\theta}\right)\theta^{A}\psi_{A}(x) \\
&\quad + (\theta\theta)\left(\bar{\theta}\bar{\theta}\right) d(x),
\end{aligned} \tag{3.46}$$

where $(\theta\theta) = \theta^{A}\theta_{A}$ and $\left(\bar{\theta}\bar{\theta}\right) = \bar{\theta}_{\dot{A}}\bar{\theta}^{\dot{A}}$. This is due to the fact that the powers of θ (and hence $\overline{\theta}$) higher than two vanishes identically due to the anti-commuting nature of θ and $\overline{\theta}$. For example,

$$\begin{aligned}
(\theta\theta)\,\theta^{1} &= \theta^{B}\theta_{B}\theta^{1} \\
&= \left(\theta^{1}\theta^{2} - \theta^{2}\theta^{1}\right)\theta^{1} \\
&= -\left(\theta^{2}\theta^{1} - \theta^{1}\theta^{2}\right)\theta^{1} \\
&= -(\theta\theta)\,\theta^{1};
\end{aligned} \tag{3.47}$$

which holds if both sides of the equation vanish.

The quantities $f(x)$, $\chi^{\dot{A}}(x)$, $m(x)$, $n(x)$, $V_{\mu}(x)$, $\lambda^{\dot{A}}(x)$, $\psi_{A}(x)$ and $d(x)$ are called component fields. With the requirement that $\Phi\left(x^{\mu}, \theta, \bar{\theta}\right)$ is a Lorentz scalar or pseudoscalar, these fields can be classified according to their Lorentz transformation properties as follows:

- $f(x)$, $m(x)$ and $n(x)$ are complex scalar or pseudoscalar fields;
- $\varphi_{A}(x)$ and $\psi_{A}(x)$ are left-handed Weyl spinor fields;
- $\bar{\chi}^{\dot{A}}(x)$ and $\bar{\lambda}^{\dot{A}}(x)$ are right-handed Weyl spinor fields;
- $V_{\mu}(x)$ is a Lorentz four-vector field;

- $d(x)$ is a scalar field.

To calculate the infinitesimal supersymmetry transformations of these component fields, one equates

$$\begin{aligned}
\delta_S\Phi\left(x^\mu,\theta,\bar{\theta}\right) &= \delta_S f(x) + \theta^A\delta_S\varphi_A(x) + \bar{\theta}_{\dot{A}}\delta_S\bar{\chi}^{\dot{A}}(x) \\
&\quad + (\theta\theta)\,\delta_S m(x) + \left(\bar{\theta}\bar{\theta}\right)\delta_S n(x) + \left(\theta\sigma^\mu\bar{\theta}\right)\delta_S V_\mu(x) \\
&\quad + (\theta\theta)\,\bar{\theta}_{\dot{A}}\delta_S\bar{\lambda}^{\dot{A}}(x) + \left(\bar{\theta}\bar{\theta}\right)\theta^A\delta_S\psi_A(x) \\
&\quad + (\theta\theta)\left(\bar{\theta}\bar{\theta}\right)\delta_S d(x) \qquad (3.48)
\end{aligned}$$

with

$$\begin{aligned}
\delta_G\left(\xi,\bar{\xi}\right)\Phi\left(x^\mu,\theta,\bar{\theta}\right) &= \left[\left(-i\xi\sigma^\mu\bar{\theta} + i\theta\sigma^\mu\bar{\xi}\right)\partial_\mu + \xi^A\partial_A + \bar{\xi}^{\dot{A}}\bar{\partial}_{\dot{A}}\right] \\
&\quad \times\Phi\left(x^\mu,\theta,\bar{\theta}\right) \\
&= \left[\left(-i\xi\sigma^\mu\bar{\theta} + i\theta\sigma^\mu\bar{\xi}\right)\partial_\mu + \xi^A\partial_A + \bar{\xi}^{\dot{A}}\bar{\partial}_{\dot{A}}\right] \\
&\quad \times[f(x) + \theta^A\varphi_A(x) + \bar{\theta}_{\dot{A}}\bar{\chi}^{\dot{A}}(x) \\
&\quad + (\theta\theta)\,m(x) + \left(\bar{\theta}\bar{\theta}\right)n(x) + \left(\theta\sigma^\mu\bar{\theta}\right)V_\mu(x) \\
&\quad + (\theta\theta)\,\bar{\theta}_{\dot{A}}\bar{\lambda}^{\dot{A}}(x) + \left(\bar{\theta}\bar{\theta}\right)\theta^A\psi_A(x) \\
&\quad + (\theta\theta)\left(\bar{\theta}\bar{\theta}\right)d(x)], \qquad (3.49)
\end{aligned}$$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$, $\partial_A = \frac{\partial}{\partial\theta^A}$, $\bar{\partial}_{\dot{A}} = \frac{\partial}{\partial\bar{\theta}^{\dot{A}}}$. Once the explicit form of the right-hand side of the above equation is obtained, one is able to compare the coefficients of the same power of θ and $\overline{\theta}$ of the above two equations to give the supersymmetry transformations of the component fields as:

$$\begin{aligned}
\delta_S f(x) &= \alpha\varphi(x) + \bar{\alpha}\bar{\chi}(x), \\
\delta_S\varphi_A(x) &= 2\alpha_A m(x) + (\sigma^\mu\bar{\alpha})_A\left[i\partial_\mu f(x) + V_\mu(x)\right], \\
\delta_S\bar{\chi}^{\dot{A}}(x) &= 2\bar{\alpha}^{\dot{A}}n(x) + (\alpha\sigma^\mu\varepsilon)^{\dot{A}}\left[i\partial_\mu f(x) - V_\mu(x)\right], \\
\delta_S m(x) &= \bar{\alpha}\bar{\lambda}(x) - \frac{i}{2}\partial_\mu\varphi(x)\,\sigma^\mu\partial_\mu\varphi(x) - \frac{i}{2}\partial_\mu\bar{\chi}(x)\,\bar{\alpha},
\end{aligned}$$

$$\begin{aligned}
\delta_S n\left(x\right) &= \alpha\psi\left(x\right) + \frac{i}{2}\alpha\sigma^\mu\partial_\mu\bar{\chi}\left(x\right), \\
\delta_S V_\mu\left(x\right) &= \alpha\sigma^\mu\bar{\lambda}\left(x\right) + \psi\left(x\right)\sigma_\mu\bar{\alpha} + \frac{i}{2}\alpha\partial_\mu\varphi - \frac{i}{2}\partial_\mu\bar{\chi}\left(x\right)\bar{\alpha}, \\
\delta_S\bar{\lambda}^{\dot{A}}\left(x\right) &= 2\bar{\alpha}^{\dot{A}}d\left(x\right) + \frac{i}{2}\bar{\alpha}^{\dot{A}}\partial^\mu V_\mu\left(x\right) + i\left(\alpha\sigma^\mu\varepsilon\right)^{\dot{A}}\partial_\mu m\left(x\right), \\
\delta_S\psi_A\left(x\right) &= 2\alpha_A d\left(x\right) - \frac{i}{2}\alpha_A\partial^\mu V_\mu\left(x\right) + i\left(\sigma^\mu\bar{\alpha}\right)_A\partial_\mu m\left(x\right), \\
\delta_S d\left(x\right) &= \frac{i}{2}\partial_\mu\psi\left(x\right)\sigma^\mu\bar{\alpha} + \frac{i}{2}\alpha\sigma^\mu\partial_\mu\bar{\lambda}\left(x\right) \\
&= \frac{i}{2}\partial_\mu\psi\left(x\right)\sigma^\mu\bar{\alpha} - \frac{i}{2}\partial_\mu\bar{\lambda}\left(x\right)\alpha\sigma^\mu.
\end{aligned} \tag{3.50}$$

One sees immediately that supersymmetry transformations transform bosons into fermions, and vice versa. Notice that $\delta_S d\left(x\right)$ is a total derivative. This is very important, because this implies that the coefficient of $\theta^2\bar{\theta}^2$ of some (composite) superfield can play the role of the Lagrangian in a supersymmetric invariant action. This point will be discussed later on in this chapter.

3.2.3 Covariant Derivatives

As in the case of gauge symmetries, one can define a covariant derivative D as an operator acting on superfields that commutes with the supersymmetry transformations δ_{SUSY}:

$$D\Phi \to D\left(\delta_{SUSY}\Phi\right) = \delta_{SUSY}\left(D\Phi\right) \tag{3.51}$$

or

$$\left[D, \delta_{SUSY}\right] = 0. \tag{3.52}$$

For the representations introduced in the previous subsection, the corresponding covariant derivatives are:

1. For the ordinary representation:

$$D_A = \partial_A + i\sigma^\mu_{A\dot{B}}\bar{\theta}^{\dot{B}}\partial_\mu, \tag{3.53}$$

$$\bar{D}_{\dot{A}} = -\bar{\partial}_{\dot{A}} - i\theta^B\sigma^\mu_{B\dot{A}}\partial_\mu. \tag{3.54}$$

2. For the L-representation:

$$D_{(L)A} = \partial_A + 2i\sigma^{\mu}_{A\dot{B}}\bar{\theta}^{\dot{B}}\partial_{\mu}, \tag{3.55}$$

$$\bar{D}_{(L)\dot{A}} = -\bar{\partial}_{\dot{A}}. \tag{3.56}$$

3. For the R-representation:

$$D_{(R)A} = \partial_A, \tag{3.57}$$

$$\bar{D}_{(R)\dot{A}} = -\bar{\partial}_{\dot{A}} - 2i\theta^{B}\sigma^{\mu}_{B\dot{A}}\partial_{\mu}. \tag{3.58}$$

One can verify that these covariant derivatives commute with all supertranslational operators $\{P, Q, \overline{A}\}$ as in (3.51), and satisfy the following algebra:

$$\begin{array}{l} \{D_A, D_B\} = \{\bar{D}_{\dot{A}}, \bar{D}_{\dot{B}}\} = 0 \\ D^3 = \bar{D}^3 = 0 \\ \{D_A, \bar{D}_{\dot{B}}\} = -2i\sigma^{\mu}_{A\dot{B}}\partial_{\mu}\,. \end{array} \tag{3.59}$$

3.3 Constrained Superfields

In the last section, it has been shown that a supersymmetry transformation transforms one component field of a superfield into other components fields with the opposite statistics. Thus the component fields of a superfield form a representation of the super Poincaré group (or supersymmetry). For a general superfield, the associated representation is normally reducible, that is, the superfield contains more independent component fields than it is necessary. To reduce the number of component fields so as to make the superfield contain only one irreducible representation of supersymmetry, the trick is to impose some appropriate constraint on the superfield, and this constraint has to commute with the supersymmetry transformations in order to retain the supersymmetry transformation rule of the component fields. Once the superfield satisfies this constraint, some component fields are no longer independent: they are expressed in terms of the others (normally nonlinear combinations of them).[1]

Note that the trick just described is independent of the dynamics of the theory, that is, one does not have to consider the Lagrangian of the theory. However, sometimes merely imposing a constraint is not sufficient, and one has to use some gauge transformations (allowed by the Lagrangian, hence the dynamics of the theory) to get rid of some more component fields so as to reduce the number of component fields down to that of an irreducible representation of supersymmetry.

There are two types of constrained superfields used in particle physics phenomenology (some other types do exist, but will not be considered here), chiral superfields and vector superfields.

3.3.1 Chiral Superfields

A *chiral superfield* (also called a left-chiral superfield) $\Phi(x^\mu, \theta, \overline{\theta})$ is defined to satisfy the constraint

$$\overline{D}_{\dot{A}}\Phi = 0 \tag{3.60}$$

with $\overline{D}_{\dot{A}}$ being a superspace covariant derivative defined in (3.54). This constraint clearly commutes with the supersymmetry transformation, since if $\overline{D}_{\dot{A}}\Phi = 0$ then $\overline{D}_{\dot{A}}\delta_S\Phi = 0$ too, as the covariant derivatives commute with all the supertranslation operators.

To see explicitly how the above constraint reduces the number of independent component fields, observe that

$$\overline{D}_{\dot{A}} = -\frac{\partial}{\partial\overline{\theta}^{\dot{A}}}\bigg|_y \tag{3.61}$$

[1]Mathematically speaking, what this constraint does is to reduce the number of dimensions of the functional space of superfields so that the constrained superfields now lives in a subspace (of the functional space) invariant under the supersymmetry transformations.

where $y^\mu = x^\mu + i\theta\sigma^\mu\overline{\theta}$ is the bosonic coordinates defined in the L-representation of supersymmetry. This implies that the coordinates $(y^\mu, \theta, \overline{\theta})$ in the L-representation are appropriate for describing a chiral superfield. Thus the chiral superfield satisfying (3.60) is simply a function of (y^μ, θ) and takes the form

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta^A\psi_A(y) + \theta^A\theta^B\epsilon_{AB}F(y) \tag{3.62}$$

without the explicit appearance of $\overline{\theta}$ (compare this with the general superfield in (3.46)). The factor of $\sqrt{2}$ as been chosen by convention. Compare the above equation with (3.46), one notices that the only spinor contained in the chiral superfield is a left-handed Weyl spinor $\psi(x)$; this leads to the prefix "chiral" in the name of this superfield. Thus a chiral superfield contains three independent component fields: two complex scalars ϕ and F, and a spinor ψ of left-chirality. These three fields form an irreducible representation of the super Poincaré group. If one were to expand all the component fields as the Taylor series with respect to the fermionic coordinates (for example, one expands $\phi(y) = \phi(x) + i\theta\sigma^\mu\overline{\theta}\partial_\mu\phi(x) + \ldots$), one would obtain the original form of the superfield (as in (3.46)) whose component fields are no longer independent. Explicitly, the result is

$$\begin{aligned}
\Phi(y,\theta) &= \phi(y) + \sqrt{2}\theta\psi(y) + (\theta\theta)F(y) \\
&= \phi\left(x + i\theta\sigma\bar{\theta}\right) + \sqrt{2}\theta\psi\left(x + i\theta\sigma\bar{\theta}\right) + (\theta\theta)F\left(x + i\theta\sigma\bar{\theta}\right) \\
&= \phi(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) + \frac{1}{2}\left(\theta\sigma^\mu\bar{\theta}\right)\left(\theta\sigma^\nu\bar{\theta}\right)\partial_\mu\partial_\nu\phi(x) \\
&\quad + \sqrt{2}\theta^A\psi_A(x) + \sqrt{2}i\theta^A\left(\theta\sigma\bar{\theta}\right)\partial_\mu\psi_A(x) + (\theta\theta)F(x) \\
&= \phi(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \frac{1}{4}(\theta\theta)\left(\bar{\theta}\bar{\theta}\right)\partial^\mu\partial_\mu\phi(x) \\
&\quad + \sqrt{2}\theta^A\psi_A(x) + \frac{i}{\sqrt{2}}(\theta\theta)\bar{\theta}_{\dot{A}}\partial_\mu\psi^A(x)\sigma^\mu_{A\dot{B}}\epsilon^{\dot{B}\dot{A}} \\
&\quad + (\theta\theta)F(x).
\end{aligned} \tag{3.63}$$

This is how the chiral constraint reduced the number of independent component fields.

The supersymmetry transformations of the component fields of a chiral superfield is obtained by first identifying the component fields in (3.63) with those in (3.46), and then using the general formulae in (3.50). One finds

$$\delta_S \phi\,(x) = \sqrt{2}\alpha\psi\,(x) \qquad \text{(boson} \to \text{fermion)} \qquad (3.64)$$

$$\delta_S \psi\,(x) = \sqrt{2}\alpha_A F\,(x) + \sqrt{2}i\sigma^{\mu}_{A\dot{A}}\bar{\alpha}^{\dot{A}}\partial_\mu\phi\,(x) \qquad \text{(fermion} \to \text{boson)} \qquad (3.65)$$

$$\delta_S F\,(x) = -\sqrt{2}i\partial_\mu\psi\,(x)\,\sigma^\mu\bar{\alpha} \qquad (F \to \text{total derivative}). \qquad (3.66)$$

Notice that the supersymmetry transformation of the component field F, which is a coefficient of θ^2, is a total spacetime derivative. As the other coefficient of θ^2 in (3.63) is already a total derivative which gives zero after integrating over a spacetime, the whole set of coefficients of θ^2 in a chiral superfield can serve as a Lagrangian in a supersymmetric invariant action. This fact will be of great use in constructing a supersymmetric Lagrangian.

An important property of chiral superfields is that a product of two chiral superfield is also a chiral superfield. This can be verified by observing that if $\overline{D}_{\dot{A}}\Phi_i = 0 = \overline{D}_{\dot{A}}\Phi_j$, then $\overline{D}_{\dot{A}}(\Phi_i\Phi_j) = 0$, or the product $\Phi_i\Phi_j$ is a chiral superfield if Φ_i and Φ_j are. More explicitly,

$$\begin{aligned}
\Phi_i\Phi_j \;=\;& \left[\phi_i\,(y) + \sqrt{2}\theta\psi_i\,(y) + (\theta\theta)\,F_i\right] \\
& \times \left[\phi_j\,(y) + \sqrt{2}\theta\psi_j\,(y) + (\theta\theta)\,F_j\right] \\
=\;& \phi_i\,(y)\,\phi_j\,(y) + 2\,(\theta\psi_i\,(y))\,(\theta\psi_j\,(y)) \\
& + \sqrt{2}\theta\phi_i\,(y)\,\psi_j\,(y) + (\theta\theta)\,\phi_i\,(y)\,F_j \\
& + \sqrt{2}\theta\psi_i\,(y)\,\phi_j\,(y) + (\theta\theta)\,F_i\phi_j\,(y) \\
=\;& \phi_i\,(y)\,\phi_j\,(y) \\
& + \sqrt{2}\theta\,[\phi_i\,(y)\,\psi_j\,(y) + \psi_i\,(y)\,\phi_j\,(y)] \\
& + (\theta\theta)\,[\phi_i\,(y)\,F_j + F_i\phi_j\,(y) - \psi_i\,(y)\,\psi_j\,(y)]\,. \qquad (3.67)
\end{aligned}$$

This result implies that an arbitrary function of chiral superfields is also a chiral superfield.

Observe that a chiral superfield is a complex superfield. It is not hard to see that the hermitian conjugate of a chiral superfield depends only on the coordinates $(z, \overline{\theta})$ in the R-representation,

$$\Phi^{\dagger} \;=\; \phi^{*}(z) - \sqrt{2}\bar{\theta}\bar{\psi}(z) + \left(\bar{\theta}\bar{\theta}\right) F^{*}(z), \tag{3.68}$$

and satisfies the constraint

$$D_A \Phi^{\dagger} = \left.\frac{\partial}{\partial \theta^A}\right|_z \Phi^{\dagger} = 0. \tag{3.69}$$

The superfield $\Phi^{\dagger}$ satisfying the constraint $D_A \Phi^{\dagger} = 0$ is called an "antichiral" or "right-chiral" superfield (since it contains only a right-handed spinor). In terms of the original superspace coordinates $(x^{\mu}, \theta, \overline{\theta})$, the antichiral superfield takes the form

$$\begin{aligned}
\Phi\left(z, \bar{\theta}\right) &= \phi^{*}(z) + \sqrt{2}\bar{\theta}\bar{\psi}(z) + \left(\bar{\theta}\bar{\theta}\right) F^{*}(z) \\
&= \phi^{*}(x) - i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}\phi^{*}(x) - \frac{1}{4}(\theta\theta)\left(\bar{\theta}\bar{\theta}\right)\partial^{\mu}\partial_{\mu}\phi^{*}(x) \\
&\quad + \sqrt{2}\bar{\theta}\bar{\psi}(x) - \frac{i}{\sqrt{2}}\left(\bar{\theta}\bar{\theta}\right)\theta\sigma^{\mu}\partial_{\mu}\bar{\psi}(x) + (\theta\theta) F^{*}(x).
\end{aligned} \tag{3.70}$$

The supersymmetry transformations of the above component fields can be obtained by performing hermitian conjugation on the previous result for a chiral superfield. Similar to the case of chiral superfields, any function of antichiral superfields is also an antichiral superfield, and the coefficients of $\overline{\theta}^2$ in the expansion of an antichiral superfield can serve as a Lagrangian in a supersymmetric invariant action.

Chiral and antichiral superfields represent supersymmetric multiplets of matter fields, such as quarks and leptons. Thus for a known spin-$\frac{1}{2}$ particle, one can construct a supersymmetry multiplet separately for its left-handed and right-handed components by associating with them a chiral and an antichiral superfield respectively. This is of great advantages in constructing the supersymmetric extension of the Standard Model in which left-handed and right-handed components of a matter spinor field transform differently under the Standard Model gauge group.

3.3.2 Vector Superfields

The aim of this chapter is to describe the Standard Model particles and their superpartners by the component fields of superfields. In the previous subsection, it is clear that a chiral superfield, which contains only spin-0 bosons and a spin-1/2 fermion, cannot describe all particles in the SM, because it does not contain any spin-1 gauge boson as its component field. Thus one has to introduce another type of constrained superfields, called a vector superfield V, as follows. First of all, a vector superfield has to satisfy a "reality" condition:

$$V\left(x,\theta,\bar{\theta}\right) = V^{\dagger}\left(x,\theta,\bar{\theta}\right). \tag{3.71}$$

Just like (3.46), one can expand the vector superfield as

$$\begin{aligned} V\left(x,\theta,\bar{\theta}\right) &= C\left(x\right) + \theta\varphi\left(x\right) + \bar{\theta}\bar{\chi}\left(x\right) \\ &\quad + \left(\theta\theta\right) M\left(x\right) + \left(\bar{\theta}\bar{\theta}\right) N\left(x\right) + \left(\theta\sigma^{\mu}\bar{\theta}\right) V_{\mu}\left(x\right) \\ &\quad + \left(\theta\theta\right)\bar{\theta}\bar{\lambda}\left(x\right) + \left(\bar{\theta}\bar{\theta}\right)\theta\psi\left(x\right) + \left(\theta\theta\right)\left(\bar{\theta}\bar{\theta}\right) D\left(x\right). \end{aligned} \tag{3.72}$$

Its hermitian conjugate is

$$\begin{aligned} V^{\dagger}\left(x,\theta,\bar{\theta}\right) &= C^{*}\left(x\right) + \bar{\theta}\bar{\varphi}\left(x\right) + \theta\chi\left(x\right) \\ &\quad + \left(\bar{\theta}\bar{\theta}\right) M^{*}\left(x\right) + \left(\theta\theta\right) N^{*}\left(x\right) + \left(\theta\sigma^{\mu}\bar{\theta}\right) V_{\mu}^{*}\left(x\right) \\ &\quad + \left(\bar{\theta}\bar{\theta}\right)\theta\lambda\left(x\right) + \left(\theta\theta\right)\bar{\theta}\bar{\psi}\left(x\right) + \left(\theta\theta\right)\left(\bar{\theta}\bar{\theta}\right) D^{*}\left(x\right). \end{aligned} \tag{3.73}$$

The reality condition is satisfied if and only if

$$\begin{aligned}
&C(x) = C^*(x) \Rightarrow C(x) \text{ is a real scalar field;} \\
&\varphi(x) = \chi(x) \\
&M(x) = N^*(x) \\
&V_\mu(x) = V_\mu^*(x) \Rightarrow V_\mu(x) \text{ is a real vector field;} \\
&\lambda(x) = \psi(x) \\
&D(x) = D^*(x) \Rightarrow D(x) \text{ is a real scalar field.}
\end{aligned} \tag{3.74}$$

Thus the expansion of a vector superfield satisfying the reality condition is

$$\begin{aligned}
V\left(x,\theta,\bar\theta\right) &= C(x) + \theta\varphi(x) + \bar\theta\bar\varphi(x) \\
&+ (\theta\theta) M(x) + \left(\bar\theta\bar\theta\right) M^*(x) + \left(\theta\sigma^\mu\bar\theta\right) V_\mu(x) \\
&+ (\theta\theta)\bar\theta\bar\lambda(x) + \left(\bar\theta\bar\theta\right)\theta\lambda(x) + (\theta\theta)\left(\bar\theta\bar\theta\right) D(x),
\end{aligned} \tag{3.75}$$

where

- $C(x)$ and $D(x)$ are real scalar fields;
- $M(x)$ is a complex scalar field;
- $\lambda(x)$ and $\varphi(x)$ are spinor fields;
- $V_\mu(x)$ is a real vector field.

Note that the above superfield contains equal numbers of bosonic and fermionic degrees of freedom as it should. Despite the simple form in (3.75), it proves convenient to rewrite the vector superfield as

$$\begin{aligned}
V\left(x,\theta,\bar\theta\right) &= \left(1 + \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial_\mu\partial^\mu\right) C(x) + \left(i\theta + \frac{1}{2}\theta\theta\sigma^\mu\bar\theta\partial_\mu\right)\varphi(x) \\
&+ \frac{i}{2}\theta\theta M(x) + \left(-i\bar\theta + \frac{1}{2}\bar\theta\bar\theta\sigma^\mu\theta\partial_\mu\right)\bar\varphi(x) - \frac{i}{2}\bar\theta\bar\theta M^* \\
&- \theta\sigma^\mu\bar\theta V_\mu(x) + \theta\theta\bar\theta\bar\lambda(x) - i\bar\theta\bar\theta\theta\lambda(x) \\
&+ \frac{1}{2}\theta\theta\bar\theta\bar\theta D(x)
\end{aligned} \tag{3.76}$$

which still satisfies the reality condition (3.71). One sees that the vector field $V_\mu(x)$, which is needed to describe a gauge field, is contained in the vector superfield. This is a good sign that one is on the right track to construct a superfield which plays the role of a gauge field.

However, the above vector superfield still contains too many component fields to form just one irreducible representation of supersymmetry (which, of course, must contain a vector field). Thus for a vector superfield to contain just one irreducible representation, an equal number of bosonic and fermionic degrees of freedom have to be eliminated. To do this, the trick is to subtract a vector superfield V using the other "real" superfield K which contains fewer independent component fields. The component fields of K have to be chosen just right that they can eliminate precisely the unwanted component fields of V. Of course, K cannot contain a vector field as its component field as it would eliminate the desired gauge field from V.

It turns out that the superfield K that does a great job takes the form $K = \Phi + \Phi^\dagger$, where Φ is a chiral superfield defined in the previous subsection, and $\Phi^\dagger$ is its hermitian conjugate. To verify our assertion, consider the transformation

$$\begin{aligned} V\left(x,\theta,\bar{\theta}\right) \;&\rightarrow\; V'\left(x,\theta,\bar{\theta}\right) \\ &= V\left(x,\theta,\bar{\theta}\right) + \Phi\left(x,\theta,\bar{\theta}\right) + \Phi^\dagger\left(x,\theta,\bar{\theta}\right) \\ &\equiv V\left(x,\theta,\bar{\theta}\right) + i\Lambda\left(x,\theta,\bar{\theta}\right) - i\Lambda^\dagger\left(x,\theta,\bar{\theta}\right). \end{aligned} \tag{3.77}$$

where $\Phi = i\Lambda$ is a chiral superfield whose explicit form was given in (3.63). Using (3.63), (3.70) and (3.76), the component fields transform under this transformation as

$$\begin{aligned}
C(x) &\to C'(x) = C(x) + \phi(x) + \phi^*(x) \\
\varphi(x) &\to \varphi'(x) = \varphi(x) + \sqrt{2}\psi(x) \\
M(x) &\to M'(x) = M(x) + F(x) \\
V_\mu(x) &\to V'_\mu(x) = V_\mu(x) + i\partial_\mu(\phi(x) - \phi^*(x)) \\
\lambda(x) &\to \lambda'(x) = \lambda(x) \\
D(x) &\to D'(x) = D(x).
\end{aligned} \tag{3.78}$$

Thus if the component fields of Φ are chosen as

$$\sqrt{2}\psi(x) = -\varphi(x) \tag{3.79}$$

$$F(x) = -M(x) \tag{3.80}$$

$$2\mathrm{Re}(\phi(x)) = \phi(x) + \phi^*(x) = -C(x), \tag{3.81}$$

then the component fields $\phi(x)$, $M(x)$ and $C(x)$ are completely eliminated, and one is left with the fields V_μ, λ and D which form an irreducible representation of supersymmetry. Note that under this transformation, λ and D are invariant while the vector field V_μ transforms as $V_\mu(x) \to V'_\mu(x) = V_\mu(x) + i\partial_\mu(\phi(x) - \phi^*(x))$, which is precisely an Abelian gauge transformation. This implies that the above transformation is a superfield version of an Abelian gauge transformation, and V_μ really plays the role of a gauge field. One sees immediately that such a process of component field elimination will work only if the theory has some gauge invariance; this has been remarked at the beginning of this section.

Once the unwanted component fields have been eliminated, the vector superfield is said to be in the *Wess-Zumino (WZ) gauge*, and takes the form

$$\begin{aligned}
V_{\mathrm{WZ}}\left(x,\theta,\bar{\theta}\right) &= \left(\theta\sigma^\mu\bar{\theta}\right)\left[V_\mu(x) + i\partial_\mu(\phi(x) - \phi^*(x))\right] \\
&\quad + (\theta\theta)\,\bar{\theta}\bar{\lambda}(x) + \left(\bar{\theta}\bar{\theta}\right)\theta\lambda(x) + (\theta\theta)\left(\bar{\theta}\bar{\theta}\right)D(x).
\end{aligned} \tag{3.82}$$

Thus an irreducible representation of supersymmetry containing an Abelian gauge field contains a gauge field $V_\mu(x)$, its fermionic superpartner $\lambda(x)$ called a gaugino, and an auxiliary field $D(x)$.

The derivation of supersymmetry transformations for a vector superfield is rather tricky. Suppose one starts with a vector superfield in the Wess-Zumino gauge and then performs a supersymmetry transformation using the method in the previous section, it is not hard to check that the transformed superfield is no longer in the Wess-Zumino gauge, that is, its $C(x)$, $\varphi(x)$, and $M(x)$ components no longer vanish. The way out of this trouble is to include a gauge transformation with "field-dependent" parameters in the definition of supersymmetry transformation so that the transformed superfield remains in the Wess-Zumino gauge. This means that one has to define a new supersymmetry transformation $\delta_{\tilde{S}}$ as $\delta_{\tilde{S}} = \delta_S + \delta_{gauge}$, where δ_S is the usual supersymmetry transformation and δ_{gauge} is a gauge transformation whose parameters are appropriately chosen (normally functions of component fields) to cancel out the unwanted component fields. The final result is

$$\delta_{\tilde{S}} V^\mu = \frac{1}{\sqrt{2}} \left(\alpha \sigma^\mu \overline{\lambda} + \lambda \sigma^\mu \overline{\alpha} \right) \tag{3.83}$$

$$\delta_{\tilde{S}} \lambda = \frac{1}{\sqrt{2}} \alpha D - \frac{i}{2\sqrt{2}} \left(\sigma^\mu \overline{\sigma}^\nu \right) \alpha F_{\mu\nu} \tag{3.84}$$

$$\delta_{\tilde{S}} D = \frac{i}{\sqrt{2}} \left(\alpha \sigma^\mu \partial_\mu \bar{\lambda} + \bar{\alpha} \overline{\sigma}^\mu \partial_\mu \lambda \right) \tag{3.85}$$

where $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ is the usual field strength of V_μ. An interesting point is that the supersymmetry variation of D is a total derivative, and therefore one can construct a supersymmetric action using the component field D of a vector superfield as the Lagrangian.

In a theory with a non-Abelian gauge symmetry, one defines a vector superfield as a real superfield V transforming as an adjoint representation of the gauge group and subject to the gauge transformation

$$e^{gV} \to e^{-ig\Lambda^\dagger} e^{gV} e^{ig\Lambda}, \tag{3.86}$$

where g is the gauge coupling and Λ is a chiral superfield in an adjoint representation of the gauge group. All the results for the non-Abelian case are the same as those in the Abelian case with the partial derivatives being replaced by the gauge covariant derivatives, so they are not repeated here.

3.4 Construction of the Supersymmetric Action

To construct a supersymmetric action, one needs the Lagrangian whose supersymmetric variation is either zero or a total spacetime derivative:

$$\delta_S \int d^4x \mathcal{L}(x) = \int d^4x \partial_\mu (\ldots) = 0. \tag{3.87}$$

According to (3.66) and (3.85), the highest order terms in θ and $\bar{\theta}$ of any chiral and real superfields satisfy this requirement. This observation is a starting point in constructing a supersymmetric Lagrangian using the superspace techniques. Consider a general superfield $S(x,\theta,\overline{\theta})$. One can extract its coefficient of $\theta^2\overline{\theta}^2$ as follows. Using the rules of integration over Grassmann variables,

$$\int d\theta_\alpha = 0, \qquad \int \theta_\alpha d\theta_\beta = \delta_{\alpha\beta}, \tag{3.88}$$

one sees that

$$\begin{aligned} \int d^2\theta d^2\overline{\theta} S\left(x,\theta,\bar{\theta}\right) &= \int d^2\theta d^2\overline{\theta} \Big[f(x) + \theta^A \varphi_A(x) + \bar{\theta}_{\dot{A}} \bar{\chi}^{\dot{A}}(x) \\ &\quad + (\theta\theta)\, m(x) + \left(\bar{\theta}\bar{\theta}\right) n(x) + \left(\theta\sigma^\mu\bar{\theta}\right) V_\mu(x) \\ &\quad + (\theta\theta)\, \bar{\theta}_{\dot{A}} \bar{\lambda}^{\dot{A}}(x) + \left(\bar{\theta}\bar{\theta}\right) \theta^A \psi_A(x) \\ &\quad + (\theta\theta)\left(\bar{\theta}\bar{\theta}\right) d(x) \Big] \\ &= d(x). \end{aligned} \tag{3.89}$$

This means that the coefficient of $\theta^2\overline{\theta}^2$ can be extracted by integrating S over the Grassmann coordinates. Similarly, one can extract the coefficient of θ^2 in a chiral superfield by integrating it over "half" of the Grassmann coordinates θ.

Using the above method, the supersymmetric Lagrangian can be constructed as follows. Define a superpotential $W(\Phi)$ as a function of chiral superfields Φ_i $(i = 1, \ldots, N)$. It is clear that W is also a chiral superfield, so the supersymmetric variation of the coefficient of θ^2 contained in it is a total spacetime derivative and therefore such a coefficient can serve as a part of a supersymmetric Lagrangian (which, of course, gives a supersymmetric invariant action by itself). This means that

$$\mathcal{L}_{int}(\Phi) = \int d^2\theta W(\Phi) + h.c. \tag{3.90}$$

which, after integrating over spacetime, is supersymmetric invariant. Since Φ_i describes a multiplet of matter fields, then its mass dimension is one. So the most general form of W that gives a renormalizable Lagrangian is

$$W(\Phi) = \sum_i k_i\Phi_i + \frac{1}{2}\sum_{ij} m_{ij}\Phi_i\Phi_j + \frac{1}{3}\sum_{ijk} g_{ijk}\Phi_i\Phi_j\Phi_k \tag{3.91}$$

where k_i, m_{ij} and g_{ijk} are constants. With the form of W, $\mathcal{L}_F(\Phi)$ is

$$\begin{aligned}\mathcal{L}_{int}(\Phi) &= \int d^2\theta \left[\sum_i k_i\Phi_i + \frac{1}{2}\sum_{ij} m_{ij}\Phi_i\Phi_j + \frac{1}{3}\sum_{ijk} g_{ijk}\Phi_i\Phi_j\Phi_k\right] \\ &= \sum_j \frac{\partial W(\phi_i)}{\partial\phi_j}F_j - \frac{1}{2}\sum_{j,k}\frac{\partial^2 W(\phi_i)}{\partial\phi_j\partial\phi_k}\psi_j\psi_k,\end{aligned} \tag{3.92}$$

where $W(\phi_i)$ is the superpotential in (3.91) with all the superfields Φ_i being replaced by their $\theta = \overline{\theta} = 0$ components ϕ_i describing the scalar fields. It will be seen below that the above Lagrangian describes the potential and non-derivative interaction terms of the theory.

The kinetic terms can be obtained by integrating a real superfield $\Phi\Phi^\dagger$, with Φ a chiral superfield as usual, over the Grassmann coordinates,

$$\begin{aligned}\mathcal{L}_{kin}(V) &= \int d^2\theta d^2\bar{\theta}\Phi\Phi^\dagger \\ &= FF^* - \phi\partial_\mu\partial^\mu\phi^* - i\bar{\psi}\sigma_\mu\partial^\mu\psi.\end{aligned} \tag{3.93}$$

It can be seen that the last two terms of this equation are kinetic terms for a scalar field and a fermionic field respectively. However, there is no kinetic term for the F-field. Thus this field has no dynamics (unphysical) and is called an *auxiliary field.*

Combinding $\mathcal{L}_{int}(\Phi)$ of (3.90) together with its hermitian conjugate, and $\mathcal{L}_{kin}(V)$ of (3.93), one can write the full Lagrangian as

$$\mathcal{L} = \sum_i \int d^2\theta d^2\bar{\theta}\Phi_i\Phi_i^\dagger + \left(\int d^2\theta W(\Phi) + h.c.\right) \tag{3.94}$$

$$= \sum_i \left[F_iF_i^* + \partial_\mu\phi_i\partial^\mu\phi_i - i\bar{\psi}_i\sigma^\mu\partial_\mu\psi_i\right] + \left[\sum_j \frac{\partial W(\phi_i)}{\partial\phi_j}F_j - \frac{1}{2}\sum_{j,k}\frac{\partial^2 W(\phi_i)}{\partial\phi_j\partial\phi_k}\psi_j\psi_k + h.c.\right]. \tag{3.95}$$

As the auxiliary fields F_i are non-dynamical, then one can eliminate them using their equations of motion[2]

$$\frac{\partial\mathcal{L}}{\partial F_j} = 0$$
$$F_j = -\left[\frac{\partial W(\phi_i)}{\partial\phi_j}\right]^*. \tag{3.96}$$

Using the above result in (3.95), the supersymmetric Lagrangian becomes

$$\mathcal{L} = \sum_i \left[F_iF_i^* + \partial_\mu\phi_i\partial^\mu\phi_i - i\bar{\psi}_i\sigma^\mu\partial_\mu\psi_i\right] + \left[-\frac{1}{2}\sum_{j,k}\frac{\partial^2 W(\phi_i)}{\partial\phi_j\partial\phi_k}\psi_j\psi_k + h.c.\right] - \sum_j\left|\frac{\partial W(\phi_i)}{\partial\phi_j}\right|^2. \tag{3.97}$$

The last term on the right-hand side above is called the F-term contribution to the potential. Using the explicit form of the superpotential, one sees that it contains the mass terms and self-interactions of the scalar fields.

[2]This process actually has a quantum mechanical origin. In the path integral quantization, one performs the functional integration of the exponential of the action. If the Lagrangian contains a non-dynamical field F, then "the functional integration over F" gives exactly the same result as "replacing F with its form, obtained from its classical equations of motion, in the Lagrangian."

To introduce gauge interactions into the supersymmetric Lagrangian, the Lagrangian of (3.94) is required to be invariant under the gauge transformation:

$$\Phi \rightarrow e^{-ig\Lambda(x)}\Phi, \tag{3.98}$$

where $\Lambda(x)$ is a chiral superfield (so that $e^{-ig\Lambda(x)}\Phi$ is still a chiral superfield). Under the transformation (3.98), $\mathcal{L}_{kin}(V)$ is not invariant:

$$\Phi^\dagger\Phi \rightarrow \Phi^\dagger e^{+ig\Lambda^\dagger} e^{-ig\Lambda}\Phi. \tag{3.99}$$

To make $\mathcal{L}_{kin}(V)$ invariant under the transformation (3.98), it needs to be modified by introducing a vector superfield into the Lagrangian:

$$\int d^2\theta d^2\bar{\theta}\Phi^\dagger\Phi \rightarrow \int d^2\theta d^2\bar{\theta}\Phi^\dagger e^{2gV}\Phi \tag{3.100}$$

and recall that e^{gV} transforms as in (3.86). As the Lagrangian is now gauge invariant, one is eligible to use the Wess-Zumino gauge in which the Lagrangian takes the form

$$\begin{aligned}\int d^2\theta d^2\bar{\theta}\Phi^\dagger e^{2gV}\Phi &= |D_\mu\phi|^2 - i\bar{\psi}\sigma_\mu D^\mu\psi + g\phi^*\sum_a D_a T^a\phi \\ &\quad + ig\sqrt{2}\left(\phi^*\lambda\psi - \bar{\lambda}\bar{\psi}\phi\right) + |F|^2,\end{aligned} \tag{3.101}$$

where $D_\mu = \partial_\mu + igA_\mu^a T^a$ and T^a are group generators. On the other hand, the superpotential in (3.91) is unchanged provided the coefficients k_i, m_{ij}, and g_{ijk} are chosen appropriately so as to make W gauge invariant.

Finally, the kinetic terms of gauge fields are considered. In the electromagnetic theory, the kinetic terms of a gauge vector field is described by the contraction of the electromagnetic field strengths:

$$\mathcal{L}_{\text{kin}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{3.102}$$

The kinetic terms for the vector superfield must therefore give rise to the above Lagrangian. Such terms can be constructed by defining a *supersymmetric field* *strength*:

$$W_A = \overline{D}^2 e^{-gV} D_A e^{gV} \tag{3.103}$$

where $\overline{D}^2 = \frac{1}{2}\epsilon^{\dot{A}\dot{B}}\overline{D}_{\dot{A}}\overline{D}_{\dot{B}}$. W_A is a chiral superfield as one can check by using (3.59), and is gauge covariant, i.e., it transforms under the gauge transformation as $W_A \to e^{-ig\Lambda}W_A e^{ig\Lambda}$. For Abelian symmetries, it reduces to

$$W_A = \overline{D}^2 D_A V \tag{3.104}$$

where the coupling constant g has been absorbed into V. Since W_A is chiral superfield, $W^A W_A$ is also a chiral superfield and its trace over the group indices is gauge invariant. Thus the action for the gauge supermultiplet takes the form

$$\begin{aligned}\int d^2\theta \left\{ \frac{1}{32g^2}\mathrm{Tr}\left(W^A W_A\right)\right\} + h.c. \quad = \quad & -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + \frac{1}{2}D_a D^a \\ & -\frac{i}{2}\lambda^a \sigma_\mu \partial^\mu \bar{\lambda}_a \\ & +\frac{1}{2}g f^{abc}\lambda_a \sigma_\mu A^\mu_b \bar{\lambda}_c + h.c.,\end{aligned} \tag{3.105}$$

where f^{abc} are the Lie algebra structure constants. One sees that in addition to the kinetic terms for the gauge field, this Lagrangian contains the kinetic terms for the gauginos λ_a ($-\frac{i}{2}\lambda^a\sigma_\mu\partial^\mu\bar{\lambda}_a$ and its hermitian conjugate) and the canonical coupling of the gauginos and the gauge fields ($\frac{1}{2}gf^{abc}\lambda_a\sigma_\mu A^\mu_b\bar{\lambda}_c$ and its hermitian conjugate).

Because there is no kinetic terms for D_a in (3.105), D_a are auxiliary fields and can be integrated out. To integrate out the D_a field, one combines (3.101) and (3.105) and then derives the equations of motion for D_a which give

$$D_a = -g\sum_{ij}\phi_i^* T_a^{ij}\phi_j, \tag{3.106}$$

where i, j are group indices. Substituting this result into the Lagrangian gives the so-called D-term contribution to the potential as

$$V_D = \frac{1}{2}\sum_a \left|\sum_{ij} g\phi_i^* T_a^{ij}\phi_j\right|^2. \tag{3.107}$$

3.5 The Minimal Supersymmetric Standard Model

3.5.1 The MSSM Particle Contents

The Minimal Supersymmetric Standard Model (MSSM) is the supersymmetric extension of the Standard Model with the smallest possible number of superfields and their interactions.

It is natural to ask "Is it possible that the particles in the Standard Model might already be superpartners of one another?" Unfortunately, the answer is "No" because the matter fermions and the gauge bosons belong to the different representations of the gauge group. Even though the Higgs boson and the neutrino have their spins different by one-half but the same gauge quantum numbers, they cannot be the superpartner of each other as this would lead to the unacceptable phenomenological results. For example, the terms required to give masses to the charged leptons explicitly break the lepton number conservation if the superpartner of a neutrino were the Higgs field.

In the SM, an $SU(2)$ scalar doublet with hypercharge $Y = 1/2$ is needed to break the $SU(2) \times U(1)$ invariance, but the MSSM needs two such doublets: one has hypercharge $Y = 1/2$ like the SM doublet while the other has hypercharge $Y = -1/2$. Each scalar doublet and its fermionic superpartners are contained in a Higgs superfield doublet. There are two important reasons for introducing two doublets in the MSSM. The first one is that only one Higgs doublet cannot give masses to all fermion matter fields after the electroweak symmetry breaking. The other one is more technical. With only one doublet, its fermionic superpartners, which are chiral fermions, contribute the extra gauge anomalies into the theory, and these anomalies would definitely spoil the renormalizability of the theory. Thus the second Higgs doublet with just the right quantum numbers has to be introduced so that the anomalies contributed by its fermionic superpartners will cancel precisely those from the first doublet.

Table 3.1: Chiral superfields of the MSSM.

Superfield	$SU(3)$	$SU(2)_L$	$U(1)_Y$	Particle Contents
$\hat{Q}$	3	2	$\frac{1}{6}$	(u_L, d_L), $(\tilde{u}_L, \tilde{d}_L)$
$\hat{U}^c$	$\overline{3}$	1	$-\frac{2}{3}$	$\bar{u}_R, \tilde{u}^*_R$
$\hat{D}^c$	$\overline{3}$	1	$\frac{1}{3}$	$\bar{d}_R, \tilde{d}^*_R$
$\hat{L}$	1	2	$-\frac{1}{2}$	(ν_L, e_L), $(\tilde{\nu}_L, \tilde{e}_L)$
$\hat{E}^c$	1	1	1	$\bar{e}_R, \tilde{e}^*_R$
$\hat{H}_1$	1	2	$\frac{1}{2}$	$(H_1, \tilde{h}_1)$
$\hat{H}_2$	1	2	$-\frac{1}{2}$	$(H_2, \tilde{h}_2)$

To represent all particles of the Standard Model and their superpartners, the matter fields and their superpartners are combined into chiral superfields while the gauge bosons and their superpartners are combined into vector superfields. For the left-handed SM fermions, the forms of the corresponding chiral superfields are quite obvious as the chiral superfields automatically contain left-handed fermions. But for a right-handed SM fermion, one needs to use its hermitian conjugate, which is a left-handed fermion, to construct the superfield. All of these superfields and their corresponding SM particles belong to the same representations of the Standard Model gauge group. For the second Higgs supermultiplet which has no SM counterpart, it is a singlet under $SU(3)_C$, a doublet under $SU(2)_L$, and has $Y = -1/2$ to cancel the gauge anomalies.

As for the names of the particles, the superpartners of quarks and leptons are called *squarks* and *sleptons* respectively. In particular, the superpartner of the top quark is called the *stop squark* and that of the electron is called the *selectron*. The fermionic superpartners of the gauge bosons are called the *gauginos*. For example, the superpartners of $W^{\pm}$ and Z^0 bosons are called *winos* and *zinos* respectively. For the Higgs bosons, their superpartners are called the *Higgsinos*. These particles are gathered in Tables 3.1 and 3.2, where all spinors are represented as two-component Weyl spinors. In Tables 3.1 and 3.2, each superfield is denoted by a capital letter with a hat (e.g., $\hat{Q}$) while the superpartner of each SM particle is denoted by the letter corresponding to the particle with a tilde

Table 3.2: Vector superfields of the MSSM.

Superfield	$SU(3)$	$SU(2)_L$	$U(1)_Y$	Particle Contents
$\hat{G}^a$	8	1	0	g, $\tilde{g}$
$\hat{W}^i$	1	3	0	W_i, $\tilde{w}_i$
$\hat{B}$	3	1	0	B, $\tilde{b}$

(e.g., $\widetilde{u}$ is the scalar superpartner of the up quark). The "c" superscript on the superfields corresponding to the right-handed fermions in the SM (namely $\hat{U}^c$, $\hat{D}^c$ and $\hat{E}^c$) indicates that it is the hermitian conjugate of the right-handed fermion that appears in each superfield; this explains the notations for the particle contents of such superfields. Notice that Table 3.1 contains only the first generation particles without their antiparticles. The superfields for the other generations can be defined in the same way.

3.5.2 The MSSM Action

In Section 3.4, the general form of the supersymmetric Lagrangian was constructed using two kinds of superfield, namely the chiral superfields and the vector superfields. The chiral superfield, which describes the multiplet of a matter field, contains a complex scalar field ϕ, a left-handed Weyl fermion ψ, and an auxiliary field F. The vector superfield, which describes the multiplet of a gauge field, contains a gauge field A_μ, a gaugino λ, and an auxiliary field D. The general gauge invariant supersymmetric action, with all the auxiliary fields eliminated by their equations of motion, is given by

$$
\begin{aligned}
S \quad = \quad & \int d^4x \left\{ \sum_i \left(\left| D_\mu \phi^i \right|^2 + i\bar{\psi}_i \sigma_\mu D^\mu \psi_i \right) \right. \\
& -\frac{1}{2} \sum_{i,j} \left(\frac{\partial^2 W}{\partial\phi_i \partial\phi_j} \psi_i \psi_j + \frac{\partial^2 \bar{W}}{\partial\phi_i \partial\phi_j} \bar{\psi}_i \bar{\psi}_j \right) - \sum_i \left| \frac{\partial W}{\partial \phi_i} \right|^2 \\
& -\frac{1}{4} \sum_a \left(F^a_{\mu\nu} F^{a\mu\nu} - i\lambda^a \sigma_\mu D^\mu \bar{\lambda}^a \right) \\
& -\sqrt{2} \sum_{a,i,j} g^a \left[\bar{\lambda}^a \left(\bar{\psi}^i T^a \phi^i \right) + \lambda^a \left(\phi^{i*} T^a \psi^i \right) \right] \\
& \left. -\frac{1}{2} \sum_a \left| \sum_{i,j} g \phi^{i*} T^a_{ij} \phi^i \right|^2 \right\}. \qquad (3.108)
\end{aligned}
$$

The indices i, j run over all the chiral multiplets in the theory. The function W is a general gauge invariant superpotential.

In order to construct the action of the MSSM, one demands that the superpotential W be invariant under $SU(3) \times SU(2)_L \times U(1)_Y$. The information concerning the gauge group representations of all superfields in Table 3.1 enables one to construct the MSSM gauge invariant superpotential as

$$
\begin{aligned}
W \quad = \quad & \sum_{\alpha,\beta=1}^{2} \mu \epsilon_{\alpha\beta} \hat{H}_1^\alpha \hat{H}_2^\beta \\
& + \sum_{i,j=1}^{3} \left[(\lambda_E)_{ij} \hat{H}_2 \hat{L}_i \hat{E}^c_j + (\lambda_D)_{ij} \hat{H}_2 \hat{Q}_i \hat{D}^c_j + (\lambda_U)_{ij} \hat{H}_1 \hat{Q}_i \hat{U}^c_j \right] \\
& + \sum_{\alpha,\beta=1}^{2} \epsilon_{\alpha\beta} \left[\lambda_1 \hat{L}^\alpha \hat{L}^\beta \hat{E}^c + \lambda_2 \hat{L}^\alpha \hat{Q}^\beta \hat{D}^c \right] + \lambda_3 \hat{U}^c \hat{D}^c \hat{D}^c \\
& + \sum_{\alpha,\beta=1}^{2} \tilde{\mu} \epsilon_{\alpha\beta} \hat{H}_2^\alpha \hat{L}^\beta, \qquad (3.109)
\end{aligned}
$$

where the $SU(2)$ indices (α, β) and the family indices (i, j) have been displayed for convenience. However, some of the contractions over $SU(2)$, $SU(3)$, and family indices were not displayed explicitly. In particular, the couplings λ_i and $\tilde{\mu}$ actually contain family indices so that, for example, the explicit form of the term $\lambda_3 \hat{U}^c \hat{D}^c \hat{D}^c$ is $(\lambda_3)_{ijk}\, \epsilon_{ABC} (\hat{U}^c)_{iA} (\hat{D}^c)_{jB} (\hat{D}^c)_{kC}$ with i, j, k and A, B, C being respectively family and $SU(3)$ indices, and ϵ_{abc} being the totally antisymmetric $SU(3)$ invariant tensor with $\epsilon_{123} = 1$.

It is not difficult to see that the interactions in the last two lines in (3.109) cause some problems phenomenologically, since they generally result in the violations of lepton or baryon numbers. A popular way to eliminate these terms is to impose a conservation of a multiplicative quantum number called R parity. The R parity is assigned to be $+1$ for the SM particles and -1 for their superpartners [10]. Requiring the R parity conservation, the MSSM superpotential becomes

$$\begin{aligned} W &= \sum_{\alpha,\beta=1}^{2} \mu\epsilon_{\alpha\beta}\hat{H}_1^\alpha \hat{H}_2^\beta + \sum_{i,j=1}^{3} (\lambda_E)_{ij}\, \hat{H}_2 \hat{L}_i \hat{E}_j^c \\ &\quad + \sum_{i,j=1}^{3} (\lambda_D)_{ij}\, \hat{H}_2 \hat{Q}_i \hat{D}_j^c + \sum_{i,j=1}^{3} (\lambda_U)_{ij}\, \hat{H}_1 \hat{Q}_i \hat{U}_j^c. \end{aligned} \tag{3.110}$$

This form of the superpotential will be used from now on.

In the discussion of the super Poincaré algebra, it was found that $P^2 = P_\mu P^\mu$ is a Casimir operator. An important consequence of this is that

$$\left(QP^2 - P^2Q\right)|\text{boson}\rangle = \left(m_{boson}^2 - m_{fermion}^2\right)|\text{fermion}\rangle = 0, \tag{3.111}$$

so all particles in the same supermultiplet have the same masses. However, since the existing experiments have so far failed to find the evidence for the superpartners of all known particles, supersymmetry must be broken in the realistic model so as to make the masses of all superpartners larger than the highest energy scale of all existing experiments.

To discuss supersymmetry breaking, one should not forget that the equality of masses within a supermultiplet was an important ingredient for solving the hierarchy problem. Thus to have a model with supersymmetry breaking, one begins with a fully supersymmetric Lagrangian and then adds extra terms into it to explicitly break supersymmetry. The forms of such extra terms have to be restricted so that they will not introduce quadratic divergences to the particle masses via loop diagrams. These terms are called the “soft supersymmetry breaking" terms. It was found that the acceptable soft supersymmetry breaking part of the Lagrangian takes the form

$$\begin{aligned}
\mathcal{L}_{\text{soft}} &= m_{\tilde{q}}^2 \left|\tilde{q}_L\right|^2 + m_{\tilde{u}}^2 \left|\tilde{u}_R^c\right|^2 + m_{\tilde{d}}^2 \left|\tilde{d}_R^c\right|^2 + m_{\tilde{l}}^2 \left|\tilde{l}_L\right|^2 + m_{\tilde{e}}^2 \left|\tilde{e}_R^c\right|^2 \\
&\quad + \left(\lambda_E A_E H \tilde{l}_L \tilde{e}_R^c + \lambda_D A_D H \tilde{q}_L \tilde{d}_R^c + \lambda_U A_U H \tilde{q}_L \tilde{u}_R^c + B\mu H_1 H_2 + h.c\right) \\
&\quad + m_{H_1}^2 \left|H_1\right|^2 + m_{H_2}^2 \left|H_2\right|^2 + \frac{1}{2} M_1 \tilde{B}\tilde{B} + \frac{1}{2} M_2 \tilde{W}\tilde{W} + \frac{1}{2} M_3 \tilde{g}\tilde{g} \qquad (3.112)
\end{aligned}$$

where $m_{\tilde{q}}^2$, $m_{\tilde{u}}^2$, $m_{\tilde{d}}^2$, $m_{\tilde{l}}^2$ and $m_{\tilde{e}}^2$ are general hermitian 3×3 matrices in the family space, and $\lambda_E A_E$, $\lambda_D A_D$ and $\lambda_U A_U$ are general 3×3 matrices also in the family space. If these parameters are complex, then (3.112) contains more than 100 unknown real parameters [11]. However, most processes are sensitive to only some (small) subset of these parameters at least at the classical level. Now the construction of all parts of the MSSM Lagrangian is completed.

In the next chapter, one of the MSSM phenomenology, the Higgs phenomenology, will be considered.

CHAPTER IV

ELECTROWEAK SYMMETRY BREAKING AND HIGGS PHENOMENOLOGY IN THE MSSM

Having defined the Minimal Supersymmetric Standard Model (MSSM) in the previous chapter, we now turn to the electroweak symmetry breaking and Higgs particles in this model. Unlike the Standard Model in which only one Higgs doublet is required to break the electroweak symmetry and give masses to all quarks and charged leptons, two Higgs doublets are required in the case of the MSSM. As will be seen, there are five physical Higgs states, which include three neutral Higgs bosons and two charged Higgs bosons. Among these five Higgs particles, the lightest one, h^0, is electrically neutral and has its tree-level mass less than that of the Z^0 boson ($m_Z \approx 91$ GeV), which is within the energy range that the Large Electron-Positron Collider (LEP) can detect (< 104 GeV). Unfortunately, there has been no experimental evidence of such Higgs particle from LEP. So if the tree-level calculation gives the correct value of h^0 mass, then the MSSM should have been ruled out by now. However, as we shall see, the h^0 mass acquires large radiative corrections proportional to the fourth power of the top quark mass, and this makes its upper bound well beyond the highest energy that LEP can reach [13]. This gives us some hope that the MSSM could survive as a phenomenologically viable theory.

In this chapter, we start with a review of the electroweak symmetry breaking in the MSSM in Section 4.1. We then go on to calculate all the Higgs masses at tree level in Section 4.2. In Section 4.3, the radiative corrections to the lightest Higgs mass m_{h^0}, using the method of effective potential [13], will be obtained and the upper bound of the lightest Higgs boson mass will be analyzed.

4.1 The Tree-Level Scalar Potential and the Conditions for Electroweak Symmetry Breaking

To spontaneously break the $SU(2)_L \times U(1)_Y$ symmetry, the scalar potential should have the absolute minimum away from the origin (in the space of the Higgs scalars). Therefore, in this section, only the part of the MSSM potential that depends on the Higgs fields will be focused. This part of the potential is comprised of three types of contributions:

1. The supersymmetric "F-terms" of the form $\sum_i \left|\frac{\partial W}{\partial \phi_i}\right|^2$ in (3.108). With the superpotential in (3.110), they contribute the mass terms:

$$\mu\left(|H_1|^2 + |H_2|^2\right). \tag{4.1}$$

2. The soft SUSY breaking part (3.112) of the Lagrangian gives the additional mass and mixing terms:

$$m_{H_1}^2 |H_1|^2 + m_{H_2}^2 |H_2|^2 \tag{4.2}$$

and

$$B\mu H_1 H_2 + h.c. = B\mu\left(H_1^+ H_2^- - H_1^0 H_2^0\right) + h.c.. \tag{4.3}$$

3. The supersymmetric "D-terms" of the form $-\frac{1}{2}\sum_a \left|\sum_{i,j} g\phi^{i*} T_{ij}^a \phi^i\right|^2$ give rise to the quadratic interactions which, after the electroweak symmetry breaking, result in the mass terms. The $SU(2)_L$ contributes the terms

$$
\begin{aligned}
&\sum_{\alpha}\left[H_u^{\dagger}\left(\tau^{\alpha}/2\right)H_u+H_d^{\dagger}\left(\tau^{\alpha}/2\right)H_d\right]\left[H_u^{\dagger}\left(\tau^{\alpha}/2\right)H_u+H_d^{\dagger}\left(\tau^{\alpha}/2\right)H_d\right]\\
&\quad=\sum_{\alpha}\left(H_u^{\dagger}\left(\tau^{\alpha}/2\right)H_u\right)\cdot\left(H_u^{\dagger}\left(\tau^{\alpha}/2\right)H_u\right)\\
&\qquad+\sum_{\alpha}\left(H_d^{\dagger}\left(\tau^{\alpha}/2\right)H_d\right)\cdot\left(H_d^{\dagger}\left(\tau^{\alpha}/2\right)H_d\right)\\
&\qquad+2\sum_{\alpha}\left(H_u^{\dagger}\left(\tau^{\alpha}/2\right)H_u\right)\cdot\left(H_d^{\dagger}\left(\tau^{\alpha}/2\right)H_d\right)\\
&\quad=\frac{1}{4}\left[\left(\left|H_1^+\right|^2+\left|H_1^0\right|^2\right)-\left(\left|H_2^0\right|^2+\left|H_2^-\right|^2\right)\right]^2\\
&\qquad+\left(H_1^+\left(H_2^0\right)^*+H_1^0\left(H_2^-\right)^*\right)\left(\left(H_1^+\right)^*H_2^0+\left(H_1^0\right)^*H_2^-\right),
\end{aligned}
\tag{4.4}
$$

while the part corresponding to $U(1)_Y$ is

$$
\begin{aligned}
\frac{1}{2}\left(g'/2\right)^2\left[H_1^{\dagger}H_1-H_2^{\dagger}H_2\right]^2 &= \frac{g'^2}{8}\left[\left(\left|H_1^+\right|^2+\left|H_1^0\right|^2\right)\right.\\
&\quad\left.-\left(\left|H_2^0\right|^2+\left|H_2^-\right|^2\right)\right]^2.
\end{aligned}
\tag{4.5}
$$

Thus the complete potential for the Higgs fields in the MSSM is

$$
\begin{aligned}
V_H &= \left(|\mu|^2+m_{H_1}^2\right)\left(\left|H_1^+\right|^2+\left|H_1^0\right|^2\right)+\left(|\mu|^2+m_{H_2}^2\right)\left(\left|H_2^0\right|^2+\left|H_2^-\right|^2\right)\\
&\quad+\left[b\left(H_1^+H_2^--H_1^0H_2^0\right)+h.c.\right]\\
&\quad+\frac{\left(g^2+g'^2\right)}{8}\left(\left|H_1^+\right|^2+\left|H_1^0\right|^2-\left|H_2^0\right|^2-\left|H_2^-\right|^2\right)^2\\
&\quad+\frac{g^2}{2}\left|H_1^+\left(H_2^0\right)^{\dagger}+H_1^0\left(H_2^-\right)^{\dagger}\right|^2,
\end{aligned}
\tag{4.6}
$$

where $b = B\mu$. Note that the terms proportional to $|\mu|^2$ came from the supersymmetric invariant part of the Lagrangian, and hence is necessarily positive, while the factors of $m_{H_1}^2$ and $m_{H_2}^2$ originated from the soft-supersymmetry breaking terms, and have the possibility of being negative via the renormalization group running.

To find the minima of the above Higgs potential after the breaking of $SU(2)_L \times U(1)_Y$ down to $U(1)_{\mathrm{EM}}$, one employs the $SU(2)_L$ degrees of freedom to choose the appropriate vacuum expectation values (VEVs) of H_1 and H_2. Thus one may choose $H_1^+ = 0$ at the minimum of the potential, which implies that

$$\left.\frac{\partial V}{\partial H_1^+}\right|_{H_1^+=0} = \left[b + \frac{g^2}{2}\left(H_2^0\right)^\dagger \left(H_1^0\right)^\dagger\right] H_2^- = 0. \tag{4.7}$$

This equation implies that either

$$H_2^- = 0 \tag{4.8}$$

or

$$\left[b + \frac{g^2}{2}\left(H_2^0\right)^\dagger \left(H_1^0\right)^\dagger\right] = 0 \tag{4.9}$$

must hold. The second choice implies that the b terms in (4.6) become

$$g^2 \left|H_1^0\right|^2 \left|H_2^0\right|^2 \tag{4.10}$$

which is positive, and hence unfavorable to symmetry breaking. Thus the proper solution for which the b terms can be negative is (4.8). Surprisingly, the fact that the choice $H_1^+ = 0$ forces H_2^- to become zero implies that the electromagnetism is not spontaneously broken. Thus to find the minima of the potential away from the origin, one can now completely ignore the charged components, and just considers the potential for the neutral fields:

$$\begin{aligned} V_{\mathrm{n}} &= \left(|\mu|^2 + m_{H_1}^2\right)\left|H_1^0\right|^2 + \left(|\mu|^2 + m_{H_2}^2\right)\left|H_2^0\right|^2 \\ &\quad - \left(bH_1^0H_2^0 + h.c.\right) + \left(\frac{g^2+g'^2}{8}\right)\left(\left|H_1^0\right|^2 - \left|H_2^0\right|^2\right)^2. \end{aligned} \tag{4.11}$$

Note that the coefficient of the quartic terms is not a free parameter, but is determined from the known electroweak couplings $\left(\frac{g^2+g'^2}{8} = 0.065\right)$ [12]. This is in contrast to the case of the Standard model in which the quartic coupling constant $\lambda/4$ is a free parameter. With a relatively small quartic coupling constant in the MSSM, one suspects the possibility of having a relatively light Higgs particle in the MSSM, at least at the tree level.

To proceed to find the potential minima, consider the b terms in (4.11) which depend on the phases of the fields. One can set b to be real and positive by absorbing its phase into the product $H_1^0 H_2^0$. As the terms other than the b terms are non-negative, then the minimization of V at the points away from the origin is possible if the product $H_1^0 H_2^0$ is real and positive, this implies that the VEVs of H_1^0 and H_2^0 have the opposite phases. Because these two Higgs fields have opposite hypercharges, one can use a $U(1)_Y$ gauge transformation to set both their phases to zero. So all VEVs and couplings can be chosen to be real, which means that the breaking of CP symmetry is not caused by the 2-Higgs potential in the MSSM.

To simplify the calculation, H_1^0 and H_2^0, which are now regarded as real, are rewritten as

$$x = \left|H_1^0\right| \qquad \text{and} \qquad y = \left|H_2^0\right|. \tag{4.12}$$

Therefore the potential for neutral fields can be rewritten as

$$\begin{aligned} V_{\mathrm{n}} &= \left(|\mu|^2 + m_{H_1}^2\right) x^2 + \left(|\mu|^2 + m_{H_2}^2\right) y^2 \\ &\quad -2bxy + \left(\frac{g^2 + g'^2}{8}\right)\left(x^2 - y^2\right)^2. \end{aligned} \tag{4.13}$$

To identify the minimum conditions of V_{n} occurring at non-zero values of x and y, one first considers a special direction $x = y$ in which the potential becomes

$$V_{\mathrm{n}} = \left[\left(|\mu|^2 + m_{H_1}^2\right) + \left(|\mu|^2 + m_{H_2}^2\right) - 2b\right] x^2. \tag{4.14}$$

One sees that V_{n} is bounded from below (i.e., the theory does not have any instability) only if

$$2\left|\mu\right|^2 + m_{H_1}^2 + m_{H_2}^2 > 2b > 0. \tag{4.15}$$

With the above condition, the origin $x = y = 0$ is the minimum point of the potential along the direction $x = y$. Since it is required that the minimum of the potential must be away from the origin for the symmetry breaking to take place, then the origin must be a saddle point. For the origin to be a saddle point, the following condition must be satisfied:

$$\left[\left(\frac{\partial^2 V_n}{\partial x^2}\right)\left(\frac{\partial^2 V_n}{\partial y^2}\right)-\left(\frac{\partial^2 V_n}{\partial x \partial y}\right)^2\right]\Bigg|_{x=y=0}<0 \tag{4.16}$$

or

$$\left(|\mu|^2+m_{H_1}^2\right)\left(|\mu|^2+m_{H_2}^2\right)<b^2. \tag{4.17}$$

One of the possible situations that satisfies (4.17) is that either $\left(|\mu|^2+m_{H_1}^2\right)$ or $\left(|\mu|^2+m_{H_2}^2\right)$ is negative. In the supergravity inspired model, the soft-SUSY breaking parameters $m_{H_1}^2$ and $m_{H_2}^2$ are equal and positive at the grand unification scale of about 10^{16} GeV. Their values at the electroweak symmetry breaking scale, however, decrease via the renormalization group running, and one can show that the parameter $m_{H_1}^2$ decreases much faster than $m_{H_2}^2$ and becomes so negative that $\left(|\mu|^2+m_{H_1}^2\right)<0$ at the electroweak symmetry breaking scale (while $m_{H_2}^2$ is still positive) [12]; this makes the electroweak symmetry breaking possible. Note that, with the conditions (4.15) and (4.17), the situation in which $\left(|\mu|^2+m_{H_1}^2\right)=\left(|\mu|^2+m_{H_2}^2\right)$ is impossible.

Having established the necessary conditions (4.15) and (4.17) for $|H_1^0|$ and $|H_2^0|$ to get non-zero VEVs, says ν_1 and ν_2 respectively, one can determine the VEVs by using the minimization conditions:

$$\frac{\partial V_{\mathrm{n}}}{\partial x}\bigg|_{x=\nu_1,y=\nu_2}=\frac{\partial V_{\mathrm{n}}}{\partial y}\bigg|_{x=\nu_1,y=\nu_2}=0 \tag{4.18}$$

which yield

$$\left(|\mu|^2+m_{H_1}^2\right)\nu_1=b\nu_2-\frac{1}{4}\left(g^2+g'^2\right)\left(\nu_2^2-\nu_1^2\right)\nu_1 \tag{4.19}$$

and

$$\left(|\mu|^2+m_{H_2}^2\right)\nu_2=b\nu_1+\frac{1}{4}\left(g^2+g'^2\right)\left(\nu_2^2-\nu_1^2\right)\nu_2. \tag{4.20}$$

Even though ν_1 and ν_2 are individually undetermined, a certain combination of ν_1 and ν_2 is fixed by the mass of the W bosons. To show this, consider the Higgs kinetic terms

$$(D_\mu H_1)^\dagger (D_\mu H_1) + (D_\mu H_2)^\dagger (D_\mu H_2) \tag{4.21}$$

where $D_\mu = D_\mu = \partial_\mu + ig\left(\frac{\tau^a}{2}\right) W_\mu^a + i\left(\frac{g'}{2}\right) yB_\mu$. Analogous to the determination of the masses of the vector bosons in the SM, one inserts the VEVs of H_1 and H_2 into the non-derivative terms above. Defining

$$Z^\mu = \frac{(-g'B^\mu + gW_3^\mu)}{(g^2 + g'^2)^{1/2}} \tag{4.22}$$

one finds that

$$m_Z^2 = \frac{1}{2}\left(g^2 + g'^2\right)\left(\nu_1^2 + \nu_2^2\right) \tag{4.23}$$

and also

$$m_W^2 = \frac{1}{2} g^2 \left(\nu_1^2 + \nu_2^2\right). \tag{4.24}$$

With the knowledge of m_W^2 and g^2, one can determine

$$\sqrt{\nu_1^2 + \nu_2^2} = \sqrt{\frac{2m_W^2}{g^2}} = 174 \text{ GeV}. \tag{4.25}$$

Thus the minimization conditions (4.19) and (4.20) can be rewritten as

$$\left(|\mu|^2 + m_{H_1}^2\right) = b\cot(\beta) + \frac{m_Z^2}{2}\cos(2\beta) \tag{4.26}$$

and

$$\left(|\mu|^2 + m_{H_2}^2\right) = b\tan(\beta) - \frac{m_Z^2}{2}\cos(2\beta), \tag{4.27}$$

where $\tan(\beta) = \nu_1/\nu_2$. The angle β lies between 0 and $\pi/2$ since both ν_1 and ν_2 are positive real numbers. Moreover, one can eliminate the parameter $|\mu|$ and b in terms of $\tan(\beta)$, but the phase of μ is still undetermined.

4.2 The Tree-Level Masses of the Scalar Higgs Bosons in the MSSM

In the SM, there are four real scalar degrees of freedom in the scalar doublet. After the electroweak symmetry breaking, three of them become the longitudinal modes of the massive vector bosons $W^{\pm}$ and Z^0, while the other becomes a neutral Higgs boson, the mass of which is determined from the terms quadratic in the fluctuations about the potential minimum.

In the MSSM with two Higgs doublets, there are 8 real scalar degrees of freedom. Three of them are eaten by the massive vector bosons $W^{\pm}$ and Z^0 as in the SM. The masses of the others are again determined by expanding the potential about the minimum, up to the second order in the field fluctuations. Though straightforward, the work is complicated by the fact that the quadratic terms are not diagonal in the fields. So one needs to diagonalize these terms in order to determine the physical Higgs masses. The general procedure is that, for any potential V of scalar fields ϕ_i, one defines the mass matrix

$$\mathbf{M}^2_{ij} = \frac{1}{2}\frac{\partial^2 V}{\partial\phi_i\partial\phi_j} \tag{4.28}$$

evaluated at the minimum of V. Then the mass terms take the form $-\xi_i\mathbf{M}^2_{ij}\xi_j/2$, where $\xi_i \equiv \phi_i - v_i$ is the fluctuation of ϕ_i about its VEV v_i. Note that the mass matrix $\mathbf{M}^2_{ij}$ is symmetric. By diagonalizing the mass matrix, one obtains the physical masses and physical fields as its eigenvalues and eigenvectors respectively.

Even though the procedure just described might be complicated in practice, the situation is not that terrible in the case of the MSSM, as the mass matrix is block diagonal in this case. Indeed, the MSSM Higgs mass matrix is a direct sum of four 2×2 blocks. The first independent block in the mass matrix corresponds to the pair of fields $(\mathrm{Im}\, H_1^0, \mathrm{Im}\, H_2^0)$. The part of the scalar potential in

(4.6) involving this pair is

$$V_{\mathrm{A}} = \left(|\mu|^2 + m_{H_1}^2\right)\left(\operatorname{Im} H_1^0\right)^2 + \left(|\mu|^2 + m_{H_2}^2\right)\left(\operatorname{Im} H_2^0\right)^2 + 2b\left(\operatorname{Im} H_1^0\right)\left(\operatorname{Im} H_2^0\right) + \frac{\left(g^2+g'^2\right)}{8}\left[\left(\operatorname{Re} H_1^0\right)^2 + \left(\operatorname{Im} H_1^0\right)^2 - \left(\operatorname{Re} H_2^0\right)^2 - \left(\operatorname{Im} H_2^0\right)^2\right]^2. \quad (4.29)$$

Using the VEVs

$$\langle H_1 \rangle = \begin{pmatrix} H_1^+ = 0 \\ H_1^0 = \nu_1 \end{pmatrix} \quad \text{and} \quad \langle H_2 \rangle = \begin{pmatrix} H_2^0 = \nu_2 \\ H_2^- = 0 \end{pmatrix}, \quad (4.30)$$

one finds the matrix elements $\mathbf{M}_{ij}^2$ of this block as follows:

$$\mathbf{M}_{11}^2 = |\mu|^2 + m_{H_1}^2 + \frac{\left(g^2+g'^2\right)}{4}\left(\nu_1^2 - \nu_2^2\right) = b\cot(\beta), \quad (4.31)$$

and

$$\mathbf{M}_{12}^2 = b, \qquad \mathbf{M}_{22}^2 = b\tan(\beta). \quad (4.32)$$

It can be easily checked that the eigenstates corresponding to the eigenvalues of this 2×2 block of the mass matrix are

$$\sqrt{2}\left[\left(\operatorname{Im} H_1^0\right)\sin\beta + \left(\operatorname{Im} H_2^0\right)\cos\beta\right] \quad (4.33)$$

and

$$\sqrt{2}\left[\left(\operatorname{Im} H_1^0\right)\cos\beta + \left(\operatorname{Im} H_2^0\right)\sin\beta\right]. \quad (4.34)$$

The eigenstate in (4.33) is a massless eigenstate and eaten by the longitudinal mode of the Z^0. On the contrary, the eigenstate in (4.34) corresponds to a scalar particle A^0 of mass

$$m_{A^0} = \sqrt{\frac{2b}{\sin 2\beta}}. \quad (4.35)$$

The next 2×2 block of the mass matrix is due to the charged pair $\left(H_1^+, \left(H_2^-\right)^\dagger\right)$. As these fields are complex, their mass matrix is defined by

$$M_{\text{charged}}^2 = \begin{pmatrix} \partial^2 V/\partial\left(H_1^+\right)^\dagger \partial H_1^+ & \partial^2 V/\partial\left(H_1^+\right)^\dagger \partial\left(H_2^-\right)^\dagger \\ \partial^2 V/\partial\left(H_2^-\right)\partial H_1^+ & \partial^2 V/\partial\left(H_2^-\right)^\dagger \partial H_2^- \end{pmatrix}, \quad (4.36)$$

where V is the potential in (4.6). Performing the differentiations and evaluating the results at the VEVs (4.30), one obtains

$$M^2_{\text{charged}} = \begin{pmatrix} b\cot\beta + \frac{g^2}{2}\nu_2^2 & b + \frac{g^2}{2}\nu_1\nu_2 \\ b + \frac{g^2}{2}\nu_1\nu_2 & b\tan\beta + \frac{g^2}{2}\nu_1^2 \end{pmatrix}. \tag{4.37}$$

The matrix M^2_{charged} has eigenvalues 0 and $m_W^2 + m_{A^0}^2$. The corresponding massless state

$$G^+ = H_1^+ \sin\beta - \left(H_2^-\right)^\dagger \cos\beta \tag{4.38}$$

is eaten to become the longitudinal mode of the W^+ boson, while the one of mass $m_W^2 + m_{A^0}^2$

$$H^+ = H_1^+ \cos\beta - \left(H_2^-\right)^\dagger \sin\beta \tag{4.39}$$

becomes a physical positively charged Higgs particle.

The hermitian conjugate of the above pair $\left(H_2^-, \left(H_1^+\right)^\dagger\right)$ gives another 2×2 block of the mass matrix with the same eigenvalues. The corresponding massless state $G^- = (G^+)^\dagger$ eaten by the W^-, while the massive one $H^- = (H^+)^\dagger$ of mass $\left(m_W^2 + m_{A^0}^2\right)^{1/2}$ becomes a physical negatively charged Higgs particle.

The last block of the mass matrix comes from the last pair ($\operatorname{Re} H_1^0 - \nu_1$, $\operatorname{Re} H_2^0 - \nu_2$), analogous to the case of ($\operatorname{Im} H_1^0, \operatorname{Im} H_2^0$). The corresponding mass matrix is

$$\mathbf{M}^2_{h,H} = \begin{pmatrix} b\cot\beta + m_Z^2\sin^2\beta & -b\beta - \frac{1}{2}m_Z^2\sin 2\beta \\ -b\beta - \frac{1}{2}m_Z^2\sin 2\beta & b\tan\beta + m_Z^2\cos^2\beta \end{pmatrix} \tag{4.40}$$

which has two eigenstates: h^0 corresponding to the eigenvalue

$$m_{h^0}^2 = \frac{1}{2}\left\{m_{A^0}^2 + m_Z^2 - \sqrt{\left(m_{A^0}^2 + m_Z^2\right)^2 - 4m_{A^0}^2 m_Z^2\cos^2 2\beta}\right\} \tag{4.41}$$

and H^0 corresponding to the eigenvalue

$$m_{H^0}^2 = \frac{1}{2}\left\{m_{A^0}^2 + m_Z^2 + \sqrt{\left(m_{A^0}^2 + m_Z^2\right)^2 - 4m_{A^0}^2 m_Z^2\cos^2 2\beta}\right\} \tag{4.42}$$

are both physical massive neutral particles.

Thus there are totally five Higgs particles in the MSSM. By examining the masses above, one observes that even though the masses m_{A^0}, m_{H^0} and $m_{H^\pm}$ are unconstrained (since they grow as $b/\sin\beta$ which are arbitrary in principle), the mass m_{h^0} has an upper bound.

To find the upper bound of m_{h^0} in a simple way, one rewrites $m^2_{A^0}$ and m^2_Z as

$$m^2_{A^0} = x \qquad \text{and} \qquad m^2_Z = a.$$

Thus $m^2_{A^0}$ becomes

$$m^2_{h^0} = \frac{1}{2}\left\{x + a - \sqrt{(x+a)^2 - 4ax\cos^2 2\beta}\right\}. \tag{4.43}$$

It is easy to see that $m^2_{h^0}$ is a strictly-increasing function of x for fixed a. For small x ($x \ll a$), one finds that

$$\begin{aligned}
m^2_{h^0} &= \frac{1}{2}\left\{x + a - a\left[\frac{(x+a)^2}{a^2} - \frac{4x}{a}\cos^2 2\beta\right]^{1/2}\right\} \\
&= \frac{1}{2}\left\{x + a - a\left[1 + \frac{2x}{a} + \frac{x^2}{a^2} - \frac{4x}{a}\cos^2 2\beta\right]^{1/2}\right\} \\
&= \frac{1}{2}\left[x + a - a\left(1 + \frac{x}{a} + \frac{x^2}{2a^2} - \frac{2x}{a}\cos^2 2\beta\right)\right] \\
&\approx x\cos^2 2\beta,
\end{aligned} \tag{4.44}$$

while for large x ($x \gg a$),

$$\begin{aligned}
m^2_{h^0} &= \frac{1}{2}\left\{x + a - x\left[\frac{(x+a)^2}{x^2} - \frac{4a}{x}\cos^2 2\beta\right]^{1/2}\right\} \\
&= \frac{1}{2}\left[x + a - x\left(1 + \frac{a}{x} + \frac{a^2}{2x^2} - \frac{2a}{x}\cos^2 2\beta\right)\right] \\
&\approx a\cos^2 2\beta.
\end{aligned} \tag{4.45}$$

Thus $m^2_{h^0}$ has its upper bound of about $a\cos^2 2\beta$. From (4.42) and (4.45), one can summarize that

$$m_{h^0} \leqslant m_Z\,|\cos 2\beta| \leqslant m_Z < m^0_H. \tag{4.46}$$

This gives the upper bound on the mass of the lightest Higgs boson in the MSSM, h^0, at the tree-level. The MSSM seems to predict that one of the neutral Higgs scalars must be lighter than the Z boson. However, the above tree-level masses of the Higgses receive significant one loop corrections, which will be discussed in the next section. Further, according to (4.46), the mass of h^0 vanishes for $\beta = \pi/4$. It is important to realize that, at tree-level, all Higgs masses and couplings depend on only two parameters, m_{A^0} and $\tan\beta$.

Before going on to the next section, we summarize the spectrum of the MSSM Higgses. In terms of the original gauge-eigenstate fields, the mass eigenstates, including the ones eaten by the vector bosons, are given by

$$\begin{pmatrix} G^0 \\ A^0 \end{pmatrix} = \sqrt{2} \begin{pmatrix} \sin\beta & -\cos\beta \\ \cos\beta & \sin\beta \end{pmatrix} \begin{pmatrix} \mathrm{Im}\, H_1^0 \\ \mathrm{Im}\, H_2^0 \end{pmatrix}, \tag{4.47}$$

$$\begin{pmatrix} G^+ \\ A^+ \end{pmatrix} = \sqrt{2} \begin{pmatrix} \sin\beta & -\cos\beta \\ \cos\beta & \sin\beta \end{pmatrix} \begin{pmatrix} H_1^+ \\ H_2^{-*} \end{pmatrix}, \tag{4.48}$$

with $G^- = G^{+*}$ and $G^- = G^{+*}$, and

$$\begin{pmatrix} h^0 \\ H^0 \end{pmatrix} = \sqrt{2} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \mathrm{Re}\, H_1^0 - \nu_1 \\ \mathrm{Re}\, H_2^0 - \nu_2 \end{pmatrix} \tag{4.49}$$

which defines a mixing angle α. The tree-level masses of these fields are

$$m_{A^0}^2 = 2b/\sin 2\beta \tag{4.50}$$

$$m_{H^\pm}^2 = m_{A^0}^2 + m_W^2 \tag{4.51}$$

$$m_{h^0}^2 = \frac{1}{2}\left\{ m_{A^0}^2 + m_Z^2 - \sqrt{\left(m_{A^0}^2 + m_Z^2\right)^2 - 4m_{A^0}^2 m_Z^2 \cos^2 2\beta} \right\} \tag{4.52}$$

$$m_{H^0}^2 = \frac{1}{2}\left\{ m_{A^0}^2 + m_Z^2 + \sqrt{\left(m_{A^0}^2 + m_Z^2\right)^2 - 4m_{A^0}^2 m_Z^2 \cos^2 2\beta} \right\}. \tag{4.53}$$

In terms of these masses, the mixing angle α in (4.49) is determined at the tree-level by

$$\sin 2\alpha = -\frac{m_{A^0}^2 + m_Z^2}{m_{H^0}^2 + m_{h^0}^2} \sin 2\beta, \qquad \cos 2\alpha = -\frac{m_{A^0}^2 - m_Z^2}{m_{H^0}^2 - m_{h^0}^2} \cos 2\beta. \tag{4.54}$$

Notice that, once the value of $\nu_1^2 + \nu_2^2$ has been fixed by the mass of the W bosons, the supersymmetric Higgs boson masses (actually for the whole Higgs sector) are then described by two additional parameters, the usual choice being $\tan\beta$ and m_{A^0}.

4.3 Radiative Corrections to the Mass of the Lightest Supersymmetric Higgs Boson

The search for Higgs bosons which are related to the origin of masses is one of the extremely important research projects in experimental particle physics. According to the last section, the neutral Higgs boson, h^0, is the lightest supersymmetric Higgs boson, and has its mass at the tree-level less than that of the Z. The upper bound of m_{h^0}, which is reached when the mass of A^0 is very much larger than M_Z (i.e., when $|\cos 2\beta| \approx 1$), implies the large value of $\tan\beta$ ($\tan\beta \approx \tan\frac{\pi}{2}$). So if the tree-level calculation really gives the correct value of m_{h^0}, the MSSM should have been ruled out by now as there has been no signal of Higgs bosons from LEP, the maximum energy of which exceeds M_Z. So one might ask if there is any possibility that the upper bound of m_{h^0} could be pushed up so that it exceeds the maximum energy available at LEP. If the answer to this question turns out to be yes, then the fact that the Higgs boson has not been found at LEP cannot be used to rule out the MSSM.

Indeed, if one includes the radiative corrections to the Higgs masses, the upper bound on m_{h^0} can be substantially increased. The calculations of such corrections are the main purpose of this section. As will be justified later on, in a model with unbroken supersymmetry, the corrections to the Higgs masses due to fermions and their superpartners cancel. But since the supersymmetry has been broken by splitting the masses of the fermions and their scalar superpartners, the corrections to the neutral Higgs masses do not vanish at one-loop level. The radiative corrections can be very large if one considers the loop diagrams due to heavy particles such as top quarks and squarks. Thus the lightest Higgs boson mass has a chance of being heavier than the mass of the Z boson.

In this section, we give a review of the radiative corrections to m_{h^0} and show that the radiative corrections can shift the maximal value of m_{h^0} from m_Z to about $m \simeq 130$ GeV. As the nicest method for calculating the radiative corrections to the Higgs masses is the method of effective potential, we start with a review of the effective potential and then go on to give some examples of the effective potential calculations at the one-loop level. The calculational method is finally applied to the calculation of the radiative corrections to the Higgs masses, thereby showing the effects of top quarks and squarks on the Higgs masses via the loop diagrams.

4.3.1 The Effective Potential

To discuss the effective potential formalism in a simple way, we consider the case of a scalar field; the generalization to the case of many scalar fields is straightforward. The dynamics of a scalar field ϕ coupled to an external source $J(x)$ is described by the Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + J\left(x\right)\phi\left(x\right). \tag{4.55}$$

In quantum field theory, the generating functional corresponding to the above Lagrangian is given by

$$W\left[J\right] = \int \left[d\phi\right] \exp\left\{ i \int d^4x \left[\mathcal{L}(\phi(x)) + J(x)\phi(x)\right]\right\} \tag{4.56}$$

which represents the vacuum-to-vacuum transition amplitude in the presence of the external source $J\left(x\right)$:

$$W\left[J\right] = \langle 0|0\rangle_J. \tag{4.57}$$

If the logarithm of $W\left[J\right]$ is expanded as a functional Taylor series in $J\left(x\right)$:

$$\ln W\left[J\right] = \sum_n \frac{1}{n!} \int d^4x_1 \cdots d^4x_n G^{(n)}(x_1, \ldots, x_n) J(x_1) \cdots J(x_n), \tag{4.58}$$

then it can be shown that the "Taylor coefficients" $G^{(n)}$, known as the *connected Green's functions*, can be calculated by summing over all connected Feynman diagrams with n external lines.

Define a "classical field," $\phi_c(x)$, as the vacuum expectation value (VEV) of the field ϕ in the presence of the external source $J(x)$,

$$\phi_c(x) = \left[\frac{\langle 0|\phi(x)|0\rangle}{\langle 0|0\rangle}\right]_J = G^{(1)}(x) = \frac{\delta \ln W}{\delta J(x)}. \tag{4.59}$$

With the above definition of $W[J]$, one can perform a Legendre transformation to obtain a functional Γ of ϕ_c as

$$\Gamma[\phi_c] = \ln W[J] - \int d^4x J(x)\phi_c(x) \tag{4.60}$$

which clearly satisfies $\delta\Gamma[\phi_c]/\delta J = 0$. From this definition, one obtains

$$\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} = -J(x) \tag{4.61}$$

which reduces to $\delta\Gamma[\phi_c]/\delta\phi_c = 0$ in the absence of the source J. The importance of this equation is as follows: Recall the definition of the classical field $\phi_c(x)$ as the VEV of $\phi(x)$ in the presence of the source J. If J is set to zero, then the classical field is indeed the VEV of $\phi(x)$ with all quantum corrections included, hence the true ground state of the theory.[1] However, despite this formal definition, one still does not know what exactly the ground state $\phi_c(x)$ of a given theory really is; it may or may not be zero, or even it is non-zero it still has a chance of being either a constant or some specific function of spacetime. What the equation $\delta\Gamma[\phi_c]/\delta\phi_c|_{J=0} = 0$ tells us is that, suppose one could find a way to calculate $\Gamma[\phi_c]$, then the ground state of the theory can be obtained by solving this equation.

A beautiful way to calculate $\Gamma[\phi_c]$ is as follows. Just like the case of $\ln W[J]$, one expands $\Gamma[\phi_c]$ in powers of ϕ_c:

$$\Gamma[\phi_c] = \sum_n \frac{1}{n!}\int d^4x_1 \cdots d^4x_n \Gamma^{(n)}(x_1, \ldots, x_n)\phi_c(x_1)\cdots\phi_c(x_n). \tag{4.62}$$

[1]This is in contrast to the conventional way of finding a tree-level VEV by minimizing the classical potential in the Lagrangian, where it was inherently assumed that the VEV of a scalar field is a constant and not a function of spacetime.

It can be shown that the coefficient $\Gamma^{(n)}(x_1 \dots x_n)$ can be calculated by summing over all the one-particle irreducible (or 1PI) Feynman diagrams with n external lines.[2] Thus one is able to calculate $\Gamma[\phi_c]$ by calculating the 1PI diagrams using the standard perturbation theory. In general, $\Gamma[\phi_c]$ takes the form

$$\Gamma[\phi_c] = \int d^4x \left[-V(\phi_c) + \frac{1}{2} (\partial_\mu \phi_c)^2 Z(\phi_c) + \dots \right] \tag{4.63}$$

where $(\dots)$ indicates terms with higher numbers of derivatives. $\Gamma[\phi_c]$ is called an *effective action* while $V(\phi_c)$ is called an *effective potential.* In most cases of interest in which the ground state ϕ_c is spacetime independent (hence its spacetime derivatives vanish), the derivative terms disappear and so one is left with only the effective potential term on the right-hand side of (4.63).

The calculation of $V(\phi_c)$ goes as the following. One considers $\Gamma^{(n)}$ in the momentum space

$$\begin{aligned} \Gamma^{(n)}(x_1, \dots, x_n) &= \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4} (2\pi)^4 \delta^4(k_1 + \dots + k_n) \\ &\quad \times \exp[i(k_1 \cdot x_1 + \dots + k_n \cdot x_n)] \Gamma^{(n)}(k_1, \dots, k_n) \end{aligned} \tag{4.64}$$

where the delta function inside the integral came from the conservation of total momentum. To calculate each term in (4.63), one expands $\Gamma^{(n)}(k_1, \dots, k_n)$ as a power series in the momenta k_i about $k_i = 0$ and rewrites the delta function as a Fourier integral:

$$\begin{aligned} \Gamma[\phi_c] &= \sum_n \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4} \\ &\quad \times \int d^4x \exp\left[-i\left(k_1 + \dots + k_n\right) \cdot x\right] \exp\left[i\left(k_1 \cdot x_1 + \dots + k_n \cdot x_n\right)\right] \\ &\quad \times \Gamma^{(n)}\left(k_1, \dots, k_n\right) \phi_c(x_1) \cdots \phi_c(x_n) \end{aligned}$$

[2] A 1PI diagram is the Feynman diagram which cannot be divided into two disconnected diagrams by cutting just one internal line.

$$= \sum_n \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4}$$
$$\times \int d^4x \exp\left[-i(k_1 + \cdots + k_n)\cdot x\right] \exp\left[i(k_1\cdot x_1 + \cdots + k_n\cdot x_n)\right]$$
$$\times \left[\Gamma^{(n)}(0,\ldots,0)\phi_c(x_1)\cdots\phi_c(x_n) + \ldots\right]$$

$$= \sum_n \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4}$$
$$\times \int d^4x \times \exp\left[-i\left(k_1\cdot(x+x_1) + \cdots + k_n\cdot(x+x_n)\right)\right]$$
$$\times \left[\Gamma^{(n)}(0,\ldots,0)\phi_c(x_1)\cdots\phi_c(x_n) + \ldots\right]$$

$$= \int d^4x \sum_n \frac{1}{n!}\left\{\Gamma^{(n)}(0,\ldots,0)[\phi_c(x)]^n \ldots\right\}. \qquad (4.65)$$

Comparing (4.63) and (4.65), one sees that the effective potential can be obtained by calculating the 1PI diagrams with zero external momenta:

$$V(\phi_c) = -\sum_n \frac{1}{n!}\Gamma^{(n)}(0,\ldots,0)[\phi_c(x)]^n. \qquad (4.66)$$

In the calculations of the 1PI diagrams, infinities generally arise and one has to add counter terms to cancel these infinities, thereby introducing the “mass scale” (i.e., the renormalization point) into the theory. In doing this, the typical procedure is to impose the renormalization conditions on the derivatives of V. For example in $\lambda\phi^4$ theory, the mass squared can be defined as the value of the inverse propagator at zero momentum

$$\Gamma^{(2)}(0) = -\mu^2 \qquad (4.67)$$

which in turn results in the condition on the second derivative of V,

$$\mu^2 = \left.\frac{d^2V}{d\phi_c^2}\right|_{\phi_c=0}. \qquad (4.68)$$

Similarly, the definition of the coupling constant as the four-point function at zero external momenta,

$$\Gamma^{(4)}(0) = -\lambda, \qquad (4.69)$$

results in the condition

$$\lambda = \left. \frac{d^4 V}{d\phi_c^4} \right|_{\phi_c = 0}. \tag{4.70}$$

Note that the definitions of mass and coupling constant in the above example originated from the assumption that the ground state of the theory is at $\phi_c = 0$, or in other words, the symmetry of the theory is not spontaneously broken.

When the spontaneous symmetry breaking occurs, the VEV of the scalar field no longer vanishes and so (4.61) (for $J = 0$)

$$\frac{\delta \Gamma[\phi_c]}{\delta \phi_c} = 0 \tag{4.71}$$

admits a non-zero solution $\phi_c \neq 0$ corresponding to the ground state of the theory. If, moreover, the VEV is translational invariant, then (4.71) reduces to

$$\frac{\delta V(\phi_c)}{\delta \phi_c} = 0 \qquad \text{with the solution } \phi_c \neq 0, \tag{4.72}$$

which is analogous to the typical condition for spontaneous symmetry breaking, except that the potential being minimized is now an effective potential which includes the radiative corrections to the classical potential. An immediate consequence of this is that there is a possibility that radiative corrections can induce spontaneous symmetry breaking even though the VEVs of the scalar fields vanish at the tree level. Such a situation is indeed possible and is called the Coleman-Weinberg mechanism [15].

In the case of the spontaneous symmetry breaking, the definition of the mass squared is almost the same as (4.68) except that the derivative of the effective potential is evaluated at the solution $\phi_c \neq 0$ of (4.72). To see this, it is appropriate to reformulate the whole thing using the so-called "background field" technique [16] as follows. Expressing $\phi(x)$ as a sum of a classical background field $v(x)$ and the field fluctuation $\eta(x)$, i.e., $\phi(x) = \eta(x) + v(x)$, and treating $\eta(x)$ as a quantum field, one finds that the corresponding generating functional $\tilde{W}[J,\upsilon]$ is related to the one in (4.56) by

$$\tilde{W}\left[J,\upsilon\right] = W\left[J\right]\exp\left\{-i\int d^4x\, J(x)\upsilon(x)\right\}. \tag{4.73}$$

Let η_c be the classical field of η, which is related to ϕ_c defined in (4.59) by $\phi_c = \eta_c + \upsilon$, then it is easy to verify that the effective potential $\tilde{\Gamma}\left[\eta_c,\upsilon\right] = \ln\tilde{W}\left[J,\upsilon\right] - \int d^4xJ(x)\eta_c(x)$ is equal to the one defined in (4.60), i.e.,

$$\tilde{\Gamma}\left[\eta_c,\upsilon\right] = \Gamma\left[\phi_c\right]. \tag{4.74}$$

By setting $\eta_c = 0$ (and hence $\upsilon = \phi_c$), one obtains an important relation:

$$\tilde{\Gamma}\left[0,\phi_c\right] = \Gamma\left[\phi_c\right]. \tag{4.75}$$

To see how this relation may be used in practical calculations, one expands the effective action $\tilde{\Gamma}\left[\eta_c,\upsilon\right]$ as

$$\tilde{\Gamma}\left[\eta_c,\upsilon\right] = \sum_n \frac{1}{n!}\int d^4x_1\cdots d^4x_n\tilde{\Gamma}^{(n)}(x_1,\ldots,x_n;\upsilon)\eta_c(x_1)\cdots\eta_c(x_n) \tag{4.76}$$

where $\tilde{\Gamma}^{(n)}(x_1,\ldots,x_n;\upsilon)$ is the sum of all 1PI diagrams with n external lines (of "η particles") whose vertices may depend on the background field υ and its derivatives. With this expansion, the relation (4.75) implies that $\Gamma[\phi_c]$ can be calculated by summing over all "vacuum-to-vacuum" diagrams (i.e., the diagrams with no external lines) in the theory with the background field $\upsilon = \phi_c$:

$$\Gamma[\phi_c] = \tilde{\Gamma}^{(0)}(\phi_c). \tag{4.77}$$

This result will be used in the next subsection.

To obtain the mass spectrum of the theory, one chooses the background field υ such that $\tilde{\Gamma}^{(1)}(x;\upsilon) = 0$. With this choice of the background field, the effective potential has its extremum point at $\eta_c = 0$, which in turn implies that the ground state is at $\eta(x) = 0$. Thus η is the quantum field which generates the particle spectrum of the theory, and its mass μ is obtained from the coefficient $\tilde{\Gamma}^{(2)}(k_1, k_2)$ in momentum space as $\mu^2 = -\tilde{\Gamma}^{(2)}(0,0)$ (see (4.67)).[3] To determine what this choice of the background field really is, consider

$$\begin{aligned} \frac{\delta\tilde{\Gamma}\left[\eta_c, \upsilon\right]}{\delta\eta_c(x)} &= \frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} \\ &= \frac{\delta\tilde{\Gamma}\left[\eta_c, \upsilon\right]}{\delta\upsilon(x)} \end{aligned} \tag{4.78}$$

where we have used (4.74) and the fact that, since $\phi_c(x) = \eta_c(x) + \upsilon(x)$, the functional derivative with respect to ϕ_c is equivalent to the one with respect to either η_c (with υ fixed) or υ (with η_c fixed). Then the extremal condition

$$\left.\frac{\delta\tilde{\Gamma}\left[\eta_c, \upsilon\right]}{\delta\eta_c(x)}\right|_{\eta_c=0} = 0 \tag{4.79}$$

and the relation (4.75) imply that such a choice of the background field is nothing but the true ground state of the original theory with the field $\phi(x)$ (i.e., the function $\tilde{\phi}_c(x)$ that extremizes $\Gamma[\phi_c]$).

To obtain the mass in terms of the effective potential, it is convenient to assume that $\tilde{\phi}_c$ that extremizes the effective action is translational invariant, that is, it is a constant. Then, with the specific choice of $\upsilon = \tilde{\phi}_c$, one has

$$\tilde{\Gamma}[\eta_c, \tilde{\phi}_c] = \int d^4x \left[-\tilde{V}(\eta_c; \tilde{\phi}_c) + \frac{1}{2}(\partial_\mu\eta_c)^2\tilde{Z}(\eta_c; \tilde{\phi}_c) + \ldots\right] \tag{4.80}$$

where

$$\tilde{V}\left(\eta_c; \tilde{\phi}_c\right) = -\sum_{n\neq 1}\frac{1}{n!}\tilde{\Gamma}^{(n)}(0, \ldots, 0; \tilde{\phi}_c)[\eta_c(x)]^n. \tag{4.81}$$

Expressing $\eta_c = \phi_c - \tilde{\phi}_c$ in (4.80) and (4.81), and using (4.74), one gets the effective action of the original action as

$$\Gamma[\phi_c] = \int d^4x \left[-V(\phi_c) + \frac{1}{2}(\partial_\mu\phi_c)^2\tilde{Z}(\phi_c - \tilde{\phi}_c; \tilde{\phi}_c) + \ldots\right] \tag{4.82}$$

[3]This comes from the fact that $\Gamma^{(2)}(k, -k)$ is the inverse of the propagator of the particle.

with

$$
\begin{aligned}
V(\phi_c) &= -\sum_{n\neq 1} \frac{1}{n!}\tilde{\Gamma}^{(n)}(0,\ldots,0;\tilde{\phi}_c)[\phi_c(x)-\tilde{\phi}_c]^n \\
&\equiv -\sum_{n\neq 1} \frac{1}{n!}\Gamma^{(n)}(0,\ldots,0)[\phi_c(x)]^n
\end{aligned}
\tag{4.83}
$$

where the second line is the usual form of the effective potential. The equality (4.74) implies that $V(\phi_c) = \tilde{V}(\eta_c;\tilde{\phi}_c)$, so that the renormalization condition $d^2\tilde{V}/d\eta_c^2|_{\eta_c=0} = \mu^2$ (used to determine the mass μ) becomes $d^2V/d\phi_c^2|_{\phi_c=\tilde{\phi}_c} = \mu^2$. This means that the mass of the particle is obtained by evaluating the second derivative of the effective potential at its extremum point $\tilde{\phi}_c$.

Although, in principle, the effective potential can be calculated using perturbation theory but the calculation involves a double sum: a sum over all 1PI Green's functions, and for each 1PI Green's function there is an expansion in powers of the coupling constant. A nice way to organize the double sum is known as the loop expansion. It is an expansion with respect to the number of independent loops of the connected Feynman diagrams. Thus the lowest order graphs are all diagrams with no closed loops (tree graphs). The next order consists of the one-loop diagrams which have one integration over the internal momentum. The usual classical potential is typically identified with the tree-level terms of $V(\phi_c)$ in the loop expansion. In fact the loop expansion is indeed an expansion in powers of the Planck's constant $\hbar$. This can be seen as follows. For a given graph, one can define the quantities:

$I =$ the number of internal lines;

$L =$ the number of independent loops;

$V =$ the number of vertices in a given Feynman diagram.

Then the number of independent loops L is equal to the number of independent internal momenta after the momentum conservation at each vertex is taken into account. Since one combination of these momentum conservations corresponds to the overall conservation of external momenta, the number of independent loops in a given Feynman diagram is given by

$$L = I - (V - 1). \tag{4.84}$$

To understand how the number of loops relates to the power of $\hbar$, one restores the Planck's constant to the Lagrangian via the relation:

$$\mathcal{L}(\phi, \partial_\mu \phi, \hbar) \equiv \hbar^{-1} \mathcal{L}(\phi, \partial_\mu \phi) \tag{4.85}$$

where $\mathcal{L}(\phi, \partial_\mu \phi)$ is the Lagrangian in the unit $\hbar = 1$. Thus every vertex carries a factor of $\hbar^{-1}$. Since the propagator is the inverse of the differential operator occurring in the quadratic terms in $\mathcal{L}$, the propagator carries a factor of $\hbar$. Then it is easy to see that

$$P = I - V = L - 1, \tag{4.86}$$

where P is the power of $\hbar$ associated with any graph.

Because $\hbar$ is a parameter that multiplies the total Lagrangian, it is unaffected by shifts of the fields and by the redefinition or division of $\mathcal{L}$ into free and interacting parts associated with such shifts. In short, it allows us to compute $V(\phi_c)$ before the shifts. This is the advantage of the loop expansion since one can investigate the theory even the radiative corrections qualitatively change the structure of the theory (e.g., by turning the minima of a classical potential into the maxima of the resulting effective potential). In other formalisms, it is much more difficult to detect the occurrence of such phenomena.

4.3.2 Effective Potential Calculations

To illustrate the calculations of the effective potential, we start with the simplest case, the $\lambda\phi^4$ theory, which contains only scalar fields. The result is then generalized to include vertices involving fermions and gauge bosons. In the calculations below, we employ the background field technique described in the previous subsection, so that the effective action is obtained by summing all the vacuum-to-vacuum diagrams with the classical field ϕ_c as the background field.

The Effective Potential of the $\lambda\phi^4$ Theory

The Lagrangian of the $\lambda\phi^4$ theory takes the form

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\left(\partial_\mu\phi\right)^2 - U(\phi) \\ &\quad -\frac{1}{2}B\phi^2 - \frac{1}{4!}C\phi^4, \end{aligned} \tag{4.87}$$

where the B and C terms on the second line have been included to take into account of the wave function, mass and coupling constant renormalization, and the classical potential takes the form

$$U(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4. \tag{4.88}$$

To make the result valid for any value of μ, the potential U in (4.87) is treated as a perturbation. After the splitting $\phi = \phi_c + \eta$, there are two vertices with two lines: μ^2 and $\frac{1}{2}\lambda\phi_c^2$ as shown in Fig. 4.1. Their combination is just the second derivative U'' evaluated at $\phi = \phi_c$. Thus one can define

$$m_s^2\left(\phi_c\right) = U'' = \mu^2 + \frac{1}{2}\lambda\phi_c^2. \tag{4.89}$$

To calculate the effective potential, the tree-level terms come from the classical potential itself, with ϕ being replaced by ϕ_c:

$$V_0 = \frac{1}{2}\mu^2\phi_c^2 + \frac{\lambda}{4!}\phi_c^4. \tag{4.90}$$

The one-loop corrections to the effective potential are obtained by summing up the diagrams in Fig. 4.2, with massless propagators and $m_s^2\left(\phi_c\right)$ vertices. At

Figure 4.1: The combination of vertices μ^2 and $\lambda\phi_c^2/2$.

Figure 4.2: One-loop diagrams which contribute to the effective potential.

one-loop, the effective potential is thus

$$\begin{aligned}
V(\phi_c) &= \frac{1}{2}\mu^2\phi_c^2 + \frac{\lambda}{4!}\phi_c^4 - \frac{1}{2}B\phi_c^2 - \frac{1}{4}C\phi_c^4 \\
&\quad + i\int\frac{d^4k}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{1}{2n}\left[\frac{m_s^2(\phi_c)}{k^2+i\epsilon}\right]^n \\
&= \frac{1}{2}\mu^2\phi_c^2 + \frac{\lambda}{4!}\phi_c^4 - \frac{1}{2}B\phi_c^2 - \frac{1}{4}C\phi_c^4 \\
&\quad + \frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\ln\left[1 - \frac{m_s^2(\phi_c)}{k^2+i\epsilon}\right] \qquad (4.91)
\end{aligned}$$

where the counterterms with parameters B and C have been included so as to make to renormalization conditions satisfied.

The integral in the last line is divergent. If the cutoff at some large momentum $k^2 = \Lambda^2$ is used to regularize the integral, one obtains

$$\begin{aligned}
V(\phi_c) &= \frac{1}{2}\mu^2\phi_c^2 + \frac{\lambda}{4!}\phi_c^4 - \frac{1}{2}B\phi_c^2 - \frac{1}{4}C\phi_c^4 + \frac{\Lambda^2}{32\pi^2}m_s^2(\phi_c) \\
&\quad + \frac{1}{64\pi^2}m_s^4(\phi_c)\left[\ln\frac{m_s^2(\phi_c)}{\Lambda^2} - \frac{1}{2}\right]. \qquad (4.92)
\end{aligned}$$

By imposing the renormalization conditions on the renormalized mass and coupling constant, the parameters in the renormalization counterterms (B and C) can be determined. A convenient choice for $\mu^2 \neq 0$ case is the conditions (4.68) and (4.70):

$$\left.\frac{d^2V}{d\phi_c^2}\right|_{\phi_c=0} = \mu^2 \qquad (4.93)$$

$$\left.\frac{d^4V}{d\phi_c^4}\right|_{\phi_c=0} = \lambda \qquad (4.94)$$

which lead to the effective potential

$$V(\phi_c) \;=\; \frac{1}{2}\mu^2\phi_c^2 + \frac{\lambda}{4!}\phi_c^4 + \frac{m_s^2(\phi_c)}{64\pi^2}\left[\ln\frac{m_s^2(\phi_c)}{\mu} + \ldots\right]. \tag{4.95}$$

With this result, the last term does not seriously affect our classical intuition.

For the case $\mu^2 = 0$, things are different. To obtain the effective potential for this case, one cannot just take the limit $\mu^2 \to 0$ in (4.95), because of the infrared singularity. To get around this difficulty, one comes back to start with (4.92) and chooses a new renormalization condition for the coupling constant at the mass scale $M \neq 0$:

$$\lambda = \left.\frac{d^4V}{d\phi_c^4}\right|_{\phi_c=M}. \tag{4.96}$$

The appearance of M should not be a surprise: In the absence of μ, the classical Lagrangian no longer contains an intrinsic mass scale. However, the renormalization process involves a large momentum cutoff, thereby introducing a mass scale into the theory. This destroys the scale invariance of the theory, and this is how the mass scale M comes to play the role in the theory.

With the above condition and restoring $m_s^2(\phi_c) = \mu^2 + \frac{1}{2}\lambda\phi_c^2$, the effective potential becomes

$$\begin{aligned} V(\phi_c) \;=\; & \frac{1}{2}\mu^2\phi_c^2 + \frac{\lambda}{4!}\phi_c^4 \\ & + \frac{1}{64\pi^2}\left\{\left(\mu^2 + \frac{\lambda}{2}\phi_c^2\right)^2 \ln\left[\frac{\mu^2 + \frac{1}{2}\lambda\phi_c^2}{\mu^2}\right]\right. \\ & \left. - \frac{1}{2}\lambda\mu^2\phi_c^2 - \frac{25}{24}\lambda^2\phi_c^4 + \frac{1}{4}\lambda^2\phi_c^4\ln\left(\frac{2\mu^2}{\lambda M^2}\right)\right\}. \end{aligned} \tag{4.97}$$

Taking the limit $\mu^2 \to 0$, one obtains

$$V(\phi_c) \;=\; \frac{\lambda\phi_c^4}{4!} + \frac{\lambda^2\phi_c^4}{256\pi^2}\left[\ln\frac{\phi_c^2}{M^2} - \frac{25}{6}\right]. \tag{4.98}$$

It can be seen that the one-loop radiative corrections turn the potential minimum at the origin $\phi = 0$ into a maximum, and generate a new minimum at

$$\lambda\ln\left(\frac{\phi_c^2}{M^2}\right) = -\frac{32}{3}\pi^2 + \mathcal{O}(\lambda). \tag{4.99}$$

This results in a spontaneous symmetry breaking induced by radiative corrections mentioned earlier. However, this new minimum lies outside of validity of perturbation theory since the higher order calculations will give the higher power of $\lambda \ln\left(\frac{\phi_c^2}{M^2}\right)$ which is bigger than one.

The General One-Loop Calculations of the Effective Potential

Having considered the special case of the ϕ^4 theory which contains only the scalar vertices, we now turn to the more general cases which involve scalar, fermion, and gauge boson loops. Again, we confine our calculations at one-loop level. The notations are as follows. The real scalar fields are denoted by ϕ^a, the fermionic fields are denoted by ψ^a, and the vector bosons are denoted by A_μ^a. The index a is typically a group index; it runs over the appropriate range for each representation of group.

The one-loop approximation to the effective potential can be written as

$$V\left(\Phi_c\right) = V_0 + V_s + V_f + V_g + V_c, \tag{4.100}$$

where V_0 is the tree-level approximation; V_s, V_f and V_g are respectively the contributions from scalar loops, fermion loops and gauge loops; and V_c is the contribution from the renormalization counterterms. Note that V_0 is just a classical potential.

To compute V_s, one needs to know every vertex of the scalar loops. Analogous to (4.89), the vertex connecting two scalar fields of types a and b is given by

$$M_{ab}^2\left(\Phi_c\right) = \frac{\partial^2 V_0}{\partial\phi_a\partial\phi_b} \tag{4.101}$$

evaluated at Φ_c. Consider the loop with n vertices as shown in Fig. 4.3. As each internal line is constructed by connecting the same type of scalar fields, then this loop contains a factor

$$M_{a_1a_2}^2 M_{a_2a_3}^2 M_{a_3a_4}^2 \cdots M_{a_{n-1}a_n}^2 M_{a_na_1}^2 \tag{4.102}$$

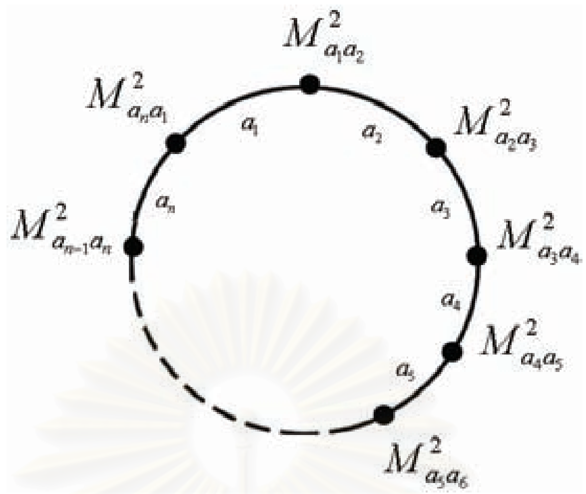

Figure 4.3: A one-loop diagram with n vertices.

apart from the scalar propagators. After summing over all types of the scalar fields, the above factor becomes the trace of the nth power of M^2:

$$\begin{aligned}\sum_{a_1 \dots a_n} M^2_{a_1a_2} M^2_{a_2a_3} M^2_{a_3a_4} \cdots M^2_{a_{n-1}a_n} M^2_{a_na_1} &= \sum_{a_1} (\overbrace{M^2 \cdots M^2}^{n \text{ times}})_{a_1a_1} \\ &= \operatorname{Tr}\left[\left(M^2\right)^n\right]. \end{aligned} \tag{4.103}$$

Since there are n propagators with the same momentum k, each factor of M^2 has to be multiplied by a propagator before integrating over the loop momentum. As V_s is obtained by summing over all scalar one-loop diagrams, then it is proportional to

$$\begin{aligned}\int d^4k \sum_n \frac{1}{2n} \operatorname{Tr}\left[\frac{\left(M^2\right)^n}{(k^2+i\epsilon)^n}\right] &= \int d^4k \operatorname{Tr}\left[\sum_n \frac{1}{2n} \frac{\left(M^2\right)^n}{(k^2+i\epsilon)^n}\right] \\ &= \operatorname{Tr} \int d^4k \sum_n \frac{1}{2n} \frac{\left(M^2\right)^n}{(k^2+i\epsilon)^n}. \end{aligned} \tag{4.104}$$

Repeating the process in the case of $\lambda\phi^4$ theory, one arrives at

$$V_s = \frac{1}{64\pi^2} \operatorname{Tr}\left[\left(M^2\right)^2 \ln M^2\right], \tag{4.105}$$

plus cutoff-dependent quadratic terms, which are absorbed in V_c.

We next consider the fermion loops. The calculation is similar to the previous case. One defines a generalized mass matrix $m(\Phi)$ whose dependence on the scalar fields arises from the Yukawa couplings:

$$\mathcal{L} = \ldots - \sum_{ab} \overline{\Psi}_a m_{ab}(\Phi) \Psi_b + \ldots . \tag{4.106}$$

With Φ replaced by the VEV Φ_c, one obtains $m_{ab}(\Phi_c)$ as a vertex with two fermion legs, a and b. Note that the matrix m is a matrix in the spinor space as well as the internal space.

The calculations go as follows. One first notes that since the trace of an odd number of Dirac γ matrices is zero, there can be only an even number of fermion propagators in each loop diagram. Thus each diagram is proportional to some power of $mm^\dagger$:

$$\cdots m \frac{1}{\gamma^\mu k_\mu} m^\dagger \frac{1}{\gamma^\mu k_\mu} \cdots = \cdots \frac{1}{k^2} m m^\dagger \cdots . \tag{4.107}$$

Then the rest of the calculations are exactly the same as the scalar case except for the overall minus sign for the fermion loops. The result is

$$V_f = -\frac{1}{64\pi^2} \mathrm{Tr}\left[\left(m(\Phi_c)\, m^\dagger(\Phi_c) \right)^2 \ln\left(m(\Phi_c)\, m^\dagger(\Phi_c) \right) \right] \tag{4.108}$$

plus some quadratic terms which can be absorbed into V_c. Note that in this equation the trace runs over spinor indices as well as internal indices. Typically $M_f \equiv mm^\dagger$ is already diagonal, and $m(\phi) = m^\dagger(\phi)$. If the spinors are all Majorana spinors as in the supersymmetric theory, then

$$V_f = -\frac{2}{64\pi^2} \left[\sum_\alpha m_\alpha^4 \ln m_\alpha^2 \right] \tag{4.109}$$

with m_α being the eigenvalues of the mass matrix. The factor of 2 on the right-hand side comes from the trace over Majorana indices.

Finally, the contribution V_g from the gauge fields is considered. At the one-loop level there are two types of diagrams containing gauge fields. The simplest ones have only gauge fields traveling around the loops. The others are gauge mixed loops, shown in Fig. 4.4. However, if one quantizes the theory in the Landau gauge, where the gauge propagator is

$$i\Delta_{\mu\nu}(k) = -i\frac{g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}}{k^2 + i\epsilon}, \tag{4.110}$$

then, as the momentum of the internal scalar field is the same as that of the internal gauge boson (since the external momentum is zero) and the vertex is proportional to the momentum of the scalar field, all the gauge mixed loops vanish due to the fact that they contain a factor

$$\begin{aligned} \sim k^\mu \Delta_{\mu\nu} &= -ik^\mu \frac{(g_{\mu\nu} - k_\mu k_\nu/k^2)}{k^2} \\ &= -i\frac{(k^\mu g_{\mu\nu} - k^\mu k_\mu k_\nu/k^2)}{k^2} \\ &= -i\frac{(k_\nu - k_\nu k^2/k^2)}{k^2} \\ &= 0. \end{aligned} \tag{4.111}$$

Thus the contributions from gauge fields come from the sum over pure gauge loops. Defining a gauge field mass matrix in terms of the nonderivative couplings of gauge fields to the scalar fields:

$$\mathcal{L} = \ldots + \frac{1}{2}\sum_{ab} M_g^2(\Phi) A_{\mu a} A_b^\mu + \ldots, \tag{4.112}$$

then the vertex is just the mass matrix $M_g^2(\Phi_c)$ evaluated at the VEV Φ_c. The calculation of V_g is then analogous to the previous cases, with the result

$$V_g = \frac{3}{64\pi^2}\mathrm{Tr}\left[\left(M_g^2\right)^2(\Phi_c)\ln M_g^2(\Phi_c)\right] \tag{4.113}$$

apart from quadratic terms which are absorbed in V_c. The extra factor of 3 comes from the trace of the Landau gauge propagator.

4.3.3 Bounds on the Mass of the Lightest Supersymmetric Higgs Boson

Equipped with the general one-loop results for the effective potential, we are ready to calculate the radiative corrections to the mass of the lightest supersymmetric

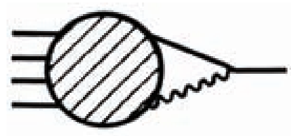

Figure 4.4: An example of gauge mixed loop.

Higgs boson, h^0. This is done by first finding the effective potential of the Higgs fields at one-loop level in terms of the classical fields (of all components of the Higgs doublets). Once the VEVs of all components of the Higgs doublets are found by minimizing the effective potential,[4] the Higgs mass matrix is obtained by evaluating the second order derivatives with respect to the classical fields of the effective potential at the VEVs. After diagonalizing the mass matrix, the masses (with radiative corrections included) of all Higgs particles are finally obtained.

We now begin the calculation. As we are interested only in the mass of h^0 which is a linear combination of $\mathrm{Re}\, H_1^0$ and $\mathrm{Re}\, H_2^0$, it is sufficient to consider only the part of the tree-level effective potential that contains $\mathrm{Re}\, H_1^0$ and $\mathrm{Re}\, H_2^0$:

$$V_0 = m_1^2 A^2 + m_2^2 B^2 + 2m_3^2 AB + \frac{g^2 + g'^2}{8}\left(A^2 - B^2\right)^2, \qquad (4.114)$$

where we have defined

$$A = \mathrm{Re}\, H_1^0, \qquad B = \mathrm{Re}\, H_2^0$$
$$m_1^2 = \left(|\mu|^2 + m_{H_1}\right), \quad m_2^2 = \left(|\mu|^2 + m_{H2}\right), \quad m_3^2 = b.$$

To find the one-loop corrections of the above potential, one notes that if all vector fields in the MSSM are quantized in the Landau gauge and since all fermions in the MSSM are Majorana fermions, then one can write the one-loop corrections to the Higgs potential, V_1, as the sum over scalar, fermion, and gauge loops:

$$\begin{aligned} V_1 &= \frac{1}{64\pi^2}\mathrm{Tr}\left[\left(M^2\right)^2\left(\ln\frac{M^2}{\Lambda^2}\right)\right] + \frac{3}{64\pi^2}\mathrm{Tr}\left[\left(M_v^2\right)^2\left(\ln\frac{M_v^2}{\Lambda^2}\right)\right] \\ &\quad - \frac{2}{64\pi^2}\mathrm{Tr}\left[\left(mm^\dagger\right)^2\left(\ln\frac{mm^\dagger}{\Lambda^2}\right)\right], \end{aligned} \qquad (4.115)$$

[4]Like the tree-level case, the only non-zero VEVs are of $\mathrm{Re}\, H_1^0$ and $\mathrm{Re}\, H_2^0$.

where M^2, M_v^2 and m are mass matrices of scalar fields, vector fields and fermionic fields, respectively, and Λ is a cutoff scale as usual. Note that (4.115) is the corrections to the effective potential before the renomalization conditions are taken into account (see (4.92)). If one observes that the coefficient of each term in (4.115) is the number of spin degrees of freedom of the fields in each loop diagram (the spin degrees of freedom of a spin-J particle is $2J+1$), then one can rewrite (4.115) as

$$V_1 = \frac{1}{64\pi^2} \sum_J (-1)^{2J} (2J+1) \operatorname{Tr} \left[\left(M_J^2\right)^2 \left(\ln \frac{M_J^2}{\Lambda^2} \right) \right], \tag{4.116}$$

where M_J^2 is the mass matrix of the fields with spin J (for spin-$\frac{1}{2}$ fields, $M_{\frac{1}{2}}^2 = mm^\dagger$). Since the trace over M_J^2, $\mathrm{Tr} M_J^2$, results in the summation over all mass eigenvalues of the fields with spin-J, one can write V_1 as a supertrace,

$$V_1 = \frac{1}{64\pi^2} \mathrm{Str} \left[\left(\mathcal{M}^2\right)^2 \left(\log \frac{\mathcal{M}^2}{\Lambda} \right) \right], \tag{4.117}$$

$$\mathrm{Str}\ f\left(\mathcal{M}^2\right) \equiv \sum_i (-1)^{2J_i} (2J_i+1) f\left(m_i^2\right), \tag{4.118}$$

where m_i is the mass of the i-th particle of spin J_i. After imposing the renormalization conditions, the effective potential V becomes

$$\begin{aligned} V(Q) &= V_0(Q) + V_1(Q) \\ &= m_1^2 A^2 + m_2^2 B^2 + 2m_3^2 AB + \frac{g^2 + g'^2}{8} \left(A^2 - B^2\right)^2 \\ &\quad + \frac{1}{64\pi^2} \mathrm{Str} \left[\left(\mathcal{M}^2\right)^2 \left(\ln \frac{\mathcal{M}^2}{Q^2} - \frac{1}{2} \right) \right], \end{aligned} \tag{4.119}$$

where Q is the renormalization scale. Note that the factor $-\frac{1}{2}$ in the logarithmic term in (4.119) has been restored (compared with (4.92)) for convenience even though it could have been absorbed into the polynomial terms.

We are now at the point to show that one-loop diagrams involving the top-quark supermultiplet induce a finite, non-negligible contribution to the Higgs potential as the top-quark mass is fairly large, $m_t \simeq 170$ GeV. Consider the interactions between the Higgs fields and other fields in the MSSM. Although the Higgs fields interact with lots of fields, the major contributions to the Higgs effective potential come from their interactions with the top-quark supermultiplet due to the large value of the corresponding Yukawa coupling λ_t. According to (3.110), only the H_1 Higgs doublet interacts with the top-quark supermultiplet. So it is sufficient to consider only the interactions coming from the term

$$\lambda_t \Phi_{\tilde{t}_R} \Phi_{q_L^3} \Phi_{H_1} \tag{4.120}$$

in the superpotential, where $\Phi_{\tilde{t}_R}$ and $\Phi_{q_L^3}$ are chiral superfields associated with the right-handed top-quark and the left-handed quark doublet in the third generation, respectively. This term implies the interactions of H_1 with top quarks and squarks with the strengths proportional to the Yukawa coupling λ_t.

To calculate the one-loop diagrams, one simplifies the calculation by taking the large $\tan\beta$ limit, which is equivalent to taking the VEVs $\nu_1 \neq 0$ and $\nu_2 \approx 0$ (so that $\tan\beta = \nu_1/\nu_2 \gg 1$). That this limit simplifies the calculation can be understood as follows. Since the tree-level mass $m^2_{A^0} = 2b/\sin 2\beta$ must be finite, then the limit of large $\tan\beta$ (or small $\sin 2\beta$) corresponds to the situation in which $b \to 0$, so one neglects the bilinear term $2m_3^2 AB$ in (4.114). Also as the quartic terms containing B (which come from the $(A^2 - B^2)^2$ term in (4.114)) give the mass matrix elements proportional to at least one factor of the VEV of B (which is $\nu_2 \approx 0$ in the large $\tan\beta$ limit), then one can neglect these terms as well. Thus in this situation, the lightest supersymmetric Higgs boson is $h^0 \equiv h \approx \sqrt{2}\,\mathrm{Re}\,H_1^0 = \sqrt{2}A$ (compare this with the form of h^0 at the end of the last chapter). Let $m_1 \to m$, $\nu_1 \to \nu$, the tree-level effective potential becomes

$$V_0 = m^2 A^2 + \frac{g^2 + g'^2}{8} A^4 \tag{4.121}$$

so that

$$\frac{\partial V_0}{\partial A} = 2m^2 A + \frac{g^2 + g'^2}{2} A^3$$
$$\frac{\partial^2 V_0}{\partial A^2} = 2m^2 + \frac{3(g^2 + g'^2)}{2} A^2. \tag{4.122}$$

Thus the tree-level minimization gives

$$m^2 = -\frac{g^2 + g'^2}{4}\nu^2 = -\frac{m_Z^2}{2} \tag{4.123}$$

and so the tree-level mass of h is

$$m_h^2 = \left[\frac{1}{2}\frac{\partial^2 V_0}{\partial A^2}\right]_{A=\nu} = m_Z^2. \tag{4.124}$$

If one assumes in addition that the two stop squarks have the same mass and do not mix with each other, and that $m_{\tilde{q}}^2, m_t^2 \gg m_Z^2$, the dominant field masses are those of the top quark t and the stop squarks $\tilde{t}$:

$$m_t^2 = \lambda_t^2 A^2, \qquad m_{\tilde{t}}^2 = \lambda_t^2 A^2 + m_{\tilde{q}}^2 \tag{4.125}$$

where $m_{\tilde{q}}$ came from the stop mass term in the soft-SUSY breaking part of the Lagrangian, and D-terms which give an additional contribution to the stop mass have been neglected due to the smallness of the gauge couplings. Thus one can approximate the one-loop correction V_1 by including only the contributions from top quark t and stop squark $\tilde{t}$ loops. The result is

$$V_1 = \frac{3}{16\pi^2}\left[m_{\tilde{t}}^4\left(\ln\frac{m_{\tilde{t}}^4}{Q^2} - \frac{1}{2}\right) - m_t^4\left(\ln\frac{m_t^4}{Q^2} - \frac{1}{2}\right)\right] \tag{4.126}$$

where the factor of 3 on the right-hand side is the number of colors of quarks. The minimization of $V = V_0 + V_1$ (i.e., $\left[\frac{\partial V}{\partial A}\right] = 0$ at $A = \nu$) gives

$$\left[\frac{\partial V_0}{\partial A} + \frac{3}{8\pi^2}\frac{\partial m_t^2}{\partial A}\left[m_{\tilde{t}}^2\left(\ln\frac{m_{\tilde{t}}^2}{Q^2}\right) - m_t^2\left(\ln\frac{m_t^2}{Q^2}\right)\right]\right]_{A=\nu} = 0 \tag{4.127}$$

where we have used the fact that $\frac{\partial m_t^2}{\partial A} = \frac{\partial m_{\tilde{t}}^2}{\partial A}$ after neglecting the D-terms. It is convenient to choose a renormalization scale Q such that the log terms in (4.127) sum up to zero. Thus, the above minimization condition becomes

$$\frac{\partial V_0}{\partial A} = 0 \tag{4.128}$$

and

$$\left[m_{\tilde{t}}^2\left(\ln\frac{m_{\tilde{t}}^2}{Q^2}\right)-m_t^2\left(\ln\frac{m_t^2}{Q^2}\right)\right]_{A=\nu}=0. \qquad (4.129)$$

The first condition in (4.128) has already been considered and gives the result in (4.123), while the second one in (4.129) allows us to evaluate the renormalization scale Q. Thus one finally obtains the mass of the lightest supersymmetric Higgs boson h^0 as

$$\begin{aligned} m_{h^0}^2 &= \left[\frac{1}{2}\frac{\partial^2 V}{\partial A^2}\right]_{A=\nu} \\ &= m_{h^0}^2|_{tree}+\frac{3}{8\pi^2}\left(4\lambda_t^4\nu^2\ln\frac{m_{\tilde{t}}^2}{m_t^2}\right), \end{aligned} \qquad (4.130)$$

where $m_{h^0}^2|_{tree}=m_Z^2$ is the mass of h^0 at tree level.

Since the condition (4.128) is exactly the same as the tree-level minimization condition, then the relation between the VEVs and the mass of W boson still holds at one-loop order. Thus, in this limit, one can still use the relation

$$\nu=\left(\sqrt{2}G_F\right)^{\frac{1}{2}} \qquad (4.131)$$

where G_F is the Fermi coupling constant. With this identification and using (4.125), the result (4.130) becomes

$$m_{h^0}^2=m_{h^0}^2|_{tree}+\frac{3G_F}{\sqrt{2}\pi^2}m_t^4\ln\frac{m_{\tilde{t}}^2}{m_t^2}. \qquad (4.132)$$

Thus one can see that, in the limit of $\tan\beta\gg 1$, the correction to the mass of h^0 grows quartically with top mass and logarithmically with the stop mass. Thus the one-loop effective potential gives a significant improvement to the mass of the lightest supersymmetric Higgs boson if the top quark is sufficiently heavy. This is indeed the case as we know nowadays that the top quark mass is about 170 GeV.

In general, one needs to compute the upper bound of m_{h^0} for given values of $\tan\beta$ and the squark masses. In doing so, one has to keep the terms involving

$B = \mathrm{Re}\, H_2^0$ in the tree-level effective potential. The one-loop correction (unrenormalized) sufficient for determining the corrections to m_{h^0} is still the same:

$$\begin{aligned} V_1 &= \frac{3}{16\pi^2}\left[\left(m_{\tilde{q}}^2 + \lambda_t^2 A^2\right)^2 \left(\ln \frac{m_{\tilde{q}}^2 + \lambda_t^2 A^2}{\Lambda^2} - \frac{1}{2}\right)\right. \\ &\quad \left. - \left(\lambda_t^2 A^2\right)^2 \left(\ln \frac{\lambda_t^2 A^2}{\Lambda^2} - \frac{1}{2}\right)\right], \end{aligned} \tag{4.133}$$

where the top quark mass and the stop mass have been expressed in their original forms as in (4.125).[5] If $m_{\tilde{q}}^2 \gg m_t^2\, (= \lambda_t^2 A^2)$, then one can approximate (4.133) as

$$\begin{aligned} V_1 &\approx \frac{3}{16\pi^2}\left[-2 m_{\tilde{q}}^2 \lambda_t^2 \left(A^2\right) \ln \frac{\Lambda^2}{m_{\tilde{q}}^2} + \frac{3}{2}\lambda_t^4 \left(A^2\right)^2\right. \\ &\quad \left. - \lambda_t^4 \left(A^2\right)^2 \ln \frac{\lambda_t^2 A^2}{m_{\tilde{q}}^2 + \lambda_t^2 A^2}\right]. \end{aligned} \tag{4.134}$$

After the renormalization is performed, the divergent mass term in (4.134) can be absorbed by the tree-level potential.

The rest of the calculation is straightforward: One just adds the one-loop correction V_1 to the tree-level effective potential, and evaluates the second order derivatives of the total effective potential at $A = \nu_1$, $B = \nu_2$ to obtain the mass matrix for the Higgs fields h^0 and H^0. It can be shown that the mass of the lightest supersymmetric Higgs boson has an upper bound:

$$m_{h^0} \leqslant \sqrt{m_{Z^0}^2 \cos^2 2\beta + \frac{3 G_F}{\sqrt{2}\pi^2} m_t^4 \ln \frac{m_{\tilde{t}}^2}{m_t^2}}, \tag{4.135}$$

which is higher than the one obtained in the previous chapter. Fig. 4.5 shows the upper bound on the lightest Higgs mass as a function of $\tan\beta$ for several choices of m_t. With the supersymmetry breaking scale $m_{\tilde{q}} = 1$ TeV, one sees that the upper bound of m_{h^0} can reach $130 - 150$ GeV in the range of $m_t = 150 - 200$ GeV. This explains why the h^0 boson has not been seen at LEP: the upper bound on m_{h^0} in the MSSM, when the one-loop radiative corrections are included, is such that the h^0 boson can be kinematically not accessible at LEP energies (≈ 104 GeV).

[5]Actually, A^2 in (4.133) should be replaced by $(\mathrm{Re}\, H_1^0)^2 + (\mathrm{Im}\, H_1^0)^2$. But since the VEV of $\mathrm{Im}\, H_1^0$ is zero, then $\mathrm{Im}\, H_1^0$ does not contribute anything to the mass of m_{h^0} and so can be ignored.

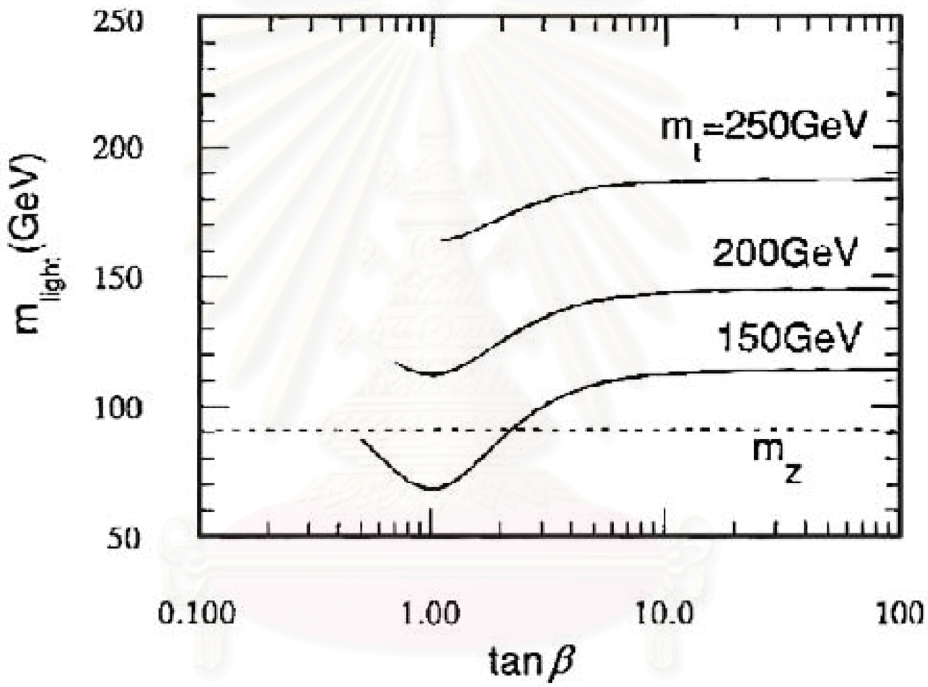

Figure 4.5: The mass of the lightest neutral Higgs boson as a function of $\tan\beta$ for various top quark masses [14]. This figure includes radiative corrections to the Higgs mass, assuming the supersymmetry breaking scale $m_{\tilde{q}} = 1$ TeV. (This figure is taken from [14]).

CHAPTER V

CONCLUSIONS

In this thesis, we have reviewed the Higgs sector of the MSSM and some aspects of the MSSM Higgs bosons. The Higgs sector of the MSSM consists of two Higgs doublets with opposite hypercharges. The tree-level potential for these scalar doublets was determined from three types of contributions:

1. The supersymmetric F-terms which give the quadratic terms involving the μ parameter, $\mu^2 \left(|H_1^2| + |H_2^2|\right)$.

2. The supersymmetric D-terms which give the Higgs self-interactions of which coupling constants are completely determined by the $SU(2)_L$ and $U(1)_Y$ gauge coupling constants.

3. The soft-supersymmetry breaking terms which give additional mass and mixing terms, $m_{H_1} |H_1|^2 + m_{H_2} |H_2|^2$ and $b \left(H_1^+ H_2^- - H_1^0 H_2^0\right)$.

Such the Higgs potential allowed us to break the electroweak symmetry if the VEVs of their electrically neutral components, ν_1 and ν_2, are non-zero. After the electroweak symmetry breaking, the physical Higgs states include three neutral Higgs bosons (h^0, H^0 and A^0) of different masses, and one pair of charged Higgs bosons ($H^{\pm}$) of equal masses. Once $\nu_1^2 + \nu_2^2$ has been fixed by the experimental values of the mass of the W bosons and the weak coupling, the MSSM Higgs sector can be described in terms of two independent parameters: the ratio of MSSM VEVs, $\tan\beta = \nu_1/\nu_2$, and the mass of the A^0 Higgs boson.

A remarkable consequence of the tree-level Higgs potential is the mass of the lightest supersymmetric Higgs boson, h^0, which is always smaller than the Z^0 boson mass. However, it is a nature of the theory that m_{h^0} receives the radiative corrections, so its upper bound can be raised substantially. Indeed, by computing the one-loop corrections due to the top quark and its scalar superpartner, stop squark, it was found that the correction to m_{h^0} (in the theory with soft-supersymmetry breaking terms) is proportional to the fourth power of the top quark mass, but vanishes if supersymmetry is unbroken. It was moreover found that the upper bound on m_{h^0} is changed by one-loop corrections into

$$m_{h^0} \leqslant \sqrt{m_{Z^0}^2 \cos^2 2\beta + \frac{3G_F}{\sqrt{2}\pi^2} m_t^4 \ln \frac{m_{\tilde{t}}^2}{m_t^2}}, \tag{5.1}$$

where one assumed that all squarks have equal masses. Thus the upper bound on m_{h^0}, in the case of the large top quark mass m_t, can exceed the highest energy accessible at LEP. For $m_t = 175$ GeV and $m_{\tilde{t}} = 1$ TeV, the upper bound of m_{h^0} varies from $130 - 150$ GeV.

As there have been more sophisticated analyses which include a variety of two-loop effects, renormalization group effects, etc., this bound can, of course, be changed. However, there is no model which has the mass upper bound larger than 150 GeV for the lightest supesymmetric Higgs boson. Such a mass scale will be accessible at LEP II or the Large Hadron Collider (LHC), so the experiments to be performed in the near future will provide a definitive test of the MSSM.

REFERENCES

1. Wess, J., From Symmetry to Supersymmetry [Online]. Available from: http://www.theorie.physik.uni-muenchen.de/wess/Wess.texhaifa2.ps [2006, February 3].

2. Lee, B. W., Quigg, C., and Thacker, H., "Strength of Weak Interaction at Very High Energies and the Higgs Boson Mass," Phys. Rev. **D16**, (1977): 1519.

3. Haag, R., Lopuszanski, J. T., and Sohnius, M., "All Posible Generators of Supersymmetries of the S-Matrix," Nucl. Phys. **B88**, (1975): 257.

4. Sohnius, M., "Introducing Supersymmetry," Phys. Rept. **128**, (1985): 39-204.

5. Nakahara, M., Geometry, Topology and Physics, (Bristol and Philadelphia: Institute of Physics Publishing, 1990).

6. Cahn, R. N., Semi-Simple Lie Algebras and Their Representations, (Menlo Park, California: Benjamin/Cummings, 1984).

7. Gieres, F., "About Symmetries in Physics," hep-th/9712154.

8. Ryder, L. H., Quantum Field Theory, 2nd ed. (Cambridge University Press, 1996).

9. Gates, S. J. Jr., Roček, M., Grisaru, M. T., and Siegel, W., Superspace or One Thousand and One Lessons in Supersymmetry, (Reading, Massachusetts: Benjamin/Cummings, 1983), hep-th/0108200.

10. Okada, Y., "SUSY Phenomenology," hep-ph/9811502.

11. Drees, M., "An Introduction to Supersymmetry," hep-ph/9611409.

12. Aitchison, I. J. R., "Supersymmetry and the MSSM: An Elementary Introduction," hep-ph/0505105.

13. Ellis, J., Ridolft, G., and Zwirner, F., "Radiative Corrections to the Masses of Supersymmetric Higgs Bosons," Phys. Lett. **B257**, (1991) 83.

14. Okada, Y., Yamaguchi, M., and Yanagida, T., "Upper Bound of the Lightest Higgs Boson Mass in the Minimal Supersymmetric Standard Model." Prog. Theor. Phys. Lett. **85**, (1991) 1.

15. Coleman, S. and Weinberg, E., "Radiative Corrections as the Origin of Spontaneous Symmetry Breaking." Phys. Rev. **D7**, (1973) 2887.

16. Rivers, R. J., Path Integral Methods in Quantum Field Theory, (Cambridge: Cambridge Univesity Press, 1987).

17. Dawson, S., "Susy and Such," hep-ph/9612229.

18. Peskin, M. E. and Schroeder, D. V., An Introduction to Quantum Field Theory, (New York: Addison-Wesley Publishing Company, 1995).

19. Cheng, T. P. and Li, L. F., Gauge Theory of Elementary Particle Physics, (New York: Oxford University Press, 1984).

20. Martin, S. P., "A Supersymmetry Primer," hep-ph/9709356.

21. Bailin, D. and Love, A., Supersymmetric Gauge Field Theory and String Theory, (Bristol and Philadelphia: Institute of Physics Publishing, 1994).

22. Itzykson, C. and Zuber, J. B., Quantum Field Theory, (New York: McGraw-Hill, 1980).

23. Pokorski, S., Gauge Field Theories, 2nd ed. (Cambridge: Cambridge Univesity Press, 2000).

24. Kaku, M., Quantum Field Theory: A Modern Introduction, (Oxford: Oxford Univesity Press, 1993).

25. Muller-Kirsten, H. J. W. and Wiedemann, A., Supersymmetry: An Introduction with Conceptual and Calculation Details, (Singapore: World Scientific, 1987).

26. Wess, J. and Bagger, J., Supersymmetry and Supergravity, (Princeton, Princeton University Press, 1983).

27. Zee, A., Quantum Field Theory in a Nutshell, (Princeton: Princeton University Prees, 2003).

28. Vissani, S. V., "What is the Standard Model of Elementary Particles and Why We Have to Modify It," hep-ph/0007040.

29. de Boer, W., "Grand Unified Theories and Supersymmetry in Particle Physics and Cosmology," hep-ph/9402266.

30. Novaes, S. F., "Standard Model: An Introduction," hep-ph/0001283.

31. Polonsky, N., Supersymmetry: Structure and Phenomena, (Heidelberg: Springer-Verlag, 2001).

32. Ketov, S. V., "Why Do We Need Supersymmetry," hep-th/0011042.

33. Huang, K., Quantum Field Theory, (New York: John Wiley Inc., 1998).

34. Hatfield, B. F., Quantum Field Theory of Point Particles and Strings, (New York: Addison-Wesley Publishing Company, 1991).

VITAE

Mr. Wirin Sonsrettee was born on September 27, 1978. He received his Bachelor of Science in Physics from Chulalongkorn University in May 2001. His research interests are in theoretical physics, especially in the areas of elementary particle physics and general relativity.

Conference Presentations

- *From Symmetry to Supersymmetry*: XI Vietnam School of Physics, Danang, Vietnam (December 31, 2004).
- *The Necessity of Supersymmetry*: Chulalongkorn University, Bangkok (May 19, 2005).

Schools and Workshops Attended

- A Short Course on Cosmology, Chulalongkorn University, Bangkok, 16-27 January 2006.
- XI Vietnam School of Physics, Danang, Vietnam, 27 December 2004 – 7 January 2005.
- A Short Course on Self-Organization in Complex Systems, Chulalongkorn University, Bangkok, 13-24 September 2004.
- A Short Course on Conformal Field Theory, Chulalongkorn University, Bangkok, 13-22 September 2004.
- The Third Thai School & The Second "ThaiPhysUniverse" Symposium in Thailand, Khonkaen University, Khonkaen, 10-20 October 2004.