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POINTWISE DIMENSION OF ARCHIMEDEAN COPULAS

Mr. Noppawit Yanpaisan

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By	Mr. Noppawit Yanpaisan
Field of Study	Mathematics
Thesis Advisor	Associate Professor Songkiat Sumetkijakan, Ph.D.
Thesis Co-advisor	Associate Professor Tippawan Santiwipanont

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master Degree

.... Dean of the Faculty of Science

(Professor Polkit Sangvanich, Ph.D.)

THESIS COMMITTEE

.... Chairman
(Associate Professor Phichet Chaoha, Ph.D.)

.... Thesis Advisor
(Associate Professor Songkiat Sumetkijakan, Ph.D.)

.... Thesis Co-advisor
(Associate Professor Tippawan Santiwipanont.)

.... Examiner
(Athipat Thamrongthanyalak, Ph.D.)

.... External Examiner
(Pongpol Ruankong, Ph.D.)

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ในวิทยานิพนธ์นี้ เราเริ่มด้วยการนิยามมิติรายจุดของคอปูลา 2 ตัวแปร C ที่จุด $(u, v) \in [0, 1]^2$ ผ่านปริมาตร-C โดยปริมาณนี้จะให้ภาพของการกระจายมวลความน่าจะเป็นของคอปูลาที่กว้างกว่าเซตค้ำจุนของคอปูลา ซึ่งนิยามเป็นเซตปิดที่เล็กที่สุดที่บรรจุมวลความน่าจะเป็นทั้งหมด ต่อมา เราศึกษาสมบัติพื้นฐานของมิติรายจุด รวมถึงแสดงการเท่ากันของมิติรายจุดและมิติเลขชี้กำลังของคอปูลาเมื่อทั้งสองปริมาณมีค่า หลังจากนั้น เราจะศึกษาพฤติกรรมของมิติรายจุดโดยคำนวณค่ามิติรายจุดของคอปูลาที่สร้างมาจากวิธีต่าง ๆ และในส่วนสุดท้าย เราจะคำนวณค่ามิติรายจุดของคอปูลาอาร์คิมีเดียน ซึ่งเป็นกลุ่มของคอปูลาที่สำคัญและมีการนำไปใช้อย่างกว้างขวาง

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In this thesis, we first introduce the pointwise dimension of a bivariate copula C at $(u, v) \in [0, 1]^2$ via its C-volume. This quantity gives a broader view of probability mass distribution than the support of a copula, the smallest closed set on which the whole mass lies. We would like to study some fundamental properties of pointwise dimension, including the equality to pointwise exponent whenever both of them exist. After that, we investigate the behavior of the pointwise dimension of copulas constructed by various methods. In the last part, we compute the pointwise dimension of Archimedean copulas, an important class of copulas used widely in many branches.

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CHAPTER I
INTRODUCTION

Copulas, defined as joint distribution functions whose one-dimensional marginals are uniformly distributed on $[0,1]$, have been successful in modeling dependence between random variables. They have been used widely in many branches, for instance, risk management, quantitative finance, geology, hydrology, including many applications involving modeling and analyzing of data. Apart from the perspective of probability theory, we can view any copula as a measure, called doubly stochastic measure, because for any given copula C, the corresponding doubly stochastic measure can be defined by the extension of its C-volume. In fact, the class of copulas and the class of doubly stochastic measures are isomorphic.

With the aspect of copulas in measure theory, they can be decomposed into an absolutely continuous part and a singular part. Absolutely continuous copulas are very well-understood and more convenient in modeling real-world data because they are presentable by joint density functions. On the other hand, copulas with no absolutely continuous part are called singular copulas, which can be quite complicated. However, in theoretical studies, singular copulas are very interesting in its own right and also useful to study because they give some strange but fascinating results connecting to other branches of mathematics. For example, in [7], for any given $s \in (1,2)$, there exists a singular copula for which the Hausdorff dimension of its support is exactly s.

In measure theory, the support of a copula can be defined as the smallest closed set covering the whole mass of copula. The support gives a crude picture of the copula mass distribution. For instance, if the support has Lebesgue measure zero then the copula is singular. However, the concept of supports is a global property which does not give any local character about points in the support. In dimension theory, pointwise dimension is a local quantity that describes roughly a character of mass distribution around a point via its value. It is defined in [1] by

$$d_C(u,v) = \lim_{h\to 0} \frac{\log V_C(R_h(u,v))}{\log(h)} \quad \text{where } R_h(u,v) = [u-h,u+h]\times[v-h,v+h],$$

if the limit exists. From the definition, larger pointwise dimension means smaller mass distribution around the point. Moreover, the concept of pointwise dimension can be used to formulate a sufficient condition to infer some statements about

Hausdorff and Box dimensions, two quantities in dimension theory that measure complexity of sets, measures, etc. For instance, any absolutely continuous copula has Hausdorff dimension 2.

One of the most popular classes of copulas is the Archimedean copulas which is defined in [9] by

$$C(u,v) = \varphi^{[-1]}\left(\varphi(u) + \varphi(v)\right) \quad \text{for } (u,v) \in \mathbb{I}^2,$$

where $\varphi\colon [0,1] \to [0,\infty]$, called Archimedean generator, is a continuous, strictly decreasing and convex function such that $\varphi(1) = 0$ and $\varphi^{[-1]}$ is the pseudo-inverse of φ. The class of Archimedean copulas is popular due to many desirable properties. For example,

1. they are constructed by a simple formula which makes their properties much easier to derive. In other words, many quantities and formulas obtaining from Archimedean copulas can be expressed explicitly in terms of Archimedean generators;

2. many Archimedean copulas form a dependence monotonic parametric family.

In [13], we computed the pointwise dimension of a few families of copulas: Clayton copulas, Marshall-Olkin copulas and copulas with fractal support defined in [7] (at some points). Especially for Clayton copulas, which are absolutely continuous Archimedean copulas defined by

$$C(u,v) = \left(\max\{u^{-\theta} + v^{-\theta} - 1, 0\}\right)^{-\frac{1}{\theta}} \quad \text{for } \theta \in (-1,\infty)\setminus\{0\},$$

we obtain the pointwise dimension in the case $\theta \in (-1,0)$ that

$$d_C(u,v) = \begin{cases} \infty & \text{if } u^{-\theta} + v^{-\theta} > 1; \\ 2 & \text{if } u^{-\theta} + v^{-\theta} < 1; \\ -\frac{1}{\theta} & \text{if } u^{-\theta} + v^{-\theta} = 1. \end{cases}$$

This shows that the behavior of pointwise dimension of absolutely continuous copulas may not be the same throughout its support.

In this thesis, we are interested in extending the result in [13] for Archimedean copulas case. That is, we find the pointwise dimension of general Archimedean copulas. Furthermore, we shall also investigate general behaviors of pointwise dimension of copulas constructed by various methods.

CHAPTER II
PRELIMINARIES

We organize the content in this chapter as follows. Section 1 lays out some background on copulas, especially Archimedean copulas and methods of constructing copulas. Section 2 lists some results on convex functions, an essential property of Archimedean generators, and conditions that are similar to differentiation. Section 3 contains some basic knowledge of regular variation, functions whose behavior is similar to polynomial in some sense, while Section 4 introduces pointwise dimension, another main concept we study in this thesis.

2.1 Background on copulas

First, we let $\mathbb{I} := [0,1]$ and define a *copula* [9] to be a function $C : \mathbb{I}^2 \to \mathbb{I}$ with the following properties:

1. For every $u, v \in \mathbb{I}$,

$$C(u,0) = 0 = C(0,v) \quad C(u,1) = u \text{ and } C(1,v) = v.$$

2. For every $u_1, u_2, v_1, v_2 \in \mathbb{I}$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$V_C([u_1,u_2] \times [v_1,v_2]) := C(u_2,v_2) - C(u_1,v_2) - C(u_2,v_1) + C(u_1,v_1) \geq 0$$

where $V_C([u_1,u_2] \times [v_1,v_2])$ is called *C-volume* of the set $[u_1,u_2] \times [v_1,v_2]$.

- The condition 2 is called “2-increasing property” and implies that any copula C is
  - increasing in each variable: for any $u, u_1, u_2, v, v_1, v_2 \in \mathbb{I}$ with $u_1 \leq u_2$ and $v_1 \leq v_2$, $C(u,v_1) \leq C(u,v_2)$ and $C(u_1,v) \leq C(u_2,v)$, and
  - Lipschitz continuous: for any $(u_1,v_1),(u_2,v_2) \in \mathbb{I}^2$, $|C(u_2,v_2) - C(u_1,v_1)| \leq |u_2 - u_1| + |v_2 - v_1|$.

  These two properties give rise to *Fréchet-Hoeffding bound* for copulas, i.e., for any copula C and $(u,v) \in \mathbb{I}^2$, $\max\{u+v-1,0\} \leq C(u,v) \leq \min\{u,v\}$, where the *Fréchet-Hoeffding lower bound* $W(u,v) := \max\{u+v-1, 0\}$ is a copula, called *countermonotonic copula* and the *Fréchet-Hoeffding upper bound* $M(u,v) := \min\{u,v\}$ is also a copula, called *comonotonic copula.*

- V_C can be extended to a doubly stochastic measure μ_C, a Borel probability measure on $\mathscr{B}(\mathbb{I}^2)$ satisfying $\mu_C(B \times \mathbb{I}) = \lambda(B) = \mu_C(\mathbb{I} \times B)$ for all $B \in \mathscr{B}(\mathbb{I})$ where λ is Lebesgue measure on $\mathbb{I}$. Conversely, for any doubly stochastic measure μ, we define $C_\mu(u,v) := \mu([0,u] \times [0,v])$, which can be easily shown to be a copula. Hence, there is a 1-1 correspondence between copulas and doubly stochastic measures.

- By the 2-increasing property above, we can show that for any $(u,v) \in \mathbb{I}^2$, $V_C(\{(u,v)\}) = 0$. Moreover, for any $u, u_1, u_2, v, v_1, v_2 \in \mathbb{I}$ with $u_1 \leq u_2$ and $v_1 \leq v_2$, $V_C(\{u\} \times [v_1, v_2]) = 0 = V_C([u_1, u_2] \times \{v\})$. Hence C-volume is invariant under replacing a closed rectangle with the open rectangle with the same vertices, and vice versa.

- The *support* of a copula C is the smallest closed set containing the whole mass distribution of copula which is defined as follows:

$$\operatorname{supp}(C) = \bigcap \left\{ R \subseteq \mathbb{I}^2 : R \text{ is closed and } \mu_C(R) = 1 \right\}.$$

Example 2.1. We consider examples of basic copulas, their doubly stochastic measures, as well as their supports.

1. $\Pi(u,v) := uv$. This copula is called the *independence copula.* Since for any rectangle $R = [a,b] \times [c,d] \subseteq \mathbb{I}^2$, $V_\Pi(R) = (b-a)(d-c)$, we have for any $E \in \mathscr{B}(\mathbb{I}^2)$, $\mu_\Pi(E)$ is equal to the area of E which implies that $\operatorname{supp}(\Pi) = \mathbb{I}^2$.

2. $M(u,v) = \min\{u,v\}$. For any rectangle $R = [a,b] \times [c,d]$, we consider 3 cases as shown in Figure 2.1.
   Hence $V_M(R) = \lambda([a,b] \cap [c,d])$ which implies that for any $E \in \mathscr{B}(\mathbb{I}^2)$, $\mu_M(E) = \lambda(\pi_1(E) \cap \pi_2(E))$ where $\pi_1(E)$ and $\pi_2(E)$ are projections of E into x-axis and y-axis, respectively. Furthermore, $\operatorname{supp}(M) = \{(x,x) : x \in \mathbb{I}\}$.

3. $W(u,v) = \max\{u+v-1, 0\}$. For any rectangle $R = [a,b] \times [c,d]$, we denote $c^* = 1-c$, $d^* = 1-d$ and consider 3 cases as shown in Figure 2.2.
   Hence $V_W(R) = \lambda([a,b] \cap [d^*, c^*])$ which implies that for any $E \in \mathscr{B}(\mathbb{I}^2)$, $\mu_M(E) = \lambda(\pi_1(E) \cap (1 - \pi_2(E)))$ where $1 - A = \{1 - a : a \in A\}$ for any set A. Furthermore, $\operatorname{supp}(W) = \{(x, 1-x) : x \in \mathbb{I}\}$.

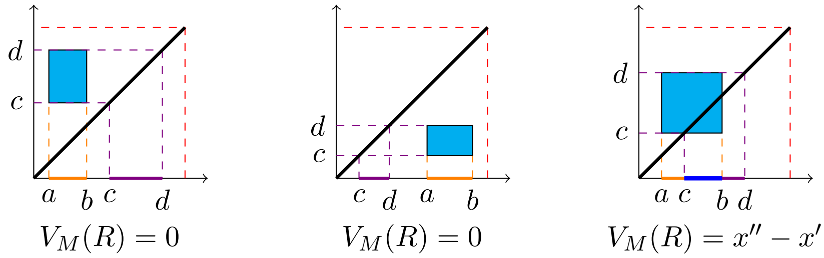

Figure 2.1: M-volume of rectangles in different positions where $x' = \max\{a, c\}$ and $x'' = \min\{b, d\}$

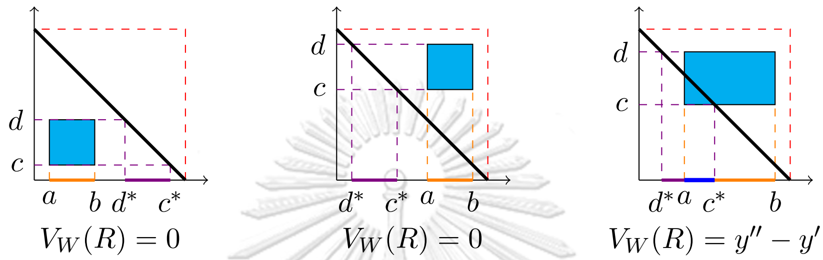

Figure 2.2: W-volume of rectangles in different positions where $y' = \max\{a, d^*\}$ and $y'' = \min\{b, c^*\}$

Next, we classify types of copulas by the Lebesgue decomposition theorem with respect to Lebesgue measure as follows.

Theorem 2.2 ([9]). *Let C be a copula and λ_2 be 2-dimensional Lebesgue measure. Then we can write $C = A_C + S_C$ where $A_C \ll \lambda_2$ and $S_C \perp \lambda_2$ in the sense that A_C and S_C induce measures μ_A and μ_S on $\mathscr{B}(\mathbb{I}^2)$ with $\mu_A \ll \lambda_2$ and $\mu_S \perp \lambda_2$. Moreover,*

$$A_C(u, v) = \int_0^u \int_0^v \frac{\partial^2 C}{\partial x \partial y}(x, y)\mathrm{d}y\mathrm{d}x \quad \text{and} \quad S_C(u, v) = C(u, v) - A_C(u, v)$$

for all $(u, v) \in \mathbb{I}^2$. We call A_C and S_C the absolutely continuous part and the singular part of C, respectively. In particular, if $C = A_C$, we call C an absolutely continuous copula and if $C = S_C$, it is a singular copula.

Before introducing Archimedean copulas, we show the following proposition that gives a relationship between C-volume of a rectangle in $\mathbb{I}^2$ and its area.

Proposition 2.3 ([9]). *Let $R = [u_1, u_2] \times [v_1, v_2]$ be a rectangle in $\mathbb{I}^2$. If $V_C(R) = \theta$ for some copula C, then $A(R)$, the area of R, satisfies $\theta^2 \leq A(R) \leq \left(\frac{1+\theta}{2}\right)^2$.*

Proof. Recall that $\theta = V_C(R) = C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) + C(u_1, v_1)$.
To show that $A(R) \geq \theta^2$, since $V_C([u_1, u_2] \times [0, v_1])$ and $V_C([u_1, u_2] \times [v_2, 1])$ are nonnegative, we have $C(u_2, v_1) - C(u_1, v_1) \geq 0$ and $u_2 - u_1 - C(u_2, v_2) + C(u_1, v_2) \geq 0$. So $u_2 - u_1 \geq C(u_2, v_2) - C(u_1, v_2) = \theta + C(u_2, v_1) - C(u_1, v_1) \geq \theta$. Similarly, $v_2 - v_1 \geq \theta$ by considering $V_C([0, u_1] \times [v_1, v_2])$ and $V_C([u_2, 1] \times [v_1, v_2])$. Hence $A(R) = (u_2 - u_1)(v_2 - v_1) \geq \theta^2$.
To show that $A(R) \leq \left(\frac{1+\theta}{2}\right)^2$, since $V_C([0, u_1] \times [0, v_1])$, $V_C([0, u_1] \times [v_2, 1])$, $V_C([u_2, 1] \times [0, v_1])$ and $V_C([u_2, 1] \times [v_2, 1])$ are nonnegative, we have

$$C(u_1, v_1) \geq 0; \quad u_1 - C(u_1, v_2) \geq 0; \quad v_1 - C(u_2, v_1) \geq 0; \quad 1 - u_2 - v_2 + C(u_2, v_2) \geq 0,$$

and so by AM-GM inequality, $1+\theta \geq (u_2 - u_1) + (v_2 - v_1) \geq 2\sqrt{(u_2 - u_1)(v_2 - v_1)}$.
Hence $\left(\frac{1+\theta}{2}\right)^2 \geq A(R)$ as desired. □

2.1.1 Archimedean copulas

Let $\varphi : \mathbb{I} \to [0, \infty]$ be a continuous and strictly decreasing function such that $\varphi(1) = 0$ and define $\varphi^{[-1]}$ to be a function from $[0, \infty]$ to $\mathbb{I}$ such that

$$\varphi^{[-1]}(t) := \inf\{x \in [0,1] \mid \varphi(x) \leq t\} = \begin{cases} \varphi^{-1}(t) & \text{if } t \in [0, \varphi(0)]; \\ 0 & \text{if } t > \varphi(0). \end{cases}$$

It can be easily shown [9] that the *pseudo-inverse* $\varphi^{[-1]}$ satisfies the following.

1. $\varphi^{[-1]}$ is continuous, decreasing on $[0, \infty]$ and strictly decreasing on $[0, \varphi(0)]$.
2. $\varphi^{[-1]}(\varphi(t)) = t$ for any $t \in \mathbb{I}$ and $\varphi\left(\varphi^{[-1]}(t)\right) = \min\{t, \varphi(0)\}$ for any $t \in [0, \infty]$.

Now, we define $C_\varphi(u, v) := \varphi^{[-1]}(\varphi(u) + \varphi(v))$ for any $(u, v) \in \mathbb{I}^2$.
By the definition above, we see that C_φ satisfies condition 1 in the definition of copula because for any $u, v \in \mathbb{I}$,

1. $C_\varphi(u, 0) \leq \varphi^{[-1]}(\varphi(0)) = 0$ and similarly, $C_\varphi(0, v) = 0$,
2. $C_\varphi(u, 1) = \varphi^{[-1]}(\varphi(u)) = u$ and similarly, $C_\varphi(1, v) = v$.

By [9], we obtain a necessary and sufficient condition on φ in order for C_φ to be a copula.

Theorem 2.4. *Let $\varphi : \mathbb{I} \to [0, \infty]$ be a continuous and strictly decreasing function such that $\varphi(1) = 0$. Let C_φ be a function given by*

$$C_\varphi(u, v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)) \quad \text{for all } u, v \in \mathbb{I}. \tag{2.1}$$

Then C_φ is a copula if and only if φ is convex.

Definition 2.5. A copula C is called an *Archimedean copula* if $C = C_\varphi$ for some function φ satisfying the condition in Theorem 2.4. The function φ is called a *generator* for C.

Many examples of Archimedean copulas are given in [9].

Note:

- If φ is an Archimedean generator and $c > 0$ is a constant, then $\phi := c\varphi$ is also an Archimedean generator and $C_\phi = C_\varphi$ because for any $x \in [0, \infty]$, $\phi^{[-1]}(x) = \varphi^{[-1]}\left(\frac{x}{c}\right)$, which implies that for any $(u, v) \in \mathbb{I}^2$,

$$\begin{aligned} C_\phi(u, v) = \phi^{[-1]}(\phi(u) + \phi(v)) &= \varphi^{[-1]}\left(\frac{c\varphi(u) + c\varphi(v)}{c}\right) \\ &= \varphi^{[-1]}(\varphi(u) + \varphi(v)) = C_\varphi(u, v). \end{aligned}$$

- From (2.1), if $(u, v) \in \mathbb{I}^2$ is such that $\varphi'(u), \varphi'(v)$ and $\varphi''(C_\varphi(u, v))$ exist, then

$$\frac{\partial^2 C_\varphi}{\partial u \partial v}(u, v) = -\frac{\varphi''(C_\varphi(u, v))\,\varphi'(u)\varphi'(v)}{[\varphi'(C_\varphi(u, v))]^3}. \tag{2.2}$$

  In particular, if φ is twice differentiable, then (2.2) holds.

- If $\varphi(t) = 1-t$, then $\varphi^{[-1]}(t) = \max\{1-t, 0\}$ and $C_\varphi(u, v) = \max\{u+v-1, 0\} = W(u, v)$.

- If $\psi(t) = -\log(t)$, then $\psi^{[-1]}(t) = e^{-t}$ and $C_\psi(u, v) = uv = \Pi(u, v)$.

Hence W and Π are Archimedean copulas.

For copula C, we define the *diagonal section* of C to be $\delta_C(x) := C(x, x)$ for any $x \in \mathbb{I}$. Note that for any copula C, $\delta_C(0) = 0$, $\delta_C(1) = 1$ and by the Fréchet-Hoeffding upper bound, $\delta_C(x) \le x$ for all $x \in (0, 1)$. By the definition of Archimedean copula, it is easy to show the following statement.

Proposition 2.6 ([9]). *Let C be an Archimedean copula. Then for any $u \in (0, 1)$, $\delta_C(u) < u$.*

Since $\delta_M(u) = u$, it is clear from Proposition 2.6 that M is not an Archimedean copula.

The reason that copulas in this class are called "Archimedean" is that they have a property which is similar to the Archimedean property : for any real number a, b with $a > 0$, there is a positive integer n such that $na > b$. The property is stated as follows.

Proposition 2.7 ([9]). *Let C be an Archimedean copula. For any $x \in (0,1)$, define $x_C^1 = x$ and $x_C^{n+1} = C(x, x_C^n)$ for any $n \in \mathbb{N}$. Then for any $u, v \in (0,1)$, there is $n \in \mathbb{N}$ such that $u_C^n < v$.*

2.1.2 Constructing methods of copulas

1. Convex sum [9]

Definition 2.8. Let $\{C_i\}_{i=1}^n$ be a collection of copulas and $\{\alpha_i\}_{i=1}^n$ be real numbers in $(0,1)$ such that $\sum_{i=1}^{n} \alpha_i = 1$. We call $C := \sum_{i=1}^{n} \alpha_i C_i$ a *convex sum* of $\{C_i\}_{i=1}^n$.

It is easy to see that the function C above is a copula because

- for any $u \in \mathbb{I}$, $C(u,0) = \sum_{i=1}^{n} \alpha_i C_i(u,0) = \sum_{i=1}^{n} \alpha_i \cdot 0 = 0$ and, similarly, $C(0,v) = 0$ for any $v \in \mathbb{I}$.
- for any $u \in \mathbb{I}$, $C(u,1) = \sum_{i=1}^{n} \alpha_i C_i(u,1) = u \sum_{i=1}^{n} \alpha_i = u$ and, similarly, $C(1,v) = v$ for any $v \in \mathbb{I}$.
- for any $u_1, u_2, v_1, v_2 \in \mathbb{I}$ such that $u_1 \le u_2$ and $v_1 \le v_2$, $V_C([u_1,u_2] \times [v_1,v_2]) = \sum_{i=1}^{n} \alpha_i V_{C_i}([u_1,u_2] \times [v_1,v_2]) \ge 0$.

An example of a convex sum is shown in Figure 2.3.

Note: it is clear that $\text{supp}(C) = \bigcup_{i=1}^{n} \text{supp}(C_i)$.

2. Ordinal sum [8]

First, we say that two distinct intervals I and J are *non-overlapping* if $I \cap J$ is empty or a singleton set.

Definition 2.9. Let $\{J_i\}_{i \in \Lambda}$, where $J_i = [a_i, b_i]$ with $a_i < b_i$ for all $i \in \Lambda \subseteq \mathbb{N}$, be a family of closed, non-overlapping, non-degenerate sub-intervals on $\mathbb{I}$ and let

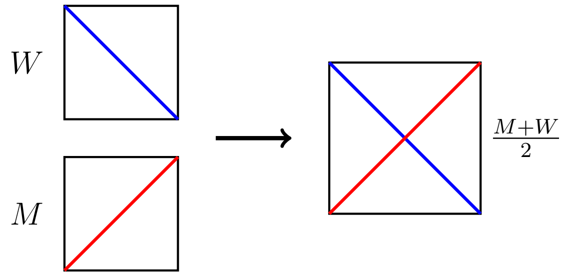

Figure 2.3: The support of a convex sum of W and M

$\{C_i\}_{i\in\Lambda}$ be a collection of copulas with the same index as J_i. Then the *ordinal sum of* $\{C_i\}$ *with respect to* $\{J_i\}$ is a copula C given by

$$C(u,v) = \begin{cases} a_i + (b_i - a_i)\, C_i\left(\dfrac{u-a_i}{b_i-a_i}, \dfrac{v-a_i}{b_i-a_i}\right) & \text{if } (u,v) \in J_i^2; \\ \min\{u,v\} & \text{otherwise.} \end{cases} \tag{2.3}$$

An example of an ordinal sum is shown in Figure 2.4.

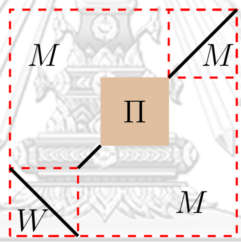

Figure 2.4: The support of an ordinal sum of $\{W, \Pi, M\}$ with respect to $\{J_1 = [0, 0.3]\,, J_2 = [0.4, 0.7]\,, J_3 = [0.7, 1]\}$

The following theorem gives a characterization of ordinal sums.

Theorem 2.10 ([9])**.** *Let C be a copula. Then C is an ordinal sum if and only if there exists $t \in (0,1)$ such that $C(t,t) = t$.*

In [8], the authors define $\mathcal{I}_C$, the *idempotent* of C, as $\mathcal{I}_C := \{x \in \mathbb{I} : C(x,x) = x\}$. By the continuity of C, $\mathcal{I}_C$ is closed. Moreover, they derive a property on the support of ordinal sums in the following statement.

Theorem 2.11 ([8])**.** *Let C be the ordinal sum of $\{C_i\}_{i\in\Lambda}$ with respect to $\{J_i\}_{i\in\Lambda}$ defined in Definition 2.9. Let $\mathcal{I}_C^2 := \{(x,x) : x \in \mathcal{I}_C\}$. Then* $\operatorname{supp}(C) \subseteq \mathcal{I}_C^2 \cup \bigcup_{i\in\Lambda} J_i^2$.

3. Patched copulas

Definition 2.12 ([7])**.** Let $T \in M_{m\times n}(\mathbb{I})$ be a matrix of the form

$$T = \begin{bmatrix} t_{1m} & t_{2m} & \dots & t_{nm} \\ \vdots & \vdots & \iddots & \vdots \\ t_{12} & t_{22} & \dots & t_{n2} \\ t_{11} & t_{21} & \dots & t_{n1} \end{bmatrix}.$$

Then T is called a *transformation matrix* if T satisfies the following properties:

- No row or column of T contains only zero entries; and
- $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{m} t_{ij} = 1.$

With the transformation matrix T, we can define a *partition on the x-axis* $P := (p_0 = 0, p_1, \dots, p_n = 1)$ and a *partition on the y-axis* $Q := (q_0 = 0, q_1, \dots, q_m = 1)$ by

$$p_k = \sum_{i=1}^{k}\sum_{j=1}^{m} t_{ij} \quad \text{for } k = 1, \dots, n \quad \text{and} \quad q_\ell = \sum_{j=1}^{\ell}\sum_{i=1}^{n} t_{ij} \quad \text{for } \ell = 1, \dots, m.$$

Note that P and Q subdivide $\mathbb{I}^2$ into a collection of non-overlapping rectangles $\{R_{ij} := [p_{i-1}, p_i] \times [q_{j-1}, q_j] : i = 1, \dots, n,\ j = 1, \dots, m\}$.

Next, we define patched copulas using transformation matrices which is a special case of the same terminology in [4].

Definition 2.13. Let $T = [t_{ij}] \in M_{m\times n}(\mathbb{I})$ be a transformation matrix with partitions on the x-axis P and the y-axis Q, respectively, and let $\{C_{ij}\}$ be a collection of copulas with the same indices as entries in T. For any $(u, v) \in \mathbb{I}^2$, define

$$C(u, v) = \sum_{i=1}^{n}\sum_{j=1}^{m} t_{ij} C_{ij}\left(F_i(u), G_j(v)\right), \tag{2.4}$$

where $F_i(u) = \min\left\{\dfrac{u - p_{i-1}}{p_i - p_{i-1}}, 1\right\} \mathbb{1}_{(p_{i-1},\infty)}(u)$ is a uniform distribution function on $[p_{i-1}, p_i]$ and $G_j(v) = \min\left\{\dfrac{v - q_{j-1}}{q_j - q_{j-1}}, 1\right\} \mathbb{1}_{(q_{j-1},\infty)}(v)$ is a uniform distribution function on $[q_{j-1}, q_j]$. Then C is called the *patched copula with respect to the transformation matrix T and the collection of copulas $\{C_{ij}\}$* or *patched copula* for short if the transformation matrix and the collection of copulas are known.

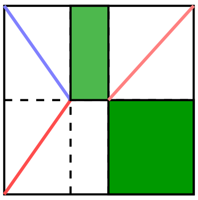

Figure 2.5: The support of the patched copula with respect to T and $\{C_{ij}\}$

An example of a patched copula is shown in Figure 2.5 where
$T = \begin{bmatrix} 0.15 & 0.2 & 0.15 \\ 0.2 & 0 & 0.3 \end{bmatrix}$ and $\{C_{ij}\} = \begin{bmatrix} W & \Pi & M \\ M & W & \Pi \end{bmatrix}$.
Note that

- if $(u, v) \in R_{k\ell}$, then (2.4) can be written as

$$C(u, v) = \sum_{i<k, j<\ell} t_{ij} + u_k \sum_{j<\ell} t_{kj} + v_\ell \sum_{i<k} t_{i\ell} + t_{k\ell} C_{k\ell}(u_k, v_\ell), \tag{2.5}$$

  where $u_k = \dfrac{u - p_{k-1}}{p_k - p_{k-1}}$ and $v_\ell = \dfrac{v - q_{\ell-1}}{q_\ell - q_{\ell-1}}$; and

- an ordinal sum of a finite collection of copulas (with respect to finite sub-intervals on $\mathbb{I}$) is a patched copula as stated in the following proposition.

Proposition 2.14. *Let $\{J_i\}_{i=1}^N$, where $J_i = [a_i, b_i]$ with $a_i < b_i$ for all $i = 1, \ldots, N$, be a family of closed, non-overlapping, non-degenerate sub-intervals on $\mathbb{I}$ and $\{C_i\}_{i=1}^N$ a collection of copulas. If C is an ordinal sum of $\{C_i\}$ with respect to $\{J_i\}$, then C is a patched copula.*

Proof. By reordering if necessary, we assume without loss of generality that $b_i \leq a_{i+1}$ for all $i = 1, \ldots, N-1$.

We define $\{J'_i\}_{i=0}^N$ by $J'_i = \begin{cases} [0, a_1] & \text{if } i = 0; \\ [b_i, a_{i+1}] & \text{if } i = 1, \ldots, N-1; \\ [b_N, 1] & \text{if } i = N. \end{cases}$

Let $S = \{J'_0\} \cup \{J_i, J'_i\}_{i=1}^N$ and define a relation $\lesssim$ on S by $I_1 \lesssim I_2$ if and only if $\min I_1 \leq \min I_2$. It is easy to see that $\lesssim$ is a total ordering on S. Next, we define a collection of closed, non-overlapping, non-degenerate sub-intervals $\{K_\ell\}$ on $\mathbb{I}$ by $K_1 = \min_{\lesssim}\{I \in S : \lambda(I) > 0\}$ and for any $\ell > 1$, $K_\ell = \min_{\lesssim}\{I \in S \setminus \bigcup_{k=1}^{\ell-1} K_k : \lambda(I) > 0\}$ where $\min_{\lesssim}$ is the minimum under the relation $\lesssim$. Note that

- $\bigcup_{\ell} K_\ell = \mathbb{I}$, that is, $\max K_\ell = \min K_{\ell+1}$ for all ℓ; and

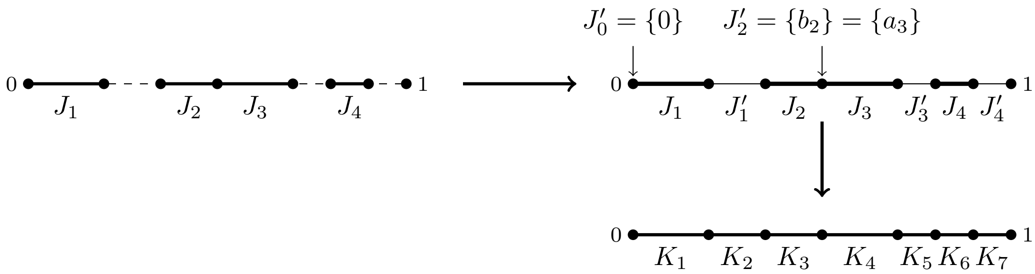

Figure 2.6: Example of defining $\{K_\ell\}_{\ell=1}^7$ from $\{J_i\}_{i=1}^4$ in Proposition 2.14

- for each $i = 1, \dots, N$, there is ℓ such that $J_i = K_\ell$.

Now, let $n := |\{K_\ell\}|$ and define a transformation matrix $T \in M_n(\mathbb{I})$ by

$$T = \begin{bmatrix} 0 & 0 & \dots & \lambda(K_n) \\ \vdots & \vdots & \iddots & \vdots \\ 0 & \lambda(K_2) & \dots & 0 \\ \lambda(K_1) & 0 & \dots & 0 \end{bmatrix}.$$

Also, we define a collection of copulas $\{C_{k\ell}\}_{k,\ell=1}^n$ by

$$C_{k\ell} = \begin{cases} C_i & \text{if } k = \ell \text{ and } K_k = J_i \text{ for some } i = 1, \dots, N; \\ M & \text{otherwise.} \end{cases}$$

Claim. *A patched copula D with respect to T and $\{C_{k\ell}\}_{k,\ell=1}^n$ is the ordinal sum C.*

To see this, let $(u, v) \in \mathbb{I}^2$. We consider 3 cases.

Case 1: $(u, v) \in J_\ell^2 = [a_\ell, b_\ell]^2 = R_{kk}$ for some $\ell = 1, \dots, N$ and $k = 1, \dots, n$. Then by (2.5),

$$\begin{aligned} D(u, v) &= \sum_{i,j<k} t_{ij} + u_k \sum_{j<k} t_{kj} + v_k \sum_{i<k} t_{ik} + t_{kk} C_{kk}(u_k, v_k) \\ &= a_\ell + u_k \cdot 0 + v_k \cdot 0 + \lambda(K_k) C_\ell(u_k, v_k) \\ &= a_\ell + (b_\ell - a_\ell) C_\ell \left(\frac{u - a_\ell}{b_\ell - a_\ell}, \frac{v - a_\ell}{b_\ell - a_\ell} \right) = C(u, v). \end{aligned}$$

Case 2: $(u, v) \in J_\ell'^2 = [b_\ell, a_{\ell+1}]^2 = R_{kk}$ for some $\ell = 0, 1, \dots, N$ and $k = 1, \dots, n$ (for convenience, $b_0 = 0$ and $a_{N+1} = 1$). Then as in the previous case,

$$D(u, v) = \sum_{i,j<k} t_{ij} + u_k \sum_{j<k} t_{kj} + v_k \sum_{i<k} t_{ik} + t_{kk} C_{kk}(u_k, v_k)$$

$$
\begin{aligned}
&= b_\ell + u_k \cdot 0 + v_k \cdot 0 + \lambda(K_k) M(u_k, v_k) \\
&= b_\ell + (a_{\ell+1} - b_\ell)\, M\left(\frac{u - b_\ell}{a_{\ell+1} - b_\ell}, \frac{v - b_\ell}{a_{\ell+1} - b_\ell}\right) = M(u,v) = C(u,v).
\end{aligned}
$$

Case 3: $(u,v) \in R_{k\ell}$ for some $k, \ell = 1, \ldots, n$ with $k \neq \ell$. Without loss of generality, we assume that $k < \ell$. Then by (2.5),

$$
\begin{aligned}
D(u,v) &= \sum_{i<k, j<\ell} t_{ij} + u_k \sum_{j<\ell} t_{kj} + v_\ell \sum_{i<k} t_{i\ell} + t_{k\ell} C_{k\ell}(u_k, v_\ell) \\
&= \sum_{i=1}^{k-1} \lambda(K_i) + u_k \lambda(K_k) + v_\ell \cdot 0 + 0 \cdot M(u_k, v_\ell) \\
&= \max K_{k-1} + \frac{u - \min K_k}{\lambda(K_k)} \cdot \lambda(K_k) = u = M(u,v) = C(u,v).
\end{aligned}
$$

Similarly, if $k > \ell$, then $D(u,v) = v = M(u,v) = C(u,v)$.
Therefore, $C = D$, i.e., C is a patched copula with respect to the transformation matrix T and the collection of copulas $\{C_{k\ell}\}_{k,\ell=1}^{n}$. □

2.2 Some theorems in real analysis

In this section, we collect some basic knowledge about convex functions, symmetric derivatives and strong differentiability, which will be used in this thesis. See [2, 5, 6, 12] for more details.

2.2.1 Convex functions

We first recall some basic properties of convex functions on a subset of $\mathbb{R}$.

Definition 2.15 ([2])**.** Let I be an interval in $\mathbb{R}$. A function $f\colon I \to \mathbb{R}$ is *convex* if for any $x, y \in I$ and $t \in [0,1]$, $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$.
Equivalently, f is convex if for any $x < y < z$ in I, $\dfrac{f(y) - f(x)}{y - x} \leq \dfrac{f(z) - f(y)}{z - y}$.

Proposition 2.16 ([6])**.** *Let I be an interval in $\mathbb{R}$ with $\operatorname{Int}(I) = (a,b)$ and $f\colon I \to \mathbb{R}$ be a convex function. Then*

(1) for any $x \in (a,b)$, the left derivative $f'(x^-) := \lim_{h \to 0^-} \dfrac{f(x+h) - f(x)}{h}$ and the right derivative $f'(x^+) := \lim_{h \to 0^+} \dfrac{f(x+h) - f(x)}{h}$ exist. Moreover, $f'(a^+)$ and $f'(b^-)$ exist.

(2) for any $x, y \in I$ such that $x < y$, $f'(x^+) \leq f'(y^-) \leq f'(y^+)$.

(3) $D := \{x \in (a,b) : f'(x)$ *exists*$\}$ *has a countable complement. That is, any convex function on* I *is differentiable at all but countably many points in* I.

(4) By (2), we see that f' *is increasing on* D. *Hence by Lebesgue Differentiation Theorem,* $D_2 := \{x \in (a,b) : f''(x)$ *exists*$\}$ *has Lebesgue measure* $\lambda(D_2) = b-a$ *and for any* $x \in D_2$, $f''(x) \geq 0$.

Now, if f is a convex differentiable function on $(a,b) \subseteq \mathbb{R}$. From Darboux's Theorem [2]: the derivative of a differentiable function on an interval has the intermediate value property, we can show the following statement.

Corollary 2.17. *Let* f *be a convex and differentiable function on* (a,b). *Then* f' *is continuous on* (a,b).

Proof. Let $x \in (a,b)$. Then by Proposition 2.16(2), $f'(a^+) \leq f'(x) \leq f'(b^-)$. Note that if $f'(a^+) = f'(b^-)$, then f' is a constant function on (a,b). That is, f' is continuous at x. With the same reason, we may assume without loss of generality that $f'(a^+) < f'(x) < f'(b^-)$. Now, let $c, d \in (f'(a^+), f'(b^-))$ be such that $c < f'(x) < d$. Then by Darboux's Theorem, there are $y \in (a,x)$ and $z \in (x,b)$ such that $f'(y) = c$ and $f'(z) = d$. This statement implies the continuity of f' at x by Proposition 2.16(2). □

2.2.2 Symmetric derivative and strong differentiability

Definition 2.18 (Symmetric derivative [12])**.** Let f be a real-valued function on an open interval D and $x \in D$. We define the first and second symmetric derivatives of f by the expressions

$$\operatorname{SD} f(x) := \lim_{t \to 0} \frac{f(x+t) - f(x-t)}{2t} \tag{2.6}$$

and

$$\operatorname{SD}_2 f(x) := \lim_{t \to 0} \frac{f(x+t) - 2f(x) + f(x-t)}{t^2}. \tag{2.7}$$

Next, we give some statements about symmetric derivatives.

Proposition 2.19 ([12])**.** *Let* f *be a real-valued function on an open interval* D *and* $x \in D$.

1. *If* $f'(x)$ *exists, then so does* $\operatorname{SD} f(x)$ *and* $\operatorname{SD} f(x) = f'(x)$.

2. *If* $f'(x^+)$ *and* $f'(x^-)$ *exist, then so does* $\operatorname{SD} f(x)$ *and* $\operatorname{SD} f(x) = \dfrac{f'(x^+) + f'(x^-)}{2}$.

3. *If $f''(x)$ exists, then so does* $\mathrm{SD}_2\, f(x)$ *and* $\mathrm{SD}_2\, f(x) = f''(x)$.

4. *If $f'(x^+)$, $f'(x^-)$ exist and* $\mathrm{SD}_2\, f(x) \in \mathbb{R}$, *then $f'(x)$ exists.*

5. *If f is convex and* $\mathrm{SD}_2\, f(x)$ *exists, then* $\mathrm{SD}_2\, f(x) \geq 0$.

However, if $\mathrm{SD}\, f(x)$ exists, it is not necessary that $f'(x)$ exists. Likewise, the existence of $\mathrm{SD}_2\, f(x)$ does not guarantee the existence of $f''(x)$. For example,

$$f(x) = |x| \text{ and } g(x) = \operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0; \\ 0 & \text{if } x = 0; \\ -1 & \text{if } x < 0. \end{cases}$$

We see that $\mathrm{SD}\, f(0) = \lim\limits_{t \to 0} \dfrac{|t| - |-t|}{2t} = 0$ but $f'(0)$ does not exist because $f'(0^+) = 1$ and $f'(0^-) = -1$.
Similarly, $\mathrm{SD}_2\, g(0) = \lim\limits_{t \to 0} \dfrac{\operatorname{sgn}(t) - 2\operatorname{sgn}(0) + \operatorname{sgn}(-t)}{t^2} = 0$ but $g'(0)$ does not exist ($\because$ g is not continuous at 0). So $g''(0)$ does not exist.

Definition 2.20 (strong differentiability [5])**.** Let f be a real-valued function on an open interval D. For $a \in D$, we say that f is *strongly differentiable* at a if the limit $\lim\limits_{\substack{(x,y)\to(a,a) \\ x \neq y}} \dfrac{f(x) - f(y)}{x - y}$ exists and is finite. We denote the limit by $f^*(a)$ and call it the *strong derivative* of f at a.

The following results from [5] will be used in the proof of Theorem 4.9(D).

Theorem 2.21. *Let f be a real-valued function on an open interval D and $a \in D$.*

1. *If $f^*(a)$ exists, then so does $f'(a)$ and $f^*(a) = f'(a)$.*

2. *If f' is continuous at a, then f is strongly differentiable at a.*

2.2.3 Derivative of measures

Let μ be a complex Borel measure on $\mathbb{R}^2$. We define the *symmetric derivative* of μ at x to be $(D\mu)(x) := \lim\limits_{r \to 0^+} \dfrac{\mu\,(B(x,r))}{\lambda_2\,(B(x,r))}$. In this subsection, we list a few theorems about Lebesgue points and symmetric derivatives of singular measures from [10] that are used to show some statements about association between pointwise dimensions and types of copulas as follows.

Theorem 2.22 ([10])**.** *If $f \in L^1(\mathbb{R}^2)$, then for λ_2-almost all $x \in \mathbb{R}^2$,*

$$\lim_{r \to 0^+} \frac{1}{\lambda_2\,(B(x,r))} \int_{B(x,r)} |f(y) - f(x)|\, \mathrm{d}\lambda(y) = 0,$$

where $B(x,r) = \{y \in \mathbb{R}^2 : ||y - x|| < r\}$. *This implies that for any complex measure* μ *such that* $\mu \ll \lambda_2$ *with Radon-Nikodym derivative* $f \in L^1(\mathbb{R}^2)$ *and for* λ_2*-almost all* $x \in \mathbb{R}^2$, $(D\mu)(x) = f(x)$.

Theorem 2.23 ([10])**.** *If* μ *is a positive Borel measure on* $\mathbb{R}^2$ *and* $\mu \perp \lambda_2$, *then for* λ_2*-almost all* $x \in \mathbb{R}^2$, $(D\mu)(x) = 0$ *and for* μ*-almost all* $x \in \mathbb{R}^2$, $(D\mu)(x) = \infty$.

2.3 Regular variation

Regular variation [3] is a subject that studies functions whose behavior at some interesting points, espectially at infinity, is similar to the behavior of a power function at those points. In this topic, we start by giving the definition of a regularly varying function at infinity and at the right of 0.

Definition 2.24 ([3])**.** A positive measurable function f defined on $[M, \infty)$ such that there exists a real number ρ satisfying $\lim\limits_{x\to\infty} \dfrac{f(\lambda x)}{f(x)} = \lambda^\rho$ for any $\lambda > 0$ is called a *regularly varying function* of *index* ρ; we write $f \in RV_\rho$.
In particular, if $\rho = 0$, we call f a *slowly varying function*.

Definition 2.25 ([3])**.** A positive measurable function f defined on $(0, N]$ such that there exists a real number ρ satisfying $\lim\limits_{x\to 0^+} \dfrac{f(\lambda x)}{f(x)} = \lambda^\rho$ for any $\lambda > 0$ is called a *regularly varying function* at the right of 0 of *index* ρ; we write $f \in RV_\rho^0$.
In particular, if $\rho = 0$, we call f a *slowly varying function at the right of* 0.

Note that $f \in RV_\rho^0$ if and only if $g\colon x \mapsto f(1/x)$ is in $RV_{-\rho}$ because for any $\lambda > 0$,

$$\begin{aligned} f \in RV_\rho^0 &\iff \lim_{x\to 0^+} \frac{f(\lambda x)}{f(x)} = \lambda^\rho \iff \lim_{x\to 0^+} \frac{g(1/\lambda x)}{g(1/x)} = \left(\frac{1}{\lambda}\right)^{-\rho} \\ &\iff \lim_{u\to\infty} \frac{g(\alpha u)}{g(u)} = \alpha^{-\rho} \quad (\text{use } u = 1/x \text{ and } \alpha = 1/\lambda) \iff g \in RV_{-\rho}. \end{aligned}$$

Example 2.26.

1. $f(x) = \arcsin(x)$ for $x \in [0,1]$ is in RV_1^0 because for any $\lambda > 0$,
$$\lim_{x\to 0^+} \frac{f(\lambda x)}{f(x)} = \lim_{x\to 0^+} \frac{\arcsin(\lambda x)}{\arcsin(x)} = \lim_{x\to 0^+} \frac{\lambda}{\sqrt{1-(\lambda x)^2}} \sqrt{1-x^2} = \lambda.$$

2. $g(x) = -\log(x)$ for $x \in (0,1]$ is in RV_0^0 because for any $\lambda > 0$,
$$\lim_{x\to 0^+} \frac{g(\lambda x)}{g(x)} = \lim_{x\to 0^+} \frac{-\log(\lambda x)}{-\log(x)} = \lim_{x\to 0^+} \frac{\log(\lambda) + \log(x)}{\log(x)} = 1 + \lim_{x\to 0^+} \frac{\log(\lambda)}{\log(x)} = 1.$$

3. $h(x) = \dfrac{1}{x - x^2}$ for $x \in (0,1)$ is in RV^0_{-1} because for any $\lambda > 0$,

$$\lim_{x\to 0^+} \frac{h(\lambda x)}{h(x)} = \lim_{x\to 0^+} \frac{\frac{1}{\lambda x - (\lambda x)^2}}{\frac{1}{x - x^2}} = \lim_{x\to 0^+} \frac{x(1-x)}{\lambda x(1-\lambda x)} = \frac{1}{\lambda} \lim_{x\to 0^+} \frac{1-x}{1-\lambda x} = \frac{1}{\lambda}.$$

From the definition, it is easy to see that $f \in RV_\rho$ if and only if $f(x) = x^\rho \ell(x)$ for some slowly varying function ℓ and the result holds for $f \in RV^0_\rho$ in a similar way. Hence we see that slowly varying functions are important in this subject. Furthermore, they are related to the most important theorem in this subject, *Uniform Convergence Theorem (UCT)*, which is stated as follows

Theorem 2.27 (Uniform Convergence Theorem [3]). *If ℓ is a slowly varying function, then* $\lim_{x\to\infty} \dfrac{\ell(\lambda x)}{\ell(x)} = 1$ *uniformly on each compact λ-set in* $(0, \infty)$.

Surely, there is a Uniform Convergence Theorem for slowly varying function at 0. UCT is used to prove the following theorem which characterizes the slowly varying functions.

Theorem 2.28 (Representation Theorem [3]). *A positive measurable function ℓ is slowly varying if and only if it can be written in the form*

$$\ell(x) = C(x) \exp\left\{ \int_a^x \frac{\varepsilon(t)}{t} \mathrm{d}t \right\} \quad (\textit{for } x \geq a)$$

for some $a > 0$, where $C(\cdot)$ is positive and measurable, and $C(x) \to c \in (0,\infty)$ and $\varepsilon(x) \to 0$ as $x \to \infty$.

See [3] for the proof of these two theorems. From Representation Theorem, we prove *Potter's Theorem* which gives bounds for the quotient of the values of a slowly varying function at different points.

Theorem 2.29 (Potter's Theorem [3]).

1. *If f is regularly varying of index ρ then for any chosen constant $A > 1$ and $\delta > 0$, there is $X = X(A, \delta)$ such that for any $x, y \geq X$,*

$$A^{-1} \left(\frac{y}{x}\right)^\rho \left(\max\left\{\frac{x}{y}, \frac{y}{x}\right\}\right)^{-\delta} \leq \frac{f(y)}{f(x)} \leq A \left(\frac{y}{x}\right)^\rho \left(\max\left\{\frac{x}{y}, \frac{y}{x}\right\}\right)^\delta. \quad (2.8)$$

2. *If f is regularly varying at 0 of index ρ then for any chosen constant $A > 1$ and $\delta > 0$, there is $Y = Y(A, \delta)$ such that for any $0 < x, y \leq Y$, (2.8) holds.*

Proof. For the first part, it suffices to show the case where f is slowly varying. To see this, assume that f is a slowly varying function. Then by Representation Theorem, $f(x) = C(x)\exp\left\{\int_a^x \frac{\eta(t)}{t}\mathrm{d}t\right\}$ for some $a > 0$, positive measurable function C with $C(x) \to c \in (0,\infty)$ and $\eta(x) \to 0$ as $x \to \infty$. Let $A > 1$ and $\delta > 0$. Choose $\varepsilon = \min\left\{\frac{A-1}{A+1}, \delta\right\} \in (0,1)$. Then by the existence of the limits of C and η as $x \to \infty$, there exists $x_0 > 0$ such that for any $x \geq x_0$, $c(1-\varepsilon) \leq C(x) \leq c(1+\varepsilon)$ and $|\eta(x)| < \varepsilon$. Let $X = X(A,\delta) = \max\{x_0, a\}$. Then for $y \geq x \geq X$,

$$\frac{f(y)}{f(x)} = \frac{C(y)}{C(x)}\exp\left\{\int_x^y \frac{\eta(t)}{t}\mathrm{d}t\right\} \leq \frac{1+\varepsilon}{1-\varepsilon}\exp\left\{\varepsilon\int_x^y \frac{\mathrm{d}t}{t}\right\}$$
$$\leq A\exp\{\varepsilon\log(y/x)\} = A(y/x)^{\varepsilon} \leq A(y/x)^{\delta}.$$

Also, for $x \geq y \geq X$,

$$\frac{f(y)}{f(x)} \leq \frac{1+\varepsilon}{1-\varepsilon}\exp\left\{-\varepsilon\int_x^y \frac{\mathrm{d}t}{t}\right\} \leq A\exp\{-\varepsilon\log(y/x)\} = A(x/y)^{\varepsilon} \leq A(x/y)^{\delta}.$$

The lower bound can be shown in a similar way.
For the second part, assume that $f \in RV_\rho^0$. Define $f_1\colon x \mapsto f(1/x)$. Then $f_1 \in RV_{-\rho}$. By the previous part, for each $A > 1$ and $\delta > 0$, there is $X = X(A,\delta)$ such that for $x, y \geq X$,

$$A^{-1}\left(\frac{y}{x}\right)^{-\rho}\left(\max\left\{\frac{x}{y}, \frac{y}{x}\right\}\right)^{-\delta} \leq \frac{f_1(y)}{f_1(x)} \leq A\left(\frac{y}{x}\right)^{-\rho}\left(\max\left\{\frac{x}{y}, \frac{y}{x}\right\}\right)^{\delta}.$$

Choose $Y = 1/X$. Then for $0 < x, y \leq Y$, we have $1/x, 1/y \geq X$, i.e.,

$$\frac{f(y)}{f(x)} = \frac{f_1(1/y)}{f_1(1/x)} \leq A\left(\frac{1/y}{1/x}\right)^{-\rho}\left(\max\left\{\frac{1/x}{1/y}, \frac{1/y}{1/x}\right\}\right)^{\delta} = A\left(\frac{y}{x}\right)^{\rho}\left(\max\left\{\frac{x}{y}, \frac{y}{x}\right\}\right)^{\delta}$$

and the lower bound of $\frac{f(y)}{f(x)}$ can be obtained similarly. □

Now, we list some properties of regularly varying functions and slowly varying functions at infinity and we give corresponding statements for regularly varying functions and slowly varying functions at the right of 0 used in this thesis.

Proposition 2.30 ([3]).

(i) If $\ell \in RV_0$, then for any $\alpha > 0$, $\lim_{x\to\infty} x^{\alpha}\ell(x) = \infty$ and $\lim_{x\to\infty} x^{-\alpha}\ell(x) = 0$.

(ii) If $\ell \in RV_0$, then $\lim_{x\to\infty} \frac{\log(\ell(x))}{\log(x)} = 0$.

(iii) Let $\alpha \in \mathbb{R}$. *If* $f \in RV_\rho$ *and* $g(x) = (f(x))^\alpha$, *then* $g \in RV_{\alpha\rho}$.

(iv) If $f \in RV_\rho$, $g \in RV_\sigma$, *then* $f + g \in RV_{\max\{\rho,\sigma\}}$ *and* $f \cdot g \in RV_{\rho+\sigma}$. *Moreover, if we assume in addition that* $\lim_{x\to\infty} g(x) = \infty$, *then* $f \circ g \in RV_{\rho\sigma}$.

Proof [11]. For (i) and (ii), by Theorem 2.28, we write

$$\ell(x) = C(x) \exp\left\{\int_a^x \frac{\varepsilon(t)}{t} \mathrm{d}t\right\} \tag{2.9}$$

for some $a > 0$, measurable functions C, ε such that C is positive, $C(x) \to c > 0$ and $\varepsilon(x) \to 0$ as $x \to \infty$.

(i) Assume that $\ell \in RV_0$ and let $\alpha > 0$. We give a proof for $\lim_{x\to\infty} x^\alpha \ell(x) = \infty$ as another case can be handled similarly. By (2.9), we see that

$$x^\alpha \ell(x) = C(x) \exp\left\{\alpha \log(x) + \int_a^x \frac{\varepsilon(t)}{t} \mathrm{d}t\right\}.$$

Since $C(x) \to c > 0$ as $x \to \infty$, C is eventually bounded away from 0. Hence it suffices to show that $\lim_{x\to\infty} \left(\alpha \log(x) + \int_a^x \frac{\varepsilon(t)}{t} \mathrm{d}t\right) = \infty$.
Let $M > 0$. Since $\varepsilon(x) \to 0$ as $x \to \infty$, there is $M_1 > 0$ such that for any $x \geq M_1$, $|\varepsilon(x)| < \frac{\alpha}{2}$. Hence

$$\begin{aligned}
\alpha \log(x) + \int_a^x \frac{\varepsilon(t)}{t} \mathrm{d}t &= \alpha \log(x) + \int_a^{M_1} \frac{\varepsilon(t)}{t} \mathrm{d}t + \int_{M_1}^x \frac{\varepsilon(t)}{t} \mathrm{d}t \\
&> \alpha \log(x) + \int_a^{M_1} \frac{\varepsilon(t)}{t} \mathrm{d}t - \frac{\alpha}{2} \int_{M_1}^x \frac{1}{t} \mathrm{d}t \\
&= \frac{\alpha}{2} \log(x) + \int_a^{M_1} \frac{\varepsilon(t)}{t} \mathrm{d}t + \frac{\alpha}{2} \log(M_1).
\end{aligned}$$

Since $C' := \int_a^{M_1} \frac{\varepsilon(t)}{t} \mathrm{d}t + \frac{\alpha}{2} \log(M_1)$ is a constant and $\lim_{x\to\infty} \log(x) = \infty$, there is $M_2 > M_1$ such that for any $x \geq M_2$, $\log(x) > \frac{2(M - C')}{\alpha}$. Thus, for any $x \geq M_2$, $\alpha \log(x) + \int_a^x \frac{\varepsilon(t)}{t} \mathrm{d}t > \frac{\alpha}{2} \cdot \frac{2(M - C')}{\alpha} + C' = M$ which implies that $\lim_{x\to\infty} x^\alpha \ell(x) = \infty$.

(ii) Assume that $\ell \in RV_0$. Then by (2.9), $\log(\ell(x)) = \log(C(x)) + \int_a^x \frac{\varepsilon(t)}{t} \mathrm{d}t$. Now, let $\delta > 0$. Since $\varepsilon(x) \to 0$ as $x \to \infty$, there is $M_1 > 0$ such that for any $x \geq M_1$, $|\varepsilon(x)| < \frac{\delta}{2}$. Hence for any $x \geq M_1$,

$$\frac{\log(\ell(x))}{\log(x)} < \frac{\log(C(x))}{\log(x)} + \frac{\int_a^{M_1} \frac{\varepsilon(t)}{t} \mathrm{d}t}{\log(x)} + \frac{\delta}{2} \frac{\int_{M_1}^x \frac{1}{t} \mathrm{d}t}{\log(x)}$$

$$= \frac{\log(C(x)) - \frac{\delta}{2}\log(M_1) + \int_a^{M_1} \frac{\varepsilon(t)}{t}\mathrm{d}t}{\log(x)} + \frac{\delta}{2}.$$

Since $C''(x) := \log(C(x)) - \frac{\delta}{2}\log(M_1) + \int_a^{M_1} \frac{\varepsilon(t)}{t}\mathrm{d}t$ is eventually bounded and $\lim_{x\to\infty} \log(x) = \infty$, there is $M_2 > M_1$ such that for any $x \geq M_2$, $\frac{C''(x)}{\log(x)} < \frac{\delta}{2}$. Hence $\frac{\log(\ell(x))}{\log(x)} < \delta$. Similarly, we can show that there is $M_3 > 0$ such that for any $x \geq M_3$, $\frac{\log(\ell(x))}{\log(x)} > -\delta$. That is, $\lim_{x\to\infty} \frac{\log(\ell(x))}{\log(x)} = 0$.

(iii) Assume that $f \in RV_\rho$ and $g(x) = (f(x))^\alpha$. Then for any $\lambda > 0$, $\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda^\rho$. Thus, $\lim_{x\to\infty} \frac{g(\lambda x)}{g(x)} = \left(\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)}\right)^\alpha = (\lambda^\rho)^\alpha = \lambda^{\alpha\rho}$, i.e., $g \in RV_{\alpha\rho}$.

(iv) Assume that $f \in RV_\rho$, $g \in RV_\sigma$ and without loss of generality, $\rho \geq \sigma$. Then $f(x) = x^\rho \ell_1(x)$ and $g(x) = x^\sigma \ell_2(x)$ for some slowly varying functions ℓ_1, ℓ_2. We divides the proof into 3 parts.

Part 1: $f \cdot g$. We see that for any $\lambda > 0$,

$$\lim_{x\to\infty} \frac{(f\cdot g)(\lambda x)}{(f\cdot g)(x)} = \lim_{x\to\infty} \frac{f(\lambda x)g(\lambda x)}{f(x)g(x)} = \lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} \lim_{x\to\infty} \frac{g(\lambda x)}{g(x)} = \lambda^\rho \lambda^\sigma = \lambda^{\rho+\sigma}.$$

Hence $f \cdot g \in RV_{\rho+\sigma}$.

Part 2: $f + g$. We see that if $\rho = \sigma$, then for any $\lambda > 0$,

$$\frac{\ell_1(\lambda x) + \ell_2(\lambda x)}{\ell_1(x) + \ell_2(x)} = \frac{\ell_1(\lambda x)}{\ell_1(x)} \frac{\ell_1(x)}{\ell_1(x) + \ell_2(x)} + \frac{\ell_2(\lambda x)}{\ell_2(x)} \frac{\ell_2(x)}{\ell_1(x) + \ell_2(x)}.$$

Now, let $\varepsilon > 0$. Since $\ell_1, \ell_2 \in RV_0$, there is $M > 0$ such that for any $x \geq M$, $\frac{\ell_1(\lambda x)}{\ell_1(x)} \in (1-\varepsilon, 1+\varepsilon)$ and $\frac{\ell_2(\lambda x)}{\ell_2(x)} \in (1-\varepsilon, 1+\varepsilon)$. Hence for any $x \geq M$, $\frac{\ell_1(\lambda x) + \ell_2(\lambda x)}{\ell_1(x) + \ell_2(x)} \in (1-\varepsilon, 1+\varepsilon)$. This implies that $\ell_1 + \ell_2 \in RV_0$ and $f(x) + g(x) = x^\rho(\ell_1(x) + \ell_2(x)) \in RV_\rho$.

If $\rho > \sigma$, then $f(x) + g(x) = x^\rho \ell_1(x) + x^\sigma \ell_2(x) = x^\rho(\ell_1(x) + x^{\sigma-\rho}\ell_2(x))$. Since

$$\lim_{x\to\infty} \frac{\ell_1(\lambda x) + (\lambda x)^{\sigma-\rho}\ell_2(\lambda x)}{\ell_1(x) + x^{\sigma-\rho}\ell_2(x)} = \lim_{x\to\infty} \frac{\ell_1(\lambda x)}{\ell_1(x)} \lim_{x\to\infty} \frac{1 + (\lambda x)^{\sigma-\rho}\frac{\ell_2(\lambda x)}{\ell_1(\lambda x)}}{1 + x^{\sigma-\rho}\frac{\ell_2(x)}{\ell_1(x)}}$$

$$= \lim_{x\to\infty} \frac{\ell_1(\lambda x)}{\ell_1(x)} = 1$$

for any $\lambda > 0$, where the second equality follows from (i),(iii) and (v) in the product part, we have $\ell_1(x) + x^{\sigma-\rho}\ell_2(x) \in RV_0$ and $f + g \in RV_\rho$.

Part 3: $f \circ g$. We suppose in addition that $\lim_{x\to\infty} g(x) = \infty$. Then from (i), $\sigma \geq 0$. Since

$$f(g(x)) = g(x)^{\rho}\ell_1(g(x)) = (x^{\sigma}\ell_2(x))^{\rho}\,\ell_1(x^{\sigma}\ell_2(x)) = x^{\rho\sigma}(\ell_2(x))^{\rho}\,\ell_1(x^{\sigma}\ell_2(x)),$$

we see that for any $\lambda > 0$, $\dfrac{\ell_1((\lambda x)^{\sigma}\ell_2(\lambda x))}{\ell_1(x^{\sigma}\ell_2(x))} = \dfrac{\ell_1\left(\left(\lambda^{\sigma}\frac{\ell_2(\lambda x)}{\ell_2(x)}\right)g(x)\right)}{\ell_1(g(x))}$. Now, let $\varepsilon > 0$. Since $\ell_2 \in RV_0$, there is $M_1 > 0$ such that for any $x \geq M_1$, $1-\varepsilon < \dfrac{\ell_2(\lambda x)}{\ell_2(x)} < 1+\varepsilon$. Next, since $K := [\lambda^{\sigma}(1-\varepsilon), \lambda^{\sigma}(1+\varepsilon)]$ is a compact subset of $(0,\infty)$, by UCT, there is $M_2 > 0$ such that for any $t \geq M_2$ and any $k \in K$, $1-\varepsilon < \dfrac{\ell_1(kt)}{\ell_1(t)} < 1+\varepsilon$. Moreover, since $\lim_{x\to\infty} g(x) = \infty$, there is $M_3 > 0$ such that for any $x \geq M_3$, $g(x) > M_2$. Hence for any $x \geq \max\{M_1, M_3\}$, $g(x) > M_2$ and $\lambda^{\sigma}\dfrac{\ell_2(\lambda x)}{\ell_2(x)} \in K$, that is, $\dfrac{\ell_1((\lambda x)^{\sigma}\ell_2(\lambda x))}{\ell_1(x^{\sigma}\ell_2(x))} \in (1-\varepsilon, 1+\varepsilon)$. This implies that $\ell_1(x^{\sigma}\ell_2(x)) \in RV_0$. Therefore, $f \circ g \in RV_{\rho\sigma}$ which follows from (iii) and (v) in the product part. $\square$

Corollary 2.31.

(i) If $\ell \in RV_0^0$, then for any $\alpha > 0$, $\lim_{x\to 0^+} x^{\alpha}\ell(x) = 0$ and $\lim_{x\to 0^+} x^{-\alpha}\ell(x) = \infty$.

(ii) If $\ell \in RV_0^0$, then $\lim_{x\to 0^+} \dfrac{\log(\ell(x))}{\log(x)} = 0$.

(iii) Let $\alpha \in \mathbb{R}$. If $f \in RV_{\rho}^0$ and $g(x) = (f(x))^{\alpha}$, then $g \in RV_{\alpha\rho}^0$.

(iv) If $f \in RV_{\rho}^0$, $g \in RV_{\sigma}^0$, then $f + g \in RV_{\min\{\rho,\sigma\}}^0$ and $f \cdot g \in RV_{\rho+\sigma}^0$. Moreover, if we assume in addition that $\lim_{x\to 0^+} g(x) = 0$, then $f \circ g \in RV_{\rho\sigma}^0$.

Proof.

(i) Assume that $\ell \in RV_0^0$ and let $\alpha > 0$. Then $\bar{\ell}\colon x \mapsto \ell(1/x)$ is in RV_0. By Proposition 2.30(i), we have $\lim_{x\to 0^+} x^{\alpha}\ell(x) = \lim_{t\to\infty}\left(\dfrac{1}{t}\right)^{\alpha}\ell\left(\dfrac{1}{t}\right) = \lim_{t\to\infty} t^{-\alpha}\bar{\ell}(t) = 0$ and similarly, $\lim_{x\to 0^+} x^{-\alpha}\ell(x) = \infty$.

(ii) Assume that $\ell \in RV_0^0$. Then $\bar{\ell}\colon x \mapsto \ell(1/x)$ is in RV_0. By Proposition 2.30(ii), $\lim_{x\to 0^+}\dfrac{\log(\ell(x))}{\log(x)} = \lim_{t\to\infty}\dfrac{\log(\ell(1/t))}{\log(1/t)} = -\lim_{t\to\infty}\dfrac{\log(\bar{\ell}(t))}{\log(t)} = 0$.

(iii) It is similar to Proposition 2.30(iii).

(iv) Assume that $f \in RV_{\rho}^{0}$, $g \in RV_{\sigma}^{0}$. Then $\bar{f}\colon x \mapsto f(1/x)$ and $\bar{g}\colon x \mapsto g(1/x)$ are in $RV_{-\rho}$ and $RV_{-\sigma}$, respectively. By Proposition 2.30(iv), $\bar{f} + \bar{g} \in RV_{\max\{-\rho,-\sigma\}}$ and $\bar{f} \cdot \bar{g} \in RV_{-\rho-\sigma}$. Since $\max\{-\rho,-\sigma\} = -\min\{\rho,\sigma\}$, we are done. Now, we suppose that $\lim\limits_{x\to 0^+} g(x) = 0$. Then $\lim\limits_{x\to\infty} \bar{g}(x) = \lim\limits_{x\to\infty} g(1/x) = \lim\limits_{t\to 0^+} g(t) = 0$. Let $\bar{h}(x) = \dfrac{1}{\bar{g}(x)}$. Then by Proposition 2.30(iii), $\bar{h} \in RV_{\sigma}$ and $\lim\limits_{x\to\infty} \bar{h}(x) = \infty$. Hence by Proposition 2.30(iv), $\bar{f} \circ \bar{h} \in RV_{-\rho\sigma}$. Since $(f \circ g)(x) = \bar{f}\left(\dfrac{1}{g(x)}\right) = \bar{f}\left(\dfrac{1}{\bar{g}(1/x)}\right) = \left(\bar{f} \circ \bar{h}\right)(1/x)$, we have $f \circ g \in RV_{\rho\sigma}^{0}$. $\square$

2.4 Pointwise dimension

In this section, we introduce the pointwise dimension [1]. This notation is used to measure a "local dimension" at each point in the domain under various measures. Moreover, it gives a sufficient condition for equality between Hausdorff dimension and box dimension, both of which are important tools in dimension theory.

Definition 2.32. Let μ be a measure on $X \subseteq \mathbb{R}^m$. For each $x \in X$, define an *upper pointwise dimension* $\overline{d}_{\mu}(x)$ and *lower pointwise dimension* $\underline{d}_{\mu}(x)$ to be

$$\overline{d}_{\mu}(x) = \limsup_{r\to 0^+} \frac{\log \mu\left(B(x,r)\right)}{\log r} \quad \text{and} \quad \underline{d}_{\mu}(x) = \liminf_{r\to 0^+} \frac{\log \mu\left(B(x,r)\right)}{\log r}.$$

If $\overline{d}_{\mu}(x) = \underline{d}_{\mu}(x) := d_{\mu}(x)$, we called $d_{\mu}(x)$ the *pointwise dimension* of x under μ.

We can write upper and lower pointwise dimensions in another form as follows.

Proposition 2.33. *For each $a > 0$ and $x \in X$, we have*
$\overline{d}_{\mu}(x) = \limsup\limits_{n\to\infty} \dfrac{\log \mu\left(B\left(x, ae^{-n}\right)\right)}{-n}$ *and* $\underline{d}_{\mu}(x) = \liminf\limits_{n\to\infty} \dfrac{\log \mu\left(B\left(x, ae^{-n}\right)\right)}{-n}$.

In the following statement, we show the relationship between the Hausdorff dimension and the lower pointwise dimension where the Hausdorff dimension of measure μ is defined to be $\dim_H \mu = \inf\left\{\dim_H Z \mid \mu\left(X \backslash Z\right) = 0\right\}$ and

$$\dim_H Z = \inf\left\{\alpha \in \mathbb{R} : \lim_{\varepsilon\to 0^+} \inf_{\mathcal{U}} \sum_{U} (\operatorname{diam} U)^{\alpha} = 0\right\}$$

is the Hausdorff dimension of $Z \subseteq X$ [1] where the infimum is taken over all countable coverings $\mathcal{U}$ of the set Z with $\sup\left\{\operatorname{diam} U : U \in \mathcal{U}\right\} \leq \varepsilon$.

Theorem 2.34. *The following statements hold:*

(i) *if* $\underline{d}_\mu(x) \geq \alpha$ *for* μ*-almost all* $x \in X$*, then* $\dim_H \mu \geq \alpha$*.*

(ii) *if* $\underline{d}_\mu(x) \leq \alpha$ *for all* $x \in Z \subseteq X$*, then* $\dim_H Z \leq \alpha$*.*

(iii) $\dim_H \mu = \operatorname{ess\,sup}\left\{\underline{d}_\mu(x) \mid x \in X\right\}$.

See [1] for the proof of this theorem.

Recall that for a finite measure μ on X, *upper box dimension* and *lower box dimension* are defined to be

$$\overline{\dim}_B\mu = \lim_{\varepsilon\to 0^+}\inf\left\{\overline{\dim}_B Z \mid \mu(Z) \geq \mu(X) - \varepsilon\right\}$$
$$\underline{\dim}_B\mu = \lim_{\varepsilon\to 0^+}\inf\left\{\underline{\dim}_B Z \mid \mu(Z) \geq \mu(X) - \varepsilon\right\}$$

where $\overline{\dim}_B Z = \limsup_{r\to 0^+}\dfrac{\log N(Z,r)}{\log r}$ and $\underline{\dim}_B Z = \liminf_{r\to 0^+}\dfrac{\log N(Z,r)}{\log r}$ are upper and lower box dimension of $Z \subseteq X$ respectively and $N(Z,r)$ is the least number of balls of radius r that are needed to cover Z. In addition, by the well-known fact in dimension theory [1] that for any measure μ in X and any $Z \subseteq X$, $\dim_H Z \leq \underline{\dim}_B Z \leq \overline{\dim}_B Z$, we have

$$\dim_H \mu \leq \underline{\dim}_B\mu \leq \overline{\dim}_B\mu.$$

These inequalities could be strict for some measures μ. However, under a condition on the pointwise dimension, these three quantities coincide.

Theorem 2.35. *If* μ *is a finite measure on* X *and there is* $d \geq 0$ *such that* $d_\mu(x) = d$ *for* μ*-almost all* $x \in X$*, then* $\dim_H \mu = \underline{\dim}_B\mu = \overline{\dim}_B\mu = d$*.*

CHAPTER III
POINTWISE DIMENSION OF COPULAS

In this chapter, we derive some properties of pointwise dimension of copulas and the equality to the pointwise exponent defined in [13]. Moreover, we study the behavior of the pointwise dimensions of copulas constructed by methods introduced in Chapter 2.

3.1 Notation and some properties of pointwise dimension

First of all, for $h > 0$, let $R_h(u, v) := [u - h, u + h] \times [v - h, v + h]$. Since there is a 1-1 correspondence between copulas and doubly stochastic measures, we use notations $\overline{d}_C \equiv \overline{d}_{\mu_C}$, $\underline{d}_C \equiv \underline{d}_{\mu_C}$ and $d_C \equiv d_{\mu_C}$ (if exists). According to Proposition 2.3, we see that $V_C(R_h(u, v)) \le 2h$ for any $h > 0$ such that $(u, v) \in \mathbb{I}^2$ and C is a copula. That is, $\underline{d}_C(u, v) \ge \liminf\limits_{h \to 0} \dfrac{\log(2h)}{\log(h)} = 1$.

From the relation $B((u, v), h) \subseteq R_h(u, v) \subseteq B((u, v), \sqrt{2}h)$, we see that the topologies generated by the collection of squares $\{R_h(u, v) \cap \mathbb{I}^2 : (u, v) \in \mathbb{I}^2, h > 0\}$ and the collection of open disks $\{B((u, v), r) \cap \mathbb{I}^2 : (u, v) \in \mathbb{I}^2, r > 0\}$ are the same. Hence we may replace $B((u, v), r)$ in Theorems 2.22 and 2.23 with $R_h(u, v)$.
Now, from Theorem 2.22, the Hausdorff dimension of every absolutely continuous copula is two.

Proposition 3.1. *Let C be an absolutely continuous copula with the corresponding doubly stochastic measure μ_C. Then $\underline{d}_C(u, v) \ge 2$ for μ_C-almost all $(u, v) \in \mathbb{I}^2$ and so $\dim_H \mu_C = 2$.*

Proof. Since $\mu_C \ll \lambda_2$ with Radon-Nikodym derivative $f(x, y) = \dfrac{\partial^2 C}{\partial x \partial y}(x, y)$, by Theorem 2.22, for λ_2-almost all $(u, v) \in \mathbb{I}^2$,

$$\lim_{h \to 0^+} \frac{\mu_C(R_h(u, v))}{\lambda_2(R_h(u, v))} = f(u, v). \tag{3.1}$$

Next, let $A = \left\{(u, v) \in \mathbb{I}^2 : (3.1) \text{ holds and } f(u, v) \in [0, \infty)\right\}$. Then $\lambda_2(\mathbb{I}^2 \backslash A) = 0$ which also implies that $\mu_C(\mathbb{I}^2 \backslash A) = 0$.
Note that for any $(u, v) \in A$, there is $\delta > 0$ such that for any $h \in (0, \delta)$,

$\dfrac{\mu_C(R_h(u,v))}{4h^2} < f(u,v)+1$. Hence

$$\begin{aligned}\underline{d}_C(u,v) &= \liminf_{h\to 0}\frac{\log \mu_C(R_h(u,v))}{\log(h)} \geq \liminf_{h\to 0}\frac{\log\left((f(u,v)+1)\cdot 4h^2\right)}{\log(h)} \\ &= \liminf_{h\to 0}\frac{\log\left(4(f(u,v)+1)\right)}{\log(h)} + 2 = 2.\end{aligned}$$

Now, we have $\underline{d}_C(u,v) \geq 2$ for all $(u,v) \in A$ which is μ_C-full measure, so by Theorem 2.34, we obtain $\dim_H \mu_C \geq 2$. But then, $\dim_H \mu_C \leq \dim_H(\mathbb{I}^2) = 2$. Thus, $\dim_H \mu_C = 2$ as desired. □

Next, we turn to the pointwise dimension of singular copulas.

Proposition 3.2. *Let C be a singular copula with the corresponding doubly stochastic measure μ_C. Then $\overline{d}_C(u,v) \leq 2$ for μ_C-almost all $(u,v) \in \mathbb{I}^2$.*

Proof. Since $\mu_C \perp \lambda_2$, by Theorem 2.23, for μ_C-almost all $(u,v) \in \mathbb{I}^2$, there is $\delta > 0$ such that for any $h \in (0,\delta)$, $\dfrac{\mu_C(R_h(u,v))}{\lambda_2(R_h(u,v))} > 1$. Hence

$$\overline{d}_C(u,v) = \limsup_{h\to 0}\frac{\log \mu_C(R_h(u,v))}{\log(h)} \leq \limsup_{h\to 0}\frac{\log(4h^2)}{\log(h)} = 2.$$ □

In [13], we introduce some notations used in this thesis as follows: for any $(u,v) \in (0,1)^2$, $\alpha > 0$ and copula C, define $\overline{D^\alpha C}(u,v) := \limsup_{h\to 0^+}\dfrac{V_C(R_h(u,v))}{(2h)^\alpha}$ and $\underline{D^\alpha C}(u,v) := \liminf_{h\to 0^+}\dfrac{V_C(R_h(u,v))}{(2h)^\alpha}$. If these values coincide, we let $D^\alpha C(u,v) := \lim_{h\to 0^+}\dfrac{V_C(R_h(u,v))}{(2h)^\alpha}$.

In the following lemma, we verify the monotonic property of $\overline{D^\alpha C}(u,v)$ and $\underline{D^\alpha C}(u,v)$ which will indicate values of $D^\beta C(u,v)$ for β on the left or right of α as follows.

Lemma 3.3. *Let C be a copula and $(u,v) \in \mathbb{I}^2$. Then both $\overline{D^\alpha C}(u,v)$ and $\underline{D^\alpha C}(u,v)$ are increasing in α. More precisely, for $0 < \beta < \alpha < \gamma$,*

(1) If $\overline{D^\alpha C}(u,v) < \infty$, then $\overline{D^\beta C}(u,v) = 0$.

(2) If $\overline{D^\alpha C}(u,v) > 0$, then $\overline{D^\gamma C}(u,v) = \infty$.

And the same statements hold for $\underline{D^\alpha C}(u,v)$. In particular,

- *if $\overline{D^\alpha C}(u,v) < \infty$, then $D^\beta C(u,v) = 0$ for all $\beta < \alpha$.*

- *if* $\underline{D^{\alpha}C}(u,v) > 0$*, then* $D^{\gamma}C(u,v) = \infty$ *for all* $\gamma > \alpha$*.*

Proof. Let $0 < \beta < \alpha < \gamma$. To show (1), assume that $\overline{D^{\alpha}C}(u,v) < \infty$. Since $\dfrac{V_C(R_h(u,v))}{(2h)^{\beta}} = \dfrac{V_C(R_h(u,v))}{(2h)^{\alpha}} \cdot (2h)^{\alpha-\beta}$, $\limsup\limits_{h\to 0^+} \dfrac{V_C(R_h(u,v))}{(2h)^{\alpha}} = \overline{D^{\alpha}C}(u,v) < \infty$ and $\lim\limits_{h\to 0^+}(2h)^{\alpha-\beta} = 0$, we have

$$\begin{aligned}\overline{D^{\beta}C}(u,v) &= \limsup_{h\to 0^+} \frac{V_C(R_h(u,v))}{(2h)^{\beta}} = \limsup_{h\to 0^+}\left(\frac{V_C(R_h(u,v))}{(2h)^{\alpha}} \cdot (2h)^{\alpha-\beta}\right)\\ &= \limsup_{h\to 0^+} \frac{V_C(R_h(u,v))}{(2h)^{\alpha}} \cdot \lim_{h\to 0^+}(2h)^{\alpha-\beta} = \overline{D^{\alpha}C}(u,v)\cdot 0 = 0.\end{aligned}$$

Next, to show (2), assume that $\overline{D^{\alpha}C}(u,v) > 0$ and let $M > 0$. Since $\limsup\limits_{h\to 0^+} \dfrac{V_C(R_h(u,v))}{(2h)^{\alpha}} = \overline{D^{\alpha}C}(u,v) > 0$ and $\lim\limits_{h\to 0^+}(2h)^{\alpha-\gamma} = \infty$, there is $\delta > 0$ such that for any $h \in (0,\delta)$, $(2h)^{\alpha-\gamma} > \dfrac{2M}{\overline{D^{\alpha}C}(u,v)}$ and there is a sequence $h_n \searrow 0$ such that $\dfrac{V_C(R_{h_n}(u,v))}{(2h_n)^{\alpha}} > \dfrac{\overline{D^{\alpha}C}(u,v)}{2}$. Thus, for any $n \in \mathbb{N}$ such that $h_n < \delta$,

$$\frac{V_C(R_{h_n}(u,v))}{(2h_n)^{\gamma}} = \frac{V_C(R_{h_n}(u,v))}{(2h_n)^{\alpha}} \cdot (2h_n)^{\alpha-\gamma} > M.$$

This implies that $\overline{D^{\gamma}C}(u,v) = \limsup\limits_{h\to 0^+} \dfrac{V_C(R_h(u,v))}{(2h)^{\gamma}} = \infty$.
Note that the proof of (1) for $\underline{D^{\alpha}C}(u,v)$ is similar to (1) for $\overline{D^{\alpha}C}(u,v)$.
However, the proof of (2) for $\underline{D^{\alpha}C}(u,v)$ is easier than (2) for $\overline{D^{\alpha}C}(u,v)$ because we can consider the reciprocal $\dfrac{(2h)^{\gamma}}{V_C(R_h(u,v))} = \left(\dfrac{V_C(R_h(u,v))}{(2h)^{\alpha}}\right)^{-1} \cdot (2h)^{\gamma-\alpha}$ where $\liminf\limits_{h\to 0^+}\left(\dfrac{V_C(R_h(u,v))}{(2h)^{\alpha}}\right)^{-1} = \left(\overline{D^{\alpha}C}(u,v)\right)^{-1}$, which is finite by assumption of (2), and $\lim\limits_{h\to 0^+}(2h)^{\gamma-\alpha} = 0$. Hence

$$\begin{aligned}\left(\underline{D^{\gamma}C}(u,v)\right)^{-1} &= \liminf_{h\to 0^+}\left(\left(\frac{V_C(R_h(u,v))}{(2h)^{\alpha}}\right)^{-1} \cdot (2h)^{\gamma-\alpha}\right)\\ &= \left(\limsup_{h\to 0^+} \frac{V_C(R_h(u,v))}{(2h)^{\alpha}}\right)^{-1} \cdot \lim_{h\to 0^+}(2h)^{\gamma-\alpha} = \left(\overline{D^{\alpha}C}(u,v)\right)^{-1} \cdot 0\\ &= 0.\end{aligned}$$

That is, $\underline{D^{\gamma}C}(u,v) = \infty$. □

By the previous lemma, with the behaviors of both $\overline{D^{\alpha}C}(u,v)$ and $\underline{D^{\alpha}C}(u,v)$, we can define *upper pointwise exponent* and *lower pointwise exponent* as follows.

- Let $\mathcal{A} := \left\{\alpha \in \mathbb{R}^+ : \overline{D^\alpha C}(u,v) = 0\right\}$ and $\mathcal{B} := \{\alpha \in \mathbb{R}^+ : \underline{D^\alpha C}(u,v) = \infty\}$.
- By Proposition 2.3 and Lemma 3.3, we see that $(0,1) \subseteq \mathcal{A}$.
- $\underline{\alpha}_C(u,v) := \sup\mathcal{A} = \sup\left\{\alpha \in \mathbb{R}^+ : \overline{D^\alpha C}(u,v) = 0\right\}$.
- $\overline{\alpha}_C(u,v) := \inf\mathcal{B} = \inf\{\alpha \in \mathbb{R}^+ : \underline{D^\alpha C}(u,v) = \infty\}$ and $\overline{\alpha}_C(u,v) = \infty$ whenever $\mathcal{B} = \varnothing$.
- These values are defined from completeness axiom of the extended real numbers. In addition, if they coincide, we denote the common value by $\alpha_C(u,v)$, and call it the *pointwise exponent* of copula C at (u,v).

Lemma 3.4. *For any copula C and $(u,v) \in \mathbb{I}^2$, $\underline{\alpha}_C(u,v) \le \overline{\alpha}_C(u,v)$.*

Proof. Let $b_0 < \underline{\alpha}_C(u,v)$. Then there is $a \in \mathcal{A}$ such that $b_0 < a$, i.e., $\overline{D^a C}(u,v) = 0$. By Lemma 3.3, $\underline{D^{b_0} C}(u,v) = D^{b_0}C(u,v) = 0$ so that $b_0 \le b$ for any $b \in \mathcal{B}$ which implies $b_0 \le \inf\mathcal{B} = \overline{\alpha}_C(u,v)$. Therefore, $\underline{\alpha}_C(u,v) = \sup\mathcal{A} \le \overline{\alpha}_C(u,v)$. □

From the definition of $\alpha_C(u,v)$, we can show that $\overline{\alpha}_C(u,v) = \overline{d}_C(u,v)$ and $\underline{\alpha}_C(u,v) = \underline{d}_C(u,v)$ as stated in the next proposition.

Proposition 3.5. *For any copula C and $(u,v) \in \mathbb{I}^2$, $\overline{\alpha}_C(u,v) = \overline{d}_C(u,v)$ and $\underline{\alpha}_C(u,v) = \underline{d}_C(u,v)$.*

Proof. We verify only that $\overline{\alpha}_C(u,v) = \overline{d}_C(u,v)$ as the other equality can be proved similarly.
($\ge$) Let $b \in \mathcal{B} = \{\alpha \in \mathbb{R}^+ : \underline{D^\alpha C}(u,v) = \infty\}$. Then there is $\delta > 0$ such that for any $h \in (0,\delta)$, $\dfrac{V_C(R_h(u,v))}{(2h)^b} > 1$. Hence for such h,

$$\frac{\log V_C(R_h(u,v))}{\log(h)} < \frac{b\log(2h)}{\log(h)} = b + \frac{b\log(2)}{\log(h)}, \quad \text{i.e.,}$$

$\overline{d}_C(u,v) = \limsup_{h\to 0} \dfrac{\log V_C(R_h(u,v))}{\log(h)} \le b$. Therefore, $\overline{d}_C(u,v) \le \inf\mathcal{B} = \overline{\alpha}_C(u,v)$.
($\le$) Note that the case $\overline{d}_C(u,v) = \infty$ is obvious, so we show this statement only in the case $\overline{d}_C(u,v) < \infty$. Let $b > \overline{d}_C(u,v)$ and $b_0 \in \left(\overline{d}_C(u,v), b\right)$. Then there is $\delta \in \left(0, \dfrac{1}{2}\right)$ such that for any $h \in (0,\delta)$, $\dfrac{\log V_C(R_h(u,v))}{\log(h)} < b_0$. Hence for such h, $(2h)^{b_0} < (2h)^{\log V_C(R_h(u,v))/\log(h)} = V_C(R_h(u,v)) \cdot 2^{\log V_C(R_h(u,v))/\log(h)}$, i.e., $\dfrac{V_C(R_h(u,v))}{(2h)^{b_0}} > 2^{-\log V_C(R_h(u,v))/\log(h)}$. It yields that

$$\underline{D^{b_0}C}(u,v) = \liminf_{h\to 0^+} \frac{V_C(R_h(u,v))}{(2h)^b} \ge \frac{1}{2^{\overline{d}_C(u,v)}} > 0.$$

Hence by Lemma 3.3(ii), $b \in \mathcal{B}$, that is, $\overline{\alpha}_C(u,v) = \inf\mathcal{B} \le \overline{d}_C(u,v)$. □

3.2 Pointwise dimension of copulas constructed by various methods

In this section, we study copulas constructed from a collection of copulas. Their pointwise dimension can be computed from the pointwise dimensions of copulas used in the construction. We first prove the following statement which tells us about a relation between the pointwise dimension of a point and the volume of copulas around the point.

Proposition 3.6. *Let C, D be copulas, $(u_1, v_1), (u_2, v_2) \in \mathbb{I}^2$ and $r, s \in \mathbb{R}^+$. Suppose that $d_C(u_1, v_1) = \alpha$ and $d_D(u_2, v_2) = \beta$.*

1. *If $\alpha < \beta$, then there exists $\delta > 0$ such that for any $h \in (0, \delta)$,*

$$V_D\left(R_{sh}\left(u_2, v_2\right)\right) < V_C\left(R_{rh}\left(u_1, v_1\right)\right)^{\log(sh)/\log(rh)}.$$

2. *If $\alpha = \beta$, then for any $\varepsilon > 0$, there is $\delta > 0$ such that for any $h \in (0, \delta)$,*

$$h^{\varepsilon}\frac{r^{\alpha+\frac{\varepsilon}{2}}}{s^{\alpha-\frac{\varepsilon}{2}}}V_D\left(R_{sh}\left(u_2, v_2\right)\right) < V_C\left(R_{rh}\left(u_1, v_1\right)\right) < h^{-\varepsilon}\frac{r^{\alpha-\frac{\varepsilon}{2}}}{s^{\alpha+\frac{\varepsilon}{2}}}V_D\left(R_{sh}\left(u_2, v_2\right)\right).$$

Proof. To prove 1., assume that $\alpha < \beta$. Since $\alpha = \lim_{h\to 0}\frac{\log V_C\left(R_h(u_1, v_1)\right)}{\log(h)}$ and $\beta = \lim_{h\to 0}\frac{\log\left(V_D\left(R_h\left(u_2, v_2\right)\right)\right)}{\log(h)}$, there is $\gamma \in (0, 1)$ such that for any $h \in (0, \gamma)$,

$$\frac{\log V_C\left(R_h(u_1, v_1)\right)}{\log(h)} - \alpha < \frac{\beta - \alpha}{2} \quad \text{and} \quad \frac{\log\left(V_D\left(R_h\left(u_2, v_2\right)\right)\right)}{\log(h)} - \beta > \frac{\alpha - \beta}{2}.$$

Let $\delta = \min\{\frac{\gamma}{r}, \frac{\gamma}{s}\}$. Then for any $h \in (0, \delta)$, we have $rh, sh \in (0, \gamma)$ and so

$$\frac{\log V_C\left(R_{rh}(u_1, v_1)\right)}{\log(rh)} < \frac{\alpha + \beta}{2} < \frac{\log\left(V_D\left(R_{sh}\left(u_2, v_2\right)\right)\right)}{\log(sh)}.$$

Since $sh \in (0, 1)$, $\log(sh) < 0$ and

$$\begin{aligned}\log\left(V_D\left(R_{sh}(u_2, v_2)\right)\right) &< \frac{\log(sh)}{\log(rh)} \cdot \log V_C\left(R_{rh}(u_1, v_1)\right) \\ &= \log V_C\left(R_{rh}(u_1, v_1)\right)^{\log(sh)/\log(rh)}.\end{aligned}$$

Thus, we obtain the desired inequality because log is a strictly increasing function. To prove 2., assume that $\alpha = \beta$ and let $\varepsilon > 0$. By the definition of $d_C(u, v)$, there is $\gamma \in (0, 1)$ such that for any $h \in (0, \gamma)$, $\frac{\log V_C\left(R_h(u_1, v_1)\right)}{\log(h)} \in \left(\alpha - \frac{\varepsilon}{2}, \alpha + \frac{\varepsilon}{2}\right)$

and $\dfrac{\log V_D\left(R_h(u_2,v_2)\right)}{\log(h)} \in \left(\alpha - \dfrac{\varepsilon}{2}, \alpha + \dfrac{\varepsilon}{2}\right)$. Since $\log(h) < 0$ and log is a strictly increasing function, we have $V_C\left(R_h(u_1,v_1)\right), V_D\left(R_h\left(u_2,v_2\right)\right) \in \left(h^{\alpha+\frac{\varepsilon}{2}}, h^{\alpha-\frac{\varepsilon}{2}}\right)$ for any $h \in (0,\gamma)$. Let $\delta = \min\{\dfrac{\gamma}{r}, \dfrac{\gamma}{s}\}$. Then for $h \in (0,\delta)$, we have $rh, sh \in (0,\gamma)$, so

$$h^{\varepsilon}\frac{r^{\alpha+\frac{\varepsilon}{2}}}{s^{\alpha-\frac{\varepsilon}{2}}} = \frac{(rh)^{\alpha+\frac{\varepsilon}{2}}}{(sh)^{\alpha-\frac{\varepsilon}{2}}} < \frac{V_C\left(R_{rh}\left(u_1,v_1\right)\right)}{V_D\left(R_{sh}\left(u_2,v_2\right)\right)} < \frac{(rh)^{\alpha-\frac{\varepsilon}{2}}}{(sh)^{\alpha+\frac{\varepsilon}{2}}} = h^{-\varepsilon}\frac{r^{\alpha-\frac{\varepsilon}{2}}}{s^{\alpha+\frac{\varepsilon}{2}}}$$

and we are done. □

Now, we are ready to find the pointwise dimension of copulas constructed by convex sum, patching, and ordinal sum as follows.

Theorem 3.7. *Let $\{C_i\}_{i=1}^n$ be a collection of copulas and $\{\alpha_i\}_{i=1}^n \subseteq (0,1)$ be such that $\sum_{i=1}^n \alpha_i = 1$. Let $C = \sum_{i=1}^n \alpha_i C_i$ be the convex sum of $\{C_i\}_{i=1}^n$. Then for any $(u,v) \in \mathbb{I}^2$, $d_C(u,v) = \min_{1\le i\le n}\{d_{C_i}(u,v)\}$.*

Proof. By rearranging, we suppose without loss of generality that $d_{C_i}(u,v) = d_{C_1}(u,v)$ for all $i = 1,\ldots,k$ and $d_{C_i}(u,v) > d_{C_1}(u,v)$ for all $i = k+1,\ldots,n$. Note that for any $h > 0$, $V_C\left(R_h(u,v)\right) = \sum_{i=1}^n \alpha_i V_{C_i}\left(R_h(u,v)\right)$. Hence by Proposition 3.6(1) and (2), for any $\varepsilon > 0$, there is $\delta \in (0,1)$ such that for any $h \in (0,\delta)$,

$$\begin{aligned} V_{C_i}\left(R_h(u,v)\right) < h^{-\varepsilon} V_{C_1}\left(R_h(u,v)\right) &\quad \text{if } i = 1,\ldots,k; \\ V_{C_i}\left(R_h(u,v)\right) < V_{C_1}\left(R_h(u,v)\right) < h^{-\varepsilon} V_{C_1}\left(R_h(u,v)\right) &\quad \text{if } i = k+1,\ldots,n. \end{aligned}$$

That is, for any $h \in (0,\delta)$,

$$\alpha_1 V_{C_1}\left(R_h(u,v)\right) \le V_C\left(R_h(u,v)\right) \le h^{-\varepsilon} V_{C_1}\left(R_h(u,v)\right)\sum_{i=1}^n \alpha_i = h^{-\varepsilon} V_{C_1}\left(R_h(u,v)\right).$$

Since

$$\lim_{h\to 0}\frac{\log\left(\alpha_1 V_{C_1}\left(R_h(u,v)\right)\right)}{\log(h)} = \lim_{h\to 0}\left(\frac{\log(\alpha_1)}{\log(h)} + \frac{\log\left(V_{C_1}\left(R_h(u,v)\right)\right)}{\log(h)}\right) = d_{C_1}(u,v)$$

and

$$\begin{aligned} \lim_{h\to 0}\frac{\log\left(h^{-\varepsilon} V_{C_1}\left(R_h(u,v)\right)\right)}{\log(h)} &= \lim_{h\to 0}\left(\frac{\log(h^{-\varepsilon})}{\log(h)} + \frac{\log\left(V_{C_1}\left(R_h(u,v)\right)\right)}{\log(h)}\right) \\ &= d_{C_1}(u,v) - \varepsilon, \end{aligned}$$

we have $d_{C_1}(u,v) - \varepsilon \le \lim_{h\to 0}\dfrac{\log V_C\left(R_h(u,v)\right)}{\log(h)} \le d_{C_1}(u,v)$. Since $\varepsilon > 0$ is arbitrary, we obtain $d_C(u,v) = d_{C_1}(u,v) = \min_{1\le i\le n}\{d_{C_i}(u,v)\}$ as desired. □

Theorem 3.8. *Let $T = [t_{ij}] \in M_{m\times n}(\mathbb{I})$ be a transformation matrix and $\{C_{ij}\}$ be a collection of copulas with the same indices as entries in T. Let C be the patched copula with respect to the transformation matrix T and the collection of copulas $\{C_{ij}\}$. Then for any $(u, v) \in \mathbb{I}^2$,*

$$d_C(u, v) = \inf_{(i,j)\in A(u,v)} \{d_{C_{ij}}(u_i, v_j)\}$$

where $A(u, v) := \{(i, j) : (u, v) \in R_{ij} = [p_{i-1}, p_i] \times [q_{j-1}, q_j], t_{ij} > 0\}$, $u_i = \dfrac{u - p_{i-1}}{p_i - p_{i-1}}$ *and* $v_j = \dfrac{v - q_{j-1}}{q_j - q_{j-1}}$.

Proof. Let $(u, v) \in \mathbb{I}^2$. For convenience, we define $\Delta p_i := p_i - p_{i-1}$ and $\Delta q_j := q_j - q_{j-1}$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.
Note that if $A(u, v) = \varnothing$, then there is $\delta > 0$ such that for any $h \in (0, \delta)$, $V_C(R_h(u, v)) = 0$ which implies that $d_C(u, v) = \infty$.
Now, we suppose that $A(u, v) \neq \varnothing$ and assume without loss of generality that $t_{ij} > 0$ for all (i, j) such that $(u, v) \in R_{ij}$. Then we can divide the proof into 3 cases as in Figure 3.1.

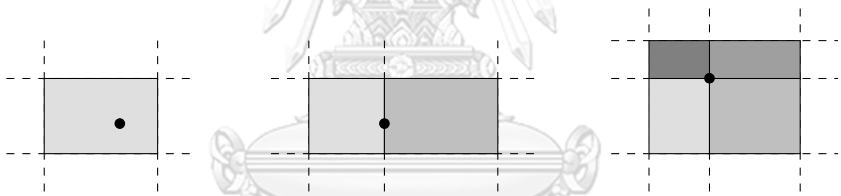

Figure 3.1: (u, v) in which $|A(u, v)| = 1$ (Left), $|A(u, v)| = 2$ (Middle) and $|A(u, v)| = 4$ (Right), respectively

Case 1: $|A(u, v)| = 1$. Then (u, v) is in the interior of $R_{k\ell}$ for some unique pair (k, ℓ) and there is $\delta > 0$ such that for any $h \in (0, \delta)$, $R_h(u, v) \subseteq R_{k\ell}$. Next, let $k_1 = \dfrac{h}{\Delta p_k}$ and $k_2 = \dfrac{h}{\Delta q_\ell}$. Note that from

$$C(x, y) = \sum_{i<k, j<\ell} t_{ij} + \frac{x - p_{k-1}}{\Delta p_k} \sum_{j<\ell} t_{kj} + \frac{y - q_{\ell-1}}{\Delta q_\ell} \sum_{i<k} t_{i\ell} + t_{k\ell} C_{k\ell}\left(\frac{x - p_{k-1}}{\Delta p_k}, \frac{y - q_{\ell-1}}{\Delta q_\ell}\right)$$

for $(x, y) \in R_{k\ell}$, we have

$$V_C(R_h(u, v)) = t_{k\ell} V_{C_{k\ell}}\Big([u_k - k_1, u_k + k_1] \times [v_\ell - k_2, v_\ell + k_2]\Big).$$

Since $R_{\min\{k_1,k_2\}}(u_k, v_\ell) \subseteq [u_k - k_1, u_k + k_1] \times [v_\ell - k_2, v_\ell + k_2] \subseteq R_{\max\{k_1,k_2\}}(u_k, v_\ell)$,

$$\lim_{h\to 0} \frac{\log(V_{C_{k\ell}}(R_{k_1}(u_k, v_\ell)))}{\log(h)} = \lim_{k_1\to 0} \frac{\log(V_{C_{k\ell}}(R_{k_1}(u_k, v_\ell)))}{\log((\Delta p_k) k_1)}$$

$$= \lim_{k_1\to 0}\left(\frac{\log\left(V_{C_{k\ell}}\left(R_{k_1}(u_k,v_\ell)\right)\right)}{\log(k_1)}\cdot\frac{1}{1+\frac{\log(\Delta p_k)}{\log(k_1)}}\right)$$

$$= \lim_{k_1\to 0}\frac{\log\left(V_{C_{k\ell}}\left(R_{k_1}(u_k,v_\ell)\right)\right)}{\log(k_1)} = d_{C_{k\ell}}(u_k,v_\ell),$$

and similarly, $\lim_{h\to 0}\frac{\log\left(V_{C_{k\ell}}\left(R_{k_2}(u_k,v_\ell)\right)\right)}{\log(h)} = d_{C_{k\ell}}(u_k,v_\ell)$, we have

$$\begin{aligned} d_C(u,v) &= \lim_{h\to 0}\frac{\log V_C(R_h(u,v))}{\log(h)} \\ &= \lim_{h\to 0}\frac{\log(t_{k\ell})}{\log(h)} + \lim_{h\to 0}\frac{\log\Big(V_{C_{k\ell}}\Big([u_k-k_1,u_k+k_1]\times[v_\ell-k_2,v_\ell+k_2]\Big)\Big)}{\log(h)} \\ &= d_{C_{k\ell}}(u_k,v_\ell). \end{aligned}$$

Case 2: $|A(u,v)| = 2$. Then $(u,v)\in R_{k\ell}\cap R_{k'\ell'}$ where $|k-k'|+|\ell-\ell'| = 1$.

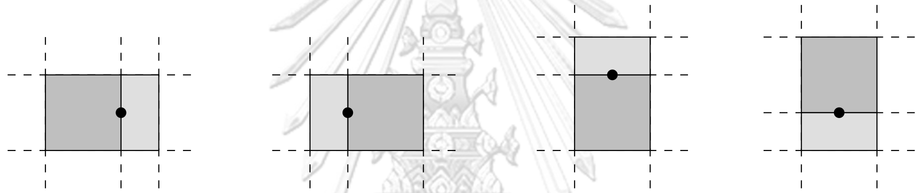

Figure 3.2: All possibilities of (u,v) in Case 2

Without loss of generality, we assume that $(u,v)\in R_{k\ell}\cap R_{(k+1)\ell}$ where $u = p_k$ and $v\in(q_{\ell-1},q_\ell)$ for some (k,ℓ). Then there is $\delta>0$ such that for any $h\in(0,\delta)$, $R_h(u,v)\subseteq R_{k\ell}\cup R_{(k+1)\ell}$. Considering $R_h(u,v)\cap R_{k\ell}$ and $R_h(u,v)\cap R_{(k+1)\ell}$, we have $[1-k_1,1]\times[v_\ell-k_2,v_\ell+k_2]\subseteq\mathbb{I}^2$ and $[0,k_3]\times[v_\ell-k_2,v_\ell+k_2]\subseteq\mathbb{I}^2$ where $k_1 = \dfrac{h}{\Delta p_k}$, $k_2 = \dfrac{h}{\Delta q_\ell}$ and $k_3 = \dfrac{h}{\Delta p_{k+1}}$. Note that

$$\begin{aligned} V_C\left(R_h(u,v)\right) &= V_C\left([u-h,u]\times[v-h,v+h]\right) + V_C\left([u,u+h]\times[v-h,v+h]\right) \\ &= t_{k\ell}V_{C_{k\ell}}\left([1-k_1,1]\times[v_\ell-k_2,v_\ell+k_2]\right) \\ &\quad + t_{(k+1)\ell}V_{C_{(k+1)\ell}}\left([0,k_3]\times[v_\ell-k_2,v_\ell+k_2]\right). \end{aligned}$$

Now, we assume without loss of generality that $d_{C_{k\ell}}(1,v_\ell)\le d_{C_{(k+1)\ell}}(0,v_\ell)$. Let $s_1 = \min\{k_1,k_2\}$, $s_2 = \max\{k_1,k_2\}$, $s_3 = \max\{k_2,k_3\}$. Then we have the following inequalities:

$$V_C\left(R_h(u,v)\right) \ge t_{k\ell}V_{C_{k\ell}}\left(R_{s_1}(1,v_\ell)\right) \tag{3.2}$$

$$V_C\left(R_h(u,v)\right) \le t_{k\ell}V_{C_{k\ell}}\left(R_{s_2}(1,v_\ell)\right) + t_{(k+1)\ell}V_{C_{(k+1)\ell}}\left(R_{s_3}(0,v_\ell)\right). \tag{3.3}$$

As in Case 1, we obtain $\lim\limits_{h\to 0} \dfrac{\log\left(V_{C_{k\ell}}\left(R_{s_1}(1,v_\ell)\right)\right)}{\log(h)} = d_{C_{k\ell}}(1,v_\ell)$, which implies that $d_C(u,v) \leq d_{C_{k\ell}}(1,v_\ell)$. We then divide into 2 subcases as follows:

- If $d_{C_{k\ell}}(1,v_\ell) < d_{C_{(k+1)\ell}}(0,v_\ell)$, then by Proposition 3.6(1), there is $\delta' < \delta$ such that for any $h \in (0,\delta')$, $V_{C_{(k+1)\ell}}\left(R_{s_3}(0,v_\ell)\right) < V_{C_{k\ell}}\left(R_{s_2}(1,v_\ell)\right)^{\frac{\log(s_3)}{\log(s_2)}}$. Hence by (3.3), for any $h \in (0,\delta')$,

$$\begin{aligned} V_C\left(R_h(u,v)\right) &\leq t_{k\ell} V_{C_{k\ell}}\left(R_{s_2}(1,v_\ell)\right) + t_{(k+1)\ell} V_{C_{k\ell}}\left(R_{s_2}(1,v_\ell)\right)^{\frac{\log(s_3)}{\log(s_2)}} \\ &\leq \begin{cases} \left(t_{k\ell} + t_{(k+1)\ell}\right) V_{C_{k\ell}}\left(R_{s_2}(1,v_\ell)\right) & \text{if } s_2 \geq s_3; \\ \left(t_{k\ell} + t_{(k+1)\ell}\right) V_{C_{k\ell}}\left(R_{s_2}(1,v_\ell)\right)^{\frac{\log(s_3)}{\log(s_2)}} & \text{if } s_2 < s_3, \end{cases} \end{aligned}$$

  where $\lim\limits_{h\to 0} \dfrac{\log\left(V_{C_{k\ell}}\left(R_{s_2}(1,v_\ell)\right)\right)}{\log(h)} = d_{C_{k\ell}}(1,v_\ell)$ and

$$\begin{aligned} \lim_{h\to 0} \frac{\log\left(V_{C_{k\ell}}\left(R_{s_2}(1,v_\ell)\right)^{\frac{\log(s_3)}{\log(s_2)}}\right)}{\log(h)} &= \lim_{h\to 0} \frac{\log(s_3)}{\log(s_2)} \cdot \frac{\log\left(V_{C_{k\ell}}\left(R_{s_2}(1,v_\ell)\right)\right)}{\log(h)} \\ &= d_{C_{k\ell}}(1,v_\ell) \cdot \lim_{h\to 0} \frac{\log\left(\frac{h}{\min\{\Delta p_{k+1},\Delta q_\ell\}}\right)}{\log\left(\frac{h}{\min\{\Delta p_k,\Delta q_\ell\}}\right)} \\ &= d_{C_{k\ell}}(1,v_\ell). \end{aligned}$$

  Hence $d_C(u,v) \geq d_{C_{k\ell}}(1,v_\ell)$, i.e.,

$$d_C(u,v) = d_{C_{k\ell}}(1,v_\ell) = \min_{(i,j)\in A(u,v)} \{d_{C_{ij}}(u_i,v_j)\}.$$

- If $d_{C_{k\ell}}(1,v_\ell) = d_{C_{(k+1)\ell}}(0,v_\ell) = \alpha$, then by Proposition 3.6(2), for each $\varepsilon > 0$, there is $\delta' < \delta$ such that for any $h \in (0,\delta')$,

$$\begin{aligned} V_{C_{(k+1)\ell}}\left(R_{s_3}(0,v_\ell)\right) &< \frac{s_3^{\alpha-\frac{\varepsilon}{2}}}{s_2^{\alpha+\frac{\varepsilon}{2}}} V_{C_{k\ell}}\left(R_{s_2}(1,v_\ell)\right) \\ &= h^{-\varepsilon} \cdot \frac{\min\{\Delta p_k,\Delta q_\ell\}^{\alpha+\frac{\varepsilon}{2}}}{\min\{\Delta p_{k+1},\Delta q_\ell\}^{\alpha-\frac{\varepsilon}{2}}} \cdot V_{C_{k\ell}}\left(R_{s_2}(1,v_\ell)\right). \end{aligned}$$

  Hence by (3.3), for any $h \in (0,\delta')$,

$$V_C\left(R_h(u,v)\right) \leq V_{C_{k\ell}}\left(R_{s_2}(1,v_\ell)\right) h^{-\varepsilon} \left[t_{k\ell} h^{\varepsilon} + t_{(k+1)\ell} \frac{\min\{\Delta p_k,\Delta q_\ell\}^{\alpha+\frac{\varepsilon}{2}}}{\min\{\Delta p_{k+1},\Delta q_\ell\}^{\alpha-\frac{\varepsilon}{2}}}\right].$$

  Now, since

$$\lim_{h\to 0} \left[t_{k\ell} h^{\varepsilon} + t_{(k+1)\ell} \frac{\min\{\Delta p_k,\Delta q_\ell\}^{\alpha+\frac{\varepsilon}{2}}}{\min\{\Delta p_{k+1},\Delta q_\ell\}^{\alpha-\frac{\varepsilon}{2}}}\right] = t_{(k+1)\ell} \frac{\min\{\Delta p_k,\Delta q_\ell\}^{\alpha+\frac{\varepsilon}{2}}}{\min\{\Delta p_{k+1},\Delta q_\ell\}^{\alpha-\frac{\varepsilon}{2}}},$$

we have

$$\lim_{h\to 0} \frac{\log\left[V_{C_{k\ell}}\left(R_{s_2}(1,v_\ell)\right) h^{-\varepsilon}\left[t_{k\ell}h^{\varepsilon} + t_{(k+1)\ell}\frac{\min\{\Delta p_k,\Delta q_\ell\}^{\alpha+\frac{\varepsilon}{2}}}{\min\{\Delta p_{k+1},\Delta q_\ell\}^{\alpha-\frac{\varepsilon}{2}}}\right]\right]}{\log(h)}$$

$$= \lim_{h\to 0}\frac{\log\left(V_{C_{k\ell}}\left(R_{s_2}(1,v_\ell)\right)\right)}{\log(h)} + \lim_{h\to 0}\frac{\log\left[t_{k\ell}h^{\varepsilon} + t_{(k+1)\ell}\frac{\min\{\Delta p_k,\Delta q_\ell\}^{\alpha+\frac{\varepsilon}{2}}}{\min\{\Delta p_{k+1},\Delta q_\ell\}^{\alpha-\frac{\varepsilon}{2}}}\right]}{\log(h)} - \varepsilon$$

$$= d_{C_{k\ell}}(1,v_\ell) - \varepsilon.$$

Thus, $d_C(u,v) \geq d_{C_{k\ell}}(1,v_\ell) - \varepsilon$.

Since ε is arbitrary, we have $d_C(u,v) = d_{C_{k\ell}}(1,v_\ell) = \min\limits_{(i,j)\in A(u,v)}\{d_{C_{ij}}(u_i,v_j)\}$.

Case 3: $|A(u,v)| = 4$. The proof in this case is similar to Case 2 above. □

Corollary 3.9. *Let $\{J_i\}_{i=1}^N$, where $J_i = [a_i,b_i]$ with $a_i < b_i$ for all $i = 1,\ldots,N$, be a family of closed, non-overlapping, non-degenerate sub-intervals on $\mathbb{I}$ and let $\{C_i\}_{i=1}^N$ be a collection of copulas. Moreover, let C be an ordinal sum of $\{C_i\}$ with respect to $\{J_i\}$. Set $A_N = \{(a_i,a_i) : i = 1,\ldots,N\}$ and $B_N = \{(b_i,b_i) : n = 1,\ldots,N\}$. Then for any $(u,v) \in \mathbb{I}^2\backslash\{(0,0),(1,1)\}$,*

$$d_C(u,v) = \begin{cases} d_{C_i}\left(\dfrac{u-a_i}{b_i-a_i}, \dfrac{v-a_i}{b_i-a_i}\right) & \textit{if } (u,v) \in J_i^2\backslash\left(A_N \cup B_N\right) \textit{ for some } i; \\ \min\{d_{C_i}(1,1), d_{C_j}(0,0)\} & \textit{if } u = v = b_i = a_j \textit{ for some } i \neq j; \\ d_M(u,v) & \textit{otherwise.} \end{cases}$$

Moreover,

$$d_C(0,0) = \begin{cases} d_{C_i}(0,0) & \textit{if } (0,0) \in J_i^2 \textit{ for some } i; \\ 1 & \textit{otherwise,} \end{cases}$$

and a similar statement holds for $d_C(1,1)$.

Proof. It follows from Proposition 2.14 and Theorem 3.8. □

CHAPTER IV
POINTWISE DIMENSION OF ARCHIMEDEAN COPULAS

Let φ be an Archimedean generator, i.e., φ is a convex, continuous and strictly decreasing on $\mathbb{I}$ such that $\varphi(1) = 0$, and $C(u,v) = \varphi^{[-1]}(\varphi(u) + \varphi(v))$, the corresponding Archimedean copula. For more details, see Subsection 2.1.1 or [9]. We divide our results on Archimedean copulas into 2 parts: piecewise linear generators case and general case.

Before we find the pointwise dimension, we can use properties of φ to compare each term in $V_C(R_h(u,v))$ as stated in the following lemma.

Lemma 4.1. *Let φ be an Archimedean generator. Suppose that $(u,v) \in (0,1)^2$ satisfies $\varphi(u)+\varphi(v) = \varphi(t) < \infty$ for some $t \in (0,1)$. Then for $h > 0$ small enough,*

- *if $u > v$, then*

  $$C(u-h, v-h) < C(u+h, v-h) \leq t \leq C(u-h, v+h) < C(u+h, v+h).$$

- *if $u < v$, then*

  $$C(u-h, v-h) < C(u-h, v+h) \leq t \leq C(u+h, v-h) < C(u+h, v+h).$$

- *if $u = v$, then*

  $$C(u-h, v-h) < C(u+h, v-h) = C(u-h, v+h) \leq t < C(u+h, v+h).$$

Furthermore, if $(u,v) \in (0,1)^2$ satisfies $\varphi(u) + \varphi(v) = \varphi(0) < \infty$, then for $h > 0$ small enough,

$$V_C(R_h(u,v)) = \begin{cases} C(u+h, v+h) - C(u-h, v+h) & \text{if } u > v; \\ C(u+h, v+h) - C(u+h, v-h) & \text{if } u < v; \\ C(u+h, v+h) & \text{if } u = v. \end{cases}$$

Proof. First, we consider the case $t \in (0,1)$. Since φ is strictly decreasing on $\mathbb{I}$ and $\{(x,y) \in (0,1)^2 : \varphi(x) + \varphi(y) < \varphi(0)\}$ is an open set, we can find $\delta \in (0, m := \min\{u, v, 1-u, 1-v\})$ such that for any $h \in (0,\delta)$, $\varphi(u+h) < \varphi(u) < \varphi(u-h)$, $\varphi(v+h) < \varphi(v) < \varphi(v-h)$ and $\varphi(u-h)+\varphi(v-h) < \varphi(0)$. Consequently,

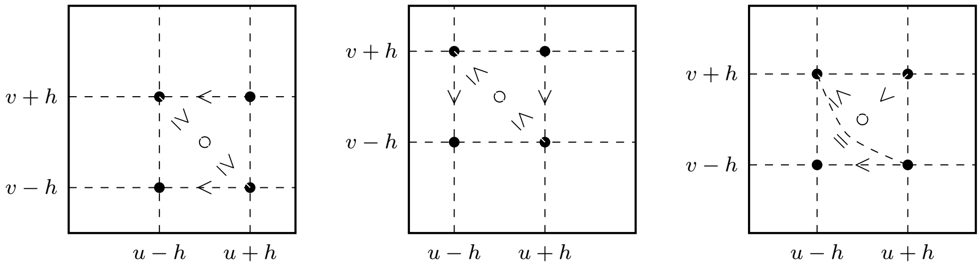

Figure 4.1: Comparing the values at the corners of $R_h(u,v)$ (for $h > 0$ small) with $C(u,v) = t > 0$ in case $u > v$ (Left), $u < v$ (Middle) and $u = v$ (Right), respectively

(1) $\varphi(u+h) + \varphi(v+h) < \varphi(t)$;

(2) $\varphi(t) < \varphi(u-h) + \varphi(v-h) < \varphi(0)$;

(3) $\varphi(u+h) + \varphi(v+h) < \min\{\varphi(u+h) + \varphi(v-h), \varphi(u-h) + \varphi(v+h)\}$; and

(4) $\varphi(u-h) + \varphi(v-h) > \max\{\varphi(u+h) + \varphi(v-h), \varphi(u-h) + \varphi(v+h)\}$.

In addition, since $\varphi^{[-1]}$ is strictly decreasing on $[0, \varphi(0)]$, we obtain

(I) $C(u+h, v+h) > t$;

(II) $C(u-h, v-h) < t$;

(III) $C(u+h, v+h) > \max\{C(u+h, v-h), C(u-h, v+h)\}$;

(IV) $C(u-h, v-h) < \min\{C(u+h, v-h), C(u-h, v+h)\}$.

Next, we consider the case $t = 0$. By the strictly decreasing property of φ, we can find $\delta > 0$ such that for any $h \in (0, \delta)$, (1) and (3) hold. However, for such h, (2) becomes $\varphi(u-h) + \varphi(v-h) > \varphi(0)$. Moreover, since $\varphi^{[-1]}$ is strictly decreasing on $[0, \varphi(0)]$ and vanishes elsewhere, we see that (I) and (III) still hold in this case but (II) becomes $C(u-h, v-h) = 0$.
Now, we compare $C\left(u+h, v-h\right)$ and $C\left(u-h, v+h\right)$ with $C(u,v) = t \in [0,1)$. We can consider 3 subcases as follows.

a. If $u > v$, then by convexity of φ, for any $h \in \left[0, \min\left\{\frac{u-v}{2}, m\right\}\right]$, $\varphi(v+h) - \varphi(v) \le \varphi(u) - \varphi(u-h)$ and $\varphi(v) - \varphi(v-h) \le \varphi(u+h) - \varphi(u)$ which implies that $\varphi(u-h) + \varphi(v+h) \le \varphi(t) \le \varphi(u+h) + \varphi(v-h)$. Since $\varphi^{[-1]}$ is decreasing, we have $C\left(u+h, v-h\right) \le t \le C(u-h, v+h)$.

b. If $u < v$, with similar argument as the previous subcase, we have $C\left(u+h, v-h\right) \geq t \geq C(u-h, v+h)$.

c. If $u = v$, then for any $h \in [0, \min\{u, 1-u\}]$, $\varphi(u) - \varphi(u-h) \leq \varphi(u+h) - \varphi(u)$ which implies that $\varphi(u+h) + \varphi(u-h) \geq \varphi(t)$ and so $C\left(u+h, u-h\right) = C\left(u-h, u+h\right) \leq t$. □

Corollary 4.2. *Let $(u, v) \in (0,1)^2$ be such that $\varphi(u) + \varphi(v) = \varphi(t)$ for some $t \in [0, 1)$.*

- *If there is $h > 0$ small enough such that $C(u-h, v+h) = C(u, v)$, then for any $h' \in [0, h]$, $C(u-h', v+h') = C(u, v)$. That is, $\varphi'(u^-) = \varphi'(v^+)$.*
- *If there is $h > 0$ small enough such that $C(u+h, v-h) = C(u, v)$, then for any $h' \in [0, h]$, $C(u+h', v-h') = C(u, v)$. That is, $\varphi'(u^+) = \varphi'(v^-)$.*

Proof. We verify only the first statement in case $u > v$ because the other cases can be handled similarly. Assume that there is $h \in \left(0, \dfrac{u-v}{2}\right)$ small enough such that $C(u-h, v+h) = C(u, v)$. Since φ is convex, for any $h' \in [0, h]$, $\varphi(v+h) - \varphi(v+h') \leq \varphi(u-h') - \varphi(u-h)$, i.e., $\varphi(u-h) + \varphi(v+h) \leq \varphi(u-h') + \varphi(v+h')$. Hence $C(u-h, v+h) \geq C(u-h', v+h') \geq C(u, v)$ by the decreasing property of $\varphi^{[-1]}$ and Lemma 4.1. Now, from $\varphi(u-h') + \varphi(v+h') = \varphi\left(C(u-h', v+h')\right) = \varphi\left(C(u, v)\right) = \varphi(u) + \varphi(v)$ for any $h' \in [0, h]$, we have

$$\varphi'(v^+) = \lim_{h' \to 0^+} \frac{\varphi(v+h') - \varphi(v)}{h'} = \lim_{h' \to 0^+} \frac{\varphi(u) - \varphi(u-h')}{h'} = \varphi'(u^-). \qquad \square$$

4.1 Simple case: piecewise linear generators

Let φ be a piecewise linear Archimedean generator. Then $\varphi'' = 0$ for all but finitely many points in $\mathbb{I}$. This implies that C_φ must be a singular copula because of (2.2) and Theorem 2.2.

Before summarizing the case, we first write the formula of φ explicitly as follows. Since φ is piecewise linear and non-negative, we can find $\{a_i\}_{i=1}^n$, $\{b_i\}_{i=1}^n \subseteq \mathbb{R}^+$ and $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ such that $\varphi(0) = a_1$ and $\varphi(t) = a_k - b_k t$ for $t \in (t_{k-1}, t_k]$, $k = 1, 2, \ldots, n$.

Note:

- Each function in φ has negative slope because φ is strictly decreasing.
- Since φ is convex (by Theorem 2.4), the slope of φ is increasing, which implies that $b_1 > b_2 > \cdots > b_n > 0$.

- Since φ is continuous, we have $a_{k+1} = a_k + (b_{k+1} - b_k)\, t_k$ for all $k = 1, \ldots, n-1$, which implies that $a_1 > a_2 > \cdots > a_n > 0$. Furthermore, for all $k = 1, 2, \ldots, n$, $a_k = \varphi(t_{k-1}) + b_k t_{k-1} = \varphi(t_k) + b_k t_k$. In particular, $a_n = b_n$ because $\varphi(1) = 0$.

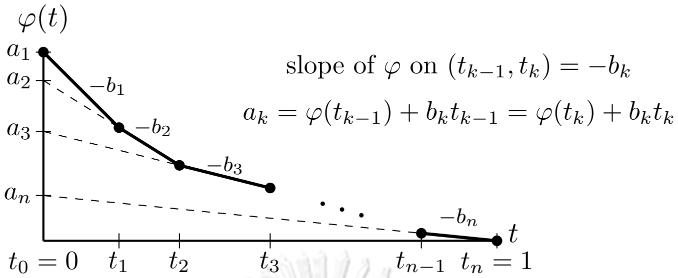

Figure 4.2: Graph of a piecewise linear Archimedean generator φ

By all properties above, we see that if $x \in [\varphi(t_k), \varphi(t_{k-1}))$ for some $k = 1, 2, \ldots, n$, $\varphi^{[-1]}(x) = \dfrac{a_k - x}{b_k} = t_k + \dfrac{\varphi(t_k) - x}{b_k} = t_{k-1} + \dfrac{\varphi(t_{k-1}) - x}{b_k}$, otherwise, $\varphi^{[-1]}(x) = 0$. In addition, we can rewrite φ and $\varphi^{[-1]}$ in another form as follows.

$$\varphi(t) = \begin{cases} \varphi(0) - b_1 t & \text{if } 0 \le t \le t_1; \\ \varphi(t_1) - b_2\,(t - t_1) & \text{if } t_1 < t \le t_2; \\ \quad\vdots & \quad\vdots \\ \varphi(t_{n-2}) - b_{n-1}\,(t - t_{n-2}) & \text{if } t_{n-2} < t \le t_{n-1}; \\ \varphi(t_{n-1}) - b_n\,(t - t_{n-1}) = b_n\,(1 - t) & \text{if } t_{n-1} < t \le 1, \end{cases} \tag{4.1}$$

and

$$\varphi^{[-1]}(x) = \begin{cases} t_{n-1} + \dfrac{\varphi(t_{n-1}) - x}{b_n} = 1 - \dfrac{x}{b_n} & \text{if } 0 \le x < \varphi(t_{n-1}); \\ t_{n-2} + \dfrac{\varphi(t_{n-2}) - x}{b_{n-1}} & \text{if } \varphi(t_{n-1}) \le x < \varphi(t_{n-2}); \\ \quad\vdots & \quad\vdots \\ t_1 + \dfrac{\varphi(t_1) - x}{b_2} & \text{if } \varphi(t_2) \le x < \varphi(t_1); \\ \dfrac{\varphi(0) - x}{b_1} & \text{if } \varphi(t_1) \le x < \varphi(0); \\ 0 & \text{if } x \ge \varphi(0). \end{cases} \tag{4.2}$$

Next, we compute the pointwise dimension of Archimedean copulas generated by piecewise linear generators in the following theorem.

Theorem 4.3. *Let C be an Archimedean copula with piecewise linear generator φ in the form* (4.1)*. Then for any $(u,v)\in(0,1)^2$,*

(A) if $\varphi(u)+\varphi(v)\neq\varphi(t_k)$ for any $k=0,1,\ldots,n$, then $d_C(u,v)=\infty$;

(B) if $\varphi(u)+\varphi(v)=\varphi(t_k)$ for some $k=0,1,\ldots,n$, then $d_C(u,v)=1$.

Proof. To prove (A), we consider 2 cases:
Case A.1 $\varphi(u)+\varphi(v)>\varphi(0)$. Since $A:=\left\{(x,y)\in(0,1)^2 \mid \varphi(x)+\varphi(y)>\varphi(0)\right\}$ is an open set and $(u,v)\in A$, we can choose $\delta>0$ small enough so that for any $h\in(0,\delta)$, $R_h(u,v)\subseteq A$. Hence for any $h\in(0,\delta)$, $V_C(R_h(u,v))=0$, i.e., $\dfrac{\log V_C(R_h(u,v))}{\log h}=\infty$. Therefore, $d_C(u,v)=\infty$.
Case A.2 $\varphi(t_{k+1})<\varphi(u)+\varphi(v)<\varphi(t_k)$ for some $k=0,1,\ldots,n-1$. Since $(u,v)\in O_k:=\{(x,y)\in(0,1)^2 \mid \varphi(x)+\varphi(y)\in(\varphi(t_{k+1}),\varphi(t_k))\}$, which is an open set, there is $\delta>0$ such that for any $h\in(0,\delta)$, $R_h(u,v)\subseteq O_k$. Hence for any $h\in(0,\delta)$, $V_C(R_h(u,v))$ equals

$$\left(t_k+\frac{\varphi(t_k)-(\varphi(u+h)+\varphi(v+h))}{b_{k+1}}\right)-\left(t_k+\frac{\varphi(t_k)-(\varphi(u-h)+\varphi(v+h))}{b_{k+1}}\right)$$
$$-\left(t_k+\frac{\varphi(t_k)-(\varphi(u+h)+\varphi(v-h))}{b_{k+1}}\right)+\left(t_k+\frac{\varphi(t_k)-(\varphi(u-h)+\varphi(v-h))}{b_{k+1}}\right)$$

which is zero, that is, $d_C(u,v)=\lim\limits_{h\to0}\dfrac{\log V_C(R_h(u,v))}{\log(h)}=\infty$.
To prove (B), we again divide into 2 cases:
Case B.1 $\varphi(u)+\varphi(v)=\varphi(0)$. By Lemma 4.1, there is $\delta>0$ such that for any $h\in(0,\delta)$, $V_C(R_h(u,v))$ is equal to

$$\begin{cases} C(u+h,v+h)-C(u-h,v+h)=\dfrac{\varphi(u-h)-\varphi(u+h)}{b_1} & \text{if } u>v;\\ C(u+h,v+h)-C(u+h,v-h)=\dfrac{\varphi(v-h)-\varphi(v+h)}{b_1} & \text{if } u<v;\\ C(u+h,u+h)=\dfrac{\varphi(0)-2\varphi(u+h)}{b_1} & \text{if } u=v,\end{cases}$$

where the right-hand sides of equations above follow from (4.2).
In the case $u\neq v$, we may assume without loss of generality that $u>v$.

- If $u\in(t_\ell,t_{\ell+1})$ for some $\ell=0,1,\ldots,n-1$, then there is $\delta'<\delta$ such that $u-h,u+h\in(t_\ell,t_{\ell+1})$ for all $h\in(0,\delta')$. Thus, for $h\in(0,\delta')$,

$$\begin{aligned} V_C(R_h(u,v)) &= \frac{1}{b_1}\left[(\varphi(t_\ell)-b_{\ell+1}(u-h-t_\ell))-(\varphi(t_\ell)-b_{\ell+1}(u+h-t_\ell))\right]\\ &=\frac{2hb_{\ell+1}}{b_1}. \end{aligned}$$

- If $u = t_\ell$ for some $\ell = 1, 2, \ldots, n-1$ ($u \neq 0, 1$), then there is $\delta' < \delta$ such that for $h \in (0, \delta')$, $u - h \in (t_{\ell-1}, t_\ell)$ and $u + h \in (t_\ell, t_{\ell+1})$. So for $h \in (0, \delta')$,

$$\begin{aligned} V_C\left(R_h(u,v)\right) &= \frac{1}{b_1}\left[\left(\varphi(t_{\ell-1}) - b_\ell\left(u - h - t_{\ell-1}\right)\right) - \left(\varphi(t_\ell) - b_{\ell+1}\left(u + h - t_\ell\right)\right)\right] \\ &= \frac{1}{b_1}\left[\left(\varphi(u) + hb_\ell\right) - \left(\varphi(u) - hb_{\ell+1}\right)\right] = \frac{h\left(b_\ell + b_{\ell+1}\right)}{b_1}. \end{aligned}$$

While in the case $u = v$, we have

$$\begin{aligned} V_C\left(R_h(u,v)\right) &= \frac{1}{b_1}\left[\varphi(0) - 2\left(\varphi(t_\ell) - b_{\ell+1}\left(u + h - t_\ell\right)\right)\right] \\ &= \frac{1}{b_1}\left[\varphi(0) - 2\left(\varphi(u) - b_{\ell+1}h\right)\right] = \frac{2hb_{\ell+1}}{b_1} \end{aligned}$$

for $h > 0$ small enough and $u \in [t_\ell, t_{\ell+1})$ for some ℓ. From all subcases, we see that $V_C\left(R_h(u,v)\right) = Kh$ for some constant K where $h > 0$ is small enough which implies that $d_C(u,v) = \lim\limits_{h \to 0} \dfrac{\log(Kh)}{\log(h)} = 1$.

Case B.2 $\varphi(u) + \varphi(v) = \varphi(t_k)$ for some $k = 1, 2, \ldots, n-1$. By Lemma 4.1, there is $\delta > 0$ such that for any $h \in (0, \delta)$, we have

- if $u > v$, $V_C\left(R_h(u,v)\right)$

$$\begin{aligned} &= \left[t_k + \frac{\varphi(t_k) - (\varphi(u+h) + \varphi(v+h))}{b_{k+1}}\right] - \left[t_k + \frac{\varphi(t_k) - (\varphi(u-h) + \varphi(v+h))}{b_{k+1}}\right] \\ &\quad - \left[t_{k-1} + \frac{\varphi(t_{k-1}) - (\varphi(u+h) + \varphi(v-h))}{b_k}\right] \\ &\quad + \left[t_{k-1} + \frac{\varphi(t_{k-1}) - (\varphi(u-h) + \varphi(v-h))}{b_k}\right] \\ &= \frac{\varphi(u-h) - \varphi(u+h)}{b_{k+1}} - \frac{\varphi(u-h) - \varphi(u+h)}{b_k} \\ &= \frac{b_k - b_{k+1}}{b_k b_{k+1}}\left(\varphi(u-h) - \varphi(u+h)\right). \end{aligned}$$

- similarly, if $u < v$, $V_C\left(R_h(u,v)\right)$

$$\begin{aligned} &= \left[t_k + \frac{\varphi(t_k) - (\varphi(u+h) + \varphi(v+h))}{b_{k+1}}\right] - \left[t_k + \frac{\varphi(t_k) - (\varphi(u+h) + \varphi(v-h))}{b_{k+1}}\right] \\ &\quad - \left[t_{k-1} + \frac{\varphi(t_{k-1}) - (\varphi(u-h) + \varphi(v+h))}{b_k}\right] \\ &\quad + \left[t_{k-1} + \frac{\varphi(t_{k-1}) - (\varphi(u-h) + \varphi(v-h))}{b_k}\right] \\ &= \frac{b_k - b_{k+1}}{b_k b_{k+1}}\left(\varphi(v-h) - \varphi(v+h)\right). \end{aligned}$$

- if $u = v$, then from $C(u+h, u-h) = C(u-h, u+h)$,

$$
\begin{aligned}
V_C\left(R_h(u,v)\right) &= \left[t_k + \frac{\varphi(t_k) - 2\varphi(u+h)}{b_{k+1}}\right] + \left[t_k + \frac{\varphi(t_k) - 2\varphi(u-h)}{b_k}\right] \\
&\quad - 2\left[t_k + \frac{\varphi(t_k) - (\varphi(u-h) + \varphi(u+h))}{b_k}\right] \\
&= \left[t_k + \frac{\varphi(t_k) - 2\varphi(u+h)}{b_{k+1}}\right] - \left[t_k + \frac{\varphi(t_k) - 2\varphi(u+h)}{b_k}\right] \\
&= \left[\frac{1}{b_{k+1}} - \frac{1}{b_k}\right]\left(\varphi(t_k) - 2\varphi(u+h)\right) = \frac{b_k - b_{k+1}}{b_k b_{k+1}}\left(\varphi(t_k) - 2\varphi(u+h)\right).
\end{aligned}
$$

Now, with a similar argument to the Case B.1, if $\max\{u,v\} \in [t_\ell, t_{\ell+1})$ for some ℓ, then for $h > 0$ small enough,

$$
V_C\left(R_h(u,v)\right) = \begin{cases} \dfrac{h\left(b_\ell + b_{\ell+1}\right)\left(b_k - b_{k+1}\right)}{b_k b_{k+1}} & \text{if } u \neq v \text{ and } \max\{u,v\} = t_\ell; \\ \dfrac{2hb_{\ell+1}\left(b_k - b_{k+1}\right)}{b_k b_{k+1}} & \text{otherwise.} \end{cases}
$$

This implies that $V_C\left(R_h(u,v)\right) = Lh$ for some constant L where $h > 0$ is small enough, and so $d_C(u,v) = \lim_{h \to 0} \dfrac{\log(Lh)}{\log(h)} = 1$. □

Example 4.4. Let $\varphi(t) = \begin{cases} 1 - 3t & \text{if } t \in \left[0, \frac{1}{4}\right]; \\ \frac{1-t}{3} & \text{if } t \in \left(\frac{1}{4}, 1\right]. \end{cases}$ The graph of φ and the support of $C = C_\varphi$ are shown in Figure 4.3.

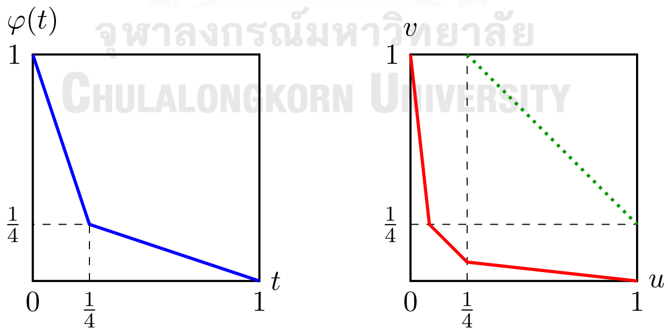

Figure 4.3: (Left) Graph of φ. (Right) Support of C in Example 4.4

By Theorem 4.3(A) and (B), we have

$$
d_C(u,v) = \begin{cases} 1 & \text{if } \varphi(u) + \varphi(v) = 1 \text{ (solid) or } \varphi(u) + \varphi(v) = \frac{1}{4} \text{ (dotted);} \\ \infty & \text{otherwise.} \end{cases}
$$

Example 4.5. Let $\phi(t) = \begin{cases} 1-2t & \text{if } t \in \left[0, \frac{1}{5}\right]; \\ \frac{4-5t}{5} & \text{if } t \in \left(\frac{1}{5}, \frac{7}{10}\right]; \\ \frac{1-t}{3} & \text{if } t \in \left(\frac{7}{10}, 1\right]. \end{cases}$ The graph of ϕ and the support of $C = C_\phi$ are shown in Figure 4.4.

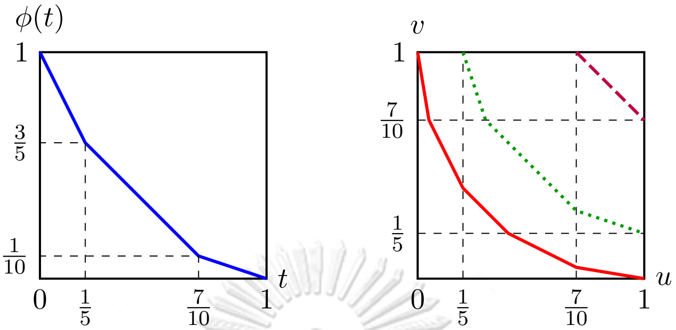

Figure 4.4: (Left) Graph of ϕ. (Right) Support of C in Example 4.5

By Theorem 4.3(A) and (B), we have

$$d_C(u,v) = \begin{cases} 1 & \text{if } \phi(u)+\phi(v) = 1 \text{ (solid) or } \phi(u)+\phi(v) = \frac{3}{5} \text{ (dotted)} \\ & \text{or } \phi(u)+\phi(v) = \frac{1}{10} \text{ (dashed);} \\ \infty & \text{otherwise.} \end{cases}$$

4.2 General case

First of all, let $\varphi'(x^-)$ and $\varphi'(x^+)$ denote the left and right derivatives of φ at x, respectively. Now, we show some statements that will be used in a part of the main theorem.

Remark 4.6. Let C be an Archimedean copula with generator φ, $(u,v) \in (0,1)^2$, $t = C(u,v)$ (i.e., $\varphi(t) = \varphi(u) + \varphi(v)$) and $*, \bullet \in \{+,-\}$. If $h > 0$ is such that $s := C(u*h, v\bullet h) - C(u,v) \neq 0$, then

$$\begin{aligned} s &= \frac{C(u*h, v\bullet h) - C(u,v)}{\varphi\left(C(u*h, v\bullet h)\right) - \varphi\left(C(u,v)\right)} \cdot \left[\varphi\left(C(u*h, v\bullet h)\right) - \varphi\left(C(u,v)\right)\right] \\ &= \left(\frac{\varphi(t+s) - \varphi(t)}{s}\right)^{-1} \cdot \left[\left(\varphi(u*h) - \varphi(u)\right) + \left(\varphi(v\bullet h) - \varphi(v)\right)\right]. \end{aligned} \tag{4.3}$$

Lemma 4.7. *Let C be an Archimedean copula with generator φ. Suppose that $(u,v) \in (0,1)^2$ satisfies $\varphi(u) + \varphi(v) = \varphi(0) < \infty$. Then for any $\alpha > 0$,*

$$\underline{D^\alpha C}\,(u,v) \geq -\varphi'\left(\max\{u,v\}^+\right) \liminf_{h\to 0^+} \gamma(h) \tag{4.4}$$

where $\gamma(h) := \dfrac{C(u+h,v+h)}{f_\varphi\left(C(u+h,v+h)\right)(2h)^{\alpha-1}}$ *and* $f_\varphi(x) = \varphi(0) - \varphi(x)$.

Proof. By assumption, $C(u,v) = 0$. If $u = v$, then by Lemma 4.1, $V_C\left(R_h\left(u,v\right)\right) = C\left(u+h,u+h\right)$ for $h > 0$ small enough. Hence by Remark 4.6,

$$\begin{aligned}\frac{V_C\left(R_h(u,v)\right)}{(2h)^\alpha} &= \left(\frac{\varphi\left(C(u+h,u+h)\right)-\varphi(0)}{C(u+h,u+h)}\right)^{-1}\left[\frac{2\left(\varphi(u+h)-\varphi(u)\right)}{2h}\right]\frac{1}{(2h)^{\alpha-1}}\\ &= -\frac{C(u+h,u+h)}{f_\varphi\left(C(u+h,u+h)\right)}\left[\frac{\varphi(u+h)-\varphi(u)}{h}\right]\frac{1}{(2h)^{\alpha-1}}.\end{aligned}$$

Taking limit inferior as $h \to 0^+$ yields

$$\underline{D^\alpha C}(u,v) \geq -\lim_{h\to 0^+}\left[\frac{\varphi(u+h)-\varphi(u)}{h}\right]\liminf_{h\to 0^+}\gamma(h) = -\varphi'(u^+)\liminf_{h\to 0^+}\gamma(h).$$

Now, we will show (4.4) in the case $u > v$ (the case $u < v$ is similar).
From Lemma 4.1, we divide into 2 subcases:
Subcase 1: $C(u-h,v+h) = 0$ for some $h > 0$ small. Then by Corollary 4.2, for any $h' \in [0,h]$, $C(u-h',v+h') = 0$. With a similar approach as in the case $u = v$, we have

$$\begin{aligned}\underline{D^\alpha C}(u,v) &\geq -\lim_{h\to 0^+}\left[\frac{\varphi(u+h)-\varphi(u)}{2h}+\frac{\varphi(v+h)-\varphi(v)}{2h}\right]\liminf_{h\to 0^+}\gamma(h)\\ &= -\frac{\varphi'(u^+)+\varphi'(v^+)}{2}\liminf_{h\to 0^+}\gamma(h) \geq -\varphi'(u^+)\liminf_{h\to 0^+}\gamma(h)\end{aligned}$$

where the last inequality follows from Proposition 2.16(2).
Subcase 2: $C(u-h,v+h) > 0$ for all $h > 0$ small. In this case, we compute each term of $V_C\left(R_h(u,v)\right)$ by using Remark 4.6 and we obtain

$$C(u+h,v+h) = \left(\frac{\varphi\left(C(u+h,v+h)\right)-\varphi(0)}{C(u+h,v+h)}\right)^{-1}\left[\left(\varphi(u+h)-\varphi(u)\right)+\left(\varphi(v+h)-\varphi(v)\right)\right] \text{ and}$$

$$C(u-h,v+h) = \left(\frac{\varphi\left(C(u-h,v+h)\right)-\varphi(0)}{C(u-h,v+h)}\right)^{-1}\left[\left(\varphi(u-h)-\varphi(u)\right)+\left(\varphi(v+h)-\varphi(v)\right)\right].$$

Since for $h > 0$ small enough,

(1) $0 < C(u-h,v+h) < C(u+h,v+h)$ and

$$C(u-h,v+h) = K(h)C(u+h,v+h) + (1-K(h))\cdot 0,$$

where $K(h) := \dfrac{C(u-h,v+h)}{C(u+h,v+h)}$;

(2) $v < v + h < u - h < u$,

we apply φ to both inequalities and, by its convexity, we obtain

(1') $\varphi\left(C(u-h, v+h)\right) \leq K(h)\varphi\left(C(u+h, v+h)\right) + \left(1 - K(h)\right)\varphi(0)$;

(2') $\dfrac{\varphi(v+h) - \varphi(v)}{h} \leq \dfrac{\varphi(u-h) - \varphi(v+h)}{u - v - 2h} \leq \dfrac{\varphi(u) - \varphi(u-h)}{h}$.

Thus, these two inequalities become

(1") $\dfrac{\varphi\left(C(u-h, v+h)\right) - \varphi(0)}{C(u-h, v+h)} \leq \dfrac{\varphi\left(C(u+h, v+h)\right) - \varphi(0)}{C(u+h, v+h)}$;

(2") $\varphi(v+h) - \varphi(v) \leq \varphi(u) - \varphi(u-h)$,

i.e., $\left(\dfrac{\varphi\left(C(u-h, v+h)\right) - \varphi(0)}{C(u-h, v+h)}\right)^{-1} \geq \left(\dfrac{\varphi\left(C(u+h, v+h)\right) - \varphi(0)}{C(u+h, v+h)}\right)^{-1}$ and $(\varphi(u-h) - \varphi(u)) + (\varphi(v+h) - \varphi(v)) \leq 0$. Hence $C(u-h, v+h) \leq \left(\dfrac{\varphi\left(C(u+h, v+h)\right) - \varphi(0)}{C(u+h, v+h)}\right)^{-1} [(\varphi(u-h) - \varphi(u)) + (\varphi(v+h) - \varphi(v))]$, i.e.,
$V_C\left(R_h(u, v)\right) \geq -\dfrac{C(u+h, v+h)}{f_\varphi\left(C(u+h, v+h)\right)} \left[(\varphi(u+h) - \varphi(u)) - (\varphi(u-h) - \varphi(u))\right]$.
Since $\lim\limits_{h\to 0^+} \left[\dfrac{\varphi(u+h) - \varphi(u)}{2h} - \dfrac{\varphi(u-h) - \varphi(u)}{2h}\right] = \dfrac{\varphi'(u^+) + \varphi'(u^-)}{2}$, we have

$$
\begin{aligned}
\underline{D^\alpha C}(u, v) &\geq -\frac{\varphi'(u^+) + \varphi'(u^-)}{2} \liminf_{h\to 0^+} \frac{C(u+h, v+h)}{f_\varphi\left(C(u+h, v+h)\right)(2h)^{\alpha-1}} \\
&= -\frac{\varphi'(u^+) + \varphi'(u^-)}{2} \liminf_{h\to 0^+} \gamma(h) \geq -\varphi'(u^+) \liminf_{h\to 0^+} \gamma(h)
\end{aligned}
$$

where the last inequality follows from Proposition 2.16(2). □

Lemma 4.8. *Let C be an Archimedean copula with generator φ and $f_\varphi(x) = \varphi(0) - \varphi(x)$ be a regularly varying function of index $\beta > 0$ at 0. Suppose that $(u, v) \in (0, 1)^2$ satisfies $\varphi(u) + \varphi(v) = \varphi(0)$. Then*

(a) $g_\varphi(x) := \varphi^{[-1]}\left(\varphi(0) - x\right) = \varphi^{-1}\left(\varphi(0) - x\right) = f_\varphi^{-1}(x) \in RV^0_{1/\beta}$, and

(b) $F(x) = C\left(u + x, v + x\right) \in RV^0_{1/\beta}$.

Proof. To prove (a), let $\lambda > 0$. Since $\lambda^{1/\beta} = \lim\limits_{x\to 0^+} \left(\dfrac{f_\varphi(g_\varphi(\lambda x))}{f_\varphi(g_\varphi(x))}\right)^{1/\beta}$, we see that for any $\varepsilon > 0$ small, there is $\delta_1 > 0$ such that for each $x \in (0, \delta_1)$,

$$
\lambda^{1/\beta} - \varepsilon < \left(\frac{f_\varphi(g_\varphi(\lambda x))}{f_\varphi(g_\varphi(x))}\right)^{1/\beta} < \lambda^{1/\beta} + \varepsilon.
$$

Now, let $A > 1$ be arbitrary and suppose that $\lambda > 1$. By Theorem 2.29, there is $X = X(A, \varepsilon)$ such that for each $0 < y \le x \le X$,

$$A^{-1}\left(\frac{x}{y}\right)^{\beta-\varepsilon} < \frac{f_\varphi(x)}{f_\varphi(y)} < A\left(\frac{x}{y}\right)^{\beta+\varepsilon}.$$

Since f_φ is strictly increasing, so is g_φ. Furthermore, from $\lambda > 1$, we have $g_\varphi(\lambda x) > g_\varphi(x)$ for all $x \in (0, \varphi(0)/\lambda)$. Next, from $\lim_{x\to 0^+} g_\varphi(\lambda x) = 0$, there is $\delta_2 \in (0, \varphi(0)/\lambda)$ such that for $x \in (0, \delta_2)$, $g_\varphi(x) < g_\varphi(\lambda x) < X$, which implies that for $x \in (0, \delta_2)$,

$$A^{-1}\left(\frac{g_\varphi(\lambda x)}{g_\varphi(x)}\right)^{\beta-\varepsilon} < \frac{f_\varphi(g_\varphi(\lambda x))}{f_\varphi(g_\varphi(x))} < A\left(\frac{g_\varphi(\lambda x)}{g_\varphi(x)}\right)^{\beta+\varepsilon}.$$

Choose $\delta = \min\{\delta_1, \delta_2\}$. Then for $x \in (0, \delta)$, we have the following two inequalities:

$$\lambda^{1/\beta} - \varepsilon < A^{1/\beta}\left(\frac{g_\varphi(\lambda x)}{g_\varphi(x)}\right)^{1+\varepsilon/\beta} \quad \text{and} \quad A^{-1/\beta}\left(\frac{g_\varphi(\lambda x)}{g_\varphi(x)}\right)^{1-\varepsilon/\beta} < \lambda^{1/\beta} + \varepsilon.$$

Since above statements hold for any $A > 1$ and $\varepsilon > 0$ small enough, we have $\lim_{x\to 0^+} \frac{g_\varphi(\lambda x)}{g_\varphi(x)} = \lambda^{1/\beta}$. The case $0 < \lambda < 1$ is similar and thus, $g_\varphi \in RV^0_{1/\beta}$ as desired.

Next, we show that (b) holds. First, we consider $d_u(x) = \varphi(u) - \varphi(u + x)$ for $x \in [0, 1-u]$ and $d_v(x) = \varphi(v) - \varphi(v + x)$ for $x \in [0, 1-v]$. Then for any $\lambda > 0$,

$$\begin{aligned}\frac{d_u(\lambda x)}{d_u(x)} &= \left[\frac{\varphi(u) - \varphi(u + \lambda x)}{\lambda x}\right]\left[\frac{\lambda x}{\varphi(u) - \varphi(u + x)}\right] \\ &= \lambda\left[\frac{\varphi(u + \lambda x) - \varphi(u)}{\lambda x}\right]\left[\frac{\varphi(u + x) - \varphi(u)}{x}\right]^{-1}.\end{aligned}$$

Hence $\lim_{x\to 0^+} \frac{d_u(\lambda x)}{d_u(x)} = \lambda \cdot \varphi'(u^+) \cdot \left[\varphi'(u^+)\right]^{-1} = \lambda$ and, similarly, $\lim_{x\to 0^+} \frac{d_v(\lambda x)}{d_v(x)} = \lambda$. Hence $d_u, d_v \in RV^0_1$, which implies that $d_u + d_v \in RV^0_1$ (by Corollary 2.31(iv)). Now, since

$$\begin{aligned}F(x) = C(u + x, v + x) &= \varphi^{[-1]}(\varphi(u + x) + \varphi(v + x)) \\ &= \varphi^{[-1]}\Big(\varphi(0) - \big((\varphi(u) + \varphi(v)) - (\varphi(u + x) + \varphi(v + x))\big)\Big) \\ &= g_\varphi\Big(\big(\varphi(u) - \varphi(u + x)\big) + \big(\varphi(v) - \varphi(v + x)\big)\Big) = (g_\varphi \circ (d_u + d_v))(x),\end{aligned}$$

by (a), $\lim_{x\to 0^+} (d_u + d_v)(x) = 0$ and Corollary 2.31(iv), we have $F \in RV^0_{1/\beta}$ as desired. □

We are now ready to consider the general case of Archimedean copulas.

Theorem 4.9. *Let C be an Archimedean copula with generator φ, a function that is convex, continuous, strictly decreasing on $\mathbb{I}$ and $\varphi(1) = 0$. Let $(u, v) \in (0, 1)^2$.*

(A) If $\varphi(u) + \varphi(v) > \varphi(0)$, then $d_C(u, v) = \infty$.

(B) If $\varphi(u) + \varphi(v) = \varphi(0)$ and $f_\varphi(x) = \varphi(0) - \varphi(x) \in RV_\beta^0$ where $\beta > 0$, then $d_C(u, v) = \frac{1}{\beta}$.

(C) If $\varphi(u) + \varphi(v) = \varphi(t)$ where $t \in (0, 1)$ and φ is not differentiable at t, then $d_C(u, v) = 1$.

(D) If $\varphi(u) + \varphi(v) = \varphi(t)$, $\mathrm{SD}_2\,\varphi(u), \mathrm{SD}_2\,\varphi(v) < \infty$, φ is differentiable at t and $\mathrm{SD}_2\,\varphi(t) \in (0, \infty)$ where $t \in (0, 1)$, then $d_C(u, v) = 2$, where $\mathrm{SD}_2\,\varphi(x)$ is the second order symmetric derivative of φ at x defined in Definition 2.18.

Note: We see that Theorem 4.9(B) and (C) are generalizations of Theorem 4.3(B) in piecewise linear generators case because if there exist a partition $\{t_i\}_{i=0}^n$ of $\mathbb{I}$ and finite subsets $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n$ of $\mathbb{R}^+$ such that φ can be expressed as in (4.1), then we have the following cases.

- If $(u, v) \in (0, 1)^2$ is such that $\varphi(u)+\varphi(v) = \varphi(0)$, then $f_\varphi(x) = \varphi(0)-\varphi(x) = b_1 x$ for $x \in [0, t_1]$ which implies that $f_\varphi \in RV_1^0$ and by Theorem 4.9(B), $d_C(u, v) = 1$.
- If $(u, v) \in (0, 1)^2$ is such that $\varphi(u) + \varphi(v) = \varphi(t_k)$ for some $k = 1, \ldots, n$, then (u, v) satisfies the assumption of Theorem 4.9(C) and so $d_C(u, v) = 1$.

By all cases in Theorem 4.9, any two points from the same *level curve* give the same pointwise dimension as in Figure 4.5.

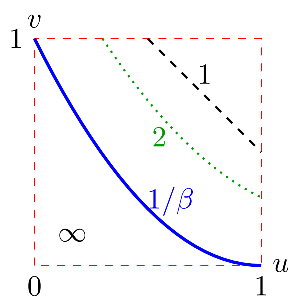

Figure 4.5: The value of $d_C(u, v)$

In this figure, the solid curve, $Z(C) := \{(u, v) : \varphi(u) + \varphi(v) = \varphi(0)\}$, is called the *zero curve* of C, while the dashed and dotted curves are *level curves* $\{(u, v) : \varphi(u)+$

$\varphi(v) = \varphi(0.5)\}$ and $\{(u, v) : \varphi(u)+\varphi(v) = \varphi(0.3)\}$, respectively (where we suppose that φ is not differentiable at 0.5 but $\varphi'(0.3)$ exists and $\mathrm{SD}_2\,\varphi(0.3) \in (0, \infty)$).
To prove this theorem, we divide the proof into several parts as follows:

Proof of Theorem 4.9(A),(B) and (C). First, for statement (A), we can prove in the same way as Theorem 4.3(A.1) and we obtain $d_C(u, v) = \infty$.
To prove (B), by Lemmas 4.1 and 4.8(b), we have $V_C\left(R_h(u, v)\right) \leq C(u+h, v+h) = h^{1/\beta}\ell(h)$ for some $\ell \in RV_0^0$. Hence

$$\underline{d}_C(u, v) \geq \liminf_{h\to 0} \frac{\log\left(h^{1/\beta}\ell(h)\right)}{\log(h)} = \liminf_{h\to 0}\left(\frac{1}{\beta} + \frac{\log\left(\ell(h)\right)}{\log(h)}\right) = \frac{1}{\beta}$$

where the last equality follows from Corollary 2.31(ii). Next, from $C(u+h, v+h) \in RV_{1/\beta}^0$ (Lemma 4.8(b)), $f_\varphi \in RV_\beta^0$, $\lim_{h\to 0^+} C(u+h, v+h) = 0$ and Corollary 2.31(iv), we have $f_\varphi\left(C(u+h, v+h)\right) \in RV_1^0$. By Corollary 2.31(iv) again, we see that for any $\alpha > 0$, $\gamma(h) = \dfrac{C(u+h, v+h)}{f_\varphi\left(C(u+h, v+h)\right)(2h)^{\alpha-1}}$ is a regularly varying function of index $\dfrac{1}{\beta} - 1 - (\alpha - 1) = \dfrac{1}{\beta} - \alpha$ at the right of 0. Now, we can rewrite (4.4) as

$$\underline{D^\alpha C}\,(u, v) \geq -\varphi'\left(\max\{u, v\}^+\right)\liminf_{h\to 0}\left(h^{1/\beta-\alpha}L(h)\right)$$

for some $L \in RV_0^0$. For $\alpha > 1/\beta$, $\lim_{h\to 0} h^{1/\beta-\alpha}L(h) = \infty$ by Corollary 2.31(i) and $\underline{D^\alpha C}\,(u, v) = \infty$ by the above inequality. This implies that

$$\overline{d}_C(u, v) = \overline{\alpha}_C(u, v) = \inf\left\{\alpha \in \mathbb{R}^+ : \underline{D^\alpha C}\,(u, v) = \infty\right\} \leq \frac{1}{\beta}.$$

Therefore, $d_C(u, v) = \dfrac{1}{\beta}$.
Next, we prove the statement (C). We first note that

$$\lim_{h\to 0^+} \frac{\varphi(u+h) - \varphi(u)}{h} = \varphi'(u^+) \text{ and } \lim_{h\to 0^+} \frac{\varphi(u-h) - \varphi(u)}{h} = -\varphi'(u^-).$$

The same statements hold for $\varphi'(v^+)$ and $\varphi'(v^-)$. Now, we define the following notations for each $h > 0$.

$$s_1(h) := C(u+h, v+h) - C(u, v), \qquad s_2(h) := C(u-h, v+h) - C(u, v),$$
$$s_3(h) := C(u+h, v-h) - C(u, v), \qquad s_4(h) := C(u-h, v-h) - C(u, v).$$

By Lemma 4.1, we see that $s_1(h) > 0$ and $s_4(h) < 0$ for all $h > 0$. Hence by Remark 4.6 and $t = C(u, v)$, $\lim_{h\to 0^+} \dfrac{s_1(h)}{h}$

$$= \lim_{h\to 0^+}\left(\frac{\varphi(t+s_1(h)) - \varphi(t)}{s_1(h)}\right)^{-1}\left[\frac{\varphi(u+h) - \varphi(u)}{h} + \frac{\varphi(v+h) - \varphi(v)}{h}\right]$$

$$= \left(\lim_{h\to 0^+} \frac{\varphi(t+s_1(h)) - \varphi(t)}{s_1(h)} \right)^{-1} \left[\lim_{h\to 0^+} \frac{\varphi(u+h)-\varphi(u)}{h} + \lim_{h\to 0^+} \frac{\varphi(v+h)-\varphi(v)}{h} \right]$$
$$= \frac{1}{\varphi'(t^+)} \left[\varphi'(u^+) + \varphi'(v^+) \right]$$

and similarly,

$$\lim_{h\to 0^+} \frac{s_4(h)}{h} = \frac{1}{\varphi'(t^-)} \left[-\varphi'(u^-) - \varphi'(v^-) \right].$$

Next, by Corollary 4.2, we consider 4 cases as follows.

Case 1: for any $h > 0$ small enough, $C(u-h, v+h) \neq C(u,v)$ and $C(u+h, v-h) \neq C(u,v)$. We find $\lim_{h\to 0^+} \frac{s_2(h)}{h}$ and $\lim_{h\to 0^+} \frac{s_3(h)}{h}$. For example, to find $\lim_{h\to 0^+} \frac{s_2(h)}{h}$, we use Remark 4.6 to obtain

$$s_2(h) = \left(\frac{\varphi(t+s_2(h)) - \varphi(t)}{s_2(h)} \right)^{-1} \left[(\varphi(u-h) - \varphi(u)) + (\varphi(v+h) - \varphi(v)) \right].$$

Later on, by taking limit $h \to 0^+$ together with using Lemma 4.1 to consider the value of $s_2(h)$, it yields that

$$\lim_{h\to 0^+} \frac{s_2(h)}{h} = \begin{cases} \frac{1}{\varphi'(t^+)} \left[-\varphi'(u^-) + \varphi'(v^+) \right] & \text{if } u > v; \\ \frac{1}{\varphi'(t^-)} \left[-\varphi'(u^-) + \varphi'(v^+) \right] & \text{if } u \leq v. \end{cases}$$

Similarly,

$$\lim_{h\to 0^+} \frac{s_3(h)}{h} = \begin{cases} \frac{1}{\varphi'(t^-)} \left[\varphi'(u^+) - \varphi'(v^-) \right] & \text{if } u \geq v; \\ \frac{1}{\varphi'(t^+)} \left[\varphi'(u^+) - \varphi'(v^-) \right] & \text{if } u < v. \end{cases}$$

Hence $D^1C(u,v) = \lim_{h\to 0^+} \frac{V_C(R_h(u,v))}{2h}$ equals

$$\begin{cases} \left(\frac{1}{\varphi'(t^+)} - \frac{1}{\varphi'(t^-)} \right) \left(\frac{\varphi'(w^+) + \varphi'(w^-)}{2} \right) & \text{if } u \neq v \text{ and } w = \max\{u,v\}; \\ \left(\frac{1}{\varphi'(t^+)} - \frac{1}{\varphi'(t^-)} \right) \varphi'(u^+) & \text{if } u = v. \end{cases}$$

To see this, we show only the case $u > v$ since the other cases are similar.

$$\begin{aligned} D^1C(u,v) &= \lim_{h\to 0^+} \frac{s_1(h)}{2h} - \lim_{h\to 0^+} \frac{s_2(h)}{2h} - \lim_{h\to 0^+} \frac{s_3(h)}{2h} + \lim_{h\to 0^+} \frac{s_4(h)}{2h} \\ &= \frac{1}{2\varphi'(t^+)} \left[\varphi'(u^+) + \varphi'(v^+) \right] - \frac{1}{2\varphi'(t^+)} \left[-\varphi'(u^-) + \varphi'(v^+) \right] \\ &\quad - \frac{1}{2\varphi'(t^-)} \left[\varphi'(u^+) - \varphi'(v^-) \right] + \frac{1}{2\varphi'(t^-)} \left[-\varphi'(u^-) - \varphi'(v^-) \right] \\ &= \left(\frac{1}{\varphi'(t^+)} - \frac{1}{\varphi'(t^-)} \right) \left(\frac{\varphi'(u^+) + \varphi'(u^-)}{2} \right). \end{aligned}$$

Case 2: for any $h > 0$, $C(u-h, v+h) \neq C(u,v)$ and there is $\delta > 0$ such that $C(u+\delta, v-\delta) = C(u,v)$. Then by Corollary 4.2, $C(u+h, v-h) = C(u,v)$ for all $h \in (0,\delta)$ and $\varphi'(u^+) = \varphi'(v^-)$. Moreover, by symmetry of Archimedean copula C, we have $u \neq v$. Since for any $h \in (0,\delta)$,

$$\begin{aligned} V_C\left(R_h(u,v)\right) &= C(u+h,v+h) - C(u-h,v+h) - C(u,v) + C(u-h,v-h) \\ &= (C(u+h,v+h) - C(u,v)) - (C(u-h,v+h) - C(u,v)) \\ &\quad + (C(u-h,v-h) - C(u,v)) \\ &= s_1(h) - s_2(h) + s_4(h), \end{aligned}$$

we see that if $u > v$, then by Lemma 4.1 and Remark 4.6,

$$\begin{aligned} D^1C(u,v) &= \lim_{h\to 0^+} \frac{s_1(h)}{2h} - \lim_{h\to 0^+} \frac{s_2(h)}{2h} + \lim_{h\to 0^+} \frac{s_4(h)}{2h} \\ &= \frac{1}{2\varphi'(t^+)}\left[\varphi'(u^+) + \varphi'(v^+)\right] - \frac{1}{2\varphi'(t^+)}\left[-\varphi'(u^-) + \varphi'(v^+)\right] \\ &\quad - \frac{1}{2\varphi'(t^-)}\left[\varphi'(u^-) + \varphi'(v^-)\right] \\ &= \frac{1}{\varphi'(t^+)}\left(\frac{\varphi'(u^+) + \varphi'(u^-)}{2}\right) - \frac{1}{\varphi'(t^-)}\left(\frac{\varphi'(u^-) + \varphi'(v^-)}{2}\right) \\ &= \left(\frac{1}{\varphi'(t^+)} - \frac{1}{\varphi'(t^-)}\right)\left(\frac{\varphi'(u^+) + \varphi'(u^-)}{2}\right). \end{aligned}$$

Similarly, if $u < v$, then $D^1C(u,v) = \left(\frac{1}{\varphi'(t^+)} - \frac{1}{\varphi'(t^-)}\right)\left(\frac{\varphi'(v^+) + \varphi'(v^-)}{2}\right)$.

Case 3: there is $\delta > 0$ such that $C(u-\delta, v+\delta) = C(u,v)$ and for any $h > 0$, $C(u+h, v-h) \neq C(u,v)$. We prove this in a similar manner as in Case 2 and obtain

$$D^1C(u,v) = \left(\frac{1}{\varphi'(t^+)} - \frac{1}{\varphi'(t^-)}\right)\left(\frac{\varphi'(w^+) + \varphi'(w^-)}{2}\right),$$

where $w = max\{u,v\}$.

Case 4: there is $\delta > 0$ such that $C(u-\delta, v+\delta) = C(u,v) = C(u+\delta, v-\delta)$. Then by Corollary 4.2, $C(u-h, v+h) = C(u,v) = C(u+h, v-h)$ for all $h \in (0,\delta)$, $\varphi'(u^+) = \varphi'(v^-)$ and $\varphi'(u^-) = \varphi'(v^+)$. Moreover, by Proposition 2.16(2), we have $\varphi'(u) = \varphi'(v)$. Since for any $h \in (0,\delta)$,

$$\begin{aligned} V_C\left(R_h(u,v)\right) &= C(u+h,v+h) - C(u,v) - C(u,v) + C(u-h,v-h) \\ &= (C(u+h,v+h) - C(u,v)) + (C(u-h,v-h) - C(u,v)) \\ &= s_1(h) + s_4(h), \end{aligned}$$

by Remark 4.6, we obtain

$$D^1C(u,v) = \lim_{h\to 0^+} \frac{s_1(h)}{2h} + \lim_{h\to 0^+} \frac{s_4(h)}{2h}$$

$$= \frac{1}{2\varphi'(t^+)}\left(\varphi'(u^+)+\varphi'(v^+)\right) - \frac{1}{2\varphi'(t^-)}\left(\varphi'(u^-)+\varphi'(v^-)\right)$$
$$= \frac{\varphi'(u)}{\varphi'(t^+)} - \frac{\varphi'(u)}{\varphi'(t^-)} = \left(\frac{1}{\varphi'(t^+)} - \frac{1}{\varphi'(t^-)}\right)\varphi'(u).$$

From all cases, since $\varphi'(t^-) < \varphi'(t^+)$, we have $D^1C(u,v) \in (0,\infty)$ which implies by Lemma 3.3 and Proposition 3.5 that $d_C(u,v) = \alpha_C(u,v) = 1$. □

In order to prove (D), we introduce some notations of quotients as follows:

- for any real-valued function f on an open interval $D \subseteq \mathbb{R}$, $a \in \mathbb{R}$ and $r \neq 0$ such that $a \pm r \in D$, define $Q_r f(a) = \dfrac{f(a+r) - 2f(a) + f(a-r)}{r^2}$.
- for any real-valued function F on an open set $D \subseteq \mathbb{I}^2$, $(u,v) \in D$ and $r \neq 0$ such that $(u \pm r, v \pm r) \in D$, define

$$Q_r^+ F(u,v) = \frac{F(u+r,v+r) - 2F(u,v) + F(u-r,v-r)}{r^2}$$
$$\text{and} \quad Q_r^- F(u,v) = \frac{F(u-r,v+r) - 2F(u,v) + F(u+r,v-r)}{r^2}.$$

Proof of Theorem 4.9(D). Since $D := \left\{(x,y) \in (0,1)^2 \mid \varphi(x) + \varphi(y) < \varphi(0)\right\}$ is an open set and $(u,v) \in D$, we can choose $\delta > 0$ small enough so that for any $h \in (0,\delta)$, $R_h(u,v) \subseteq D$. From $t = C(u,v)$, we set the following notations for each $h \in (0,\delta)$.

$$s_1 := s_1(h) = C(u+h,v+h) - C(u,v), \quad s_2 := s_2(h) = C(u-h,v+h) - C(u,v),$$
$$s_3 := s_3(h) = C(u+h,v-h) - C(u,v), \quad s_4 := s_4(h) = C(u-h,v-h) - C(u,v),$$

and $K_i(h) := \dfrac{\varphi(t+s_{5-i}) - \varphi(t-s_i)}{s_i + s_{5-i}}$ for $i = 1,2,3,4$ with $s_i + s_{5-i} \neq 0$.
Note that $s_1 > 0$, $s_4 < 0$ and $s_1 + s_4 \geq s_2 + s_3$ for all $h \in (0,\delta)$.
Since $\mathrm{SD}_2\,\varphi(u), \mathrm{SD}_2\,\varphi(v) < \infty$, by Proposition 2.19(4), we have $\varphi'(u)$ and $\varphi'(v)$ exist. Moreover, since φ is convex and differentiable at t and $\mathrm{SD}_2\,\varphi(t) \in (0,\infty)$, by Corollary 4.2, we have the following results.

Remark 4.10. For all $i \in \{1,2,3,4\}$,

(a) $\lim\limits_{h\to 0^+} Q_{s_i}\varphi(t) = \lim\limits_{s_i \to 0} Q_{s_i}\varphi(t) = \mathrm{SD}_2\,\varphi(t)$ if $s_i \neq 0$ for all $h \in (0,\delta)$;

(b) $\lim\limits_{h\to 0^+} \dfrac{s_i}{\varphi(t+s_i) - \varphi(t)} = \dfrac{1}{\lim\limits_{s_i\to 0} \frac{\varphi(t+s_i)-\varphi(t)}{s_i}} = \dfrac{1}{\varphi'(t)}$ if $s_i \neq 0$ for all $h \in (0,\delta)$;

(c) By Theorem 2.21, φ is strongly differentiable at t, i.e., for all $i \in \{1,2,3,4\}$,

$$\lim_{\substack{h\to 0^+\\ s_i+s_{5-i}\neq 0}} K_i(h) = \lim_{\substack{(s_i,s_{5-i})\to(0,0)\\ s_i+s_{5-i}\neq 0}} \frac{\varphi(t+s_{5-i})-\varphi(t-s_i)}{s_i+s_{5-i}} = \varphi'(t).$$

Next, consider

$$\begin{aligned}
Q_h^+(\varphi\circ C)(u,v) &= \frac{\varphi\left(C(u+h,v+h)\right)-2\varphi\left(C(u,v)\right)+\varphi\left(C(u-h,v-h)\right)}{h^2}\\
&= Q_{s_1}\varphi(t)\cdot\left(\frac{s_1}{h}\right)^2+\frac{\varphi(t+s_4)-\varphi(t-s_1)}{h^2}\\
&= \begin{cases} Q_{s_1}\varphi(t)\cdot\left(\frac{s_1}{h}\right)^2+\frac{\varphi(t+s_4)-\varphi(t-s_1)}{s_1+s_4}\cdot Q_h^+C(u,v) & \text{if } s_1+s_4\neq 0;\\ Q_{s_1}\varphi(t)\cdot\left(\frac{s_1}{h}\right)^2 & \text{if } s_1+s_4=0\end{cases}\\
&= \begin{cases} Q_{s_1}\varphi(t)\cdot\left(\frac{s_1}{h}\right)^2+K_1(h)\cdot Q_h^+C(u,v) & \text{if } s_1+s_4\neq 0;\\ Q_{s_1}\varphi(t)\cdot\left(\frac{s_1}{h}\right)^2 & \text{if } s_1+s_4=0.\end{cases}
\end{aligned}$$

Since $Q_h^+\left(\varphi\circ C\right)(u,v) = Q_h\varphi(u)+Q_h\varphi(v)$, by Remark 4.6,

$$\begin{aligned}
&Q_h\varphi(u)+Q_h\varphi(v)\\
&-Q_{s_1}\varphi(t)\cdot\left[\frac{s_1}{\varphi(t+s_1)-\varphi(t)}\cdot\left(\frac{\varphi(u+h)-\varphi(u)}{h}+\frac{\varphi(v+h)-\varphi(v)}{h}\right)\right]^2\\
&=\begin{cases} K_1(h)\cdot Q_h^+C(u,v) & \text{if } s_1+s_4\neq 0;\\ 0 & \text{if } s_1+s_4=0.\end{cases}
\end{aligned} \tag{4.5}$$

Moreover, from $\mathrm{SD}_2\,\varphi(u), \mathrm{SD}_2\,\varphi(v)<\infty$, we see that $Q_h^+C(u,v)$ is bounded in h. Now, we can divide the proof into 3 cases.

Case 1: $s_2(\delta')=s_3(\delta')=0$ for some $\delta'\in(0,\delta)$. Then by Corollary 4.2, $C(u-h,v+h)=C(u,v)=C(u+h,v-h)$ for all $h\in(0,\delta')$ and $\varphi'(u)=\varphi'(v)$. In this case, if $s_1+s_4=0$ for some $h\in(0,\delta')$, then $V_C\left(R_h(u,v)\right)=0$ which implies that $s_1+s_4=s_2+s_3=0$ for all $h'\in(0,h)$ and

$$\begin{aligned}
\mathrm{SD}_2\,\varphi(t) &= \lim_{s_1\to 0}\frac{\varphi(t+s_1)-2\varphi(t)+\varphi(t-s_1)}{s_1^2}\\
&= \lim_{s_1\to 0}\frac{\varphi(t+s_1)-\varphi(t+s_2)-\varphi(t+s_3)+\varphi(t+s_4)}{s_1^2}=0,
\end{aligned}$$

which contradicts the assumption. Hence $s_1+s_4\neq 0$ for all $h\in(0,\delta')$. From (4.5), we obtain

$$Q_h\varphi(u)+Q_h\varphi(v)$$

$$
\begin{aligned}
&- Q_{s_1}\varphi(t) \cdot \left[\frac{s_1}{\varphi(t+s_1)-\varphi(t)} \cdot \left(\frac{\varphi(u+h)-\varphi(u)}{h} + \frac{\varphi(v+h)-\varphi(v)}{h}\right)\right]^2 \\
&= K_1(h) \cdot Q_h^+ C(u,v) = K_1(h) \cdot \frac{V_C\left(R_h(u,v)\right)}{h^2} \qquad (4.6)
\end{aligned}
$$

for all $h \in (0, \delta')$. Since $\lim_{h \to 0^+} \left(Q_h\varphi(u) + Q_h\varphi(v)\right)$

$$
\begin{aligned}
&= \lim_{h\to 0^+} \frac{\varphi(u+h) - 2\varphi(u) + \varphi(u-h) + \varphi(v+h) - 2\varphi(v) + \varphi(v-h)}{h^2} \\
&= \lim_{h\to 0^+} \frac{\varphi(C(u+h, v+h)) - 2\varphi(t) + \varphi(C(u-h, v-h))}{h^2} \\
&= \lim_{h\to 0^+} \frac{\varphi(t+s_1) - \varphi(t+s_2) - \varphi(t+s_3) + \varphi(t+s_4)}{h^2} = 0,
\end{aligned}
$$

by Remark 4.10 above, we take limit as $h \to 0^+$ of (4.6) on both sides and obtain

$$
\begin{aligned}
\varphi'(t) \lim_{h\to 0^+} \frac{V_C\left(R_h(u,v)\right)}{h^2} &= -\frac{\mathrm{SD}_2\, \varphi(t)}{\left(\varphi'(t)\right)^2} \left(\varphi'(u^+) + \varphi'(v^+)\right)^2 \\
&= -\frac{4\, \mathrm{SD}_2\, \varphi(t) \left(\varphi'(u)\right)^2}{\left(\varphi'(t)\right)^2}.
\end{aligned}
$$

Thus, $D^2C(u,v) = \lim_{h\to 0^+} \dfrac{V_C\left(R_h(u,v)\right)}{(2h)^2} = -\dfrac{\mathrm{SD}_2\, \varphi(t) \left(\varphi'(u)\right)^2}{\left(\varphi'(t)\right)^3} \in (0, \infty)$, which can be summarized by Lemma 3.3 and Proposition 3.5 that $d_C(u,v) = \alpha_C(u,v) = 2$.

Case 2: $s_2 \neq 0$ for all $h \in (0, \delta)$. In a similar manner as in (4.5), by considering $Q_h^-(\varphi \circ C)(u,v)$ instead, we have

$$
\begin{aligned}
&Q_h\varphi(u) + Q_h\varphi(v) \\
&- Q_{s_2}\varphi(t) \cdot \left[\frac{s_2}{\varphi(t+s_2)-\varphi(t)} \cdot \left(\frac{\varphi(u-h)-\varphi(u)}{h} + \frac{\varphi(v+h)-\varphi(v)}{h}\right)\right]^2 \\
&= \begin{cases} K_2(h) \cdot Q_h^- C(u,v) & \text{if } s_2 + s_3 \neq 0; \\ 0 & \text{if } s_2 + s_3 = 0. \end{cases} \qquad (4.7)
\end{aligned}
$$

Now, we see that the limit of the left-hand sides of (4.5)-(4.7) when $h \to 0^+$ is

$$
\begin{aligned}
&\frac{\mathrm{SD}_2\, \varphi(t)}{(\varphi'(t))^2} \left[\left(-\varphi'(u^-) + \varphi'(v^+)\right)^2 - \left(\varphi'(u^+) + \varphi'(v^+)\right)^2\right] \\
&= \frac{\mathrm{SD}_2\, \varphi(t)}{(\varphi'(t))^2} \left[\left(-\varphi'(u) + \varphi'(v)\right)^2 - \left(\varphi'(u) + \varphi'(v)\right)^2\right] = -\frac{4\, \mathrm{SD}_2\, \varphi(t)}{\left(\varphi'(t)\right)^2} \varphi'(u)\varphi'(v).
\end{aligned}
$$

To consider the right-hand sides of (4.5)-(4.7), we first see that if there is $h \in (0, \delta)$ such that $s_1(h) + s_4(h) = s_2(h) + s_3(h)$, then $V_C\left(R_h(u,v)\right) = 0$ which implies that

$s_1(h') + s_4(h') = s_2(h') + s_3(h')$ for all $h' \in (0,h)$. Hence the right-hand sides of (4.5)-(4.7) become

$$\begin{cases} [K_1(h') - K_2(h')]\, Q^+_{h'}C(u,v) & \text{if } s_1+s_4 = s_2+s_3 \neq 0; \\ 0 & \text{if } s_1+s_4 = s_2+s_3 = 0, \end{cases}$$

for any $h' \in (0,h)$, which converges to 0 as $h' \to 0^+$ because of Remark 4.10(c) and boundedness of $Q^+_hC(u,v)$. That is, $-\dfrac{4\,\mathrm{SD}_2\,\varphi(t)}{(\varphi'(t))^2}\varphi'(u)\varphi'(v) = 0$.
But then, since $\varphi'(x) < 0$ for $x = u, v, t$, we have $\mathrm{SD}_2\,\varphi(t) = 0$ which contradicts the assumption. Hence the right-hand sides of (4.5)-(4.7) become

$$\begin{cases} K_2(h)\frac{V_C(R_h(u,v))}{h^2} + [K_1(h) - K_2(h)]\cdot Q^+_hC(u,v) & \text{if } s_1+s_4 \neq 0, s_2+s_3 \neq 0; \\ K_1(h)\frac{V_C(R_h(u,v))}{h^2} & \text{if } s_1+s_4 \neq 0, s_2+s_3 = 0; \\ K_2(h)\frac{V_C(R_h(u,v))}{h^2} & \text{if } s_1+s_4 = 0, s_2+s_3 \neq 0, \end{cases}$$

which converges to $\varphi'(t)\lim\limits_{h\to 0^+}\dfrac{V_C(R_h(u,v))}{h^2}$ as $h \to 0^+$ by Remark 4.10 and boundedness of $Q^+_hC(u,v)$. Thus, $D^2C(u,v) = -\dfrac{\mathrm{SD}_2\,\varphi(t)}{(\varphi'(t))^3}\varphi'(u)\varphi'(v) \in (0,\infty)$, which can be summarized by Lemma 3.3 and Proposition 3.5 that $d_C(u,v) = \alpha_C(u,v) = 2$.
Case 3: $s_3 \neq 0$ for all $h \in (0,\delta)$. In this case, we can prove in a similar manner as in Case 2, by using s_3 instead of s_2 in (4.7), to show that $d_C(u,v) = 2$. □

Example 4.11. We consider a family of *Clayton copulas* which are in the form $C_\theta(u,v) = \left[\max\left(u^{-\theta} + v^{-\theta} - 1, 0\right)\right]^{-\frac{1}{\theta}}$ for $\theta \in [-1,\infty)\setminus\{0\}$. For each θ, its generator is $\varphi_\theta(t) = \dfrac{1}{\theta}\left(t^{-\theta} - 1\right)$. We see that if $\theta = -1$, then $C_{-1}(u,v) = \max(u+v-1, 0) = W(u,v)$. It is easy to show that

$$d_W(u,v) = \begin{cases} 1 & \text{if } u+v = 1; \\ \infty & \text{otherwise.} \end{cases}$$

Now, we suppose that $\theta > -1$. Then we see that $\varphi_\theta(0) = \begin{cases} \infty & \text{if } \theta > 0; \\ -1/\theta & \text{if } \theta \in (-1,0), \end{cases}$
φ_θ is twice differentiable and $\varphi''_\theta(t) = (\theta+1)\,t^{-\theta-2} \in (0,\infty)$ for $t \in (0,1)$, so we consider 2 cases as follows.

- If $\theta > 0$, then $\varphi_\theta(u) + \varphi_\theta(v) < \varphi(0)$ for any $(u,v) \in (0,1)^2$. Hence by Theorem 4.9(D), we have $d_{C_\theta}(u,v) = 2$ for any $(u,v) \in (0,1)^2$.

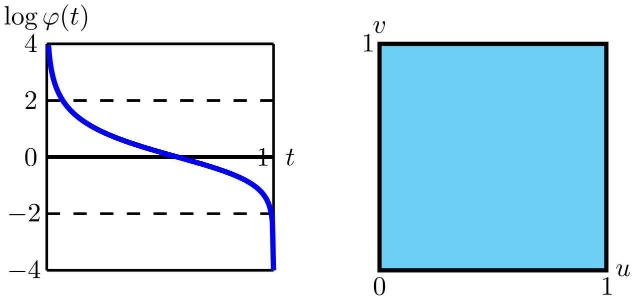

Figure 4.6: (Left) Graph of $\log(\varphi) = \log(\varphi_2)$. (Right) Support of C_2

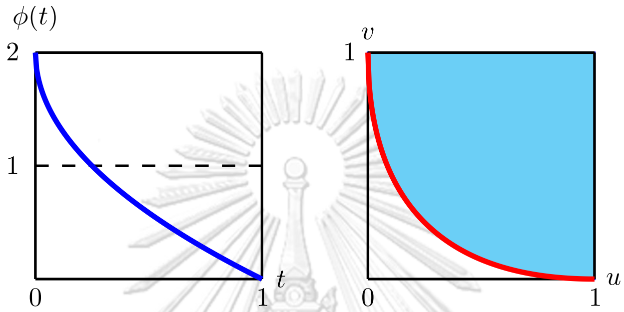

Figure 4.7: (Left) Graph of $\phi = \varphi_{-\frac{1}{2}}$. (Right) Support of $C_{-\frac{1}{2}}$

- If $\theta \in (-1, 0)$, then by Theorem 4.9(A) and (D),

$$d_{C_\theta}(u, v) = \begin{cases} \infty & \text{if } \varphi_\theta(u) + \varphi_\theta(v) > -1/\theta; \\ 2 & \text{if } \varphi_\theta(u) + \varphi_\theta(v) < -1/\theta. \end{cases}$$

To consider the case $\varphi_\theta(u) + \varphi_\theta(v) = -\dfrac{1}{\theta}$, since $f_\theta(t) = \varphi_\theta(0) - \varphi_\theta(t) = -\dfrac{1}{\theta}t^{-\theta} \in RV^0_{-\theta}$, by Theorem 4.9(B), we have $d_{C_\theta}(u, v) = -\dfrac{1}{\theta}$.

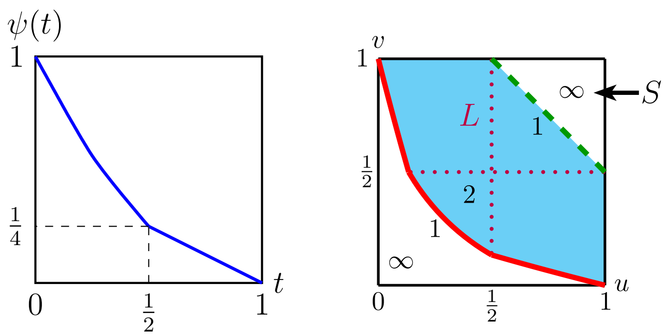

Figure 4.8: (Left) Graph of ψ. (Right) Support of C and $d_C(u, v)$ in Example 4.12

Example 4.12. Let $\psi(t) = \begin{cases} (1-t)^2 & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ \frac{1-t}{2} & \text{if } t \in \left(\frac{1}{2}, 1\right]. \end{cases}$ The graph of ψ and the support of $C = C_\psi$ are shown in Figure 4.8. We see that ψ is not differentiable at $\frac{1}{2}$ but twice differentiable elsewhere with $\psi''(t) = \begin{cases} 2 & \text{if } t \in \left(0, \frac{1}{2}\right); \\ 0 & \text{if } t \in \left(\frac{1}{2}, 1\right). \end{cases}$

Moreover, $f_\psi(t) = \begin{cases} 1-(1-t)^2 = 2t - t^2 & \text{if } t \in \left[0, \frac{1}{2}\right]; \\ 1 - \frac{1-t}{2} = \frac{1+t}{2} & \text{if } t \in \left(\frac{1}{2}, 1\right], \end{cases}$ which lies in RV_1^0.

Hence by Theorem 4.9(A),(B),(C) and (D), respectively, we have

$$d_C(u,v) = \begin{cases} \infty & \text{if } \psi(u) + \psi(v) > 1; \\ 1 & \text{if } \psi(u) + \psi(v) = \frac{1}{4} \text{ (dashed)}; \\ 1 & \text{if } \psi(u) + \psi(v) = 1 \text{ (solid)}; \\ 2 & \text{if } \frac{1}{4} < \psi(u) + \psi(v) < 1 \text{ (shaded)}. \end{cases}$$

We see that Theorem 4.9 cannot be applied on regions

$$S = \left\{(u,v) : \psi(u) + \psi(v) \in \left(0, \frac{1}{4}\right)\right\} \text{ and}$$
$$L = \left(\left\{\frac{1}{2}\right\} \times \left(1 - \frac{\sqrt{3}}{2}, 1\right)\right) \cup \left(\left(1 - \frac{\sqrt{3}}{2}, 1\right) \times \left\{\frac{1}{2}\right\}\right)$$

because

1. for any $(u,v) \in S$, $\psi(t) = \psi(u) + \psi(v) \in \left(0, \frac{1}{4}\right)$, i.e., $t \in \left(\frac{1}{2}, 1\right)$ and $\mathrm{SD}_2\,\psi(t) = 0$ for all $t \in \left(\frac{1}{2}, 1\right]$;

2. we see that

$$\lim_{h \to 0^+} \frac{\psi\left(\frac{1}{2}+h\right) - 2\psi\left(\frac{1}{2}\right) + \psi\left(\frac{1}{2}-h\right)}{h^2} = \lim_{h \to 0^+} \frac{\left(\frac{1}{4} - \frac{h}{2}\right) - 2\left(\frac{1}{4}\right) + \left(\frac{1}{2}+h\right)^2}{h^2}$$
$$= \lim_{h \to 0^+} \frac{h^2 + \frac{h}{2}}{h^2} = \infty$$

and $\lim_{h \to 0^-} \frac{\psi\left(\frac{1}{2}+h\right) - 2\psi\left(\frac{1}{2}\right) + \psi\left(\frac{1}{2}-h\right)}{h^2} = \infty$, i.e., $\mathrm{SD}_2\,\psi\left(\frac{1}{2}\right) = \infty$.

To find $d_C(u,v)$ for $(u,v) \in S$, since $C(u,v) = u + v - 1$ for all $(u,v) \in S$, it is easy to show that for such (u,v), $d_C(u,v) = \infty$.

Next, we find $d_C(u,v)$ for $(u,v) \in L$. By symmetry of C, it suffices to find $d_C\left(\frac{1}{2}, v\right)$ where $v \in \left(1-\frac{\sqrt{3}}{2}, 1\right)$. For any $h > 0$ small, let

$$\begin{aligned}
A_1(h) &:= \psi\left(\frac{1}{2}+h\right) = \frac{1}{4}-\frac{h}{2}, \\
A_2(h) &:= \psi\left(\frac{1}{2}-h\right) = \left(\frac{1}{2}+h\right)^2, \\
B_1(h) &:= \psi(v+h) = \begin{cases} (1-v-h)^2 & \text{if } v < \frac{1}{2}; \\ \frac{1-v-h}{2} & \text{if } v \geq \frac{1}{2}, \end{cases} \quad \text{and} \\
B_2(h) &:= \psi(v-h) = \begin{cases} (1-v+h)^2 & \text{if } v \leq \frac{1}{2}; \\ \frac{1-v+h}{2} & \text{if } v > \frac{1}{2}. \end{cases}
\end{aligned}$$

Then

$$A_2(h) - A_1(h) = h^2 + \frac{3h}{2} \quad \text{and} \quad B_2(h) - B_1(h) = \begin{cases} 4h(1-v) & \text{if } v < \frac{1}{2}; \\ h^2 + \frac{3h}{2} & \text{if } v = \frac{1}{2}; \\ h & \text{if } v > \frac{1}{2}, \end{cases}$$

which implies that $V_C\left(R_h\left(\frac{1}{2}, v\right)\right)$

$$\begin{aligned}
&= \left(1-\sqrt{A_1(h)+B_1(h)}\right) - \left(1-\sqrt{A_2(h)+B_1(h)}\right) - \left(1-\sqrt{A_1(h)+B_2(h)}\right) \\
&\quad + \left(1-\sqrt{A_2(h)+B_2(h)}\right) \\
&= \sqrt{A_2(h)+B_1(h)} + \sqrt{A_1(h)+B_2(h)} - \sqrt{A_1(h)+B_1(h)} - \sqrt{A_2(h)+B_2(h)} \\
&= \frac{A_2(h)-A_1(h)}{\sqrt{A_2(h)+B_1(h)}+\sqrt{A_1(h)+B_1(h)}} - \frac{A_2(h)-A_1(h)}{\sqrt{A_1(h)+B_2(h)}+\sqrt{A_2(h)+B_2(h)}}.
\end{aligned}$$

Next, we consider $\dfrac{V_C\left(R_h\left(\frac{1}{2},v\right)\right)}{A_2(h)-A_1(h)} = \dfrac{V_C\left(R_h\left(\frac{1}{2},v\right)\right)}{h^2+\frac{3h}{2}}$

$$\begin{aligned}
&= \frac{\sqrt{A_1(h)+B_2(h)} + \sqrt{A_2(h)+B_2(h)} - \sqrt{A_2(h)+B_1(h)} - \sqrt{A_1(h)+B_1(h)}}{\left(\sqrt{A_2(h)+B_1(h)}+\sqrt{A_1(h)+B_1(h)}\right)\left(\sqrt{A_1(h)+B_2(h)}+\sqrt{A_2(h)+B_2(h)}\right)} \\
&= \frac{\frac{B_2(h)-B_1(h)}{\sqrt{A_1(h)+B_2(h)}+\sqrt{A_1(h)+B_1(h)}} + \frac{B_2(h)-B_1(h)}{\sqrt{A_2(h)+B_2(h)}+\sqrt{A_2(h)+B_1(h)}}}{\left(\sqrt{A_2(h)+B_1(h)}+\sqrt{A_1(h)+B_1(h)}\right)\left(\sqrt{A_1(h)+B_2(h)}+\sqrt{A_2(h)+B_2(h)}\right)}.
\end{aligned}$$

Note that $\lim_{h\to 0^+}\left(\sqrt{A_i(h)+B_j(h)}\right) = \sqrt{\frac{1}{4}+(1-v)^2}$ for all $i,j \in \{1,2\}$. By the value of $B_2(h)-B_1(h)$, we divide the value of v into 3 cases.

Case 1: $v < \dfrac{1}{2}$. Then

$$\begin{aligned} D^2C\left(\frac{1}{2}, v\right) &= \lim_{h\to 0^+} \frac{V_C\left(R_h\left(\frac{1}{2}, v\right)\right)}{(2h)^2} \\ &= \frac{1}{32\left(\frac{1}{4}+(1-v)^2\right)^{3/2}} \lim_{h\to 0^+} \frac{\left(h^2+\frac{3h}{2}\right)(4h(1-v))}{h^2} \\ &= \frac{1-v}{8\left(\frac{1}{4}+(1-v)^2\right)^{3/2}} \lim_{h\to 0^+} \left(h+\frac{3}{2}\right) = \frac{3(1-v)}{16\left(\frac{1}{4}+(1-v)^2\right)^{3/2}} \in (0,\infty). \end{aligned}$$

Case 2: $v = \dfrac{1}{2}$. Then

$$\begin{aligned} D^2C\left(\frac{1}{2}, \frac{1}{2}\right) &= \frac{1}{32\left(\frac{1}{4}+\frac{1}{4}\right)^{3/2}} \lim_{h\to 0^+} \frac{\left(h^2+\frac{3h}{2}\right)^2}{h^2} = \frac{1}{8\sqrt{2}} \lim_{h\to 0^+} \left(h+\frac{3}{2}\right)^2 \\ &= \frac{9}{32\sqrt{2}} \in (0,\infty). \end{aligned}$$

Case 3: $v > \dfrac{1}{2}$. Then

$$\begin{aligned} D^2C\left(\frac{1}{2}, v\right) &= \frac{1}{32\left(\frac{1}{4}+(1-v)^2\right)^{3/2}} \lim_{h\to 0^+} \frac{\left(h^2+\frac{3h}{2}\right)h}{h^2} \\ &= \frac{3}{64\left(\frac{1}{4}+(1-v)^2\right)^{3/2}} \in (0,\infty). \end{aligned}$$

Therefore, we conclude that $d_C(u, v) = 2$ for all $(u, v) \in L$.

CHAPTER V
CONCLUSION

5.1 Our results

In our thesis, we investigate the pointwise dimension

$$d_C(u,v) = \lim_{h\to 0} \frac{\log V_C\left(R_h(u,v)\right)}{\log(h)}$$

of copulas constructed from other copulas by simple methods and compute the pointwise dimension of some well-known copulas.

In Chapter 3, we obtain formulas of the pointwise dimension of copulas constructed by joining finitely many copulas via 3 methods: convex sum, patching and ordinal sum as restated in the following statements. In a nutshell, the pointwise dimension of a constructed copula at a point is the minimum of those of ingredient copulas at the corresponding point.

Theorem 3.7. *Let* $\{C_i\}_{i=1}^n$ *be a collection of copulas and* $\{\alpha_i\}_{i=1}^n \subseteq (0,1)$ *be such that* $\sum_{i=1}^n \alpha_i = 1$*. Let* $C = \sum_{i=1}^n \alpha_i C_i$ *be the convex sum of* $\{C_i\}_{i=1}^n$*. Then for any* $(u,v) \in \mathbb{I}^2$, $d_C(u,v) = \min_{1\le i\le n}\{d_{C_i}(u,v)\}$.

Theorem 3.8. *Let* $T = [t_{ij}] \in M_{m\times n}(\mathbb{I})$ *be a transformation matrix and* $\{C_{ij}\}$ *be a collection of copulas with the same indices as entries in* T*. Let* C *be the patched copula with respect to the transformation matrix* T *and the collection of copulas* $\{C_{ij}\}$*. Then for any* $(u,v) \in \mathbb{I}^2$,

$$d_C\left(u,v\right) = \inf_{(i,j)\in A(u,v)}\{d_{C_{ij}}(u_i,v_j)\}$$

where $A(u,v) := \{(i,j) : (u,v) \in R_{ij} = [p_{i-1},p_i] \times [q_{j-1},q_j]\,,\, t_{ij} > 0\}$, $u_i = \dfrac{u-p_{i-1}}{p_i - p_{i-1}}$ *and* $v_j = \dfrac{v-q_{j-1}}{q_j-q_{j-1}}$.

Corollary 3.9. *Let* $\{J_i\}_{i=1}^N$*, where* $J_i = [a_i,b_i]$ *with* $a_i < b_i$ *for all* $i = 1,\ldots,N$*, be a family of closed, non-overlapping, non-degenerate sub-intervals on* $\mathbb{I}$ *and let* $\{C_i\}_{i=1}^N$ *be a collection of copulas. Moreover, let* C *be an ordinal sum of* $\{C_i\}$ *with respect* *to* $\{J_i\}$. *Set* $A_N = \{(a_i, a_i) : i = 1, \ldots, N\}$ *and* $B_N = \{(b_i, b_i) : n = 1, \ldots, N\}$. *Then for any* $(u,v) \in \mathbb{I}^2 \backslash \{(0,0),(1,1)\}$,

$$d_C(u,v) = \begin{cases} d_{C_i}\left(\dfrac{u-a_i}{b_i-a_i}, \dfrac{v-a_i}{b_i-a_i}\right) & \textit{if } (u,v) \in J_i^2 \backslash (A_N \cup B_N) \textit{ for some } i; \\ \min\{d_{C_i}(1,1), d_{C_j}(0,0)\} & \textit{if } u = v = b_i = a_j \textit{ for some } i \neq j; \\ d_M(u,v) & \textit{otherwise.} \end{cases}$$

Moreover,

$$d_C(0,0) = \begin{cases} d_{C_i}(0,0) & \textit{if } (0,0) \in J_i^2 \textit{ for some } i; \\ 1 & \textit{otherwise,} \end{cases}$$

and a similar statement holds for $d_C(1,1)$.

In Chapter 4, we compute the pointwise dimension of Archimedean copulas which is divided into 2 cases: piecewise linear and general generators. We restate the following theorems.

Theorem 4.3. *Let* C *be an Archimedean copula with piecewise linear generator* φ *in the form* (4.1). *Then for any* $(u,v) \in (0,1)^2$,

(A) if $\varphi(u) + \varphi(v) \neq \varphi(t_k)$ *for any* $k = 0, 1, \ldots, n$, *then* $d_C(u,v) = \infty$;

(B) if $\varphi(u) + \varphi(v) = \varphi(t_k)$ *for some* $k = 0, 1, \ldots, n$, *then* $d_C(u,v) = 1$.

Recall that φ in (4.1) can be written as $\varphi(t) = \varphi(t_k) - b_k(t - t_{k-1})$ for $t \in (t_{k-1}, t_k]$ and $k = 1, \ldots, n$ where $\{b_k\}_{k=1}^n$ is a strictly decreasing sequence in $\mathbb{R}^+$ and $\{t_k\}_{k=0}^n$ is a strictly increasing sequence in $\mathbb{I}$ such that $t_0 = 0$ and $t_n = 1$. Before we restate the main theorem, recall

$$RV_\beta^0 = \left\{ f : (0,N) \to [0,\infty) \mid N \in \mathbb{R}^+ \text{ and } \lim_{x \to 0^+} \frac{f(\lambda x)}{f(x)} = \lambda^\beta \text{ for all } \lambda > 0 \right\}.$$

Theorem 4.9. *Let* C *be an Archimedean copula with generator* φ, *a function that is convex, continuous, strictly decreasing on* $\mathbb{I}$ *and* $\varphi(1) = 0$. *Let* $(u,v) \in (0,1)^2$.

(A) If $\varphi(u) + \varphi(v) > \varphi(0)$, *then* $d_C(u,v) = \infty$.

(B) If $\varphi(u) + \varphi(v) = \varphi(0)$ *and* $f_\varphi(x) = \varphi(0) - \varphi(x) \in RV_\beta^0$ *where* $\beta > 0$, *then* $d_C(u,v) = \dfrac{1}{\beta}$.

(C) If $\varphi(u) + \varphi(v) = \varphi(t)$ *where* $t \in (0,1)$ *and* φ *is not differentiable at* t, *then* $d_C(u,v) = 1$.

(D) If $\varphi(u)+\varphi(v)=\varphi(t)$*,* $\mathrm{SD}_2\,\varphi(u), \mathrm{SD}_2\,\varphi(v) < \infty$*,* φ *is differentiable at* t *and* $\mathrm{SD}_2\,\varphi(t) \in (0,\infty)$ *where* $t \in (0,1)$*, then* $d_C(u,v) = 2$*, where* $\mathrm{SD}_2\,\varphi(x)$ *is the second order symmetric derivative of* φ *at* x *defined in Definition 2.18.*

5.2 Further studies

1. Compute the pointwise dimension of copulas constructed by more complicated methods, such as

   - ordinal sum of countably many copulas,
   - mixing distribution: $C_\Lambda(u,v) = \int_{\mathbb{R}} C_\theta(u,v)\mathrm{d}\Lambda(\theta)$ where Λ is a distribution function of a random variable Θ with value θ and $\{C_\theta\}$ is a collection of copulas. If Λ is a distribution function of probability function $f(i) = \alpha_i$ for $i = 1,\ldots,n$, then $C_\Lambda = \sum_{i=1}^{n} \alpha_i C_i$. Hence, this method is a generalization of convex sum in Definition 2.8,
   - $*$-product: $(C * D)(u,v) = \int_0^1 \partial_2 C(u,t)\partial_1 D(t,v)\,\mathrm{d}t$ where C, D are copulas. By testing with a few pairs of copulas, we have a conjecture about pointwise dimension of product of copulas as follows.

     Conjecture 1. For any copulas C, D and $(u,v) \in \mathbb{I}^2$, $d_{C*D}(u,v) = \inf_{t\in\mathbb{I}} \{d_C(u,t), d_D(t,v)\}$.

2. Compute the pointwise dimension of Archimedean copulas in the case that $\varphi(u)+\varphi(v) = \varphi(t)$ where $t \in (0,1)$, $\varphi'(t)$ exists and at least one of the following conditions hold:

   - $\mathrm{SD}_2\,\varphi(t) \in \{0,\infty\}$;
   - $\mathrm{SD}_2\,\varphi(u) = \infty$ or $\mathrm{SD}_2\,\varphi(v) = \infty$;
   - $\mathrm{SD}_2\,\varphi(t), \mathrm{SD}_2\,\varphi(u)$ or $\mathrm{SD}_2\,\varphi(v)$ does not exist.

   Is there a possibility that $d_C(u,v) \notin \mathbb{Z}\cup\{\infty\}$ in this case ?

3. From Theorem 4.9, we see that for any Archimedean copula C and $(u,v) \in (0,1)^2$, $d_C(u,v)$ depends on the value of $C(u,v)$ but not on u and v. Hence we have the following conjecture.

   Conjecture 2. For any Archimedean copula C with generator φ and $(u,v) \in (0,1)^2$ such that $\varphi(u)+\varphi(v) \le \varphi(0)$, $d_C(u,v) = d_C(t,t)$ where $t = \varphi^{-1}\left(\frac{\varphi(C(u,v))}{2}\right)$.

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VITA

Name	:	Mister Noppawit Yanpaisan
Date of Birth	:	15 August 1995
Place of Birth	:	Phang-nga, Thailand
Education	:	B.Sc. (Mathematics), (First Class Honors), Chulalongkorn University, 2018
Scholarship	:	Science Administrator Scholarship of Thailand (SAST)