CHAPTER II

THEORETICAL CONSIDERATIONS

2.1 Basic Equations and General Solutions

The constitutive relation for a homogeneous poroelastic material with compressible constituents can be expressed with respect to the conventional cylindrical coordinate system (r,θ,z), shown in Fig. 1, by using the standard indicial notation[12] as

$$\sigma_{ij} = 2\mu\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon\right] - \frac{3(\nu_u - \nu)}{B(1-2\nu)(1+\nu_u)}\delta_{ij}p \qquad i,j = r,\theta,z \qquad (2.1)$$

In the above equation, σ_{ij} is the total stress component of the bulk material; ε_{ij} and ε are the strain component and the dilatation of the solid matrix, respectively; p is defined as the excess pore fluid pressure (suction is considered negative); μ, ν and ν_u denote the shear modulus, drained and undrained Poisson's ratios, respectively. In addition, B is Skempton's pore pressure coefficient and δ_{ij} is the Kronecker delta.

It is noted that $0 \le B \le 1$ and $\nu \le \nu_u \le 0.5$ for all poroelastic materials[10]. The limiting cases of a poroelastic solid with incompressible constituents and a dry elastic material are obtained when $\nu_u = 0.5$ and $B = 1$, and $B \to 0$, respectively. The excess pore fluid pressure can be expressed as

$$p = -\frac{2\mu B(1+\nu_u)}{3(1-2\nu_u)}\varepsilon + \frac{2\mu B^2(1-2\nu)(1+\nu_u)^2}{9(1-2\nu_u)(\nu_u-\nu)}\zeta \qquad (2.2)$$

where ζ is the variation of fluid volume per unit reference volume. Let u_i and ψ_i denote the average displacement of solid matrix and the fluid displacement relative to the solid matrix, respectively, in the i^{th} direction $(i = r, \theta, z)$. Then,

$$\psi_i = \int_0^t q_i dt \qquad (2.3)$$

where q_i is the fluid discharge in the i^{th} direction defined as

$$q_i = -\kappa\frac{\partial p}{\partial i} \qquad i = r,z \qquad (2.4)$$

$$q_\theta = -\kappa\frac{\partial p}{r\partial\theta} \qquad (2.5)$$

In addition, κ is the coefficient of permeability of the medium.

The quasi-static governing equations for a poroelastic medium with compressible constituents, expressed in terms of stresses and pore pressure as basic variables(12), can be transformed into Navier equations with coupling terms and a diffusion equation by treating the displacements, u_i, and the variation of fluid volume, ζ, as the basic unknowns as(10)

$$\nabla^2 u_r + \frac{1}{1-2\nu_u}\frac{\partial\varepsilon}{\partial r} - \frac{1}{r}\left[\frac{2}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}\right] - \frac{2B(1+\nu_u)}{3(1-2\nu_u)}\frac{\partial\zeta}{\partial r} = 0 \qquad (2.6)$$

$$\nabla^2 u_\theta + \frac{1}{1-2\nu_u}\frac{\partial \varepsilon}{r\partial \theta} - \frac{1}{r}\left[\frac{u_\theta}{r} - \frac{2}{r}\frac{\partial u_r}{\partial \theta}\right] - \frac{2B(1+\nu_u)}{3(1-2\nu_u)}\frac{1}{r}\frac{\partial \zeta}{\partial \theta} = 0 \qquad (2.7)$$

$$\nabla^2 u_z + \frac{1}{1-2\nu_u}\frac{\partial \varepsilon}{\partial z} - \frac{2B(1+\nu_u)}{3(1-2\nu_u)}\frac{\partial \zeta}{\partial z} = 0 \qquad (2.8)$$

$$\nabla^2 \zeta = \frac{\partial \zeta}{c\partial t} \qquad (2.9)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \qquad (2.10)$$

$$\varepsilon = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \qquad (2.11)$$

$$c = \frac{2\mu\kappa B^2(1-\nu)(1+\nu_u)^2}{9(1-\nu_u)(\nu_u-\nu)} \qquad (2.12)$$

It can be shown[10] that the general solutions for eqns (2.6) - (2.9) can be derived by applying Fourier expansion, Laplace and Hankel transforms with respect to the circumferential, time and radial coordinates, respectively.

The application of the Fourier expansion with respect to the circumferential coordinate θ for the displacements and the variation of fluid volume results in

$$u_i(r,\theta,z,t) = \sum_{m=0}^{\infty} u_{im}(r,z,t)f(\theta) - \sum_{m=0}^{\infty} \hat{u}_{im}(r,z,t)f'(\theta) \qquad (2.13)$$

$$\zeta(r,\theta,z,t) = \sum_{m=0}^{\infty}\zeta_m(r,z,t)\cos(m\theta) + \sum_{m=0}^{\infty}\hat{\zeta}_m(r,z,t)\sin(m\theta) \quad (2.14)$$

where

$$f(\theta) = \begin{cases} \cos(m\theta) & i \neq \theta \\ \sin(m\theta) & i = \theta \end{cases} \quad (2.15)$$

In eqns (2.13) and (2.14), u_{im} and ζ_m are symmetric components and u_{im} and ζ_m are anti-symmetric components corresponding to the m^{th} harmonic. The term $f'(\theta)$ denotes the derivative of $f(\theta)$ with respect to the circumferential coordinate θ.

The m^{th}-order Laplace-Hankel transform of a function $\phi(r,z,t)$ with respect to the variables t and r, respectively, is defined by[13]

$$\overline{H}_m\{\phi(r,z,t)\} = \int_0^{\infty}\int_0^{\infty}\phi(r,z,t)e^{-st}J_m(\xi r)rdrdt \quad (2.16)$$

and the inverse relationship is given by

$$\phi(r,z,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\int_0^{\infty}\overline{H}_m\{\phi(r,z,t)\}e^{st}J_m(\xi r)d\xi ds \quad (2.17)$$

In the above equations, $J_m(\xi r)$ denotes the Bessel function[14] of the first kind of order m and ξ is the Hankel transform parameter. It should be noted that γ is greater than the real part of all singularities of $\overline{H}_m\{\phi(r,z,t)\}$ and $i=\sqrt{-1}$.

The general solutions for the m^{th} harmonic of solid and fluid displacements, pore pressure and stresses in the Laplace-Hankel transform space can be expressed in the following matrix form[9]

$$\mathbf{v}(\xi, z, s) \;=\; \mathbf{R}(\xi, z, s)\mathbf{X}(\xi, s) \qquad (2.18)$$

$$\mathbf{f}(\xi, z, s) \;=\; \mathbf{S}(\xi, z, s)\mathbf{X}(\xi, s) \qquad (2.19)$$

in which

$$\mathbf{v}(\xi, z, s) \;=\; \langle v_i(\xi, z, s)\rangle^T \qquad i = 1,2,3,4 \qquad (2.20)$$

$$\mathbf{f}(\xi, z, s) \;=\; \langle f_i(\xi, z, s)\rangle^T \qquad i = 1,2,3,4 \qquad (2.21)$$

$$v_1(\xi, z, s) \;=\; \frac{1}{2}\left[\overline{H}_{m+1}(u_{rm} + u_{\theta m}) - \overline{H}_{m-1}(u_{rm} - u_{\theta m})\right] \qquad (2.22)$$

$$v_2(\xi, z, s) \;=\; \frac{1}{2}\left[\overline{H}_{m+1}(u_{rm} + u_{\theta m}) + \overline{H}_{m-1}(u_{rm} - u_{\theta m})\right] \qquad (2.23)$$

$$v_3(\xi, z, s) \;=\; \overline{H}_m(u_{zm}) \qquad (2.24)$$

$$v_4(\xi, z, s) \;=\; \overline{H}_m(p_m) \qquad (2.25)$$

$$f_1(\xi, z, s) \;=\; \frac{1}{2}\left[\overline{H}_{m+1}(\sigma_{zrm} + \sigma_{z\theta m}) - \overline{H}_{m-1}(\sigma_{zrm} - \sigma_{z\theta m})\right] \qquad (2.26)$$

$$f_2(\xi, z, s) \;=\; \frac{1}{2}\left[\overline{H}_{m+1}(\sigma_{zrm} + \sigma_{z\theta m}) + \overline{H}_{m-1}(\sigma_{zrm} - \sigma_{z\theta m})\right] \qquad (2.27)$$

$$f_3(\xi, z, s) \;=\; \overline{H}_m(\sigma_{zzm}) \qquad (2.28)$$

$$f_4(\xi, z, s) = \overline{H}_m(\psi_{zm}) \tag{2.29}$$

$$\mathbf{X}(\xi, s) = \left\langle \mathbf{A}_m \ \ \mathbf{B}_m \ \ \mathbf{C}_m \ \ \mathbf{D}_m \ \ \mathbf{E}_m \ \ \mathbf{F}_m \ \ \mathbf{G}_m \ \ \mathbf{H}_m \right\rangle^T \tag{2.30}$$

and the matrices $\mathbf{R}(\xi,z,s)$ and $\mathbf{S}(\xi,z,s)$ in eqns (2.18) and (2.19) are explicitly given by eqns (A-1) to (A-6) in Appendix A. The arbitrary functions $\mathbf{A}_m(\xi,s)$, $\mathbf{B}_m(\xi,s)$, ..., $\mathbf{H}_m(\xi,s)$ appearing in $\mathbf{X}(\xi,s)$ are to be determined by employing appropriate boundary and/or continuity conditions.

2.2 Stiffness Matrix

A multilayered system with a total of N poroelastic layers overlying a poroelastic half-space is considered in this section. Layers and interfaces are numbered as shown in Fig. 2. A subscript "n" is used to denote quantities associated with the n^{th} layer (n = 1,2,...,N). For the n^{th} layer, the following relationships can be established by using eqns (2.18) and (2.19):

$$\mathbf{U}^{(n)} = \begin{bmatrix} \mathbf{R}^{(n)}(\xi, z_n, s) \\ \cdots\cdots\cdots\cdots \\ \mathbf{R}^{(n)}(\xi, z_{n+1}, s) \end{bmatrix} \mathbf{X}^{(n)}(\xi, s) \tag{2.31}$$

$$\mathbf{F}^{(n)} = \begin{bmatrix} -\mathbf{S}^{(n)}(\xi, z_n, s) \\ \cdots\cdots\cdots\cdots \\ \mathbf{S}^{(n)}(\xi, z_{n+1}, s) \end{bmatrix} \mathbf{X}^{(n)}(\xi, s) \tag{2.32}$$

where

$$\mathbf{U}^{(n)} = \left\langle \mathbf{v}^{(n)}(\xi, z_n, s) \quad \mathbf{v}^{(n)}(\xi, z_{n+1}, s) \right\rangle^T \tag{2.33}$$

$$\mathbf{F}^{(n)} = \left\langle -\mathbf{f}^{(n)}(\xi, z_n, s) \quad \mathbf{f}^{(n)}(\xi, z_{n+1}, s) \right\rangle^T \tag{2.34}$$

From eqns (2.31) to (2.34), $\mathbf{U}^{(n)}$ denotes a vector of generalized displacement for the n^{th} layer whose elements are related to the Laplace-Hankel transforms of the m^{th} Fourier harmonic of displacements and pore pressure of the top and bottom surfaces of the n^{th} layer. Similarly, $\mathbf{F}^{(n)}$ denotes a generalized force vector whose elements are related to the Laplace-Hankel transforms of the m^{th} harmonic of tractions and fluid displacement of the top and bottom surfaces of the n^{th} layer.

The vectors $\mathbf{v}^{(n)}$ and $\mathbf{f}^{(n)}$ in eqns (2.33) and (2.34) are identical to $\mathbf{v}$ and $\mathbf{f}$ defined in eqns (2.20) and (2.21) except that the material properties of the n^{th} layer are employed in the definition and $z = z_n$ or z_{n+1}. Equation (2.31) can be inverted to express $\mathbf{C}^{(n)}$ in terms of $\mathbf{U}^{(n)}$ and substitution into eqn (2.32) yields

$$\mathbf{F}^{(n)} = \mathbf{K}^{(n)}\mathbf{U}^{(n)} \tag{2.35}$$

where $\mathbf{K}^{(n)}$ can be considered as an exact stiffness matrix in the Laplace-Hankel transform space describing the relationship between the generalized displacement vector $\mathbf{U}^{(n)}$ and the generalized force vector $\mathbf{F}^{(n)}$ for the n^{th} layer.

In eqn (2.35), the layer stiffness matrix $\mathbf{K}^{(n)}$is an 8x8 symmetric matrix and its elements, $\mathbf{k}_{ij}$, are functions of layer thickness, $h^{(n)}$, layer material properties and Laplace and Hankel transform parameters, s and ξ, respectively. Only negative exponentials that decrease rapidly with increasing ξ, s and $h^{(n)}$ are involved in k_{ij}.

For the underlying half-space, the stiffness matrix for the bottom half-space can be expressed as

$$\mathbf{F}^{(N+1)} = \mathbf{K}^{(N+1)}\mathbf{U}^{(N+1)} \qquad (2.36)$$

where

$$\mathbf{U}^{(N+1)} = \left\langle \mathbf{v}^{(N+1)}(\xi, z_{N+1}, s)\right\rangle^{T} \qquad (2.37)$$

$$\mathbf{F}^{(N+1)} = \left\langle -\mathbf{f}^{(N+1)}(\xi, z_{N+1}, s)\right\rangle^{T} \qquad (2.38)$$

Due to the regularity condition at $z \to \infty$, the matrix $\mathbf{K}^{(N+1)}$ is a 4x4 symmetric matrix. It is noted that exponential terms of ξ and s are not involved in the expression of $\mathbf{K}^{(N+1)}$ and its elements depend only on the material properties of the underlying half-space and the Laplace and Hankel transform parameters s and ξ respectively. The elements of $\mathbf{K}^{(N)}$ and $\mathbf{K}^{(N+1)}$ are explicitly given by Senjuntichai and Rajapakse[9].

2.3 Global Stiffness Matrix

The global stiffness matrix of a multilayered half-space, shown in Fig. 2, is assembled by using layer and half-space stiffness matrices together with the continuity conditions of tractions and fluid flow at the layer interfaces. For example, the continuity conditions at the n^{th} interface can be expressed as

$$\mathbf{f}^{(n-1)}(\xi,z_n,s) - \mathbf{f}^{(n)}(\xi,z_n,s) = \mathbf{T}^{(n)} \tag{2.39}$$

where $\mathbf{f}^{(n-1)}$ and $\mathbf{f}^{(n)}$ are as defined in eqn (2.21) and

$$\mathbf{T}^{(n)} = \left\langle T_1^{(n)} \quad T_2^{(n)} \quad T_3^{(n)} \quad \frac{Q^{(n)}}{s} \right\rangle^T \tag{2.40}$$

in which

$$T_1^{(n)} = \frac{1}{2}\left[\overline{H}_{m+1}(T_{rm}^{(n)} + T_{\theta m}^{(n)}) - \overline{H}_{m-1}(T_{rm}^{(n)} - T_{\theta m}^{(n)})\right] \tag{2.41}$$

$$T_2^{(n)} = \frac{1}{2}\left[\overline{H}_{m+1}(T_{rm}^{(n)} + T_{\theta m}^{(n)}) + \overline{H}_{m-1}(T_{rm}^{(n)} - T_{\theta m}^{(n)})\right] \tag{2.42}$$

$$T_3^{(n)} = \overline{H}_m(T_{zm}^{(n)}) \tag{2.43}$$

$$Q^{(n)} = \overline{H}_m(Q_m^{(n)}) \tag{2.44}$$

where $T_{im}^{(n)}$ $(i = r,\theta,z)$ and $Q_m^{(n)}$ denote the m^{th} Fourier harmonic of the tractions and fluid source applied at the n^{th} interface, respectively.

Consideration of eqn (2.40) at each layer interface together with the layer and bottom half-space stiffness matrices defined in eqns (2.35) and (2.36) results in the following global stiffness equation of order 4(N+1):

$$\begin{bmatrix} \mathbf{K}^{(1)} & & & & \\ & \mathbf{K}^{(2)} & & & \\ & & \ddots & & \\ & & & \mathbf{K}^{(N)} & \\ & & & & \mathbf{K}^{(N+1)} \end{bmatrix} \begin{Bmatrix} \mathbf{U}^{(1)} \\ \mathbf{U}^{(2)} \\ \\ \mathbf{U}^{(N)} \\ \mathbf{U}^{(N+1)} \end{Bmatrix} = \begin{Bmatrix} \mathbf{T}^{(1)} \\ \mathbf{T}^{(2)} \\ \\ \mathbf{T}^{(N)} \\ \mathbf{T}^{(N+1)} \end{Bmatrix} \tag{2.45}$$

2.4 Variational Formulation

Consider a multilayered poroelastic half-space with an embedded elastic bar as shown in Fig. 3. The bar is subjected to an axial load V_0 and is assumed to be perfectly bonded to the surrounding medium along its contact surface. Since we are concerned with the deformation of an elastic bar which has a large length-to-radius ratio, it is well justified to assume one-dimensional behavior for the bar[2]. The state of deformation of the bar represented by the one-dimensional theory can be expressed in the form

$$w(z,t) = \sum_{k=1}^{K} \alpha_k(t) e^{-(k-1)z/h_b} \tag{2.46}$$

In the foregoing equation, $\alpha_k(t)$ is the arbitrary coefficients, K denotes the number of terms which are used to represent the deformation of the bar and h_b is the total length of the bar. The displacement profile along the bar in eqn (2.46) is indeterminate within the arbitrary coefficients $\alpha_1(t)$, $\alpha_2(t)$, ..., $\alpha_K(t)$. By using Laplace transformation, eqn (2.46) can be transformed into the Laplace domain and rewritten as

$$\bar{w}(z,s) = \sum_{k=1}^{K} \bar{\alpha}_k(s) e^{-(k-1)z/h_b} \tag{2.47}$$

where $\overline{w}(z,s)$ and $\overline{\alpha}_k(s)$ are the displacement profile and a set of the arbitrary coefficients in the Laplace domain, respectively. The strain energy of the elastic bar corresponding to the assumed displacement function in the Laplace domain can be expressed as

$$U_b = \sum_{j=1}^{K}\sum_{k=1}^{K} D_{jk}\overline{\alpha}_j(s)\overline{\alpha}_k(s) \tag{2.48}$$

$$D_{jk} = \sum_{i=1}^{N_t} \frac{\pi E^{(i)*}(k-1)(j-1)}{2h_b}\left[\frac{a^2\left(e^{-(k+j-2)(z_i-\Delta t_i/2)/h_b} - e^{-(k+j-2)(z_i+\Delta t_i/2)/h_b}\right)}{(k+j-2)}\right]$$

$$\text{for} \quad k+j \neq 2 \tag{2.49}$$

$$E^{(i)*} = E_b - E_h^{(i)} \qquad \text{for } i = 1,2,3,\ldots\ldots,N_t \tag{2.50}$$

$$D_{11} = 0 \tag{2.51}$$

where U_b denotes the strain energy functional of the elastic bar, a is the radius of the bar, E_b and $E_h^{(i)}$ are the moduli of elasticity of the elastic bar and the i^{th} layer of the multilayered half-space, respectively. In addition, N_t is the total number of elements used for discretizing the bar and Δt_i denotes the thickness of the i^{th} element of the bar as shown in Fig. 4. The derivation of D_{jk} is explicitly shown in Appendix B.

The strain energy of the multilayered poroelastic half-space due to the deformation imposed along the contact surface can be determined if the body forces acting through the volume enclosed by the contact surface, S, are known as shown in Fig. 4. These body forces can be computed from the following flexibility equations

$$[f_{ij}]\{B_{kj}\} = \{w_{ki}\} \quad i,j = 1,2,\ldots\ldots,N_t \text{ and } k = 1,2,\ldots\ldots,K \tag{2.52}$$

$$\{B_{kj}\} = \langle B_{k1},\ldots\ldots,B_{ki},\ldots\ldots,B_{kN_t}\rangle^T \tag{2.53}$$

$$\{w_{kj}\} = \langle \beta_{k1},\ldots\ldots,\beta_{ki},\ldots\ldots,\beta_{kN_t}\rangle^T \tag{2.54}$$

$$\beta_{ki} = e^{-(k-1)z_i/h_b} \tag{2.55}$$

From eqns (2.52) to (2.55), $[f_{ij}]$ is the flexibility matrix which can be determined by solving the global stiffness equation of the multilayered half-space defined by eqn (2.45) and its elements, f_{ij}, denote the vertical displacements at point $P_i(r_i, z_i)$ on the contact surface, S, due to a body force of unit intensity distributed through the volume of the j^{th} element. Vector B_{kj} is defined as the body force acting through the j^{th} element which causes the displacement w_{ki}. In addition, w_{ki} denotes the displacement corresponding to each term of eqn (2.47) with $\bar{\alpha}_1(s)$, $\bar{\alpha}_2(s), \ldots, \bar{\alpha}_K(s)$ equal to unity.

By using the body force B_{kj}, obtained from eqn (2.52), the strain energy of the multilayered half-space can finally be expressed as

$$U_{HS} = \frac{\pi}{2}\sum_{j=1}^{K}\sum_{k=1}^{K}\sum_{i=1}^{N_t} \bar{\alpha}_j(s)\bar{\alpha}_k(s)B_{ji}a^2 e^{-(k-1)z_i/h_b}\Delta t_i \tag{2.56}$$

The potential energy of the axial load V_0 at the top of the bar in Fig. 3 due to the assumed displacement function in the Laplace transform space can be expressed as

$$W = -V_0 \sum_{k=1}^{K} \bar{\alpha}_k(s) \qquad (2.57)$$

In view of eqns (2.48), (2.56) and (2.57), the total potential energy functional of the bar-multilayered media system, U_T, can be written as

$$U_T = U_b + U_{HS} - V_0 \sum_{k=1}^{K} \bar{\alpha}_k(s) \qquad (2.58)$$

The minimization of the total potential energy functional given by eqn (2.58) (i.e., $\partial U_T / \partial \alpha_k = 0$ and $k = 1,2,\ldots,K$) results in the following linear simultaneous equations

$$\sum_{k=1}^{K} \bar{\alpha}_k(s) \left\{ 2D_{ki} + \frac{\pi}{2} \sum_{j=1}^{N_L} \left[a^2 \left(B_{kj} e^{-(i-1)z_j/h_b} + B_{ij} e^{-(k-1)z_j/h_b} \right) \Delta t_j \right] \right\} = 0 \qquad (2.59)$$

and $i = 1,2,\ldots,K$

The above equation represents the equilibrium equation of the bar-multilayered media system. The solution of the system of simultaneous equations, eqn (2.59), results in the numerical values of the arbitrary functions $\bar{\alpha}_1(s)$, $\bar{\alpha}_2(s)$, . . . , $\bar{\alpha}_K(s)$ in the Laplace transform space. The inverse Laplace transformation is applied to transform those arbitrary functions from the Laplace domain to the time domain. Finally, back substitution of

$\alpha_1(t)$, $\alpha_2(t)$,...,$\alpha_K(t)$ into eqn (2.46) results in the time histories of displacement profiles of the bar.