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BINOMIAL CHARACTERIZATIONS OF INNER PRODUCT SPACES

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เราศึกษาเงื่อนไขจำเป็นสำหรับการที่ปริภูมินอร์มจะเป็นปริภูมิผลคูณภายใน พบว่า มีลักษณะสมบัติของปริภูมิผลคูณภายในซึ่งอยู่ในรูป $\sum_{j=0}^{k} a_j \|b_j x + c_j y\|^2 = 0$ โดยที่ a_j, b_j และ c_j สอดคล้องเงื่อนไขทางพีชคณิตบางประการ จากนั้น เราเสนอลักษณะสมบัติของปริภูมิผลคูณภายในซึ่งเกี่ยวข้องกับเอกลักษณ์ทวินาม และสรุปผลการศึกษาได้ว่าปริภูมินอร์มเชิงเส้น $(X, \|\cdot\|)$ เป็นปริภูมิผลคูณภายใน ถ้ามีจำนวนเต็มบวก k และ m โดยที่ $2m < k$ ที่ทำให้ทุก $x, y \in X$,

$$\sum_{j=0}^{k} \binom{k}{j} (-1)^j \|x + j^m y\|^2 = 0.$$

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต ....

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We study the sufficient conditions for a normed space to be an inner product space. We found that some characterizations of inner product spaces are in a form $\sum_{j=0}^{k} a_j \|b_j x + c_j y\|^2 = 0$ where a_j, b_j and c_j satisfy some algebraic conditions. We then introduce some characterizations of inner product spaces relating to the binomial identity and conclude that a normed linear space $(X, \|\cdot\|)$ is an inner product space if there exist positive integers k and m with $2m < k$ such that

$$\sum_{j=0}^{k} \binom{k}{j} (-1)^j \|x + j^m y\|^2 = 0 \text{ for all } x, y \in X.$$

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CHAPTER I
INTRODUCTION

Normed spaces are vector spaces endowed with a map called the norm, which acts as the role of the modulus. In fact, there are a lot of significant natural geometric properties which fail in general normed spaces as non Euclidean spaces. Some of these interesting properties hold just when the space is an inner product space. This is the most important motivation for study of characterizations of inner product spaces.

The classic characterization of inner product spaces is the following property called the *parallelogram law* :

$$\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2 \text{ for all } x,y\in X.$$

In 1935, Jordan and von Neumann [1] proved that the parallelogram law is a necessary and sufficient condition for the norm to be induced by an inner product. This result was generalized by Carlsson [3] by assuming some algebraic conditions. Later in 1973, Johnson [4] introduced an interesting characterization of inner product spaces involving the binomial coefficients.

As we pointed out, these motivated us to find other conditions, which are more general than Johnson' s condition and more explicit than Carlsson's condition. Moreover, since some algebraic conditions are released by the binomial identity, we also give a new characterization of inner product space.

In Chapter II, we recall some basic definitions and theorems involving the von Neumann's characterization.

In Chapter III, we study some characterizations of inner product spaces involving the binomial coefficients. We present the result of Johnson which motivates our work and we also show that the proof is specific .

In the last chapter, we prove some binomial identities which are used for the proof of our main theorem and give a new characterization generalizing the result of Johnson.

CHAPTER II
PRELIMINARIES

In this chapter, we recall some definitions, notations and theorems used throughout this research. We also review some literature related to our research.

Definition 2.1. Let X be a complex vector space over a field $\mathbb{C}$. A *norm* on X is a map $\|\cdot\| : X \to [0, \infty)$ that satisfies the following three properties.

i $\|x\| = 0$ if and only if $x = 0$.

ii $\|\alpha x\| = |\alpha| \cdot \|x\|$ for all $x \in X$ and $\alpha \in \mathbb{C}$.

iii $\|x + y\| \leqslant \|x\| + \|y\|$ for all $x, y \in X$.

A *normed linear space* is a pair $(X, \|\cdot\|)$, where X is a vector space and $\|\cdot\|$ is a norm on X.

Definition 2.2. An *inner product* on a vector space X over a field $\mathbb{C}$ is a function that takes each ordered pair (x, y) of elements of X to a number $\langle x, y\rangle \in \mathbb{C}$ and has the following properties.

i $\langle x, x\rangle$ is not any negative real number for all $x \in X$ and
$\langle x, x\rangle = 0$ if and only if $x = \bar{0}$.

ii $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ for all $x, y, z \in X$.

iii $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$ for all $x, y \in X$ and $\alpha \in \mathbb{C}$.

iv $\langle x, y\rangle = \overline{\langle y, x\rangle}$ for all $x, y \in X$.

The vector space X with an inner product is called an *inner product space*.

It is clear that every inner product space X is a normed linear space because one can define a norm on X as follows :

Proposition 2.3. *If* $\langle\ ,\ \rangle : X \times X \to \mathbb{C}$ *is an inner product on a vector space* X*, then the function* $\|\cdot\| : X \to \mathbb{R}$*, defined by*

$$\|x\| = \sqrt{\langle x, x\rangle} \text{ for } x \in X,$$

is a norm on X*. This norm is called the norm induced by the inner product* $\langle\cdot,\cdot\rangle$*.*

In Chapter I, we mentioned the property that does not hold in a general normed linear space.

Proposition 2.4. *If* X *is an inner product space, then for every* $x, y \in X$*,*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

This property is called the *parallelogram law*. The next proposition shows that for an inner product space, we can write the inner product in terms of the corresponding induced norm.

Proposition 2.5. *Let* $(X, \langle\ ,\ \rangle)$ *be a complex inner product space and* $\|\cdot\|$ *the corresponding norm, then for every* $x, y \in X$*,*

$$\langle x, y\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \|x + i^k y\|^2.$$

Proof. For any $x, y \in X$, we have

$$\begin{aligned}
\|x+y\|^2 + i^2 \left\|x+i^2y\right\|^2 &= \|x+y\|^2 - \|x-y\|^2 \\
&= \langle x+y, x+y\rangle - \langle x-y, x-y\rangle \\
&= \langle x, x+y\rangle + \langle y, x+y\rangle - \langle x, x-y\rangle + \langle y, x-y\rangle \\
&= \overline{\langle x+y, x\rangle} + \overline{\langle x+y, y\rangle} - \overline{\langle x-y, x\rangle} + \overline{\langle x-y, y\rangle} \\
&= \overline{\langle x+y, x\rangle} - \overline{\langle x-y, x\rangle} + \overline{\langle x+y, y\rangle} + \overline{\langle x-y, y\rangle} \\
&= \overline{\langle 2y, x\rangle} + \overline{\langle 2x, y\rangle} \\
&= 2\langle x, y\rangle + 2\overline{\langle x, y\rangle} \\
&= 4\mathrm{Re}\langle x, y\rangle,
\end{aligned}$$

and

$$\begin{aligned}
i\|x+iy\|^2 + i^3 \left\|x+i^3y\right\|^2 &= i\|x+iy\|^2 - i\|x-iy\|^2 \\
&= i\left(\|x+iy\|^2 - \|x-iy\|^2\right) \\
&= i\left(\langle x+iy, x+iy\rangle - \langle x-iy, x-iy\rangle\right) \\
&= i\left(\langle x, x+iy\rangle + i\langle y, x+iy\rangle - \langle x, x-iy\rangle + i\langle y, x-iy\rangle\right) \\
&= i\left(\overline{\langle x+iy, x\rangle} + i\overline{\langle x+iy, y\rangle} - \overline{\langle x-iy, x\rangle} + i\overline{\langle x-iy, y\rangle}\right) \\
&= i\left(\overline{\langle x+iy, x\rangle} - \overline{\langle x-iy, x\rangle} + i\overline{\langle x+iy, y\rangle} + i\overline{\langle x-iy, y\rangle}\right) \\
&= i\left(\overline{\langle 2iy, x\rangle} + i\overline{\langle 2x, y\rangle}\right) \\
&= i\left(-2i\langle x, y\rangle + 2i\overline{\langle x, y\rangle}\right) \\
&= 2\langle x, y\rangle - 2\overline{\langle x, y\rangle} \\
&= 4\mathrm{Im}\langle x, y\rangle.
\end{aligned}$$

Since $\sum_{k=0}^{3} i^k\|x+i^ky\|^2 = \|x+y\|^2 + i\|x+iy\|^2 + i^2\left\|x+i^2y\right\|^2 + +i^3\left\|x+i^3y\right\|^2$, it follows that $\langle x, y\rangle = \frac{1}{4}\left(4\mathrm{Re}\langle x, y\rangle + 4\mathrm{Im}\langle x, y\rangle\right) = \frac{1}{4}\sum_{k=0}^{3} i^k\|x+i^ky\|^2$. □

This identity is called the *complex polarization identity*. For an inner product space over real numbers, the polarization identity becomes

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$$

since the imaginary part is zero.

In fact, there is a normed linear space which is not an inner product space. For example, the p-norm on a vector space X of dimension n of $x = (x_1, x_2, \ldots, x_n)$ defined by $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$ where $p \neq 2$.

Example 2.6. Let $p \neq 2$. Consider the $\|\cdot\|_p$ on l^p. Choose $x = (1, 1, 0, \ldots) \in l^p$ and $y = (1, -1, 0, \ldots) \in l^p$. It is easy to see that $\|x+y\|_p = 2 = \|x-y\|_p$ and $\|x\|_p = 2^{\frac{1}{p}} = \|y\|_p$. Consider

$$\|x+y\|_p^2 + \|x-y\|_p^2 = 4 + 4 \text{ and } 2\|x\|_p^2 + 2\|y\|_p^2 = 4 \cdot 2^{\frac{2}{p}}.$$

Hence the parallelogram law holds if $8 = 4 \cdot 2^{\frac{2}{p}}$. Since $p \neq 2$, the parallelogram law fails, i.e., the norm $\|\cdot\|_p$ is not induced by inner product.

The first literature begins with that by Jordan and von Neumann who prove that the parallelogram law is a necessary and sufficient conditions for the norm to be induced by an inner product in 1935.

Theorem 2.7. *(von Neumann) Let X be a linear space. A norm on X is induced by an inner product on X if and only if it satisfies the parallelogram law. Moreover, if a norm on X satisfies the parallelogram law, then the unique inner product that induces this norm is given by the polarization identity.*

Proof. Assume that a norm $\|\cdot\|$ is induced by an inner product on X. By proposition 2.3, we have

$$\begin{aligned}
&\|x+y\|^2+\|x-y\|^2\\
&=\langle x+y,x+y\rangle+\langle x-y,x-y\rangle\\
&=\langle x,x+y\rangle+\langle y,x+y\rangle+\langle x,x-y\rangle-\langle y,x-y\rangle\\
&=\langle x+y,x\rangle+\langle x+y,y\rangle+\langle x-y,x\rangle-\langle x-y,y\rangle\\
&=\langle x,x\rangle+\langle y,x\rangle+\langle x,y\rangle+\langle y,y\rangle+\langle x,x\rangle-\langle y,x\rangle-\langle x,y\rangle+\langle y,y\rangle\\
&=2\langle x,x\rangle+2\langle y,y\rangle\\
&=2\|x\|^2+2\|y\|^2.
\end{aligned}$$

To prove that a norm on X is induced by an inner product on X if it satisfies the parallelogram law, we verify each propery in definition 2.2. Let $x, y, z \in X$. Assume that a norm $\|\cdot\|$ on X satisfies $\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2$ and a function $\langle\ ,\ \rangle$ mapping from $X\times X$ to $\mathbb{C}$ is defined by $\langle x,y\rangle=\frac{1}{4}\sum_{k=0}^{3} i^k\|x+i^k y\|^2$ for all $x, y \in X$. (i) For $x \in X$,

$$\begin{aligned}
\langle x,x\rangle&=\frac{1}{4}\sum_{k=0}^{3} i^k\|x+i^k x\|^2\\
&=\frac{\|x+x\|^2+i\|x+ix\|^2-\|x-x\|^2-i\|x-ix\|^2}{4}\\
&=\frac{4\|x\|^2+i\|x+ix\|^2-i\|x-ix\|^2}{4}\\
&=\|x\|^2+\frac{i\,|i^2|\,\|-ix+x\|^2-i\|x-ix\|^2}{4}\\
&=\|x\|^2.
\end{aligned}$$

Since $\|x\|^2 \geqslant 0$ for all $x \in X$, $\langle x,x\rangle$ is a non-negative real number. For ($\Rightarrow$), assume that $\langle x,x\rangle=0$. Since $\mathrm{Re}\langle x,x\rangle=\|x\|^2$ and $\mathrm{Im}\langle x,x\rangle=0$, x has to be zero vector. ($\Leftarrow$) Assume $x=0$. Then $\langle x,x\rangle=\langle 0,0\rangle=0$. Thus $\langle\ ,\ \rangle$ has a positive definiteness. (ii) By the parallelogram law, we have

$$\|(x+z)+y\|^2 + \|(x+z)-y\|^2 = 2\|x+z\|^2 + 2\|y\|^2 \tag{2.1}$$

$$\|(x-z)+y\|^2 + \|(x-z)-y\|^2 = 2\|x-z\|^2 + 2\|y\|^2 \tag{2.2}$$

$$\|(x+iz)+y\|^2 + \|(x+iz)-y\|^2 = 2\|x+iz\|^2 + 2\|y\|^2 \tag{2.3}$$

$$\|(x-iz)+y\|^2 + \|(x-iz)-y\|^2 = 2\|x-iz\|^2 + 2\|y\|^2 \tag{2.4}$$

Substracting equation 2.2 from equation 2.1, we obtain

$$\begin{aligned}
&2\left(\|x+z\|^2 + \|y\|^2 - \|x-z\|^2 - \|y\|^2\right)\\
&= \|(x+z)+y\|^2 + \|(x+z)-y\|^2 - \|(x-z)+y\|^2 - \|(x-z)-y\|^2\\
&= \left(\|(x+z)+y\|^2 - \|(x-z)+y\|^2\right) + \left(\|(x+z)-y\|^2 - \|(x-z)-y\|^2\right)\\
&= \left(\|(x+y)+z\|^2 - \|(x+y)-z\|^2\right) + \left(\|(x-y)+z\|^2 - \|(x-y)-z\|^2\right).
\end{aligned}$$

Similarly, substracting equation 2.4 from equation 2.3, we obtain

$$\begin{aligned}
&2\left(\|x+iz\|^2 + \|y\|^2 - \|x-iz\|^2 - \|y\|^2\right)\\
&= \left(\|(x+y)+iz\|^2 - \|(x+y)-iz\|^2\right) + \left(\|(x-y)+iz\|^2 - \|(x-y)-iz\|^2\right).
\end{aligned}$$

Multiplying both sides by i, yields

$$\begin{aligned}
&2i\left(\|x+iz\|^2 - \|x-iz\|^2\right)\\
&= i\left(\|(x+y)+iz\|^2 - \|(x+y)-iz\|^2\right) + i\left(\|(x-y)+iz\|^2 - \|(x-y)-iz\|^2\right).
\end{aligned}$$

Since $\langle\ ,\ \rangle$ is defined by $\langle x,y\rangle = \dfrac{1}{4}\displaystyle\sum_{k=0}^{3} i^k\|x+i^k y\|^2$ for all $x, y \in X$, we have that

$$\begin{aligned}
&8\mathrm{Re}\langle x,z\rangle = 4\mathrm{Re}\langle x+y,z\rangle + 4\mathrm{Re}\langle x-y,z\rangle \text{ and}\\
&8\mathrm{Im}\langle x,z\rangle = 4\mathrm{Im}\langle x+y,z\rangle + 4\mathrm{Im}\langle x-y,z\rangle.
\end{aligned}$$

Adding two above equations and dividing both sides by 4, we obtain

$$2\langle x, z\rangle = \langle x + y, z\rangle + \langle x - y, z\rangle. \tag{2.5}$$

Then letting $x', y' \in X$ such that $x = \dfrac{x'+y'}{2}$ and $y = \dfrac{x'-y'}{2}$, and substituting there in to equation 2.5 yields

$$\begin{aligned} 2\left\langle \frac{x'+y'}{2}, z\right\rangle &= \left\langle \frac{x'+y'}{2} + \frac{x'-y'}{2}, z\right\rangle + \left\langle \frac{x'+y'}{2} - \frac{x'-y'}{2}, z\right\rangle \\ \langle x'+y', z\rangle &= \langle x', z\rangle + \langle y', z\rangle. \end{aligned}$$

Hence $\langle\ ,\ \rangle$ satisfies additivity in first slot. (iii) For $\alpha \in \mathbb{C}$, we can write $\alpha = a + ib$ where $a, b \in \mathbb{R}$. Then $\langle \alpha x, y\rangle = \langle ax + ibx, y\rangle$. By the closure property of vector space X and ii), we have $\langle \alpha x, y\rangle = \langle ax, y\rangle + \langle ibx, y\rangle$. To prove that $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$ for all $\alpha \in \mathbb{C}$, we start with showing that $\langle rx, y\rangle = r\langle x, y\rangle$ for all $r \in \mathbb{R}$. Let $S = \{\lambda \in \mathbb{R} \mid \langle \lambda x, y\rangle = \lambda\langle x, y\rangle\}$. Clearly $\langle 1 \cdot x, y\rangle = \langle x, y\rangle = 1 \cdot \langle x, y\rangle$ and thus $1 \in S$. Suppose that $a, b \in S$, then

$$\begin{aligned} (a \pm b)\langle x, y\rangle &= a\langle x, y\rangle \pm b\langle x, y\rangle \\ &= \langle ax, y\rangle \pm \langle bx, y\rangle \\ &\overset{\text{(ii)}}{=} \langle ax \pm bx, y\rangle \\ &= \langle (a \pm b)x, y\rangle. \end{aligned}$$

This means that $a \pm b \in S$, and therefore $\mathbb{Z} \subset S$. Let $a, b \in S$ and $b \neq 0$. Then

$$\frac{a}{b}\langle x, y\rangle = \frac{1}{b} \cdot a\langle x, y\rangle = \frac{1}{b}\langle ax, y\rangle = \frac{1}{b}\left\langle \frac{b}{b} \cdot ax, y\right\rangle = \frac{b}{b}\left\langle \frac{1}{b}ax, y\right\rangle = \left\langle \frac{a}{b}x, y\right\rangle.$$

This means that $\dfrac{a}{b} \in S$, and therefore $\mathbb{Q} \subset S$. To show that $\mathbb{R} \subset S$, let f and g be real-valued functions such that $f(\lambda) = \lambda\langle x, y\rangle$ and $g(\lambda) = \langle \lambda x, y\rangle$. By the property $\dfrac{a}{b}\langle x, y\rangle = \left\langle \dfrac{a}{b}x, y\right\rangle$, it is clear that $f(\lambda) = g(\lambda)$ for $\lambda \in \mathbb{Q}$. Furthermore, both f and g are continuous because f is linear and g is a composition of continuous functions as we define the function $\langle\ ,\ \rangle$. Since if values of two real continuous functions coincide for every rational number, the values must coincide for each irrational number as well. This implies that $f = g$ for all $\lambda \in \mathbb{R}$. Therefore $\mathbb{R} \subset S$, and hence $S = \mathbb{R}$. It suffices to show that

$$\begin{aligned}
\langle ix, y\rangle &= \frac{1}{4}\sum_{k=0}^{3} i^k \|ix + i^k y\|^2 \\
&= \frac{\|ix+y\|^2 + i\,\|ix+iy\|^2 - \|ix-y\|^2 - i\,\|ix-iy\|^2}{4} \\
&= \frac{|i^2|\,\|x-iy\|^2 + i\cdot|i^2|\,\|x+y\|^2 - |i^2|\,\|x+iy\|^2 - i\cdot|i^2|\,\|x-y\|^2}{4} \\
&= \frac{\|x-iy\|^2 + i\,\|x+y\|^2 - \|x+iy\|^2 - i\,\|x-y\|^2}{4} \\
&= \frac{i\left(-i\,\|x-iy\|^2 + \|x+y\|^2 + i\,\|x+iy\|^2 - \|x-y\|^2\right)}{4} \\
&= i\langle x, y\rangle.
\end{aligned}$$

Hence $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$ for all $\alpha \in \mathbb{C}$. (iv) We have

$$\begin{aligned}
\langle x, y\rangle &= \frac{1}{4}\sum_{k=0}^{3} i^k \|x + i^k y\|^2 \\
&= \frac{\|x+y\|^2 + i\,\|x+iy\|^2 - \|x-y\|^2 - i\,\|x-iy\|^2}{4} \\
&= \frac{\|y+x\|^2 + i\cdot i^2\,\|y-ix\|^2 - \|y-x\|^2 - i\cdot i^2\,\|y+ix\|^2}{4} \\
&= \frac{\|y+x\|^2 - i\,\|y-ix\|^2 - \|y-x\|^2 + i\,\|y+ix\|^2}{4} \\
&= \overline{\langle y, x\rangle}.
\end{aligned}$$

Hence $\langle\ ,\ \rangle$ is an inner product. □

In 1961, Sten Olof Carlsson worked on tuples (a_j, b_j, c_j) of real numbers that satisfy $\sum_{j=0}^{k} a_j b_j^2 = 0$, $\sum_{j=0}^{k} a_j b_j c_j = 0$ and $\sum_{j=0}^{k} a_j c_j^2 = 0$. Then he gave a generalization of the Jordan-von Neumann result.

Theorem 2.8. *For a positive interger k, let $j = 0, 1, \ldots, k$ and $a_j \neq 0, b_j, c_j$ be* *real numbers such that* (b_i, c_i) *and* (b_j, c_j) *are linearly independent for* $i \neq j$. *If* X *is a normed linear space satisfying the condition*

$$\sum_{j=0}^{k} a_j \|b_j x + c_j y\|^2 = 0 \textit{ for all } x, y \in X,$$

then the norm on X *is induced by an inner product.*

However, the characterization of inner product spaces given by Sten Olof Carlsson needs some redundant algebraic conditions and it is difficult to compute the explicit values of a_j, b_j and c_j. A special case of Theorem 2.8 will be studied in the next chapter where some algebraic conditions are replaced by a binomial identity.

CHAPTER III
GORDON G. JOHNSON'S CHARACTERIZATION OF INNER PRODUCT SPACES

As we mentioned in Chapter II, the charactrization of inner product spaces given by Sten Olof Carlsson generalizes the characterization by pararellogram law. However, it is difficult to find the explicit example for the condition. In this chapter, we restate the result which is a special case of Theorem 2.8 which motivates our work.

An interesting characterization of inner product spaces was introduced by Gordon G. Johnson in 1973. The characterization is an equation relating to a binomial coefficients and its proof shows that the equation can be reformed to the parallelogram law.

Theorem 3.1. *If X is a linear space with norm $\|\cdot\|$ and, for some integer $k \geqslant 3$,*

$$\sum_{t=0}^{k} \binom{k}{t} (-1)^t \|x + ty\|^2 = 0$$

for all $x, y \in X$ then $\sum_{t=0}^{k} \binom{k}{t} (-1)^t \|x + ty\|^2 = 0$ for every integer $k \geqslant 3$ and the norm $\|\cdot\|$ is induced by an inner product on X.

We observe that this identity is a special case of Theorem 2.8 obtained by setting $a_j = \binom{k}{j}(-1)^j, b_j = 1$ and $c_j = j$ for $j = 0, 1, \ldots, k$.

Proof. For $k, n \in \mathbb{N}$, we define

$$D_k^n(x, y) = \sum_{t=0}^{k} \binom{k}{t} (-1)^t \|x + (t+n)y\|^2 \text{ for all } x, y \in X.$$

Suppose $k \geqslant 3$ and $D_k^0(x,y) = 0$ for all $x, y \in X$. Since $x + ny \in X$ for all $n \in \mathbb{N}$,

$$\begin{aligned} D_k^n(x,y) &= \sum_{t=0}^{k} \binom{k}{t} (-1)^t \|x + (t+n)y\|^2 \\ &= \sum_{t=0}^{k} \binom{k}{t} (-1)^t \|(x+ny) + ty\|^2 \\ &= D_k^0(x+ny, y) \\ &= 0. \end{aligned}$$

This means that $D_k^n(x,y) = 0$ for all $n \in \mathbb{N}$.

Consider $D^{n+1}_{k-1}(x,y) - D^{n}_{k-1}(x,y)$

$$
\begin{aligned}
&= \sum_{t=0}^{k-1}\binom{k-1}{t}(-1)^t\|x+(t+n+1)y\|^2 - \sum_{t=0}^{k-1}\binom{k-1}{t}(-1)^t\|x+(t+n)y\|^2 \\
&= \binom{k-1}{k-1}(-1)^{k-1}\|x+(k+n)y\|^2 + \sum_{t=0}^{k-2}\binom{k-1}{t}(-1)^t\|x+(t+n+1)y\|^2 \\
&\quad - \sum_{t=1}^{k-1}\binom{k-1}{t}(-1)^t\|x+(t+n)y\|^2 - \binom{k-1}{0}(-1)^0\|x+ny\|^2 \\
&= \binom{k-1}{k-1}(-1)^{k-1}\|x+(k+n)y\|^2 + \sum_{t=1}^{k-1}\binom{k-1}{t-1}(-1)^{t-1}\|x+(t+n)y\|^2 \\
&\quad - \sum_{t=1}^{k-1}\binom{k-1}{t}(-1)^t\|x+(t+n)y\|^2 - \binom{k-1}{0}(-1)^0\|x+ny\|^2 \\
&= \binom{k-1}{k-1}(-1)^{k-1}\|x+(k+n)y\|^2 - \binom{k-1}{0}(-1)^0\|x+ny\|^2 \\
&\quad + \sum_{t=1}^{k-1}\left[\binom{k-1}{t-1}(-1)^{t-1} - \binom{k-1}{t}(-1)^t\right]\|x+(t+n)y\|^2 \\
&= \binom{k-1}{k-1}(-1)^{k-1}\|x+(k+n)y\|^2 - \binom{k-1}{0}(-1)^0\|x+ny\|^2 \\
&\quad - \sum_{t=1}^{k-1}\left[\binom{k-1}{t-1}(-1)^{t} + \binom{k-1}{t}(-1)^t\right]\|x+(t+n)y\|^2 \\
&= \binom{k-1}{k-1}(-1)^{k-1}\|x+(k+n)y\|^2 - \sum_{t=1}^{k-1}\binom{k}{t}(-1)^t\|x+(t+n)y\|^2 \\
&\quad - \binom{k-1}{0}(-1)^0\|x+ny\|^2 \\
&= -\binom{k}{k}(-1)^{k}\|x+(k+n)y\|^2 - \sum_{t=1}^{k-1}\binom{k}{t}(-1)^t\|x+(t+n)y\|^2 \\
&\quad - \binom{k}{0}(-1)^0\|x+ny\|^2 \\
&= -\sum_{t=0}^{k}\binom{k}{t}(-1)^t\|x+(t+n)y\|^2 \\
&= -D^n_k(x,y).
\end{aligned}
$$

That is $D^{n+1}_{k-1}(x,y) - D^{n}_{k-1}(x,y) = -D^n_k(x,y) = 0$ and hence, $D^{n+1}_{k-1}(x,y) = D^{n}_{k-1}(x,y)$

for all $n \in \mathbb{N}$. Let m be a positive integer not exceeding k. We will prove that $D^n_{k-m}(x,y) = \sum_{t=0}^{m-1} \binom{n}{t} (-1)^t D^0_{k-m+t}(x,y)$ by using mathematical induction.

Base case, set $m = 1$.

$$\sum_{t=0}^{0} \binom{n}{t} (-1)^t D^0_{k-1+t}(x,y) = \binom{n}{0} (-1)^0 D^0_{k-1+0}(x,y) = D^0_{k-1}(x,y) = D^n_{k-1}(x,y)$$

That is the equation holds for $m = 1$. For $m = 2$, we have

$$\begin{aligned}
&D^{n+1}_{k-2}(x,y) - D^n_{k-2}(x,y) \\
&= \sum_{t=0}^{k-2} \binom{k-2}{t} (-1)^t \|x + (t+n+1)y\|^2 - \sum_{t=0}^{k-2} \binom{k-2}{t} (-1)^t \|x + (t+n)y\|^2 \\
&= \binom{k-2}{k-2} (-1)^{k-2} \|x + (k-2+n+1)y\|^2 + \sum_{t=0}^{k-3} \binom{k-2}{t} (-1)^t \|x + (t+n+1)y\|^2 \\
&\quad - \sum_{t=1}^{k-2} \binom{k-2}{t} (-1)^t \|x + (t+n)y\|^2 - \binom{k-2}{0} (-1)^0 \|x + ny\|^2 \\
&= \binom{k-2}{k-2} (-1)^{k-2} \|x + (k-2+n+1)y\|^2 + \sum_{t=1}^{k-2} \binom{k-2}{t-1} (-1)^{t-1} \|x + (t+n)y\|^2 \\
&\quad - \sum_{t=1}^{k-2} \binom{k-2}{t} (-1)^t \|x + (t+n)y\|^2 - \binom{k-2}{0} (-1)^0 \|x + ny\|^2 \\
&= \binom{k-2}{k-2} (-1)^{k-2} \|x + (k-2+n+1)y\|^2 - \binom{k-2}{0} (-1)^0 \|x + ny\|^2 \\
&\quad + \sum_{t=1}^{k-2} \left[-\binom{k-2}{t-1} - \binom{k-2}{t} \right] (-1)^{t-1} \|x + (t+n)y\|^2 \\
&= -\binom{k-1}{k-1} (-1)^{k-1} \|x + (k-1+n)y\|^2 - \binom{k-1}{0} (-1)^0 \|x + ny\|^2 \\
&\quad - \sum_{t=1}^{k-2} \binom{k-1}{t} (-1)^t \|x + (t+n)y\|^2 \\
&= -\sum_{t=1}^{k-2} \binom{k-1}{t} (-1)^t \|x + (t+n)y\|^2 \\
&= -D^n_{k-1}(x,y)
\end{aligned}$$

That is $D^{n+1}_{k-2}(x,y) - D^n_{k-2}(x,y) = -D^n_{k-1}(x,y)$ for $n = 0, 1, 2, \ldots$. Substituting

n by $n-1$, we get $D^n_{k-2}(x,y) - D^{n-1}_{k-2}(x,y) = -D^{n-1}_{k-1}(x,y)$ for all $n \geqslant 2$. Then $D^n_{k-2}(x,y) = D^{n-1}_{k-2}(x,y) - D^{n-1}_{k-1}(x,y)$. By recurrence relation and $D^n_{k-1}(x,y) = D^0_{k-1}(x,y)$ for $n = 0, 1, 2, \ldots$,

$$\begin{aligned}
D^n_{k-2}(x,y) &= D^{n-1}_{k-2}(x,y) - D^0_{k-1}(x,y)\\
&= D^{n-2}_{k-2}(x,y) - D^{n-2}_{k-1}(x,y) - D^0_{k-1}(x,y)\\
&= D^{n-2}_{k-2}(x,y) - 2D^0_{k-1}(x,y)\\
&= D^{n-3}_{k-2}(x,y) - D^{n-3}_{k-1}(x,y) - 2D^0_{k-1}(x,y)\\
&= D^{n-3}_{k-2}(x,y) - 3D^0_{k-1}(x,y)\\
&\ \vdots\\
&= D^{n-(n-1)}_{k-2}(x,y) - (n-1)D^0_{k-1}(x,y)\\
&= D^{n-n}_{k-2}(x,y) - D^{n-n}_{k-1}(x,y) - (n-1)D^0_{k-1}(x,y)\\
&= D^0_{k-2}(x,y) - nD^0_{k-1}(x,y).
\end{aligned}$$

That is $D^n_{k-2}(x,y) = D^0_{k-2}(x,y) - nD^0_{k-1}(x,y)$. Since $\binom{n}{1} = n$, $D^n_{k-2}(x,y) = \sum_{t=0}^{1}\binom{n}{t}(-1)^t D^0_{k-2+t}(x,y)$. Thus $D^n_{k-m}(x,y) = \sum_{t=0}^{m-1}\binom{n}{t}(-1)^t D^0_{k-m+t}(x,y)$ holds for $m = 2$. We now suppose that $D^n_{k-m}(x,y) = \sum_{t=0}^{m-1}\binom{n}{t}(-1)^t D^0_{k-m+t}(x,y)$ is true for $m = k-1$. That is,

$$D^n_1(x,y) = \sum_{t=0}^{k-2}\binom{n}{t}(-1)^t D^0_{1+t}(x,y).$$

For $m = k$, we have

$$
\begin{aligned}
D_0^n(x,y) &= D_0^{n-1}(x,y) - D_1^{n-1}(x,y)\\
&= \left[D_0^{n-2}(x,y) - D_1^{n-2}(x,y)\right] - D_1^{n-1}(x,y)\\
&= \left[D_0^{n-3}(x,y) - D_1^{n-3}(x,y)\right] - D_1^{n-2}(x,y) - D_1^{n-1}(x,y)\\
&\ \vdots\\
&= D_0^{n-(n-1)}(x,y) - D_1^{n-(n-1)}(x,y) - \ldots - D_1^{n-2}(x,y) - D_1^{n-1}(x,y)\\
&= D_0^{1}(x,y) - D_1^{1}(x,y) - \ldots - D_1^{n-2}(x,y) - D_1^{n-1}(x,y)\\
&= \left[D_0^{0}(x,y) - D_1^{0}(x,y)\right] - D_1^{1}(x,y) - \ldots - D_1^{n-2}(x,y) - D_1^{n-1}(x,y)\\
&= D_0^{0}(x,y) - D_1^{0}(x,y) - D_1^{1}(x,y) - \ldots - D_1^{n-2}(x,y) - D_1^{n-1}(x,y)\\
&= D_0^{0}(x,y) - \sum_{t=0}^{k-2}\binom{0}{t}(-1)^t D_{1+t}^0(x,y) - \sum_{t=0}^{k-2}\binom{1}{t}(-1)^t D_{1+t}^0(x,y) - \ldots\\
&\quad - \sum_{t=0}^{k-2}\binom{n-2}{t}(-1)^t D_{1+t}^0(x,y) - \sum_{t=0}^{k-2}\binom{n-1}{t}(-1)^t D_{1+t}^0(x,y)\\
&= D_0^{0}(x,y) - \sum_{t=0}^{k-2}\left[\binom{0}{t} + \binom{1}{t} + \ldots + \binom{n-2}{t} + \binom{n-1}{t}\right](-1)^t D_{1+t}^0(x,y)\\
&= D_0^{0}(x,y) - \sum_{t=0}^{k-2}\binom{n}{t+1}(-1)^t D_{1+t}^0(x,y)\\
&= D_0^{0}(x,y) - \sum_{t=1}^{k-1}\binom{n}{t}(-1)^{t-1} D_{t}^0(x,y)\\
&= \binom{n}{0}(-1)^0 D_0^0(x,y) + \sum_{t=1}^{k-1}\binom{n}{t}(-1)^t D_t^0(x,y)\\
&= \sum_{t=0}^{k-1}\binom{n}{t}(-1)^t D_t^0(x,y).
\end{aligned}
$$

We have that $D_{k-m}^n(x,y) = \sum_{t=0}^{m-1}\binom{n}{t}(-1)^t D_{k-m+t}^0(x,y)$ is true for $m = k$. By mathematical induction, $D_{k-m}^n(x,y) = \sum_{t=0}^{m-1}\binom{n}{t}(-1)^t D_{k-m+t}^0(x,y)$ for $m \leqslant k$. Hence $D_0^n(x,y) = \sum_{t=0}^{k-1}\binom{n}{t}(-1)^t D_t^0(x,y)$ for $n = 0, 1, 2, \ldots$. By definition, $D_0^n(x,y) =$

$\|x+ny\|^2$. Thus

$$\begin{aligned}\|\frac{1}{n}x+y\|^2 &= \left(\frac{1}{n^2}\right)\|x+ny\|^2\\ &= \frac{1}{n^2}\sum_{t=0}^{k-1}\binom{n}{t}(-1)^t D_t^0(x,y)\\ &= \sum_{t=0}^{k-1}\binom{n}{t}\frac{(-1)^t}{n^2}D_t^0(x,y).\end{aligned}$$

We have that $\lim_{n\to\infty}\|\frac{1}{n}x+y\|^2=\|y\|^2$. Hence

$$\begin{aligned}\|y\|^2 &= \lim_{n\to\infty}\|\frac{1}{n}x+y\|^2\\ &= \lim_{n\to\infty}\sum_{t=0}^{k-1}\binom{n}{t}\frac{(-1)^t}{n^2}D_t^0(x,y)\\ &= \lim_{n\to\infty}\Bigg[D_0^0(x,y)/n^2-\left(\binom{n}{1}\Big/n^2\right)D_1^0(x,y)+\left(\binom{n}{2}\Big/n^2\right)D_2^0(x,y)\\ &\quad -\left(\binom{n}{3}\Big/n^2\right)D_3^0(x,y)+\ldots+\left(\binom{n}{k-1}\Big/n^2\right)(-1)^{k-1}D_{k-1}^0(x,y)\Bigg]\\ &= \lim_{n\to\infty}\Bigg[\frac{1}{n^2}D_0^0(x,y)-\frac{1}{n}D_1^0(x,y)+\frac{n(n-1)}{2!\cdot n^2}D_2^0(x,y)-\frac{n(n-1)(n-2)}{3!\cdot n^2}D_3^0(x,y)\\ &\quad +\ldots+\frac{n(n-1)\cdots(n-k+1)}{(k-1)!\cdot n^2}D_{k-1}^0(x,y)\Bigg].\end{aligned}$$

Since $\lim_{n\to\infty}\|\frac{1}{n}x+y\|^2$ exists, this forces $D_l^0(x,y)=0$ if $3\leqslant l\leqslant k-1$. Therefore $\|y\|^2=\frac{1}{2}\left[\|x+2y\|^2-2\|x+y\|^2+\|x\|^2\right]$ for all x and y in X. Considering $a=x+y$ and $b=y$, the equation becomes $2\|b\|^2=\|a+b\|^2-2\|a\|^2+\|a-b\|^2$. $\square$

On the other hand, does the equation $\sum_{t=0}^{k}\binom{k}{t}(-1)^t\|x+ty\|^2=0$ hold for every inner product space? We now let X be an inner product space and a norm

$\|\cdot\|$ on X is induced by the inner product $\langle\cdot,\cdot\rangle$. Then for $x, y \in X$ and $k \in \mathbb{N}$,

$$\begin{aligned}\sum_{t=0}^{k}\binom{k}{t}(-1)^t\|x+ty\|^2 &= \sum_{t=0}^{k}\binom{k}{t}(-1)^t\langle x+ty, x+ty\rangle \\ &= \sum_{t=0}^{k}\binom{k}{t}(-1)^t\left[\langle x, x+ty\rangle + \langle ty, x+ty\rangle\right] \\ &= \sum_{t=0}^{k}\binom{k}{t}(-1)^t\left[\langle x, x\rangle + \langle x, ty\rangle + \overline{\langle x, ty\rangle} + \langle ty, ty\rangle\right] \\ &= \sum_{t=0}^{k}\binom{k}{t}(-1)^t\left[\|x\|^2 + t\langle x, y\rangle + t\langle y, x\rangle t^2\|y\|^2\right] \\ &= \|x\|^2\sum_{t=0}^{k}\binom{k}{t}(-1)^t + \left[\langle x, y\rangle + \langle y, x\rangle\right]\sum_{t=0}^{k}\binom{k}{t}(-1)^t t \\ &\quad + \|y\|^2\sum_{t=0}^{k}\binom{k}{t}(-1)^t t^2.\end{aligned}$$

Considering the binomial expansion $(a+b)^k = \sum_{t=0}^{k}\binom{k}{t}a^{k-t}b^t$ where $a = 1$ and $b = -1$, we have

$$\sum_{t=0}^{k}\binom{k}{t}(-1)^t = \sum_{t=0}^{k}\binom{k}{t}1^{k-t}(-1)^t = [1+(-1)]^k = 0.$$

Then the first term $\|x\|^2 \sum_{t=0}^{k} \binom{k}{t} (-1)^t = 0$. For the second term, we have that

$$\begin{aligned}
\sum_{t=0}^{k} \binom{k}{t} (-1)^t t &= \sum_{t=0}^{k} (-1)^t \binom{k}{t} t \\
&= \sum_{t=1}^{k} (-1)^t \binom{k}{t} t \\
&= \sum_{t=1}^{k} (-1)^t \frac{k!}{(k-t)!t!} t \\
&= \sum_{t=1}^{k} (-1)^t \frac{k \cdot (k-1)!}{(k-t)!((t-1)!} \\
&= k \sum_{t=1}^{k} (-1)^t \binom{k-1}{t-1}.
\end{aligned}$$

Similarly, we have $\sum_{t=1}^{k} (-1)^t \binom{k-1}{t-1} = 0$. Since $[\langle x, y\rangle + \langle y, x\rangle] \sum_{t=0}^{k} \binom{k}{t} (-1)^t t = k\,[\langle x, y\rangle + \langle y, x\rangle] \sum_{t=1}^{k} (-1)^t \binom{k-1}{t-1}$, the second term becomes zero. Hence, it suffices to verify that the last term is also zero to show the converse of Theorem 3.1 is true. However, the proof of $\sum_{t=0}^{k} \binom{k}{t} (-1)^t t^2 = 0$ is not obvious. We now study some binomial identities for the sake of the above observation and geralization of the characterization given by Gordon G. Johnson

CHAPTER IV
BINOMIAL CHARACTERIZATIONS OF INNER PRODUCT SPACES

In this chapter, we present a condition generalizing the result of Gordon G. Johnson. First, we need to prove some binomial identities which are used for verifying that the algebraic conditions supposed by Sten Olof Carsson are satisfied. Then we show that the condition is a characterization of inner product spaces.

Lemma 4.1. *For an integer $n \geqslant 0$ and a real number x,*

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (x-i)^n = n!.$$

Proof. For each non-negative integer n, define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (x-i)^n \text{ for } x \in \mathbb{R}.$$

It is easy to see that $f_0(x) = (-1)^0 \binom{0}{0} (x-0)^0 = 1 = 0!$ To show that $f_n(x) = n!$ for all $n \geqslant 0$, let k be a non-negative integer such that $f_k(x) = k!$ for every $x \in \mathbb{R}$.

Considering $f_{k+1}(x) = \sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} (x-i)^{k+1}$. Then

$$
\begin{aligned}
\frac{d}{dx} f_{k+1}(x) &= \frac{d}{dx} \sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} (x-i)^{k+1} \\
&= (k+1) \sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} (x-i)^{k} \\
&= (k+1) \sum_{i=0}^{k} (-1)^i \binom{k+1}{i} (x-i)^{k} + (k+1)(-1)^{k+1} \binom{k+1}{k+1} (x-(k+1))^k \\
&= (k+1) \sum_{i=0}^{k} (-1)^i \binom{k}{i} \left(1 + \frac{i}{k+1-i}\right) (x-i)^{k} \\
&\quad + (k+1)(-1)^{k+1} \binom{k+1}{k+1} (x-(k+1))^k \\
&= (k+1) f_k(x) + (k+1) \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{i}{k+1-i} (x-i)^{k} \\
&\quad + (k+1)(-1)^{k+1} \binom{k+1}{k+1} (x-(k+1))^k \\
&= (k+1) f_k(x) + (k+1) \sum_{i=1}^{k} (-1)^i \binom{k}{i-1} (x-i)^{k} \\
&\quad + (k+1)(-1)^{k+1} \binom{k+1}{k+1} (x-(k+1))^k \\
&= (k+1) f_k(x) + (k+1) \sum_{i=1}^{k} (-1)^i \binom{k}{i-1} (x-i)^{k} \\
&\quad + (k+1)(-1)^{k+1} \binom{k}{k} (x-(k+1))^k \\
&= (k+1) f_k(x) + (k+1) \sum_{i=1}^{k+1} (-1)^i \binom{k}{i-1} (x-i)^{k} \\
&= (k+1) f_k(x) + (k+1) \sum_{j=0}^{k+1} (-1)^{j+1} \binom{k}{j} (x-j-1)^{k} \text{ where } j = i-1 \\
&= (k+1) f_k(x) + (k+1) f_k(x-1) \\
&= (k+1)k! - (k+1)k! \\
&= 0.
\end{aligned}
$$

This implies that $f_{k+1}(x)$ is constant. So

$$\begin{aligned}
f_{k+1}(x) &= f_{k+1}(k-1) \\
&= \sum_{i=1}^{k+1}(-1)^i\binom{k+1}{i}(k+1-i)^{k+1} \\
&= \sum_{i=1}^{k}(-1)^i\binom{k+1}{i}(k+1-i)^{k+1} \\
&= \sum_{i=1}^{k}(-1)^i\frac{(k+1)!}{i!(k+1-i)!}(k+1-i)^{k+1} \\
&= \sum_{i=1}^{k}(-1)^i\frac{(k+1)!}{i!(k-i)!}(k+1-i)^{k} \\
&= (k+1)\sum_{i=1}^{k}(-1)^i\frac{k!}{i!(k-i)!}(k+1-i)^{k} \\
&= (k+1)f_k(k+1) \\
&= (k+1)k! \\
&= (k+1)!
\end{aligned}$$

Therefore $f_n(x) = n!$ for all $x \in \mathbb{R}$ for all $n \geqslant 0$. □

Corollary 4.2. *For an integer $n \geqslant 0$ and a real number x,*

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}(x-i)^m = 0$$

where $0 \leqslant m < n$.

Proof. Let n be a non-negative integer, $0 \leqslant m < n$ and $x \in \mathbb{R}$. Then $m = n - j$ for some $1 \leqslant j \leqslant n$. From lemma 4.1, we define $f_n(x) = \sum_{i=0}^{n}(-1)^i\binom{n}{i}(x-i)^n$. Differentiating j times, we get

$$f_n^{(j)}(x) = n(n-1)\cdots[n-(j-1)]\sum_{i=0}^{n}(-1)^i\binom{n}{i}(x-i)^{n-j}.$$

Since $f_n(x)$ is constant, $f_n^{(j)}(x) = 0$ and hence $\sum_{i=0}^{n}(-1)^i\binom{n}{i}(x-i)^{n-j} = 0$ for all $x \in \mathbb{R}$. Therefore $\sum_{i=0}^{n}(-1)^i\binom{n}{i}(x-i)^m = 0$ where $0 \leqslant m < n$. $\square$

Theorem 4.3. *Let X be a normed linear space. Let $k \geqslant 3$ and m a positive integer such that $2m < k$. If*

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^j \|x + j^m y\|^2 = 0$$

for all $x, y \in X$, then X is an inner product space and hence, for $l, n \in \mathbb{N}$, $\sum_{j=0}^{l}\binom{l}{j}(-1)^j \|x + j^n y\|^2 = 0$ for all $x, y \in X$.

Proof. For an integer $k \geqslant 3$ and m a positive integer such that $2m < k$, assume that $\sum_{j=0}^{k}\binom{k}{j}(-1)^j \|x + j^m y\|^2 = 0$ for all $x, y \in X$. Comparing with Theorem 2.8, we have $a_i = \binom{k}{j}(-1)^j$, $b_j = 1$ and $c_j = j^m$ for $j = 0, 1, \ldots, k$. Then $\sum_{j=0}^{k} a_j b_j^2 = \sum_{j=0}^{k}\binom{k}{j}(-1)^j$, $\sum_{j=0}^{k} a_j b_j c_j = \binom{k}{j}(-1)^j j^m$, and $\sum_{j=0}^{k} a_j c_j^2 = \binom{k}{j}(-1)^j j^{2m}$. It is easy to see that (b_i, c_i) and (b_j, c_j) are linearly independent for $i \neq j$ since $b_j = 1$ for all $j = 0, 1, \ldots, k$. By corollary 4.2, $\sum_{j=0}^{k} a_j b_j^2 = 0$, $\sum_{j=0}^{k} a_j b_j c_j = 0$, and $\sum_{j=0}^{k} a_j c_j^2 = 0$. Hence X is an inner product space.

Moreover, for vectors x, y in an inner product space X,

$$\begin{aligned}\sum_{j=0}^{l}\binom{l}{j}(-1)^j \|x + j^n y\|^2 &= \sum_{j=0}^{l}\binom{l}{j}(-1)^j \langle x + j^n y, x + j^n y\rangle \\ &= \sum_{j=0}^{l}\binom{l}{j}(-1)^j \|x\|^2 + 2\sum_{j=0}^{l}\binom{l}{j}(-1)^j j^n \langle x, y\rangle \\ &\quad + \sum_{j=0}^{l}\binom{l}{j}(-1)^j j^{2n} \|y\|^2\end{aligned}$$

since the norm on X is defined by $\|x\| = \sqrt{\langle x, x\rangle}$ for all $x \in X$. By corollary 4.2,

$$\sum_{j=0}^{l} \binom{l}{j} (-1)^j \|x + j^n y\|^2 = 0 \text{ for all } x, y \in X \text{ for } l \geqslant 3 \text{ and } 2n < l. \qquad \square$$

Finally, we observe that Theorem 4.3 generalizes the result of Jonhson and also presents the explicit condition of the characterization of Carlsson.

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