# CHAPTER III

# INDEPENDENCE

A criterion for linear independence of continued fraction expansions over the rational numbers was derived by J. Hančl [11] in 2002. As to algebraic independence, sufficient conditions were given in [1], [3] and [14]. Criteria for algebraic independence of elements in the field of $p$-adic numbers were derived by V. Laohakosol and P. Ubolsri [15] in 1987. In 2007, similar criteria in the field of formal Laurent series in $x^{-1}$ were established by T. Chaichana and V. Laohakosol, [6].

In this chapter, criteria for algebraic and linear independence of elements in $F((\pi(x)))$, as expounded in Example 2.13 of Chapter 2, are given in the first section. In the second section, we use this criteria to derive sufficient conditions for algebraic and linear independence of elements represented by JR-continued fraction expansions.

## 3.1 Criteria

In this section, we start with definitions.

**Definition 3.1.** Let $E$ be an extension field of a field $K$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in E$. Then $\alpha_1, \alpha_2, \ldots, \alpha_n$ are *algebraically independent* over $K$, if for all $f(T_1, T_2, \ldots, T_n) \in K[T_1, T_2, \ldots, T_n]$,

$$f(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0 \Longrightarrow f \equiv 0.$$

**Definition 3.2.** Let $E$ be an extension field of a field $K$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in E$. Then $\alpha_1, \alpha_2, \ldots, \alpha_n$ are *linearly independent* over $K$, if for all $a_1, a_2, \ldots, a_n \in K$

$$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = 0 \Longrightarrow a_1 = a_2 = \cdots = a_n = 0.$$

Let $F$ be a field and let $\pi(x)$ be a monic irreducible polynomial in $F[x]$. In order to prove the main results we need Uchiyama's Theorem ([31]) states as follows:

**Theorem 3.3.** *Let $\alpha$ be an element of $F((\pi(x)))$ algebraic of degree $m \geq 2$ over $F(x)$. Then there is a constant $K > 0$ such that*

$$|C - D\alpha|_\pi \geq \frac{K}{M^m} \tag{3.1}$$

*for all pairs of polynomials $C, D \in F[x]$ with $M > 0$, where*

$$M := \max\{|C|_\infty, |D|_\infty\}.$$

*If $F$ is of characteristic greater than zero, the inequality (3.1) cannot be improved in general.*

The following main results give criteria for algebraic and linear independence.

**Theorem 3.4.** *Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in F((\pi(x))) \setminus \{0\}$. Assume that there are polynomials $C_{N,j}, D_{N,j}(\neq 0) \in F[x]$ $(N \in \mathbb{N},\ j = 1, 2, \ldots, n)$ such that*

$$D_{N,j}\alpha_j \neq C_{N,j}, \quad M_{N,j} := \max\{|C_{N,j}|_\infty, |D_{N,j}|_\infty\} \to \infty \quad as \quad N \to \infty,$$

*and*

$$\lim_{N \to \infty} \frac{\left|\alpha_{j-1} - \dfrac{C_{N,j-1}}{D_{N,j-1}}\right|_\pi}{\left|\alpha_j - \dfrac{C_{N,j}}{D_{N,j}}\right|_\pi} = 0 \qquad (j = 2, \ldots, n) \tag{3.2}$$

*provided $n \geq 2$. Assume further that for each positive real number $E$, there is an $N_0 = N_0(E) \in \mathbb{N}$ such that*

$$\left|\alpha_j - \frac{C_{N,j}}{D_{N,j}}\right|_\pi \leq \frac{1}{(M_{N,1}M_{N,2}\cdots M_{N,j})^E} \qquad (N \geq N_0,\ j = 1, 2, \ldots, n). \tag{3.3}$$

*Then $\alpha_1, \alpha_2, \ldots, \alpha_n$ are algebraically independent over $F(x)$.*

*Proof.* We proceed by induction on $n$.

For $n = 1$, suppose that $\alpha_1$ is algebraic of degree $m \geq 1$ over $F(x)$.
If $m = 1$, then $\alpha_1 = \dfrac{P}{Q}$ for some $P, Q \in F[x] \setminus \{0\}$. For all $N \in \mathbb{N}$, by the product formula, we get

$$0 \neq |D_{N,1}\alpha_1 - C_{N,1}|_\pi = \frac{|D_{N,1}P - C_{N,1}Q|_\pi}{|Q|_\pi} \geq \frac{1}{|D_{N,1}P - C_{N,1}Q|_\infty \, |Q|_\pi}.$$

We also have

$$|D_{N,1}P - C_{N,1}Q|_\infty \leq \max\{|D_{N,1}P|_\infty, |C_{N,1}Q|_\infty\} \leq M_{N,1}K,$$

$K := \max\{|P|_\infty, |Q|_\infty\}$. Then

$$|D_{N,1}\alpha_1 - C_{N,1}|_\pi \geq \frac{1}{M_{N,1}K|Q|_\pi} = \frac{K_1}{M_{N,1}}, \quad K_1 := \frac{1}{K|Q|_\pi}.$$

By (3.3), there is an $N_1 = N_1(2)$ such that, for all $N \geq N_1$,

$$\frac{K_1}{M_{N,1}} \leq |D_{N,1}\alpha_1 - C_{N,1}|_\pi = |D_{N,1}|_\pi \left|\alpha_1 - \frac{C_{N,1}}{D_{N,1}}\right|_\pi \leq \frac{|D_{N,1}|_\pi}{M_{N,1}^2} \leq \frac{1}{M_{N,1}^2},$$

which is a contradiction.

For $m > 1$, by Theorem 3.3, for $F((\pi(x)))$ there is a constant $K_2 > 0$ such that

$$|D_{N,1}\alpha_1 - C_{N,1}|_\pi \geq \frac{K_2}{M_{N,1}^m} \quad \text{for all} \quad N \in \mathbb{N}.$$

By (3.3), there is an $N_2 = N_2(m+1)$ such that, for all $N \geq N_2$,

$$\frac{K_2}{M_{N,1}^m} \leq |D_{N,1}\alpha_1 - C_{N,1}|_\pi \leq \frac{|D_{N,1}|_\pi}{M_{N,1}^{m+1}} \leq \frac{1}{M_{N,1}^{m+1}},$$

which is a contradiction. Thus, $\alpha_1$ is transcendental, and we are done in the case $n = 1$.

Now consider $n > 1$. Assume the assertion of the theorem holds up to $n - 1$, but is false for $n$. Then there would exist a polynomial $f(T_1, T_2, \ldots, T_n) \in F[x][T_1, T_2, \ldots, T_n] \setminus \{0\}$ of minimal total degree such that $f(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0$.

Expanding $f$ about $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, we get

$$f(T_1, T_2, \ldots, T_n) = \sum_{(\nu)} h_{(\nu)} (T_1 - \alpha_1)^{\nu_1} \cdots (T_n - \alpha_n)^{\nu_n},$$

where $(\nu) = (\nu_1, \nu_2, \ldots, \nu_n)$, and

$$h_{(\nu)} := h_{(\nu_1,\nu_2,\ldots,\nu_n)} = \frac{1}{(\nu_1 + \nu_2 + \cdots + \nu_n)!} \frac{\partial^{\nu_1+\nu_2+\cdots+\nu_n} f(\alpha_1, \alpha_2, \ldots, \alpha_n)}{\partial T_1^{\nu_1} \partial T_2^{\nu_2} \cdots \partial T_n^{\nu_n}}.$$

Clearly,

$$h_{(0,\ldots,0)} = f(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0.$$

Set

$$\mathcal{H}_n(T_1, T_2, \ldots, T_n) := \frac{\partial}{\partial T_n} f(T_1, T_2, \ldots, T_n),$$

$$H_i := h_{(0,\ldots,0,1,0,\ldots,0)} \text{ for } i = 1, 2, \ldots, n,$$

where the digit 1 is at the $i^{\text{th}}$ position. Observe that $T_n$ occurs in $f$. Thus, $\mathcal{H}_n(T_1, T_2, \ldots, T_n) \not\equiv 0$ and $H_n = \mathcal{H}_n(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

Next, we show that $H_n \neq 0$. Suppose not. If $T_n$ occurs in $\mathcal{H}_n(T_1, , T_2, \ldots, T_n)$, then $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a root of a nonzero polynomial whose degree is lower than that of $f$, which is a contradiction. Thus, $T_n$ does not occur in $\mathcal{H}_n(T_1, T_2, \ldots, T_n)$. This means that $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are algebraically dependent, contradicting the induction hypothesis. Thus, $H_n \neq 0$.

For $j = 1, 2, \ldots, n$, let

$$\delta_j(N) = \frac{C_{N,j}}{D_{N,j}} - \alpha_j.$$

Since $D_{N,j}\alpha_j \neq C_{N,j}$, we get $|\delta_n(N)|_\pi \neq 0$. Now

$$\begin{aligned} f\left(\frac{C_{N,1}}{D_{N,1}}, \frac{C_{N,2}}{D_{N,2}}, \ldots, \frac{C_{N,n}}{D_{N,n}}\right) &= \sum_{(\nu)} h_{(\nu)} \delta_1(N)^{\nu_1} \cdots \delta_n(N)^{\nu_n} \\ &= \sum_{i=1} H_i \delta_1(N) + \sum_{\nu_1+\cdots+\nu_n \geq 2} h_{(\nu)} \delta_1(N)^{\nu_1} \cdots \delta_n(N)^{\nu_n} \end{aligned}$$

$$= \delta_n(N)\left(\left(H_1\frac{\delta_1(N)}{\delta_n(N)}+\cdots+H_{n-1}\frac{\delta_{n-1}(N)}{\delta_n(N)}+H_n\right)+O\left(|\delta_n(N)|_\pi\right)\right).$$

By hypotheses (3.2) and (3.3), we see that

$$\begin{aligned}&\left|H_1\frac{\delta_1(N)}{\delta_n(N)}+\cdots+H_{n-1}\frac{\delta_{n-1}(N)}{\delta_n(N)}+O\left(|\delta_n(N)|_\pi\right)\right|_\pi\\&\quad\le\max\left\{\left|H_1\frac{\delta_1(N)}{\delta_n(N)}\right|_\pi,\ldots,\left|H_{n-1}\frac{\delta_{n-1}(N)}{\delta_n(N)}\right|_\pi,O\left(|\delta_n(N)|_\pi\right)\right\}\to 0\quad\text{as}\quad N\to\infty,\end{aligned}$$

which yields, when $N$ is large enough,

$$\left|f\left(\frac{C_{N,1}}{D_{N,1}},\frac{C_{N,2}}{D_{N,2}},\ldots,\frac{C_{N,n}}{D_{N,n}}\right)\right|_\pi=|\delta_n(N)H_n|_\pi\neq 0.$$

Let $m_1, m_2, \ldots, m_n$ be the degrees of $f$ in $T_1, T_2, \ldots, T_n$, respectively. Then

$$D_{N,1}^{m_1}D_{N,2}^{m_2}\cdots D_{N,n}^{m_n}f\left(\frac{C_{N,1}}{D_{N,1}},\frac{C_{N,2}}{D_{N,2}},\ldots,\frac{C_{N,n}}{D_{N,n}}\right)\in F[x]\setminus\{0\},$$

and so

$$\begin{aligned}0&<\left|D_{N,1}^{m_1}D_{N,2}^{m_2}\cdots D_{N,n}^{m_n}\,f\left(\frac{C_{N,1}}{D_{N,1}},\frac{C_{N,2}}{D_{N,2}},\ldots,\frac{C_{N,n}}{D_{N,n}}\right)\right|_\infty\\&=\left|D_{N,1}^{m_1}D_{N,2}^{m_2}\cdots D_{N,n}^{m_n}\,\sum_{(\nu)}h_{(\nu)}\left(\frac{C_{N,1}}{D_{N,1}}-\alpha_1\right)^{\nu_1}\left(\frac{C_{N,2}}{D_{N,2}}-\alpha_2\right)^{\nu_2}\cdots\left(\frac{C_{N,n}}{D_{N,n}}-\alpha_n\right)^{\nu_n}\right|_\infty\\&=\left|\sum_{(\nu)}h_{(\nu)}D_{N,1}^{m_1-\nu_1}D_{N,2}^{m_2-\nu_2}\cdots D_{N,n}^{m_n-\nu_n}\left(C_{N,1}-\alpha_1D_{N,1}\right)^{\nu_1}\cdots\left(C_{N,n}-\alpha_nD_{N,n}\right)^{\nu_n}\right|_\infty\\&\le\max_{(\nu)}\left\{\left|h_{(\nu)}D_{N,1}^{m_1-\nu_1}D_{N,2}^{m_2-\nu_2}\cdots D_{N,n}^{m_n-\nu_n}\left(C_{N,1}-\alpha_1D_{N,1}\right)^{\nu_1}\cdots\left(C_{N,n}-\alpha_nD_{N,n}\right)^{\nu_n}\right|_\infty\right\}\\&\le\max_{(\nu)}\left\{M_{N,1}^{m_1-\nu_1}M_{N,2}^{m_2-\nu_2}\cdots M_{N,n}^{m_n-\nu_n}\left|h_{(\nu)}\right|_\infty\prod_{j=1}^{n}\max_j\left\{|C_{N,j}|_\infty^{\nu_j},|\alpha_jD_{N,j}|_\infty^{\nu_j}\right\}\right\}\\&\le M_{N,1}^{m_1}M_{N,2}^{m_2}\cdots M_{N,n}^{m_n}\max_{(\nu)}\left\{\left|h_{(\nu)}\right|_\infty\prod_{j=1}^{n}\max_j\left\{1,|\alpha_j|_\infty^{\nu_j}\right\}\right\}\\&=KM_{N,1}^{m_1}M_{N,2}^{m_2}\cdots M_{N,n}^{m_n},\end{aligned}$$

where $K := \max_{(\nu)}\left\{\left|h_{(\nu)}\right|_\infty\prod_{j=1}^{n}\max_j\left\{1,|\alpha_j|_\infty^{\nu_j}\right\}\right\}$ is a positive constant independent of $N$.

By the product formula, we get

$$\begin{aligned}\left|f\left(\frac{C_{N,1}}{D_{N,1}},\frac{C_{N,2}}{D_{N,2}},\dots,\frac{C_{N,n}}{D_{N,n}}\right)\right|_\pi &\geq \left|D_{N,1}^{m_1}D_{N,2}^{m_2}\cdots D_{N,n}^{m_n}\, f\left(\frac{C_{N,1}}{D_{N,1}},\frac{C_{N,2}}{D_{N,2}},\dots,\frac{C_{N,n}}{D_{N,n}}\right)\right|_\pi \\ &\geq \left|D_{N,1}^{m_1}D_{N,2}^{m_2}\cdots D_{N,n}^{m_n}\, f\left(\frac{C_{N,1}}{D_{N,1}},\frac{C_{N,2}}{D_{N,2}},\dots,\frac{C_{N,n}}{D_{N,n}}\right)\right|_\infty^{-1} \\ &\geq \left(KM_{N,1}^{m_1}M_{N,2}^{m_2}\cdots M_{N,n}^{m_n}\right)^{-1}.\end{aligned}$$

Choosing $E = \max\{m_1, m_2, \dots, m_n\} + 1$, by (3.3), there exists $N_3 = N_3(E)$ such that for all $N \geq N_3$,

$$\begin{aligned}\frac{1}{KM_{N,1}^{m_1}M_{N,2}^{m_2}\cdots M_{N,n}^{m_n}} &\leq \left|f\left(\frac{C_{N,1}}{D_{N,1}},\frac{C_{N,2}}{D_{N,2}},\dots,\frac{C_{N,n}}{D_{N,n}}\right)\right|_\pi \\ &= |\delta_n(N)H_n|_\pi \\ &\leq \frac{|H_n|_\pi}{(M_{N,1}M_{N,2}\cdots M_{N,n})^E},\end{aligned}$$

i.e.,

$$|H_n|_\pi \geq \frac{M_{N,1}^{E-m_1}M_{N,2}^{E-m_2}\cdots M_{N,n}^{E-m_n}}{K} \to \infty \quad \text{as} \quad N \to \infty,$$

which is a contradiction. □

**Theorem 3.5.** *Let* $\alpha_1, \alpha_2, \dots, \alpha_n \in F((\pi(x))) \setminus \{0\}$. *Assume that there are polynomials* $C_{N,j}, D_{N,j}\ (\neq 0)$ $(N \in \mathbb{N},\ j = 1, 2, \dots, n)$ *in* $F[x]$ *such that*

$$D_{N,j}\alpha_j \neq C_{N,j}, \quad M_{N,j} := \max\{|C_{N,j}|_\infty, |D_{N,j}|_\infty\} \to \infty \quad as \quad N \to \infty,$$

*such that if* $n \geq 2$,

$$\lim_{N\to\infty} \frac{\left|\alpha_{j-1} - \dfrac{C_{N,j-1}}{D_{N,j-1}}\right|_\pi}{\left|\alpha_j - \dfrac{C_{N,j}}{D_{N,j}}\right|_\pi} = 0 \qquad (j = 2, \dots, n). \tag{3.4}$$

*Assume further that there is a positive-valued function $g$ of natural argument, with*

$$g(N) \to \infty \quad as \quad N \to \infty,$$

*and there is an $N_0 = N_0(g) \in \mathbb{N}$ such that*

$$\left|\alpha_j - \frac{C_{N,j}}{D_{N,j}}\right|_\pi \leq \frac{1}{M_{N,1}M_{N,2}\cdots M_{N,j}g(N)} \quad (N \geq N_0,\ j = 1, 2, \ldots, n). \tag{3.5}$$

*Then $1, \alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent over $F(x)$.*

*Proof.* Suppose that $1, \alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly dependent over $F(x)$. Then there are $A_1, A_2, \ldots, A_{n+1} \in F[x]$ not all zero such that

$$A_{n+1} + A_n\alpha_n + \cdots + A_1\alpha_1 = 0.$$

Without loss of generality assume that $A_n \neq 0$, for otherwise there is $1 \leq k < n$ such that $A_k \neq 0$ and we can replace $A_n$ by $A_k$. Let

$$f(T_1, T_2, \ldots, T_n) := A_{n+1} + A_nT_n + \cdots + A_1T_1 \in F[x][T_1, T_2, \ldots, T_n].$$

Then

$$f(T_1, T_2, \ldots, T_n) = A_n(T_n - \alpha_n) + \cdots + A_1(T_1 - \alpha_1).$$

Set

$$\delta_j(N) := \frac{C_{N,j}}{D_{N,j}} - \alpha_j \quad \text{for} \quad j = 1, 2, \ldots, n.$$

Since $D_{N,j}\alpha_j \neq C_{N,j}$, we get $|\delta_n(N)|_\pi \neq 0$. Now

$$f\left(\frac{C_{N,1}}{D_{N,1}}, \frac{C_{N,2}}{D_{N,2}}, \ldots, \frac{C_{N,n}}{D_{N,n}}\right) = \delta_n(N)\left(A_n + A_{n-1}\frac{\delta_{n-1}(N)}{\delta_n(N)} + \cdots + A_1\frac{\delta_1(N)}{\delta_n(N)}\right).$$

By hypothesis (3.4), we see that

$$\left|A_{n-1}\frac{\delta_{n-1}(N)}{\delta_n(N)} + \cdots + A_1\frac{\delta_1(N)}{\delta_n(N)}\right|_\pi \leq \max\left\{\left|A_{n-1}\frac{\delta_{n-1}(N)}{\delta_n(N)}\right|_\pi, \ldots, \left|A_1\frac{\delta_1(N)}{\delta_n(N)}\right|_\pi\right\}$$

$$\to 0 \text{ as } N \to \infty,$$

which yields, when $N$ is sufficiently large,

$$\left| f\left(\frac{C_{N,1}}{D_{N,1}}, \frac{C_{N,2}}{D_{N,2}}, \ldots, \frac{C_{N,n}}{D_{N,n}}\right)\right|_\pi = |\delta_n(N)A_n|_\pi \neq 0.$$

We have

$$D_{N,1}D_{N,2}\cdots D_{N,n}f\left(\frac{C_{N,1}}{D_{N,1}}, \frac{C_{N,2}}{D_{N,2}}, \ldots, \frac{C_{N,n}}{D_{N,n}}\right) \in F[x] \setminus \{0\},$$

and so

$$\begin{aligned}
0 &< \left| D_{N,1}D_{N,2}\cdots D_{N,n}\, f\left(\frac{C_{N,1}}{D_{N,1}}, \frac{C_{N,2}}{D_{N,2}}, \ldots, \frac{C_{N,n}}{D_{N,n}}\right)\right|_\infty \\
&= \left| D_{N,1}D_{N,2}\cdots D_{N,n}\, \left\{ A_n\left(\frac{C_{N,n}}{D_{N,n}} - \alpha_n\right) + \cdots + A_1\left(\frac{C_{N,1}}{D_{N,1}} - \alpha_1\right)\right\}\right|_\infty \\
&\le \max_{1\le j\le n}\left\{ |D_{N,1}\cdots D_{N,j-1}D_{N,j+1}\cdots D_{N,n}A_j\left(C_{N,j} - \alpha_j D_{N,j}\right)|_\infty \right\} \\
&\le M_{N,1}\cdots M_{N,j-1}M_{N,j+1}\cdots M_{N,n}\max_{1\le j\le n}\left\{ |A_jC_{N,j}|_\infty, |A_j\alpha_jD_{N,j}|_\infty\right\} \\
&\le M_{N,1}\cdots M_{N,j-1}M_{N,j}M_{N,j+1}\cdots M_{N,n}\max_{1\le j\le n}\left\{ |A_j|_\infty, |A_j\alpha_j|_\infty\right\} \\
&= KM_{N,1}M_{N,2}\cdots M_{N,n},
\end{aligned}$$

where $K := \max_{1\le j\le n}\left\{|A_j|_\infty, |A_j\alpha_j|_\infty\right\}$ is a positive constant independent of $N$.

By the product formula, we get

$$\begin{aligned}
\left| f\left(\frac{C_{N,1}}{D_{N,1}}, \frac{C_{N,2}}{D_{N,2}}, \ldots, \frac{C_{N,n}}{D_{N,n}}\right)\right|_\pi &\ge \left| D_{N,1}D_{N,2}\cdots D_{N,n}\, f\left(\frac{C_{N,1}}{D_{N,1}}, \frac{C_{N,2}}{D_{N,2}}, \ldots, \frac{C_{N,n}}{D_{N,n}}\right)\right|_\pi \\
&\ge \left| D_{N,1}D_{N,2}\cdots D_{N,n}\, f\left(\frac{C_{N,1}}{D_{N,1}}, \frac{C_{N,2}}{D_{N,2}}, \ldots, \frac{C_{N,n}}{D_{N,n}}\right)\right|_\infty^{-1} \\
&\ge \left(KM_{N,1}M_{N,2}\cdots M_{N,n}\right)^{-1}.
\end{aligned}$$

Then, by (3.5), for $N$ sufficiently large,

$$\frac{1}{KM_{N,1}M_{N,2}\cdots M_{N,n}} \le \left| f\left(\frac{C_{N,1}}{D_{N,1}}, \frac{C_{N,2}}{D_{N,2}}, \ldots, \frac{C_{N,n}}{D_{N,n}}\right)\right|_\pi \le \frac{|A_n|_\pi}{M_{N,1}M_{N,2}\cdots M_{N,n}g(N)},$$

i.e.,

$$|A_n|_\pi \geq \frac{g(N)}{K} \to \infty \quad \text{as} \quad N \to \infty,$$

which is a contradiction. □

## 3.2 Applications

In this section, we apply Theorems 3.4 and 3.5 to derive sufficient conditions for elements in $F((\pi(x))) \setminus \{0\}$ represented by JR-continued fraction expansions of Example 2.13 in Chapter 2 to be algebraically and linearly independent over $F(x)$. After that, criteria for algebraic and linear independence of $\pi$-adic Ruban and $\pi$-adic Schneider continued fraction expansions are given in Subsections 3.2.1 and 3.2.2, respectively.

Throughout we assume (without loss of generality) that $|\alpha|_\pi < 1$. For $N \in \mathbb{N}$, let

$$\frac{C_N}{D_N} = \frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\cdots\frac{b_N}{a_N}$$

be the $N^{\text{th}}$ convergent of the JR-continued fraction expansion of $\alpha$. By Propositions 2.3 and 2.4, we have

$$|C_1|_\pi = |b_1|_\pi, \; |C_i|_\pi = |b_1 a_2 a_3 \cdots a_i|_\pi \;\; (i \geq 2), \; |D_i|_\pi = |a_1 a_2 \cdots a_i|_\pi \;\; (i \geq 1),$$

which implies that $|D_i|_\pi = \left|\frac{a_1 C_i}{b_1}\right|_\pi > |C_i|_\pi$ for all $i \geq 1$. Since $C_N, D_N$ do not necessarily belong to $F[x]$, to apply the results of Theorems 3.4 and 3.5, we need to convert the JR-continued fraction expansion of Example 2.13 in Chapter 2 into an equivalent continued fraction expansion.

Throughout, we focus only on the case when the sequence $\{b_i\}$ is a subset of

$$\Big\{\alpha = \frac{c_{-r}}{\pi^r} + \cdots + \frac{c_{-1}}{\pi} + c_0 + c_1\pi + \cdots + c_s\pi^s \in F((\pi(x))) \setminus \{0\} : \\ r, s \in \mathbb{N} \cup \{0\}, \; c_i \in F[x], \; \deg c_i < \deg \pi\Big\}.$$

For each $i \in \mathbb{N}$, write

$$a_i := \frac{a_i'}{\pi^{n_i}} \quad \text{and} \quad b_i := \frac{b_i'}{\pi^{m_i}},$$

where $n_i \in \mathbb{N} \cup \{0\}, m_i \in \mathbb{Z}$, and $a_i', b_i' \in F[x]$ are both relatively prime to $\pi(x)$, so that

$$|a_i'|_\pi = 1 = |b_i'|_\pi, \ |a_i|_\pi = 2^{n_i \deg \pi}, \ |b_i|_\pi = 2^{m_i \deg \pi}.$$

From the ab-condition, we have $n_i > m_i$ for all $i \geq 1$. It is convenient to introduce an associated JR-continued fraction expansion

$$\frac{\gamma_1}{\beta_1+} \frac{\gamma_2}{\beta_2+} \cdots \frac{\gamma_i}{\beta_i+} \cdots, \tag{3.6}$$

where

$$\gamma_1 = b_1' \pi^{n_1 - m_1}, \ \gamma_{i+1} = b_{i+1}' \pi^{n_i + n_{i+1} - m_{i+1}}, \quad \beta_i = a_i' \ \text{ for all } \ i \in \mathbb{N}.$$

Clearly, the partial numerators $\gamma_i$ and the partial denominators $\beta_i$ of the associated continued fraction expansion (3.6) are in $F[x]$ and $|\gamma_i|_\infty > 2^{\deg \pi}$ for all $i \geq 1$. For $N \in \mathbb{N}$, we similarly define the $N^{\text{th}}$ convergent of (3.6) to be

$$\frac{\mathcal{C}_N}{\mathcal{D}_N} = \frac{\gamma_1}{\beta_1+} \frac{\gamma_2}{\beta_2+} \cdots \frac{\gamma_N}{\beta_N},$$

where

$$\mathcal{C}_{-1} = 1, \quad \mathcal{C}_0 = 0, \quad \mathcal{C}_{i+1} = \beta_{i+1}\mathcal{C}_i + \gamma_{i+1}\mathcal{C}_{i-1} \ \text{ for all } \ i \geq 0$$

$$\mathcal{D}_{-1} = 0, \quad \mathcal{D}_0 = 1, \quad \mathcal{D}_{i+1} = \beta_{i+1}\mathcal{D}_i + \gamma_{i+1}\mathcal{D}_{i-1} \ \text{ for all } \ i \geq 0.$$

The JR-continued fraction expansion (2.10) and its associated continued fraction expansion (3.6) are equivalent in the sense that $\dfrac{C_N}{D_N} = \dfrac{\mathcal{C}_N}{\mathcal{D}_N}$ for all $N \in \mathbb{N}$, which can be shown by induction. Clearly, $\mathcal{C}_N$ and $\mathcal{D}_N$ are in $F[x]$.

Now, we consider two sets of relationships between $\gamma_i$ and $\beta_i$ as follows:

(1)

$$|\beta_i \beta_{i+1}|_\infty > |\gamma_{i+1}|_\infty \text{ for all } i \geq 1, \tag{3.7}$$

which is equivalent to $|a_i a_{i+1}|_\infty > |b_{i+1}|_\infty$ for all $i \geq 1$;

(2)

$$|\gamma_2|_\infty > |\beta_2\beta_1|_\infty \text{ and}$$
$$|\gamma_3\beta_1|_\infty > |\beta_3\gamma_2|_\infty,\ |\gamma_4\beta_2|_\infty > |\beta_4\gamma_3|_\infty,\ |\gamma_{i+2}\beta_i|_\infty \geq |\beta_{i+2}\gamma_{i+1}|_\infty \text{ for all } i \geq 3, \tag{3.8}$$

which are equivalent to

$$|b_2|_\infty > |a_2 a_1|_\infty \text{ and}$$
$$|b_3 a_1|_\infty > |a_3 b_2|_\infty,\ |b_4 a_2|_\infty > |a_4 b_3|_\infty,\ |b_{i+2} a_i|_\infty \geq |a_{i+2} b_{i+1}|_\infty \text{ for all } i \geq 3. \tag{3.9}$$

**Remark 3.6.** *From condition* (3.8), *we get* $|\gamma_{i+1}|_\infty > |\beta_i\beta_{i+1}|_\infty$ *for all* $i \geq 1$, *which implies that the two classes do not overlap.*

Under the above set up, the next lemmas summarize basic properties of $\mathcal{C}_N$ and $\mathcal{D}_N$, whose inductive proofs are omitted.

**Lemma 3.7.** *If* (3.7) *holds, then*

*(i)* $|\mathcal{C}_1|_\infty = |\gamma_1|_\infty$, $|\mathcal{C}_N|_\infty = |\gamma_1\beta_2\cdots\beta_N|_\infty$ *for all* $N \geq 2$;

*(ii)* $|\mathcal{D}_N|_\infty = |\beta_1\beta_2\cdots\beta_N|_\infty$ *for all* $N \geq 1$.

**Lemma 3.8.** *If* (3.8) *holds, then*

*(i)* $|\mathcal{C}_1|_\infty = |\gamma_1|_\infty$, $|\mathcal{C}_2|_\infty = |\beta_2\gamma_1|_\infty$ *and for all* $\ell \geq 1$
$|\mathcal{C}_{2\ell+1}|_\infty = |\gamma_{2\ell+1}\gamma_{2\ell-1}\cdots\gamma_3\gamma_1|_\infty$, $|\mathcal{C}_{2\ell+2}|_\infty = |\gamma_{2\ell+2}\gamma_{2\ell}\cdots\gamma_4\beta_2\gamma_1|_\infty$;

*(ii)* $|\mathcal{D}_1|_\infty = |\beta_1|_\infty$, $|\mathcal{D}_2|_\infty = |\gamma_2|_\infty$ *and for all* $\ell \geq 1$
$|\mathcal{D}_{2\ell+1}|_\infty = |\gamma_{2\ell+1}\gamma_{2\ell-1}\cdots\gamma_3\beta_1|_\infty$, $|\mathcal{D}_{2\ell+2}|_\infty = |\gamma_{2\ell+2}\gamma_{2\ell}\cdots\gamma_4\gamma_2|_\infty$.

From the preceding discussion, let $\alpha_j \in \pi(x)F[[\pi(x)]] \setminus \{0\}$ $(j = 1, 2, \ldots, k)$ with *infinite* associated JR-continued fraction expansions

$$\alpha_j = \frac{\gamma_{1,j}}{\beta_{1,j}+}\,\frac{\gamma_{2,j}}{\beta_{2,j}+}\cdots\frac{\gamma_{i,j}}{\beta_{i,j}+}\cdots,$$

and, for all $N \in \mathbb{N}$, let their corresponding $N^{\text{th}}$ convergents be

$$\frac{\mathcal{C}_{N,j}}{\mathcal{D}_{N,j}} = \frac{\gamma_{1,j}}{\beta_{1,j}+} \frac{\gamma_{2,j}}{\beta_{2,j}+} \cdots \frac{\gamma_{N,j}}{\beta_{N,j}}.$$

If the requirement (3.7) or (3.8) holds for each $j \in \{1, 2, \ldots, k\}$, then Lemma 3.7 or 3.8 yields

$$M_{N,j} = \max\{|\mathcal{C}_{N,j}|_\infty, |\mathcal{D}_{N,j}|_\infty\} \longrightarrow \infty \quad \text{as} \quad N \longrightarrow \infty.$$

Since each associated JR-continued fraction expansion is infinite, we infer that

$$\mathcal{C}_{N,j} - \mathcal{D}_{N,j}\alpha_j \neq 0 \;\; \text{for all} \;\; j = 1, 2, \ldots, k \text{ and } N \in \mathbb{N}.$$

If the condition (3.7) is replaced by the condition (3.8), we still get the same results. We begin with criterion for algebraic independence. Again, we are under the same set up.

**Theorem 3.9.** *Assume that*

*I. the condition* (3.7) (*or* (3.8)) *is fulfilled for each* $j \in \{1, 2, \ldots, k\}$;

*II. the limiting values*

$$\lim_{i \to \infty} \frac{|b_{1,j-1} b_{2,j-1} \cdots b_{i+1,j-1}|_\pi |D_{i,j} D_{i+1,j}|_\pi}{|b_{1,j} b_{2,j} \cdots b_{i+1,j}|_\pi |D_{i,j-1} D_{i+1,j-1}|_\pi} = 0 \qquad (j = 2, 3, \ldots, k) \quad (3.10)$$

*hold and*

*III. there exists* $g : \mathbb{N} \longrightarrow \mathbb{Z}$ *with* $g(i) \longrightarrow \infty$ *as* $i \longrightarrow \infty$ *such that*

$$\frac{|D_{i,j} D_{i+1,j}|_\pi}{|b_{1,j} b_{2,j} \cdots b_{i+1,j}|_\pi} \geq (M_{i,1} M_{i,2} \cdots M_{i,j})^{g(i)} \quad (i \in \mathbb{N}, \; j = 1, 2, \ldots, k). \quad (3.11)$$

*Then* $\alpha_1, \alpha_2, \ldots, \alpha_k$ *are algebraically independent over* $F(x)$.

*Proof.* For a fixed $E > 0$, from $g(i) \longrightarrow \infty$ as $i \longrightarrow \infty$, there is $N_0 \in \mathbb{N}$ such that for all $N > N_0$, we have $g(N) > E$. For $j \in \{1, 2, \ldots, k\}$ and $N > N_0$, applying

(3.11), we get

$$\begin{aligned}\left|\alpha_j - \frac{\mathcal{C}_{N,j}}{\mathcal{D}_{N,j}}\right|_\pi &= \left|\alpha_j - \frac{C_{N,j}}{D_{N,j}}\right|_\pi \\ &= \frac{|b_{1,j}b_{2,j}\cdots b_{N+1,j}|_\pi}{|D_{N,j}D_{N+1,j}|_\pi} \\ &\le \frac{1}{(M_{N,1}M_{N,2}\cdots M_{N,j})^{g(N)}} \\ &< \frac{1}{(M_{N,1}M_{N,2}\cdots M_{N,j})^{E}}. \end{aligned} \tag{3.12}$$

From (3.10), for $j \ge 2$, we get

$$\begin{aligned}\frac{\left|\alpha_{j-1} - \frac{\mathcal{C}_{N,j-1}}{\mathcal{D}_{N,j-1}}\right|_\pi}{\left|\alpha_j - \frac{\mathcal{C}_{N,j}}{\mathcal{D}_{N,j}}\right|_\pi} &= \frac{\left|\alpha_{j-1} - \frac{C_{N,j-1}}{D_{N,j-1}}\right|_\pi}{\left|\alpha_j - \frac{C_{N,j}}{D_{N,j}}\right|_\pi} \\ &= \frac{\dfrac{|b_{1,j-1}b_{2,j-1}\cdots b_{N+1,j-1}|_\pi}{|D_{N,j-1}D_{N+1,j-1}|_\pi}}{\dfrac{|b_{1,j}b_{2,j}\cdots b_{N+1,j}|_\pi}{|D_{N,j}D_{N+1,j}|_\pi}} \\ &\to 0 \quad \text{as} \quad N \to \infty. \end{aligned} \tag{3.13}$$

Noting (3.12) and (3.13), Theorem 3.4 yields the desired result. □

As for linear independence, we have the following result.

**Theorem 3.10.** *Assume that*

I. *the condition* (3.7) (*or* (3.8)) *is fulfilled for each* $j \in \{1, 2, \ldots, k\}$;

II. *the limiting values*

$$\lim_{i\to\infty} \frac{|b_{1,j-1}b_{2,j-1}\cdots b_{i+1,j-1}|_\pi |D_{i,j}D_{i+1,j}|_\pi}{|b_{1,j}b_{2,j}\cdots b_{i+1,j}|_\pi |D_{i,j-1}D_{i+1,j-1}|_\pi} = 0 \qquad (j = 2, 3, \ldots, k) \tag{3.14}$$

*hold and*

*III. there exists $g : \mathbb{N} \to \mathbb{Z}$ with $g(i) \to \infty$ as $i \to \infty$ such that*

$$\frac{|D_{i,j}D_{i+1,j}|_\pi}{|b_{1,j}b_{2,j}\cdots b_{i+1,j}|_\pi} \geq g(i)\,(M_{i,1}M_{i,2}\cdots M_{i,j}) \quad (i \in \mathbb{N},\ j = 1, 2, \ldots, k). \tag{3.15}$$

*Then $1, \alpha_1, \alpha_2, \ldots, \alpha_k$ are linearly independent over $F(x)$.*

*Proof.* From (3.15), for $j \geq 1$ and $N \in \mathbb{N}$, we get

$$\begin{aligned} \left|\alpha_j - \frac{\mathcal{C}_{N,j}}{\mathcal{D}_{N,j}}\right|_\pi &= \left|\alpha_j - \frac{C_{N,j}}{D_{N,j}}\right|_\pi \\ &= \frac{|b_{1,j}b_{2,j}\cdots b_{N+1,j}|_\pi}{|D_{N,j}D_{N+1,j}|_\pi} \\ &\leq \frac{1}{g(N)\,(M_{N,1}M_{N,2}\cdots M_{N,j})}. \end{aligned} \tag{3.16}$$

For $j \geq 2$ and applying (3.14), we get

$$\begin{aligned} \frac{\left|\alpha_{j-1} - \dfrac{\mathcal{C}_{N,j-1}}{\mathcal{D}_{N,j-1}}\right|_\pi}{\left|\alpha_j - \dfrac{\mathcal{C}_{N,j}}{\mathcal{D}_{N,j}}\right|_\pi} &= \frac{\left|\alpha_{j-1} - \dfrac{C_{N,j-1}}{D_{N,j-1}}\right|_\pi}{\left|\alpha_j - \dfrac{C_{N,j}}{D_{N,j}}\right|_\pi} \\ &= \frac{\dfrac{|b_{1,j-1}b_{2,j-1}\cdots b_{N+1,j-1}|_\pi}{|D_{N,j-1}D_{N+1,j-1}|_\pi}}{\dfrac{|b_{1,j}b_{2,j}\cdots b_{N+1,j}|_\pi}{|D_{N,j}D_{N+1,j}|_\pi}} \\ &\to 0 \quad \text{as} \quad N \to \infty. \end{aligned} \tag{3.17}$$

Noting (3.16) and (3.17), Theorem 3.5 yields the desired result. □

### 3.2.1 The $\pi$-adic Ruban continued fraction expansion

The $\pi$-adic Ruban continued fraction expansion of $\alpha \in \pi(x)F[[\pi(x)]] \setminus \{0\}$ is of the form

$$\alpha = \frac{1}{a_1+}\,\frac{1}{a_2+}\cdots\frac{1}{a_i+}\cdots,$$

where the $a_i$'s are nonconstant elements in the set of head parts. This is a JR-continued fraction expansion with all $b_i = 1$. To apply Theorems 3.9 and 3.10 to $\pi$-adic Ruban continued fraction expansions, since $|b_i|_\pi = 1 = |b_i|_\infty$ for all $i \in \mathbb{N}$, we obtain:

**Corollary 3.11.** *Let* $\alpha_j = \frac{1}{a_{1,j}+}\,\frac{1}{a_{2,j}+}\cdots\frac{1}{a_{i,j}+}\cdots$ $(j = 1, 2, \ldots, k)$ *be* $k$ *infinite* $\pi$*-adic Ruban continued fraction expansions and use the preceding notation and restrictions. Assume that*

*I. the condition* (3.7) (*or* (3.8)) *is fulfilled for each* $j \in \{1, 2, \ldots, k\}$*;*

*II. the limiting values*

$$\lim_{i\to\infty} \frac{|D_{i,j}D_{i+1,j}|_\pi}{|D_{i,j-1}D_{i+1,j-1}|_\pi} = 0 \qquad (j = 2, 3, \ldots, k) \tag{3.18}$$

*hold and*

*III. there exists* $g : \mathbb{N} \to \mathbb{R}$ *with* $g(i) \to \infty$ *as* $i \to \infty$ *such that*

$$|D_{i,j}D_{i+1,j}|_\pi \geq (M_{i,1}M_{i,2}\cdots M_{i,j})^{g(i)} \quad (i \in \mathbb{N},\ j = 1, 2, \ldots, k). \tag{3.19}$$

*Then* $\alpha_1, \alpha_2, \ldots, \alpha_k$ *are algebraically independent over* $F(x)$.

**Corollary 3.12.** *Let* $\alpha_j = \frac{1}{a_{1,j}+}\,\frac{1}{a_{2,j}+}\cdots\frac{1}{a_{i,j}+}\cdots$ $(j = 1, 2, \ldots, k)$ *be* $k$ *infinite* $\pi$*-adic Ruban continued fraction expansions and use the preceding notation and restrictions. Assume that*

*I. the condition* (3.7) (*or* (3.8)) *is fulfilled for each* $j \in \{1, 2, \ldots, k\}$*;*

*II. the limiting values*

$$\lim_{i\to\infty} \frac{|D_{i,j}D_{i+1,j}|_\pi}{|D_{i,j-1}D_{i+1,j-1}|_\pi} = 0 \qquad (j = 2, 3, \ldots, k)$$

*hold and*

*III. there exists $g : \mathbb{N} \to \mathbb{R}$ with $g(i) \to \infty$ as $i \to \infty$ such that*

$$|D_{i,j}D_{i+1,j}|_\pi \geq g(i)\,(M_{i,1}M_{i,2}\cdots M_{i,j}) \quad (i \in \mathbb{N},\ j = 1, 2, \ldots, k). \tag{3.20}$$

*Then $1, \alpha_1, \alpha_2, \ldots, \alpha_k$ are linearly independent over $F(x)$.*

The following are some examples.

**Example 3.13.** Let $f : \mathbb{N} \to \mathbb{R}_{\geq 1}$ be a real valued function with $f(i) \to \infty$ as $i \to \infty$ and let $\{n_i'\}_{i\geq 1}$ and $\{n_i\}_{i\geq 1}$ be sequences of positive integers such that

$$n_i' > n_i \quad \text{and} \quad n_{i+1} \geq 2f(i)(n_1' + n_2' + \cdots + n_i') \quad \text{for all} \quad i \in \mathbb{N}. \tag{3.21}$$

Let $\alpha_1$ and $\alpha_2$ be nonzero elements in $F((\pi(x)))$ having infinite $\pi$-adic Ruban continued fraction expansions of the form

$$\alpha_j = \frac{1}{a_{1,j}+}\,\frac{1}{a_{2,j}+}\cdots\frac{1}{a_{i,j}+}\cdots \qquad (i \in \mathbb{N},\ j = 1, 2),$$

where

$$a_{i,1} = \frac{c_{-i}}{\pi^{n_i'}},\ a_{i,2} = \frac{d_{-i}}{\pi^{n_i}} \quad \text{and} \quad c_{-i}, d_{-i} \in F \setminus \{0\}.$$

Then $\alpha_1$ and $\alpha_2$ are algebraically independent over $F(x)$.

*Proof.* For $i \in \mathbb{N}$, we have

$$|a_{i,1}|_\pi = 2^{n_i' \deg \pi} = |a_{i,1}|_\infty^{-1},$$
$$\text{and} \quad |a_{i,2}|_\pi = 2^{n_i \deg \pi} = |a_{i,2}|_\infty^{-1}.$$

From (3.21), we get $n_{i+1}' > n_i'$ and $n_{i+1} > n_i$ for all $i \in \mathbb{N}$.
We obtain

$$|a_{2,1}a_{1,1}|_\infty = 2^{(-n_2'-n_1')\deg \pi} < 1, \quad |a_{i,1}|_\infty = 2^{-n_i' \deg \pi} > 2^{-n_{i+2}' \deg \pi} = |a_{i+2,1}|_\infty,$$

and

$$|a_{2,2}a_{1,2}|_\infty = 2^{(-n_2-n_1)\deg\pi} < 1, \quad |a_{i,2}|_\infty = 2^{-n_i\deg\pi} > 2^{-n_{i+2}\deg\pi} = |a_{i+2,2}|_\infty$$

and so $\alpha_1$ and $\alpha_2$ satisfy (3.9) which is equivalent to (3.8).

By Proposition 2.4, for all $i \in \mathbb{N}$, we get

$$|D_{i,1}|_\pi = 2^{(n_1'+n_2'+\cdots+n_i')\deg\pi} \text{ and } |D_{i,2}|_\pi = 2^{(n_1+n_2+\cdots+n_i)\deg\pi}. \tag{3.22}$$

By the hypotheses (3.21) and (3.22), we see that

$$\frac{|D_{i,2}D_{i+1,2}|_\pi}{|D_{i,1}D_{i+1,1}|_\pi} = \frac{2^{(2n_1+2n_2+\cdots+2n_i+n_{i+1})\deg\pi}}{2^{(2n_1'+2n_2'+\cdots+2n_i'+n_{i+1}')\deg\pi}} \to 0 \text{ as } i\to\infty,$$

so $\alpha_1$ and $\alpha_2$ satisfy (3.18).

Note that, for $i \in \mathbb{N}$,

$$\gamma_{1,1} = \pi^{n_1'}, \quad \gamma_{i+1,1} = \pi^{n_i'+n_{i+1}'}, \quad \beta_{i,1} = c_{-i},$$

and

$$\gamma_{1,2} = \pi^{n_1}, \quad \gamma_{i+1,2} = \pi^{n_i+n_{i+1}}, \quad \beta_{i,2} = d_{-i},$$

so

$$|\gamma_{1,1}|_\infty = 2^{n_1'\deg\pi}, \quad |\gamma_{i+1,1}|_\infty = 2^{(n_i'+n_{i+1}')\deg\pi}, \quad |\beta_{i,1}|_\infty = 1,$$

and

$$|\gamma_{1,2}|_\infty = 2^{n_1\deg\pi}, \quad |\gamma_{i+1,2}|_\infty = 2^{(n_i+n_{i+1})\deg\pi}, \quad |\beta_{i,2}|_\infty = 1.$$

By Lemma 3.8, we get

$$M_{i,1} = \begin{cases} |\mathcal{C}_{i,1}|_\infty = 2^{(n_1'+n_2'+\cdots+n_i')\deg\pi} & \text{if } i \text{ is odd} \\ |\mathcal{D}_{i,1}|_\infty = 2^{(n_1'+n_2'+\cdots+n_i')\deg\pi} & \text{if } i \text{ is even,} \end{cases}$$

and

$$M_{i,2} = \begin{cases} |\mathcal{C}_{i,2}|_\infty = 2^{(n_1+n_2+\cdots+n_i)\deg\pi} & \text{if } i \text{ is odd} \\ |\mathcal{D}_{i,2}|_\infty = 2^{(n_1+n_2+\cdots+n_i)\deg\pi} & \text{if } i \text{ is even.} \end{cases}$$

Next, we would like to show that $\alpha_1$ and $\alpha_2$ satisfy (3.19). By (3.21), for all $i \in \mathbb{N}$, we get

$$\begin{aligned} |D_{i,1}D_{i+1,1}|_\pi &= 2^{(2n_1'+2n_2'+\cdots+2n_i'+n_{i+1}')\deg\pi} \\ &> 2^{n_{i+1}'\deg\pi} \\ &> 2^{n_{i+1}\deg\pi} \\ &> 2^{f(i)(n_1'+n_2'+\cdots+n_i')\deg\pi} \\ &= M_{i,1}^{f(i)}, \end{aligned}$$

and

$$\begin{aligned} |D_{i,2}D_{i+1,2}|_\pi &= 2^{(2n_1+2n_2+\cdots+2n_i+n_{i+1})\deg\pi} \\ &\geq 2^{(2n_1+2n_2+\cdots+2n_i)\deg\pi+2f(i)(n_1'+n_2'+\cdots+n_i')\deg\pi} \\ &> 2^{2f(i)(n_1'+n_2'+\cdots+n_i')\deg\pi} \\ &> 2^{f(i)(n_1'+n_2'+\cdots+n_i'+n_1+n_2+\cdots+n_i)\deg\pi} \\ &= (M_{i,1}M_{i,2})^{f(i)}. \end{aligned}$$

Now, all the hypotheses of Corollary 3.11 are verified. This implies that $\alpha_1$ and $\alpha_2$ are algebraically independent over $F(x)$. □

**Example 3.14.** As in the previous example, if the condition (3.21) is replaced by

$$n_i' > n_i \quad \text{and} \quad n_{i+1} \geq n_1' + n_2' + \cdots + n_i' \quad \text{for all} \quad i \in \mathbb{N}.$$

Then $1, \alpha_1$ and $\alpha_2$ are linearly independent over $F(x)$.

*Proof.* It suffices to show that $\alpha_1$ and $\alpha_2$ satisfy (3.20).

Define $g : \mathbb{N} \to \mathbb{R}$ by $g(i) = 2^{(n_1+n_2+\cdots+n_i)\deg\pi}$ for all $i \in \mathbb{N}$. Since $n_i \in \mathbb{N}$ for all $i \in \mathbb{N}$, $g(i) \to \infty$ as $i \to \infty$. Consider

$$\begin{aligned}
|D_{i,1}D_{i+1,1}|_\pi &= 2^{(2n_1'+2n_2'+\cdots+2n_i'+n_{i+1}')\deg\pi} \\
&= 2^{(n_1'+n_2'+\cdots+n_i'+n_{i+1}')\deg\pi} M_{i,1} \\
&> 2^{(n_1+n_2+\cdots+n_i)\deg\pi} M_{i,1} \\
&= g(i)M_{i,1},
\end{aligned}$$

and

$$\begin{aligned}
|D_{i,2}D_{i+1,2}|_\pi &= 2^{(2n_1+2n_2+\cdots+2n_i+n_{i+1})\deg\pi} \\
&= 2^{(n_1+n_2+\cdots+n_i)\deg\pi+(n_1+n_2+\cdots+n_i+n_{i+1})\deg\pi} \\
&\geq g(i)2^{(n_1+n_2+\cdots+n_i+n_1'+n_2'+\cdots+n_i')\deg\pi} \\
&= g(i)\left(M_{i,1}M_{i,2}\right).
\end{aligned}$$

By Corollary 3.12, we have $1, \alpha_1$ and $\alpha_2$ are linearly independent over $F(x)$. □

### 3.2.2 The $\pi$-adic Schneider continued fraction expansion

The $\pi$-adic Schneider continued fraction expansion of $\alpha \in \pi(x)F[[\pi(x)]] \setminus \{0\}$ is of the form

$$\alpha = \frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\cdots\frac{b_i}{a_i+}\cdots,$$

where the partial denominators and numerators are as the following:

$$a_i \in F[x] \setminus \{0\},\ \deg a_i < \deg\pi,\ b_i = \pi^{s_i},\ s_i \in \mathbb{N} \text{ for all } i \geq 1$$

and each $b_i$ is uniquely determined form $\alpha$ and previously known $a_j, b_j$ $(j < i)$.

For all $i \in \mathbb{N}$, we have $|b_i|_\pi = 2^{-s_i\deg\pi} = |b_i|_\infty^{-1}$ and by Proposition 2.4, we get $|D_i|_\pi = |a_1a_2\cdots a_i|_\pi = 1$. Apply Theorems 3.9 and 3.10 to $\pi$-adic Schneider continued fraction expansions, we get Corollaries 3.15 and 3.16 as follows:

**Corollary 3.15.** *Let* $\alpha_j = \frac{b_{1,j}}{a_{1,j}+}\frac{b_{2,j}}{a_{2,j}+}\cdots\frac{b_{i,j}}{a_{i,j}+}\cdots$ $(j = 1, 2, \ldots, k)$ *be* $k$ *infinite* $\pi$*-adic Schneider continued fraction expansions and use the preceding notation and restrictions. Assume that*

*I. the condition* (3.7) (*or* (3.8)) *is fulfilled for each* $j \in \{1, 2, \ldots, k\}$;

*II. the limiting values*

$$\lim_{i\to\infty} \frac{|b_{1,j}b_{2,j}\cdots b_{i+1,j}|_\infty}{|b_{1,j-1}b_{2,j-1}\cdots b_{i+1,j-1}|_\infty} = 0 \qquad (j = 2, 3, \ldots, k) \tag{3.23}$$

*hold and*

*III. there exists* $g : \mathbb{N} \to \mathbb{R}$ *with* $g(i) \to \infty$ *as* $i \to \infty$ *such that*

$$|b_{1,j}b_{2,j}\cdots b_{i+1,j}|_\infty \geq (M_{i,1}M_{i,2}\cdots M_{i,j})^{g(i)} \quad (i \in \mathbb{N},\ j = 1, 2, \ldots, k). \tag{3.24}$$

*Then* $\alpha_1, \alpha_2, \ldots, \alpha_k$ *are algebraically independent over* $F(x)$.

**Corollary 3.16.** *Let* $\alpha_j = \frac{b_{1,j}}{a_{1,j}+}\frac{b_{2,j}}{a_{2,j}+}\cdots\frac{b_{i,j}}{a_{i,j}+}\cdots$ $(j = 1, 2, \ldots, k)$ *be* $k$ *infinite* $\pi$*-adic Schneider continued fraction expansions and use the preceding notation and restrictions. Assume that*

*I. the condition* (3.7) (*or* (3.8)) *is fulfilled for each* $j \in \{1, 2, \ldots, k\}$;

*II. the limiting values*

$$\lim_{i\to\infty} \frac{|b_{1,j}b_{2,j}\cdots b_{i+1,j}|_\infty}{|b_{1,j-1}b_{2,j-1}\cdots b_{i+1,j-1}|_\infty} = 0 \qquad (j = 2, 3, \ldots, k)$$

*hold and*

*III. there exists* $g : \mathbb{N} \to \mathbb{R}$ *with* $g(i) \to \infty$ *as* $i \to \infty$ *such that*

$$|b_{1,j}b_{2,j}\cdots b_{i+1,j}|_\infty \geq g(i)\,(M_{i,1}M_{i,2}\cdots M_{i,j}) \quad (i \in \mathbb{N},\ j = 1, 2, \ldots, k). \tag{3.25}$$

*Then* $1, \alpha_1, \alpha_2, \ldots, \alpha_k$ *are linearly independent over* $F(x)$.

**Remark 3.17.** *For the $\pi$-adic Schneider continued fraction expansion, its associated continued fraction expansion is identical with itself, that is*

$$\gamma_i = b_i \quad \text{and} \quad \beta_i = a_i \quad \text{for all} \quad i \in \mathbb{N}.$$

Next, we illustrate some examples.

**Example 3.18.** Let $f : \mathbb{N} \to \mathbb{R}_{\geq 1}$ be a real valued function with $f(i) \to \infty$ as $i \to \infty$ and let $\{m_i'\}_{i\geq 1}$ and $\{m_i\}_{i\geq 1}$ be sequences of positive integers such that

$$m_i' > m_i \quad \text{and} \quad m_{i+1} \geq \begin{cases} 2f(i)(m_1' + m_3' + \cdots + m_{i-2}' + m_i') & \text{if } i \text{ is odd} \\ 2f(i)(m_2' + m_4' + \cdots + m_{i-2}' + m_i') & \text{if } i \text{ is even.} \end{cases} \tag{3.26}$$

Let $\alpha_1$ and $\alpha_2$ be nonzero elements in $F((\pi(x)))$ having infinite $\pi$-adic Schneider continued fraction expansions of the form

$$\alpha_j = \frac{b_{1,j}}{a_{1,j}+} \frac{b_{2,j}}{a_{2,j}+} \cdots \frac{b_{i,j}}{a_{i,j}+} \cdots \qquad (i \in \mathbb{N},\ j = 1, 2),$$

where

$$a_{i,1}, a_{i,2} \in F \setminus \{0\}, \quad b_{i,1} = \pi^{m_i'} \quad \text{and} \quad b_{i,2} = \pi^{m_i}.$$

Then $\alpha_1$ and $\alpha_2$ are algebraically independent over $F(x)$.

*Proof.* For $i \in \mathbb{N}$, we have

$$\begin{aligned} |a_{i,1}|_\pi = |a_{i,2}|_\pi = 1 &= |a_{i,1}|_\infty = |a_{i,2}|_\infty, \\ |b_{i,1}|_\pi = 2^{-m_i' \deg \pi} &= |b_{i,1}|_\infty^{-1}, \\ \text{and} \quad |b_{i,2}|_\pi = 2^{-m_i \deg \pi} &= |b_{i,2}|_\infty^{-1}. \end{aligned}$$

From (3.26), we get $m_{i+1}' > m_i'$ and $m_{i+1} > m_i$ for all $i \in \mathbb{N}$.

We obtain

$$|b_{i+1,1}|_\infty = 2^{m'_{i+1}\deg\pi} > 1 = |a_{i+1,1}a_{i,1}|_\infty,$$
$$|b_{i+2,1}a_{i,1}|_\infty = 2^{m'_{i+2}\deg\pi} > 2^{m'_{i+1}\deg\pi} = |a_{i+2,1}b_{i+1,1}|_\infty,$$

and

$$|b_{i+1,2}|_\infty = 2^{m_{i+1}\deg\pi} > 1 = |a_{i+1,2}a_{i,2}|_\infty,$$
$$|b_{i+2,2}a_{i,2}|_\infty = 2^{m_{i+2}\deg\pi} > 2^{m_{i+1}\deg\pi} = |a_{i+2,2}b_{i+1,2}|_\infty$$

and so $\alpha_1$ and $\alpha_2$ satisfy (3.9) which equivalent to (3.8).

By the hypothesis (3.26), we get

$$\frac{|b_{1,2}b_{2,2}\cdots b_{i+1,2}|_\infty}{|b_{1,1}b_{2,1}\cdots b_{i+1,1}|_\infty} = \frac{2^{(m_1+m_2+\cdots+m_{i+1})\deg\pi}}{2^{(m'_1+m'_2+\cdots+m'_{i+1})\deg\pi}} \to 0 \quad \text{as} \quad i \to \infty,$$

so $\alpha_1$ and $\alpha_2$ satisfy (3.23).

Note that, for $i \in \mathbb{N}$,

$$\gamma_{i,1} = \pi^{m'_i}, \quad \beta_{i,1} = a_{i,1},$$

and

$$\gamma_{i,2} = \pi^{m_i}, \quad \beta_{i,2} = a_{i,2},$$

so

$$|\gamma_{i,1}|_\infty = 2^{m'_i\deg\pi}, \quad |\beta_{i,1}|_\infty = 1,$$

and

$$|\gamma_{i,2}|_\infty = 2^{m_i\deg\pi}, \quad |\beta_{i,2}|_\infty = 1.$$

By Lemma 3.8, for $i \in \mathbb{N}$, we get

$$M_{i,1} = \begin{cases} |\mathcal{C}_{i,1}|_\infty = 2^{(m'_1+m'_3+\cdots+m'_{i-2}+m'_i)\deg\pi} & \text{if } i \text{ is odd} \\ |\mathcal{D}_{i,1}|_\infty = 2^{(m'_2+m'_4+\cdots+m'_{i-2}+m'_i)\deg\pi} & \text{if } i \text{ is even,} \end{cases}$$

and

$$M_{i,2} = \begin{cases} |\mathcal{C}_{i,2}|_\infty = 2^{(m_1+m_3+\cdots+m_{i-2}+m_i)\deg \pi} & \text{if } i \text{ is odd} \\ |\mathcal{D}_{i,2}|_\infty = 2^{(m_2+m_4+\cdots+m_{i-2}+m_i)\deg \pi} & \text{if } i \text{ is even.} \end{cases}$$

Next, we would like to show that $\alpha_1$ and $\alpha_2$ satisfy (3.24). By (3.26), for all $i \in \mathbb{N}$, we get

$$\begin{aligned} |b_{1,1}b_{2,1}\cdots b_{i+1,1}|_\infty &= 2^{(m_1'+m_2'+\cdots+m_{i+1}')\deg \pi} \\ &> 2^{m_{i+1}'\deg \pi} \\ &> 2^{m_{i+1}\deg \pi} \\ &\geq \begin{cases} 2^{2f(i)(m_1'+m_3'+\cdots+m_{i-2}'+m_i')\deg \pi} & \text{if } i \text{ is odd} \\ 2^{2f(i)(m_2'+m_4'+\cdots+m_{i-2}'+m_i')\deg \pi} & \text{if } i \text{ is even} \end{cases} \\ &= M_{i,1}^{2f(i)} \\ &> M_{i,1}^{f(i)}, \end{aligned}$$

and

$$\begin{aligned} |b_{1,2}b_{2,2}\cdots b_{i+1,2}|_\infty &= 2^{(m_1+m_2+\cdots+m_{i+1})\deg \pi} \\ &> 2^{m_{i+1}\deg \pi} \\ &\geq \begin{cases} 2^{2f(i)(m_1'+m_3'+\cdots+m_{i-2}'+m_i')\deg \pi} & \text{if } i \text{ is odd} \\ 2^{2f(i)(m_2'+m_4'+\cdots+m_{i-2}'+m_i')\deg \pi} & \text{if } i \text{ is even} \end{cases} \\ &> \begin{cases} 2^{f(i)(m_1'+m_3'+\cdots+m_{i-2}'+m_i'+m_1+m_3+\cdots+m_{i-2}+m_i)\deg \pi} & \text{if } i \text{ is odd} \\ 2^{f(i)(m_2'+m_4'+\cdots+m_{i-2}'+m_i'+m_2+m_4+\cdots+m_{i-2}+m_i)\deg \pi} & \text{if } i \text{ is even} \end{cases} \\ &= (M_{i,1}M_{i,2})^{f(i)}. \end{aligned}$$

Now, all the hypotheses of Corollary 3.15 are verified. This implies that $\alpha_1$ and $\alpha_2$ are algebraically independent over $F(x)$. □

**Example 3.19.** As in the previous example, if the condition (3.26) is replaced by

$$m_i' > m_i \quad \text{and} \quad m_{i+1} \geq \begin{cases} m_1' + m_3' + \cdots + m_{i-2}' + m_i' & \text{if } i \text{ is odd} \\ m_2' + m_4' + \cdots + m_{i-2}' + m_i' & \text{if } i \text{ is even.} \end{cases}$$

Then $1, \alpha_1$ and $\alpha_2$ are linearly independent over $F(x)$.

*Proof.* It suffices to show that $\alpha_1$ and $\alpha_2$ satisfy (3.25).
Define $g : \mathbb{N} \to \mathbb{R}$ by

$$g(i) = \begin{cases} 2^{(m_2+m_4+\cdots+m_{i-1}) \deg \pi} & \text{if } i \text{ is odd} \\ 2^{(m_1+m_3+\cdots+m_{i-1}) \deg \pi} & \text{if } i \text{ is even.} \end{cases}$$

Since $m_i \in \mathbb{N}$ for all $i \in \mathbb{N}$, $g(i) \to \infty$ as $i \to \infty$. For $i \in \mathbb{N}$, consider

$$\begin{aligned} |b_{1,1}b_{2,1}\cdots b_{i+1,1}|_\infty &= 2^{(m_1'+m_2'+\cdots+m_{i+1}') \deg \pi} \\ &= \begin{cases} 2^{(m_1'+m_3'+\cdots+m_i'+m_2'+m_4'+\cdots+m_{i-1}'+m_{i+1}') \deg \pi} & \text{if } i \text{ is odd} \\ 2^{(m_2'+m_4'+\cdots+m_i'+m_1'+m_3'+\cdots+m_{i-1}'+m_{i+1}') \deg \pi} & \text{if } i \text{ is even} \end{cases} \\ &\geq \begin{cases} 2^{(m_1'+m_3'+\cdots+m_i'+m_2+m_4+\cdots+m_{i-1}) \deg \pi} & \text{if } i \text{ is odd} \\ 2^{(m_2'+m_4'+\cdots+m_i'+m_1+m_3+\cdots+m_{i-1}) \deg \pi} & \text{if } i \text{ is even} \end{cases} \\ &= g(i)M_{i,1}, \end{aligned}$$

and

$$\begin{aligned} &|b_{1,2}b_{2,2}\cdots b_{i+1,2}|_\infty \\ &= 2^{(m_1+m_2+\cdots+m_{i+1}) \deg \pi} \\ &= \begin{cases} 2^{(m_1+m_3+\cdots+m_i+m_2+m_4+\cdots+m_{i-1}+m_{i+1}) \deg \pi} & \text{if } i \text{ is odd} \\ 2^{(m_2+m_4+\cdots+m_i+m_1+m_3+\cdots+m_{i-1}+m_{i+1}) \deg \pi} & \text{if } i \text{ is even} \end{cases} \\ &\geq \begin{cases} 2^{(m_1+m_3+\cdots+m_i+m_2+m_4+\cdots+m_{i-1}+m_1'+m_3'+\cdots+m_{i-2}'+m_i') \deg \pi} & \text{if } i \text{ is odd} \\ 2^{(m_2+m_4+\cdots+m_i+m_1+m_3+\cdots+m_{i-1}+m_2'+m_4'+\cdots+m_{i-2}'+m_i') \deg \pi} & \text{if } i \text{ is even} \end{cases} \end{aligned}$$

$$= g(i)\left(M_{i,1}M_{i,2}\right).$$

By Corollary 3.16, we have $1, \alpha_1$ and $\alpha_2$ are linearly independent over $F(x)$. $\square$