CHAPTER IV

EXPLICIT CONTINUED FRACTION EXPANSIONS

In this chapter, we work on the field $\mathbb{F}_q((x^{-1}))$ of formal Laurent series over the finite field $\mathbb{F}_q$, where q is a prime power, equipped by a degree valuation $|\cdot|_\infty$. The first section deals with notation and preliminaries. The explicit Ruban continued fraction expansions of e and other interesting elements in $\mathbb{F}_q((x^{-1}))$ are given in the last section.

4.1 Notation and preliminaries

It is known from Chapter 2, every element $\xi \in \mathbb{F}_q((x^{-1}))$ can be uniquely written as a Ruban continued fraction expansion of the form

$$\xi = a_0 + \frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots,$$

where $a_0 \in \mathbb{F}_q[x]$ and $a_n \in \mathbb{F}_q[x] \setminus \mathbb{F}_q$ $(n \geq 1)$, and that the continued fraction expansion of ξ is finite if and only if $\xi \in \mathbb{F}_q(x)$. In this chapter we use the notation

$$[a_0; a_1, a_2, a_3, \ldots] := [a_0; 1, a_1; 1, a_2; 1, a_3; \ldots],$$

for the above continued fraction expansion and $\dfrac{C_n}{D_n}$ for its n^{th} convergent.

By induction, we have

Proposition 4.1. *For any $n \geq 1$, let $[a_0; a_1, a_2, \ldots, a_n] = \dfrac{C_n}{D_n}$. Then we get*

(1) $[a_n; a_{n-1}, \ldots, a_2, a_1] = \dfrac{D_n}{D_{n-1}}$ *for all $n \geq 1$,*

(2) $[a_n; a_{n-1}, \ldots, a_3, a_2] = \dfrac{C_n}{C_{n-1}}$ *for all $n \geq 2$, if $a_0 = 0$.*

Let $a_0 \in \mathbb{F}_q[x]$ and $\{a_i\}_{i\geq 1}$ be a sequence of nonzero polynomial over $\mathbb{F}_q$, and let $\overrightarrow{X}_n$ denote the word $a_1, a_2, \ldots, a_n$. We put

$$[a_0; \overrightarrow{X}_n] = [a_0; a_1, a_2, \ldots, a_n],$$

$$[a_0; -\overrightarrow{X}_n] = [a_0; -a_1, -a_2, \ldots, -a_n],$$

$$[a_0; \overleftarrow{X}_n] = [a_0; a_n, a_{n-1}, \ldots, a_1],$$

and

$$[a_0; -\overleftarrow{X}_n] = [a_0; -a_n, -a_{n-1}, \ldots, -a_1].$$

The notation and basic results follow closely those in Carlitz [4]. For a positive integer i, let

$$[i] := x^{q^i} - x, \text{ and } d_0 := 1, \; d_i := [i]d_{i-1}^q. \tag{4.1}$$

It is known that $[i]$ is the product of monic irreducible polynomials in $\mathbb{F}_q[x]$ of degree dividing i, and d_i is the product of all monic polynomials in $\mathbb{F}_q[x]$ of degree i.

Remark 4.2. *From recursive definition* (4.1), *for all $i \geq 1$, we have the following two identities.*

(1) $d_i = [1][2]\cdots[i]d_1^{q-1}d_2^{q-1}\cdots d_{i-1}^{q-1}$.

(2) $d_i = [i][i-1]^q[i-2]^{q^2}\cdots[2]^{q^{i-2}}[1]^{q^{i-1}}$.

Let

$$e(z) := \sum_{i=0}^{\infty} \frac{z^{q^i}}{d_i},$$

known as the exponential element for $\mathbb{F}_q[x]$. For brevity, put $e := e(1)$. For many analogies with the properties of the classical exponential, we refer to [29]. The following result gives the continued fraction expansion for $\sum_{i=0}^{n+1} \frac{z^{q^i}}{d_i}$ if the continued fraction expansion for $\sum_{i=0}^{n} \frac{z^{q^i}}{d_i}$ is known. In particular, the continued fraction expansion for e in $\mathbb{F}_2((x^{-1}))$ is shown as follows:

Proposition 4.3. *Define a sequence x_n with $x_1 = [0; z^{-q}[1]]$ and if $x_n = [a_0; a_1, \ldots, a_{2^n-1}]$, then set*

$$x_{n+1} = \left[a_0; a_1, \ldots, a_{2^n-1}, \frac{-z^{-q^n(q-2)}d_{n+1}}{d_n^2}, -a_{2^n-1}, \ldots, -a_1\right].$$

We have

$$x_n = \sum_{i=1}^{n} \frac{z^{q^i}}{d_i}.$$

In particular, $e(z) = z + \lim_{n\to\infty} x_n$ and for $q = 2$,

$$e = [1; \underbrace{\underbrace{\underbrace{[1]}, [2], \underbrace{[1]}}, [3], \underbrace{[1], [2], [1]}}, [4], \underbrace{[1], [2], [1], [3], [1], [2], [1]}, [5], \ldots].$$

(More explicitly, for $n > 0$ the n^{th} partial quotient is $x^{2^{u_n}} - x$ with u_n being the exponent of the highest power of 2 dividing $2n$).

In 1996, Thakur [30] gave the pattern in the general case which is more subtle when $q = 2$.

Proposition 4.4. *Let $q = 2$. Then for $m > 2$, with $\overrightarrow{X}_{(m)}$ defined through $\sum_{i=0}^{m-2} \frac{1}{d_i x^m} = [0, \overrightarrow{X}_{(m)}]$ we have,*

$$\frac{e}{x^m} = [0; \overrightarrow{X}_{(m)}, x^{2^{m-1}-m}, \overrightarrow{X}_{(m)}, x^{2^m-m}, \overrightarrow{X}_{(m)}, x^{2^{m-1}-m}, \overrightarrow{X}_{(m)}, x^{2^{m+1}-m}, \ldots].$$

Also, with $\overrightarrow{X}$ denote the word $x^2+1, x, x+1$, we have

$$\frac{e}{x^2} = [0; \overrightarrow{X}, x^2, \overrightarrow{X}, x^6, \overrightarrow{X}, x^2, \overrightarrow{X}, x^{14}, \ldots].$$

In this chapter, we give explicit Ruban continued fraction expansions of $\frac{e}{f(x)^m}$, where $m \in \mathbb{N}$ and $f(x)$ is a nonconstant monic polynomial over a finite field $\mathbb{F}_q$ satisfying $f(x) \mid [1]$. Since

$$\begin{aligned}[1] &= x^q - x \\ &= x\left(x^{q-1} - 1\right)\end{aligned}$$

$$= x(x-1)\left(x^{q-2}+x^{q-3}+\cdots+1\right),$$

this leads naturally to consider the polynomials appearing in the following table.

$f(x)$ $(m \geq 2)$	$q=2$	$q \geq 3$
x	Corollary 4.9	Corollary 4.9
x^m	Thakur (1996)	Corollary 4.10
$x^{q-1}-1$	Corollary 4.9	Corollary 4.9
$(x^{q-1}-1)^m$	Theorem 4.12	Corollary 4.11
$x-1$	Corollary 4.9	Corollary 4.9
$(x-1)^m$	Theorem 4.12	Corollary 4.10
$x^{q-2}+x^{q-3}+\cdots+1$	Thakur (1992)	Corollary 4.9
$(x^{q-2}+x^{q-3}+\cdots+1)^m$	Thakur (1992)	Corollary 4.11
$x(x-1)$	Corollary 4.9	Corollary 4.9
$(x(x-1))^m$	Theorem 4.13	Corollary 4.10
$x(x^{q-2}+x^{q-3}+\cdots+1)$	Corollary 4.9	Corollary 4.9
$(x(x^{q-2}+x^{q-3}+\cdots+1))^m$	Thakur (1996)	Corollary 4.11
$[1]$	Corollary 4.9	Corollary 4.9
$[1]^m$	Theorem 4.13	Corollary 4.11

4.2 Main results

In this section, we find explicit Ruban continued fraction expansions of $\dfrac{e}{f(x)^m}$, where $m \in \mathbb{N}$ and $f(x)$ are polynomials appearing in the above table. The proofs of our works are based on calculation abstracted in the following lemma.

Lemma 4.5. *Let $y \in \mathbb{F}_q[x] \setminus \{0\}$ and $\dfrac{C_n}{D_n} := [0; a_1, a_2, \ldots, a_n] = [0; \overrightarrow{X}_n]$. Then*

(1) $[0; \overrightarrow{X}_n, y, \overrightarrow{X}_n] = \dfrac{C_n}{D_n} + \dfrac{(-1)^n}{D_n^2\left(y + \dfrac{C_n + D_{n-1}}{D_n}\right)},$

(2) $[0; \overrightarrow{X}_n, y, -\overleftarrow{X}_n] = \dfrac{C_n}{D_n} + \dfrac{(-1)^n}{D_n^2 y}.$

Proof. By Propositions 4.1, 2.1 and 2.2, we get (1)

$$\begin{aligned}
[0; \overrightarrow{X}_n, y, \overrightarrow{X}_n] &= [0; a_1, a_2, \ldots, a_n, y, a_1, a_2, \ldots, a_n] \\
&= [0; a_1, a_2, \ldots, a_n, y + \frac{C_n}{D_n}] \\
&= \frac{\left(y + \frac{C_n}{D_n}\right) C_n + C_{n-1}}{\left(y + \frac{C_n}{D_n}\right) D_n + D_{n-1}} \\
&= \frac{(D_n y + C_n) C_n + D_n C_{n-1}}{(D_n y + C_n) D_n + D_n D_{n-1}} \\
&= \frac{(D_n y + C_n) C_n + C_n D_{n-1} + (-1)^n}{(D_n y + C_n) D_n + D_n D_{n-1}} \\
&= \frac{C_n (D_n y + C_n + D_{n-1}) + (-1)^n}{D_n (D_n y + C_n + D_{n-1})} \\
&= \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2 \left(y + \frac{C_n + D_{n-1}}{D_n}\right)};
\end{aligned}$$

and (2)

$$\begin{aligned}
[a_0; \overrightarrow{X}_n, y, -\overleftarrow{X}_n] &= [0; a_1, a_2, \ldots, a_n, y, -a_n, -a_{n-1}, \ldots, -a_1] \\
&= [0; a_1, a_2, \ldots, a_n, y - \frac{D_{n-1}}{D_n}] \\
&= \frac{\left(y - \frac{D_{n-1}}{D_n}\right) C_n + C_{n-1}}{\left(y - \frac{D_{n-1}}{D_n}\right) D_n + D_{n-1}} \\
&= \frac{(D_n y - D_{n-1}) C_n + D_n C_{n-1}}{(D_n y - D_{n-1}) D_n + D_n D_{n-1}} \\
&= \frac{(D_n y - D_{n-1}) C_n + C_n D_{n-1} + (-1)^n}{(D_n y - D_{n-1}) D_n + D_n D_{n-1}} \\
&= \frac{C_n (D_n y) + (-1)^n}{D_n (D_n y)} \\
&= \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2 y}.
\end{aligned}$$

This proves our lemma. □

Lemma 4.5 (2) known as the Folding Lemma, first appeared in [19].

Lemma 4.6. *Let $m, t \in \mathbb{N}$. If $f(x)$ is a nonconstant monic polynomial over the finite field $\mathbb{F}_q$, where q is a prime power, such that $f(x) \mid [1]$, then*

$$\gcd\left(d_t + \frac{d_t}{d_1} + \frac{d_t}{d_2} + \cdots + \frac{d_t}{d_{t-1}} + 1, f(x)^m d_t\right) = 1.$$

Proof. Suppose that

$$\gcd\left(d_t + \frac{d_t}{d_1} + \frac{d_t}{d_2} + \cdots + \frac{d_t}{d_{t-1}} + 1, f(x)^m d_t\right) \neq 1.$$

Then there exists a prime $p \in \mathbb{F}_q[x]$ such that

$$p \mid \left(d_t + \frac{d_t}{d_1} + \frac{d_t}{d_2} + \cdots + \frac{d_t}{d_{t-1}} + 1\right) \quad \text{and} \quad p \mid f(x)^m d_t.$$

We have from Remark 4.2 (1) that $d_t = [1][2]\cdots[t]d_1^{q-1}d_2^{q-1}\cdots d_{t-1}^{q-1}$.
Since $p \mid f(x)^m d_t$, we get

$$p \mid f(x) \text{ or } p \mid [r] \text{ for some } 1 \leq r \leq t \text{ or } p \mid d_s \text{ for some } 1 \leq s \leq t-1.$$

Again, Remark 4.2 (1) leads to

$$\begin{aligned} & d_t + \frac{d_t}{d_1} + \frac{d_t}{d_2} + \cdots + \frac{d_t}{d_{t-1}} + 1 \\ &= \left([1][2]\cdots[t]d_1^{q-1}d_2^{q-1}d_3^{q-1}\cdots d_{t-1}^{q-1}\right) + \left([1][2]\cdots[t]d_1^{q-2}d_2^{q-1}d_3^{q-1}\cdots d_{t-1}^{q-1}\right) \\ &\quad + \left([1][2]\cdots[t]d_1^{q-1}d_2^{q-2}d_3^{q-1}\cdots d_{t-1}^{q-1}\right) + \cdots \\ &\quad + \left([1][2]\cdots[t]d_1^{q-1}d_2^{q-1}\cdots d_{t-2}^{q-1}d_{t-1}^{q-2}\right) + 1. \end{aligned}$$

If $p \mid f(x)$ or $p \mid [r]$ for some $1 \leq r \leq t$, then $p \mid 1$ which is a contradiction. Assume that $p \mid d_s$ for some $1 \leq s \leq t-1$. We treat two separate cases.

- **case** $q \geq 3$: We get $p \mid 1$, which is a contradiction.
- **case** $q = 2$: Then the above equation becomes

$$\begin{aligned} & d_t + \frac{d_t}{d_1} + \frac{d_t}{d_2} + \cdots + \frac{d_t}{d_{t-1}} + 1 \\ &= \left([1][2]\cdots[t]d_1 d_2 d_3\cdots d_{t-1}\right) + \left([1][2]\cdots[t]d_2 d_3 d_4\cdots d_{t-1}\right) \end{aligned}$$

$$+ ([1][2]\cdots[t]d_1d_3d_4\cdots d_{t-2}d_{t-1}) + \cdots + ([1][2]\cdots[t]d_1d_2\cdots d_{t-2}) + 1.$$

If $1 \le s \le t-2$ and since $d_i \mid d_{i+1}$ for all $i \ge 0$, then $p \mid 1$ which is a contradiction.

Assume that $s = t-1$. We apply Remark 4.2 (1) again and get $p \mid [r]$ for some $1 \le r \le t-1$ or $p \mid d_s$ for some $1 \le s \le t-2$ which is impossible.

Thus

$$\gcd\left(d_t + \frac{d_t}{d_1} + \frac{d_t}{d_2} + \cdots + \frac{d_t}{d_{t-1}} + 1, f(x)^m d_t\right) = 1.$$

Thus the proof of Lemma 4.6 is completed. □

Our first objective here is to extend Proposition 4.4 of Thakur by proving

Theorem 4.7. *Let $\{Q_i\}_{i=1}^{\infty}$ be a sequence of nonconstant monic polynomials over the finite field $\mathbb{F}_q$, where q is a prime power. Assume that there exists $N \in \mathbb{N} \cup \{0\}$ such that*

(i)

$$Q_1Q_2\cdots Q_{j+1} \mid Q_{j+2} \quad \text{for all } j \ge N \tag{4.2}$$

and

(ii) if $N \ge 1$, then

$$\gcd\left((Q_2\cdots Q_{N+1}) + (Q_3\cdots Q_{N+1}) + \cdots + Q_{N+1} + 1, Q_1Q_2\cdots Q_{N+1}\right) = 1. \tag{4.3}$$

If $\sum_{i=1}^{N+\ell} \dfrac{1}{Q_1Q_2\cdots Q_i} = [0; a_1, a_2, \ldots, a_{k_\ell}]$ $(\ell \ge 1)$, then

$$\sum_{i=1}^{N+\ell+1} \frac{1}{Q_1Q_2\cdots Q_i} = \left[0; a_1, a_2, \ldots, a_{k_\ell}, \frac{(-1)^{k_\ell} Q_{N+\ell+1}}{Q_1Q_2\cdots Q_{N+\ell}}, -a_{k_\ell}, \ldots, -a_2, -a_1\right].$$

Proof. For $\ell \ge 1$, let $\dfrac{C_{k_\ell}}{D_{k_\ell}} := [0; \overrightarrow{X}_{k_\ell}]$ be the k_ℓ^{th} convergent of the continued fraction expansion of $\sum_{i=1}^{N+\ell} \dfrac{1}{Q_1Q_2\cdots Q_i}$.

We observe that both C_{k_ℓ} and D_{k_ℓ} are monic. Consider

$$\sum_{i=1}^{N+\ell} \frac{1}{Q_1Q_2\cdots Q_i} = \frac{1}{Q_1} + \frac{1}{Q_1Q_2} + \cdots + \frac{1}{Q_1Q_2\cdots Q_{N+\ell}}$$
$$= \frac{(Q_2Q_3\cdots Q_{N+\ell}) + (Q_3Q_4\cdots Q_{N+\ell}) + \cdots + Q_{N+\ell} + 1}{Q_1Q_2\cdots Q_{N+\ell}}.$$

We assert that

$$\gcd\left((Q_2Q_3\cdots Q_{N+\ell}) + (Q_3Q_4\cdots Q_{N+\ell}) + \cdots + Q_{N+\ell} + 1, Q_1Q_2\cdots Q_{N+\ell}\right) = 1.$$

If $N \geq 1$ and $\ell = 1$, then it is obvious from assumption (4.3).

Next, we treat the other two cases.

Suppose there exists a prime $p \in \mathbb{F}_q[x]$ such that

$$p \mid ((Q_2Q_3\cdots Q_{N+\ell}) + (Q_3Q_4\cdots Q_{N+\ell}) + \cdots + Q_{N+\ell} + 1) \text{ and } p \mid Q_1Q_2\cdots Q_{N+\ell}.$$

case $N = 0$: By (4.2), we have $Q_1Q_2\cdots Q_i \mid Q_{i+1}$ for all $i \in \mathbb{N}$.
Since $p \mid (Q_1Q_2\cdots Q_\ell)$, $p \mid Q_k$ for some $1 \leq k \leq \ell$ and so $p \mid Q_jQ_{j+1}\cdots Q_\ell$ for all $2 \leq j \leq k$. Since $Q_1Q_2\cdots Q_k \mid Q_{k+t}$ for all $1 \leq t \leq \ell - k$, we have $Q_k \mid Q_{k+t}\cdots Q_\ell$ for all $1 \leq t \leq \ell - k$ and so $p \mid Q_{k+t}\cdots Q_\ell$ for all $1 \leq t \leq \ell - k$. Since $p \mid ((Q_2Q_3\cdots Q_\ell) + (Q_3Q_4\cdots Q_\ell) + \cdots + Q_\ell + 1)$, then we get $p \mid 1$, which is a contradiction. Thus

$$\gcd\left((Q_2Q_3\cdots Q_\ell) + (Q_3Q_4\cdots Q_\ell) + \cdots + Q_\ell + 1, Q_1Q_2\cdots Q_\ell\right) = 1.$$

case $N \geq 1$ **and** $\ell \geq 2$: Since $p \mid (Q_1Q_2\cdots Q_{N+\ell})$, $p \mid Q_k$ for some $1 \leq k \leq N + \ell$.
If $p \mid Q_{N+\ell}$, since $p \mid ((Q_2Q_3\cdots Q_{N+\ell}) + (Q_3Q_4\cdots Q_{N+\ell}) + \cdots + Q_{N+\ell} + 1)$, then $p \mid 1$ which is a contradiction.

Assume that $p \mid Q_k$ for some $1 \leq k \leq N + \ell - 1$. Using (4.2) when $j = N + \ell - 2 \geq N$, we get $Q_1Q_2\cdots Q_{N+\ell-1} \mid Q_{N+\ell}$, which implies that $p \mid Q_{N+\ell}$, again we have a contradiction. Thus

$$\gcd\left((Q_2Q_3\cdots Q_{N+\ell})+(Q_3Q_4\cdots Q_{N+\ell})+\cdots+Q_{N+\ell}+1, Q_1Q_2\cdots Q_{N+\ell}\right)=1.$$

Since C_{k_ℓ} and D_{k_ℓ} are relatively prime, and all Q_i are monic, then $D_{k_\ell} = Q_1Q_2\cdots Q_{N+\ell}$.
For $\ell \geq 1$, using (4.2) when $j = N+\ell-1 \geq N$, we get $\dfrac{(-1)^{k_\ell}Q_{N+\ell+1}}{Q_1Q_2\cdots Q_{N+\ell}} \in \mathbb{F}_q[x]\setminus\{0\}$.
Applying Lemma 4.5 (2), we get

$$\begin{aligned}
&[0; a_1, a_2, \ldots, a_{k_\ell}, \frac{(-1)^{k_\ell}Q_{N+\ell+1}}{Q_1Q_2\cdots Q_{N+\ell}}, -a_{k_\ell}, \ldots, -a_2, -a_1] \\
&= \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{(-1)^{k_\ell}}{D_{k_\ell}^2 \dfrac{(-1)^{k_\ell}Q_{N+\ell+1}}{Q_1Q_2\cdots Q_{N+\ell}}} \\
&= \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{(-1)^{k_\ell}}{(Q_1Q_2\cdots Q_{N+\ell})^2 \dfrac{(-1)^{k_\ell}Q_{N+\ell+1}}{Q_1Q_2\cdots Q_{N+\ell}}} \\
&= \sum_{i=1}^{N+\ell} \frac{1}{Q_1Q_2\cdots Q_i} + \frac{1}{Q_1Q_2\cdots Q_{N+\ell}Q_{N+\ell+1}} \\
&= \sum_{i=1}^{N+\ell+1} \frac{1}{Q_1Q_2\cdots Q_i},
\end{aligned}$$

and the proof is complete. □

Theorem 4.7 is contained in the following proposition which appeared in [21]. However, for convenience, we use the version of Theorem 4.7.

Proposition 4.8. *Let I be a fixed positive integer, $\{k_i\}_{i\geq 1}$ a sequence of positive integers, $\{c_i\}_{i\geq I}$ a sequence of nonzero polynomials over $\mathbb{F}_q$, subject to the condition that if $I = 1$, then c_1 and those c_i $(i \geq 2)$ for which $k_i = 2$ are nonconstant polynomials over $\mathbb{F}_q$. Let the sequence $\{P_i\}_{i\geq 1}$ be defined by*

$$P_1 = 1, \ \ P_2, P_3, \ldots, P_I \in \mathbb{F}_q[x] \setminus \mathbb{F}_q;$$
$$P_u = c_{u-1}P_{u-1}^{k_{u-1}}P_{u-2}^{k_{u-2}}\cdots P_{u-I}^{k_{u-I}} \quad (u \geq I+1),$$

and let

$$E(u) = \sum_{i=1}^{u} \frac{1}{P_i} \quad (u \in \mathbb{N}).$$

Assume that

(i) if $I \geq 2$, then $P_2 \mid P_3 \mid \cdots \mid P_I$;

(ii) $k_i \geq 2$ for all $i \geq I$.

If $E(u) = [a_0; a_1, a_2, \ldots, a_n]$ $(u \geq I+1)$, then there exists $\beta \in \mathbb{F}_q \setminus \{0\}$ such that

$$E(u+1) = [a_0; a_1, a_2, \ldots, a_n, \beta s_u, -a_n, \ldots, -a_2, -a_1],$$

where $s_u = \dfrac{c_u P_u^{k_u - 1}}{c_{u-1} P_{u-I}^{k_{u-I}}}$.

Now we apply Theorem 4.7 to show the explicit Ruban continued fraction expansions of $\dfrac{e}{f(x)}$, where $f(x)$ be a nonconstant monic polynomial such that $f(x) \mid [1]$.

Corollary 4.9. *Let $f(x)$ be a nonconstant monic polynomial over the finite field $\mathbb{F}_q$, where q is a prime power, If $f(x) \mid [1]$, then*

$$\frac{e}{f(x)} = [0; \underbrace{\underbrace{f(x)}, \frac{-[1]}{f(x)}, \underbrace{-f(x)}}, \frac{-[2]d_1^{q-2}}{f(x)}, \underbrace{f(x), \frac{[1]}{f(x)}, -f(x)}, \frac{-[3]d_2^{q-2}}{f(x)}, \ldots].$$

Proof. Let $Q_1 = f(x)$ and $Q_{i+1} = \dfrac{d_i}{d_{i-1}}$ for all $i \in \mathbb{N}$. For $i \in \mathbb{N}$, we consider

$$Q_{i+1} = \frac{d_i}{d_{i-1}} = \frac{[i]d_{i-1}^q}{d_{i-1}} = [i]d_{i-1}^{q-1} \in \mathbb{F}_q[x] \setminus \mathbb{F}_q,$$

so $Q_i \in \mathbb{F}_q[x] \setminus \mathbb{F}_q$. Since $Q_2 = d_1 = [1]$ and $f(x) \mid [1]$, $Q_1 \mid Q_2$. For $i \geq 2$, consider

$$Q_1 Q_2 Q_3 \cdots Q_i = f(x) \frac{d_1}{d_0} \frac{d_2}{d_2} \cdots \frac{d_{i-1}}{d_{i-2}} = f(x) d_{i-1}$$

and $Q_{i+1} = \dfrac{d_i}{d_{i-1}}$, so

$$\frac{Q_{i+1}}{Q_1Q_2Q_3\cdots Q_i} = \frac{d_i/d_{i-1}}{f(x)d_{i-1}} = \frac{[i]d_{i-1}^q}{f(x)d_{i-1}^2} = \frac{[i]d_{i-1}^{q-2}}{f(x)}.$$

We treat two separate cases.

case $q \geq 3$: Since $f(x) \mid [1]$ and $[1] \mid d_i$ for all $i \in \mathbb{N}$, $f(x) \mid d_i$ for all $i \in \mathbb{N}$ which implies that $Q_1Q_2Q_3\cdots Q_i \mid Q_{i+1}$.

case $q = 2$: Consider

$$\begin{aligned}
[1] &= x^2 - x = x(x-1) \\
[2] &= x^{2^2} - x = x(x^{2^2-1} - 1) = x(x-1)(x^{2^2-2} + x^{2^2-3} + \cdots + x + 1) \\
[3] &= x^{2^3} - x = x(x^{2^3-1} - 1) = x(x-1)(x^{2^3-2} + x^{2^3-3} + \cdots + x + 1) \\
&\vdots
\end{aligned}$$

so $[1] \mid [i]$ for all $i \in \mathbb{N}$, which implies that $f(x) \mid [i]$. Then we get $Q_1Q_2Q_3\cdots Q_i \mid Q_{i+1}$.

Applying Theorem 4.7 when $N = 0$, we get

$$\begin{aligned}
\frac{1}{f(x)} &= [0; f(x)] \\
\frac{1}{f(x)} + \frac{1}{f(x)d_1} &= [0; f(x), \frac{-[1]}{f(x)}, -f(x)] \\
\frac{1}{f(x)} + \frac{1}{f(x)d_1} + \frac{1}{f(x)d_2} &= [0; f(x), \frac{-[1]}{f(x)}, -f(x), \frac{-[2]d_1^{q-2}}{f(x)}, f(x), \frac{[1]}{f(x)}, -f(x)] \\
&\vdots
\end{aligned}$$

Consequently,

$$\frac{e}{f(x)} = [0; f(x), \frac{-[1]}{f(x)}, -f(x), \frac{-[2]d_1^{q-2}}{f(x)}, f(x), \frac{[1]}{f(x)}, -f(x), \frac{-[3]d_2^{q-2}}{f(x)}, \ldots].$$

This completes the proof. □

Using Corollary 4.9, we get explicit Ruban continued fraction expansions of

$\frac{e}{x}, \frac{e}{x^{q-1}-1}, \frac{e}{x-1}, \frac{e}{x^{q-2}+x^{q-3}+\cdots+1}, \frac{e}{x(x-1)}, \frac{e}{x\left(x^{q-2}+x^{q-3}+\cdots+1\right)}$ and $\frac{e}{[1]}$ for a prime power $q \geq 2$. Next, we find explicit Ruban continued fraction expansions of $\frac{e}{f(x)}$ for the remaining polynomials by treating three appropriate partitions of positive integers.

4.2.1 Partition 1

In this subsection, applying Theorem 4.7, we determine explicit Ruban continued fraction expansions of $\frac{e}{x^m}, \frac{e}{(x-1)^m}$ and $\frac{e}{(x(x-1))^m}$, for a prime power $q \geq 3$ and $m \in \mathbb{N}_{\geq 2}$.

For a prime power $q \geq 3$, let

$$
\begin{aligned}
L_1 &= 2 & R_1 &= q-1 \\
L_2 &= q & R_2 &= q^2-q-1 \\
L_3 &= q^2-q & R_3 &= q^3-q^2-q-1 \\
&\vdots & &\vdots \\
L_N &= q^{N-1}-q^{N-2}-\cdots-q^2-q & R_N &= q^N-q^{N-1}-\cdots-q-1 \quad (N \geq 3).
\end{aligned}
$$

Observe that $\mathbb{N}_{\geq 2} = \left(\cup_{N \geq 1}[L_N, R_N]\right) \cap \mathbb{N}$ and $[L_N, R_N] \cap [L_M, R_M] = \varnothing$ for all $M \neq N$.

Let m be a fixed positive integer greater than 1. Then there exists a unique N in $\mathbb{N}$ such that $m \in [L_N, R_N]$.

Corollary 4.10. *Let q be a prime power greater than 2. We have*

(1)

$$
\frac{e}{x^m} = [0; \underbrace{\underbrace{\overrightarrow{X}_{k_1}}, u_1, \underbrace{-\overleftarrow{X}_{k_1}}}, u_2, \underbrace{\overrightarrow{X}_{k_1}, -u_1, -\overleftarrow{X}_{k_1}}, u_3, \ldots],
$$

where $\overrightarrow{X}_{k_1}$ defined by $[0; \overrightarrow{X}_{k_1}] := \sum_{i=0}^{N} \frac{1}{x^m d_i}$ and

$u_\ell := \frac{(-1)^{k_\ell}[N+\ell] d_{N+\ell-1}^{q-2}}{x^m}$ *for $\ell \in \mathbb{N}$;*

(2)

$$\frac{e}{(x-1)^m} = [0; \underbrace{\underbrace{\overrightarrow{Y}_{k_1}}, v_1, \underbrace{-\overleftarrow{Y}_{k_1}}}, v_2, \underbrace{\overrightarrow{Y}_{k_1}, -v_1, -\overleftarrow{Y}_{k_1}}, v_3, \ldots],$$

where $\overrightarrow{Y}_{k_1}$ *defined by* $[0; \overrightarrow{Y}_{k_1}] := \sum_{i=0}^{N} \frac{1}{(x-1)^m d_i}$ *and*

$v_\ell := \frac{(-1)^{k_\ell}[N+\ell]d_{N+\ell-1}^{q-2}}{(x-1)^m}$ *for* $\ell \in \mathbb{N}$;

(3)

$$\frac{e}{(x(x-1))^m} = [0; \underbrace{\underbrace{\overrightarrow{Z}_{k_1}}, w_1, \underbrace{-\overleftarrow{Z}_{k_1}}}, w_2, \underbrace{\overrightarrow{Z}_{k_1}, -w_1, -\overleftarrow{Z}_{k_1}}, w_3, \ldots],$$

where $\overrightarrow{Z}_{k_1}$ *defined by* $[0; \overrightarrow{Z}_{k_1}] := \sum_{i=0}^{N} \frac{1}{(x(x-1))^m d_i}$ *and*

$w_\ell := \frac{(-1)^{k_\ell}[N+\ell]d_{N+\ell-1}^{q-2}}{(x(x-1))^m}$ *for* $\ell \in \mathbb{N}$.

Proof. (1) Let $Q_1 = x^m$ and $Q_{i+1} = \frac{d_i}{d_{i-1}}$ for $i \in \mathbb{N}$. For $i \in \mathbb{N}$, we consider

$$Q_{i+1} = \frac{d_i}{d_{i-1}} = \frac{[i]d_{i-1}^q}{d_{i-1}} = [i]d_{i-1}^{q-1} \in \mathbb{F}_q[x] \setminus \mathbb{F}_q,$$

so $Q_i \in \mathbb{F}_q[x] \setminus \mathbb{F}_q$ for all $i \geq 1$.

For $j \geq N$, we write $j = N + h$ where $h \geq 0$, so we get

$$\frac{Q_{N+h+2}}{Q_1 Q_2 \cdots Q_{N+h+1}} = \frac{\frac{d_{N+h+1}}{d_{N+h}}}{x^m \frac{d_1}{d_0}\frac{d_2}{d_1} \cdots \frac{d_{N+h}}{d_{N+h-1}}} = \frac{d_{N+h+1}/d_{N+h}}{x^m d_{N+h}} = \frac{[N+h+1]d_{N+h}^{q-2}}{x^m}.$$

First, we show that $x^m \mid [N+h+1]d_{N+h}^{q-2}$ for all $h \geq 0$.

By Remark 4.2 (2), we have

$$[N+h+1]d_{N+h}^{q-2} = [N+h+1]\left([N+h][N+h-1]^q[N+h-2]^{q^2} \cdots [1]^{q^{N+h-1}}\right)^{q-2}.$$

Since $x \mid [i]$ for all $i \in \mathbb{N}$,

$$x^{(q-2)\left(q^{N+h-1}+q^{N+h-2}+\cdots+q+1\right)+1} \mid [N+h+1]d_{N+h}^{q-2}.$$

For all $h \geq 0$, since

$$\begin{aligned}(q-2)\left(q^{N+h-1}+q^{N+h-2}+\cdots+q+1\right)+1 &\geq (q-2)\left(q^{N-1}+q^{N-2}+\cdots+q+1\right)+1\\ &= q^N - q^{N-1} - \cdots - q - 1\\ &\geq m,\end{aligned}$$

we have $x^m \mid [N+h+1]d_{N+h}^{q-2}$, which implies that Q_i satisfy (4.2).

Using Lemma 4.6, we get

$$\begin{aligned}&\gcd\left((Q_2Q_3\cdots Q_{N+1})+(Q_3Q_4\cdots Q_{N+1})+\cdots+Q_{N+1}+1, Q_1Q_2\cdots Q_{N+1}\right)\\ &=\gcd\left(\left(\frac{d_1}{d_0}\frac{d_2}{d_1}\cdots\frac{d_N}{d_{N-1}}\right)+\left(\frac{d_2}{d_1}\frac{d_3}{d_2}\cdots\frac{d_N}{d_{N-1}}\right)+\cdots+\frac{d_N}{d_{N-1}}+1, x^m\frac{d_1}{d_0}\frac{d_2}{d_1}\cdots\frac{d_N}{d_{N-1}}\right)\\ &=\gcd\left(d_N+\frac{d_N}{d_1}+\frac{d_N}{d_2}+\cdots+\frac{d_N}{d_{N-1}}+1, x^m d_N\right)=1.\end{aligned}$$

For $\ell \geq 1$, consider

$$\frac{(-1)^{k_\ell}Q_{N+\ell+1}}{Q_1Q_2\cdots Q_{N+\ell}}=\frac{(-1)^{k_\ell}[N+\ell]d_{N+\ell-1}^{q-2}}{x^m}=u_\ell.$$

Applying Theorem 4.7, we get

$$\begin{aligned}\sum_{i=1}^{N+1}\frac{1}{Q_1Q_2\cdots Q_i}&=\sum_{i=0}^{N}\frac{1}{x^m d_i} &&=[0;\overrightarrow{X}_{k_1}]\\ \sum_{i=1}^{N+2}\frac{1}{Q_1Q_2\cdots Q_i}&=\sum_{i=0}^{N+1}\frac{1}{x^m d_i} &&=[0;\overrightarrow{X}_{k_1},\frac{(-1)^{k_1}[N+1]d_N^{q-2}}{x^m},-\overleftarrow{X}_{k_1}]\\ &&&\vdots\end{aligned}$$

Consequently,

$$\frac{e}{x^m}=[0;\overrightarrow{X}_{k_1},u_1,-\overleftarrow{X}_{k_1},u_2,\overrightarrow{X}_{k_1},-u_1,-\overleftarrow{X}_{k_1},u_3,\ldots].$$

The proofs of (2) and (3) are done by similar arguments but setting $Q_1=(x-1)^m$ and $Q_1=(x(x-1))^m$, respectively. $\square$

4.2.2 Partition 2

In this subsection, we determine explicit Ruban continued fraction expansions of $\frac{e}{(x^{q-1}-1)^m}$, $\frac{e}{(x^{q-2}+x^{q-3}+\cdots+1)^m}$, $\frac{e}{(x(x^{q-2}+x^{q-3}+\cdots+1))^m}$ and $\frac{e}{[1]^m}$, for a prime power $q \geq 3$ and $m \in \mathbb{N}_{\geq 2}$, by applying Theorem 4.7.

For a prime power $q \geq 3$, let

$$\begin{aligned}
\mathcal{L}_1 &= 1 & \mathcal{R}_1 &= q-2\\
\mathcal{L}_2 &= q-1 & \mathcal{R}_2 &= q^2-2q\\
\mathcal{L}_3 &= q^2-2q+1 & \mathcal{R}_3 &= q^3-2q^2\\
&\vdots & &\vdots\\
\mathcal{L}_N &= q^{N-1}-2q^{N-2}+1 & \mathcal{R}_N &= q^N-2q^{N-1} \quad (N \geq 3).
\end{aligned}$$

Observe that $\mathbb{N} = (\cup_{N\geq 1}[\mathcal{L}_N, \mathcal{R}_N]) \cap \mathbb{N}$ and $[\mathcal{L}_N, \mathcal{R}_N] \cap [\mathcal{L}_M, \mathcal{R}_M] = \varnothing$ for all $M \neq N$.

Let m be a fixed positive integer greater than 1. Then there exists a unique N in $\mathbb{N}$ such that $m \in [\mathcal{L}_N, \mathcal{R}_N]$.

Corollary 4.11. *Let q be a prime power greater than 2. We have*

(1)

$$\frac{e}{(x^{q-1}-1)^m} = [0; \underbrace{\underbrace{\overrightarrow{W}_{k_1}}, u_1, \underbrace{-\overleftarrow{W}_{k_1}}}, u_2, \underbrace{\overrightarrow{W}_{k_1}, -u_1, -\overleftarrow{W}_{k_1}}, u_3, \ldots],$$

where $\overrightarrow{W}_{k_1}$ *defined by* $[0; \overrightarrow{W}_{k_1}] := \sum_{i=0}^{N} \frac{1}{(x^{q-1}-1)^m d_i}$ *and*

$u_\ell := \frac{(-1)^{k_\ell}[N+\ell]d_{N+\ell-1}^{q-2}}{(x^{q-1}-1)^m}$ *for* $\ell \in \mathbb{N}$;

(2)

$$\frac{e}{(x^{q-2}+x^{q-3}+\cdots+1)^m} = [0; \underbrace{\underbrace{\overrightarrow{X}_{k_1}}, v_1, \underbrace{-\overleftarrow{X}_{k_1}}}, v_2, \underbrace{\overrightarrow{X}_{k_1}, -v_1, -\overleftarrow{X}_{k_1}}, v_3, \ldots],$$

where $\overrightarrow{X}_{k_1}$ *defined by* $[0; \overrightarrow{X}_{k_1}] := \sum_{i=0}^{N} \frac{1}{(x^{q-2}+x^{q-3}+\cdots+1)^m d_i}$ *and*

$v_\ell := \frac{(-1)^{k_\ell}[N+\ell]d_{N+\ell-1}^{q-2}}{(x^{q-2}+x^{q-3}+\cdots+1)^m}$ *for* $\ell \in \mathbb{N}$;

(3)

$$\frac{e}{(x\,(x^{q-2}+x^{q-3}+\cdots+1))^m} = [0; \underbrace{\underbrace{\overrightarrow{Y}_{k_1}}, w_1, \underbrace{-\overleftarrow{Y}_{k_1}}}, w_2, \underbrace{\overrightarrow{Y}_{k_1}, -w_1, -\overleftarrow{Y}_{k_1}}, w_3, \ldots],$$

where $\overrightarrow{Y}_{k_1}$ *defined by* $[0; \overrightarrow{Y}_{k_1}] := \sum_{i=0}^{N} \frac{1}{(x\,(x^{q-2}+x^{q-3}+\cdots+1))^m\, d_i}$ *and*
$w_\ell := \frac{(-1)^{k_\ell}[N+\ell]d_{N+\ell-1}^{q-2}}{(x\,(x^{q-2}+x^{q-3}+\cdots+1))^m}$ *for* $\ell \in \mathbb{N}$;

(4)

$$\frac{e}{[1]^m} = [0; \underbrace{\underbrace{\overrightarrow{Z}_{k_1}}, y_1, \underbrace{-\overleftarrow{Z}_{k_1}}}, y_2, \underbrace{\overrightarrow{Z}_{k_1}, -y_1, -\overleftarrow{Z}_{k_1}}, y_3, \ldots],$$

where $\overrightarrow{Z}_{k_1}$ *defined by* $[0; \overrightarrow{Z}_{k_1}] := \sum_{i=0}^{N} \frac{1}{[1]^m d_i}$ *and*
$y_\ell := \frac{(-1)^{k_\ell}[N+\ell]d_{N+\ell-1}^{q-2}}{[1]^m}$ *for* $\ell \in \mathbb{N}$.

Proof. (1) Let $Q_1 = (x^{q-1}-1)^m$ and $Q_{i+1} = \frac{d_i}{d_{i-1}}$ for $i \in \mathbb{N}$. For $i \in \mathbb{N}$, we consider

$$Q_{i+1} = \frac{d_i}{d_{i-1}} = \frac{[i]d_{i-1}^q}{d_{i-1}} = [i]d_{i-1}^{q-1} \in \mathbb{F}_q[x] \setminus \mathbb{F}_q,$$

so $Q_i \in \mathbb{F}_q[x] \setminus \mathbb{F}_q$ for all $i \geq 1$.

For $j \geq N$, we write $j = N+h$ where $h \geq 0$, so we get

$$\begin{aligned}\frac{Q_{N+h+2}}{Q_1 Q_2 \cdots Q_{N+h+1}} &= \frac{\dfrac{d_{N+h+1}}{d_{N+h}}}{(x^{q-1}-1)^m \dfrac{d_1}{d_0}\dfrac{d_2}{d_1}\cdots\dfrac{d_{N+h}}{d_{N+h-1}}} \\ &= \frac{d_{N+h+1}/d_{N+h}}{(x^{q-1}-1)^m\, d_{N+h}} \\ &= \frac{[N+h+1]d_{N+h}^{q-2}}{(x^{q-1}-1)^m}.\end{aligned}$$

First, we show that $(x^{q-1}-1)^m \mid [N+h+1]d_{N+h}^{q-2}$ for all $h \geq 0$.
By Remark 4.2 (2), we have

$$[N+h+1]d_{N+h}^{q-2} = [N+h+1]\left([N+h][N+h-1]^q[N+h-2]^{q^2}\cdots[1]^{q^{N+h-1}}\right)^{q-2}.$$

Since $(x^{q-1}-1) \mid [1]$,

$$\left(x^{q-1}-1\right)^{(q-2)q^{N+h-1}} \mid [N+h+1]d_{N+h}^{q-2}.$$

For all $h \geq 0$, since

$$(q-2)q^{N+h-1} \geq (q-2)q^{N-1} = q^N - 2q^{N-1} \geq m,$$

then $(x^{q-1}-1)^m \mid [N+h+1]d_{N+h}^{q-2}$, which implies that Q_i satisfy (4.2).

Using Lemma 4.6, we get

$$\begin{aligned}
&\gcd\left((Q_2Q_3\cdots Q_{N+1}) + (Q_3Q_4\cdots Q_{N+1}) + \cdots + Q_{N+1} + 1, Q_1Q_2\cdots Q_{N+1}\right)\\
&= \gcd\left(\left(\frac{d_1}{d_0}\frac{d_2}{d_1}\cdots\frac{d_N}{d_{N-1}}\right) + \cdots + \frac{d_N}{d_{N-1}} + 1, \left(x^{q-1}-1\right)^m \frac{d_1}{d_0}\frac{d_2}{d_1}\cdots\frac{d_N}{d_{N-1}}\right)\\
&= \gcd\left(d_N + \frac{d_N}{d_1} + \frac{d_N}{d_2} + \cdots + \frac{d_N}{d_{N-1}} + 1, \left(x^{q-1}-1\right)^m d_N\right) = 1.
\end{aligned}$$

For $\ell \geq 1$, consider

$$\frac{(-1)^{k_\ell} Q_{N+\ell+1}}{Q_1Q_2\cdots Q_{N+\ell}} = \frac{(-1)^{k_\ell}[N+\ell]d_{N+\ell-1}^{q-2}}{\left(x^{q-1}-1\right)^m} = u_\ell.$$

Applying Theorem 4.7, we get

$$\begin{aligned}
\sum_{i=1}^{N+1}\frac{1}{Q_1Q_2\cdots Q_i} &= \sum_{i=0}^{N}\frac{1}{\left(x^{q-1}-1\right)^m d_i} &&=[0;\overrightarrow{W}_{k_1}]\\
\sum_{i=1}^{N+2}\frac{1}{Q_1Q_2\cdots Q_i} &= \sum_{i=0}^{N+1}\frac{1}{\left(x^{q-1}-1\right)^m d_i} &&=[0;\overrightarrow{W}_{k_1}, \frac{(-1)^{k_1}[N+1]d_N^{q-2}}{\left(x^{q-1}-1\right)^m}, -\overleftarrow{W}_{k_1}]\\
&&&\vdots
\end{aligned}$$

Finally, we have

$$\frac{e}{\left(x^{q-1}-1\right)^m} = [0;\overrightarrow{W}_{k_1}, u_1, -\overleftarrow{W}_{k_1}, u_2, \overrightarrow{W}_{k_1}, -u_1, -\overleftarrow{W}_{k_1}, u_3, \ldots].$$

The proofs of (2), (3) and (4) follow via similar arguments but setting $Q_1 =$

$(x^{q-2}+x^{q-3}+\cdots+1)^m$, $Q_1=(x(x^{q-2}+x^{q-3}+\cdots+1))^m$ and $Q_1=[1]^m$, respectively. □

4.2.3 Partition 3

In this subsection, explicit Ruban continued fraction expansions of $\frac{e}{(x+1)^m}$ and $\frac{e}{(x(x+1))^m}$ for $q=2$ and $m\in\mathbb{N}_{\geq 2}$, are determined. The proof of Theorem 4.12 and 4.13 are extended to become Proposition 4.4.

Let

$$\begin{aligned}
\mathbf{L}_1 &= 2 & \mathbf{R}_1 &= 2\\
\mathbf{L}_2 &= 2+1 & \mathbf{R}_2 &= 2^2\\
\mathbf{L}_3 &= 2^2+1 & \mathbf{R}_3 &= 2^3\\
&\vdots & &\vdots\\
\mathbf{L}_N &= 2^{N-1}+1 & \mathbf{R}_N &= 2^N \quad (N\geq 2).
\end{aligned}$$

Observe that $\mathbb{N}_{\geq 2}=(\cup_{N\geq 1}[\mathbf{L}_N,\mathbf{R}_N])\cap\mathbb{N}$ and $[\mathbf{L}_N,\mathbf{R}_N]\cap[\mathbf{L}_M,\mathbf{R}_M]=\varnothing$ for all $M\neq N$.

Let m be a fixed positive integer greater than 1. Then there exists a unique N in $\mathbb{N}$ such that $m\in[\mathbf{L}_N,\mathbf{R}_N]$.

Theorem 4.12. *Let $q=2$. If $\sum_{i=0}^{(N-1)+\ell}\frac{1}{(x+1)^m d_i}=:[0;\overrightarrow{X}_{k_\ell}]$ for $\ell\geq 1$, then*

$$\sum_{i=0}^{(N-1)+\ell+1}\frac{1}{(x+1)^m d_i}=[0;\overrightarrow{X}_{k_\ell},\frac{[N+\ell]}{(x+1)^m}+\frac{1}{(x+1)^{m-1}},\overrightarrow{X}_{k_\ell}].$$

In particular,

$$\frac{e}{(x+1)^m}=[0;\underbrace{\underbrace{\overrightarrow{X}_{k_1}},\frac{[N+1]}{(x+1)^m}+\frac{1}{(x+1)^{m-1}},\underbrace{\overrightarrow{X}_{k_1}}},\frac{[N+2]}{(x+1)^m}+\frac{1}{(x+1)^{m-1}},\ldots].$$

Proof. For $\ell\geq 1$, let $\frac{C_{k_\ell}}{D_{k_\ell}}:=[0;\overrightarrow{X}_{k_\ell}]$ be the k_ℓ^{th} convergent of the continued fraction expansion of $\sum_{i=0}^{(N-1)+\ell} \dfrac{1}{(x+1)^m d_i}$. Consider

$$\begin{aligned}\sum_{i=0}^{(N-1)+\ell} \frac{1}{(x+1)^m d_i} &= \frac{1}{(x+1)^m} + \frac{1}{(x+1)^m d_1} + \frac{1}{(x+1)^m d_2} + \cdots + \frac{1}{(x+1)^m d_{(N-1)+\ell}} \\ &= \frac{d_{(N-1)+\ell} + \frac{d_{(N-1)+\ell}}{d_1} + \frac{d_{(N-1)+\ell}}{d_2} + \cdots + \frac{d_{(N-1)+\ell}}{d_{(N-1)+\ell-1}} + 1}{(x+1)^m d_{(N-1)+\ell}}.\end{aligned}$$

Using Lemma 4.6 and since C_{k_ℓ} and D_{k_ℓ} are relatively prime, $d_{(N-1)+\ell} + \dfrac{d_{(N-1)+\ell}}{d_1} + \dfrac{d_{(N-1)+\ell}}{d_2} + \cdots + \dfrac{d_{(N-1)+\ell}}{d_{(N-1)+\ell-1}} + 1$ and $(x+1)^m d_{(N-1)+\ell}$ are monic polynomials over $\mathbb{F}_2$, we get

$$\begin{aligned} C_{k_\ell} &= d_{(N-1)+\ell} + \frac{d_{(N-1)+\ell}}{d_1} + \frac{d_{(N-1)+\ell}}{d_2} + \cdots + \frac{d_{(N-1)+\ell}}{d_{(N-1)+\ell-1}} + 1 \\ &= d_{(N-1)+\ell}\left(1 + \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_{(N-1)+\ell-1}} + \frac{1}{d_{(N-1)+\ell}}\right) \quad \text{and} \\ D_{k_\ell} &= (x+1)^m d_{(N-1)+\ell}. \end{aligned}$$

We now claim that $D_{k_\ell - 1} = (x+1)d_{(N-1)+\ell} + C_{k_\ell}$ for all $\ell \geq 1$.
For all $\ell \geq 1$, let $Q = (x+1)d_{(N-1)+\ell} + C_{k_\ell}$ and $P = \dfrac{1 + C_{k_\ell}Q}{D_{k_\ell}}$. Then

$$PD_{k_\ell} - QC_{k_\ell} = \left(\frac{1 + C_{k_\ell}Q}{D_{k_\ell}}\right) D_{k_\ell} - QC_{k_\ell} = 1.$$

We first show that $P \in \mathbb{F}_2[x]$. Note that, from Remark 4.2 (2) and since $(x+1) \mid [i]$ for all $i \in \mathbb{N}$, we have

$$(x+1)^{2^i - 1} \mid d_i \quad \text{for all} \quad i \in \mathbb{N}. \tag{4.4}$$

Now we consider

$$\begin{aligned} P &= \frac{1 + C_{k_\ell}Q}{D_{k_\ell}} \\ &= \frac{1 + C_{k_\ell}\left((x+1)d_{(N-1)+\ell} + C_{k_\ell}\right)}{D_{k_\ell}} \end{aligned}$$

$$
\begin{aligned}
&= \Bigg\{1 + (x+1)d_{(N-1)+\ell}^2\left(1 + \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_{(N-1)+\ell-1}} + \frac{1}{d_{(N-1)+\ell}}\right) \\
&\quad + d_{(N-1)+\ell}^2\left(1 + \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_{(N-1)+\ell-1}} + \frac{1}{d_{(N-1)+\ell}}\right)^2\Bigg\} / (x+1)^m d_{(N-1)+\ell} \\
&= \Bigg\{1 + (x+1)d_{(N-1)+\ell}^2\left(1 + \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_{(N-1)+\ell-1}} + \frac{1}{d_{(N-1)+\ell}}\right) \\
&\quad + d_{(N-1)+\ell}^2\left(1 + \frac{1}{d_1^2} + \frac{1}{d_2^2} + \cdots + \frac{1}{d_{(N-1)+\ell-1}^2}\right) + 1\Bigg\} / (x+1)^m d_{(N-1)+\ell} \\
&= \Bigg\{(x+1)d_{(N-1)+\ell}\left(1 + \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_{(N-1)+\ell-1}} + \frac{1}{d_{(N-1)+\ell}}\right) \\
&\quad + d_{(N-1)+\ell}\left(1 + \frac{1}{d_1^2} + \frac{1}{d_2^2} + \cdots + \frac{1}{d_{(N-1)+\ell-1}^2}\right)\Bigg\} / (x+1)^m \\
&= \Bigg\{(x+1)d_{(N-1)+\ell}\left(1 + \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_{(N-1)+\ell-1}} + \frac{1}{d_{(N-1)+\ell}}\right) \\
&\quad + \left(d_{(N-1)+\ell} + \frac{d_{(N-1)+\ell}}{d_1^2} + \frac{d_{(N-1)+\ell}}{d_2^2} + \cdots + \frac{d_{(N-1)+\ell}}{d_{(N-1)+\ell-1}^2}\right)\Bigg\} / (x+1)^m.
\end{aligned} \tag{4.5}
$$

By (4.4), it follows that for all $\ell \geq 1$,

$$
(x+1)d_{(N-1)+\ell} \equiv 0 \mod (x+1)^{2^{(N-1)+\ell}}
$$

and

$$
\begin{aligned}
\frac{(x+1)d_{(N-1)+\ell}}{d_j} + \frac{d_{(N-1)+\ell}}{d_{j-1}^2} &= \frac{d_{(N-1)+\ell}}{d_j}\left((x+1) + [j]\right) \\
&= \frac{d_{(N-1)+\ell}}{d_j}\left((x+1) + (x^{2^j} + x)\right) \\
&= \frac{d_{(N-1)+\ell}}{d_j}\left(x^{2^j} + 1\right) \\
&= \frac{d_{(N-1)+\ell}}{d_j}(x+1)^{2^j} \\
&\equiv 0 \mod (x+1)^{2^{(N-1)+\ell}}
\end{aligned}
$$

for all $j \in \{1, 2, \ldots, (N-1)+\ell\}$.

Since $m \leq 2^N \leq 2^{(N-1)+\ell}$ for all $\ell \in \mathbb{N}$, we get $P \in \mathbb{F}_2[x]$.

Now $PD_{k_\ell} - QC_{k_\ell} = 1$ and we have $C_{k_\ell - 1}D_{k_\ell} - D_{k_\ell - 1}C_{k_\ell} = 1$. Then

$$PD_{k_\ell} - QC_{k_\ell} = C_{k_\ell-1}D_{k_\ell} - D_{k_\ell-1}C_{k_\ell},$$
$$C_{k_\ell}\left(D_{k_\ell-1} - Q\right) = D_{k_\ell}\left(C_{k_\ell-1} - P\right).$$

We know C_{k_ℓ} and D_{k_ℓ} are relatively prime and by (4.5)

$$\deg P = \deg d_{(N-1)+\ell} + 1 - m \leq \deg d_{(N-1)+\ell} - 1 < \deg d_{(N-1)+\ell} = \deg C_{k_\ell}.$$

By definition, the degree of $C_{k_\ell-1} - P$ is less than that of C_{k_ℓ}. Thus

$$C_{k_\ell-1} = P \text{ so } D_{k_\ell-1} = Q = (x+1)d_{(N-1)+\ell} + C_{k_\ell}$$

and the claim is proved.

Next, we show that $\frac{[N+\ell]}{(x+1)^m} + \frac{1}{(x+1)^{m-1}} \in \mathbb{F}_2[x] \setminus \{0\}$ for all $\ell \geq 1$. Consider

$$\begin{aligned}\frac{[N+\ell]}{(x+1)^m} + \frac{1}{(x+1)^{m-1}} &= \frac{[N+\ell] + (x+1)}{(x+1)^m}\\ &= \frac{\left(x^{2^{N+\ell}} + x\right) + (x+1)}{(x+1)^m}\\ &= \frac{x^{2^{N+\ell}} + 1}{(x+1)^m}\\ &= \frac{(x+1)^{2^{N+\ell}}}{(x+1)^m},\end{aligned}$$

since $2^{N+\ell} \geq 2^{N+1} > m$ for all $\ell \geq 1$, which implies that $\frac{[N+\ell]}{(x+1)^m} + \frac{1}{(x+1)^{m-1}} \in \mathbb{F}_2[x] \setminus \{0\}$ for all $\ell \geq 1$.

Applying Lemma 4.5 (1), we get

$$\begin{aligned}&[0; \overrightarrow{X}_{k_\ell}, \frac{[N+\ell]}{(x+1)^m} + \frac{1}{(x+1)^{m-1}}, \overrightarrow{X}_{k_\ell}]\\ &= \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{1}{D_{k_\ell}^2\left(\left(\frac{[N+\ell]}{(x+1)^m} + \frac{1}{(x+1)^{m-1}}\right) + \left(\frac{C_{k_\ell} + D_{k_\ell-1}}{D_{k_\ell}}\right)\right)}\end{aligned}$$

$$
\begin{aligned}
&= \frac{C_{k_\ell}}{D_{k_\ell}} + \cfrac{1}{D_{k_\ell}^2\left(\frac{[N+\ell]}{(x+1)^m} + \frac{1}{(x+1)^{m-1}} + \frac{C_{k_\ell}+(x+1)d_{(N-1)+\ell}+C_{k_\ell}}{D_{k_\ell}}\right)} \\
&= \frac{C_{k_\ell}}{D_{k_\ell}} + \cfrac{1}{\left((x+1)^m d_{(N-1)+\ell}\right)^2\left(\frac{[N+\ell]}{(x+1)^m} + \frac{1}{(x+1)^{m-1}} + \frac{(x+1)d_{(N-1)+\ell}}{(x+1)^m d_{(N-1)+\ell}}\right)} \\
&= \frac{C_{k_\ell}}{D_{k_\ell}} + \cfrac{1}{\left((x+1)^m d_{(N-1)+\ell}\right)^2\left(\frac{[N+\ell]}{(x+1)^m} + \frac{1}{(x+1)^{m-1}} + \frac{1}{(x+1)^{m-1}}\right)} \\
&= \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{1}{(x+1)^m d^2_{(N-1)+\ell}[N+\ell]} \\
&= \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{1}{(x+1)^m d^2_{(N-1)+\ell}[(N-1)+(\ell+1)]} \\
&= \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{1}{(x+1)^m d_{(N-1)+(\ell+1)}} \\
&= \sum_{i=0}^{(N-1)+\ell} \frac{1}{(x+1)^m d_i} + \frac{1}{(x+1)^m d_{(N-1)+(\ell+1)}} \\
&= \sum_{i=0}^{(N-1)+\ell+1} \frac{1}{(x+1)^m d_i}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\sum_{i=0}^{(N-1)+1} \frac{1}{(x+1)^m d_i} &= [0; \overrightarrow{X}_{k_\ell}] \\
\sum_{i=0}^{(N-1)+2} \frac{1}{(x+1)^m d_i} &= [0; \overrightarrow{X}_{k_1}, \frac{[N+1]}{(x+1)^m} + \frac{1}{(x+1)^{m-1}}, \overrightarrow{X}_{k_1}] \\
&\vdots
\end{aligned}
$$

Consequently,

$$
\frac{e}{(x+1)^m} = [0; \overrightarrow{X}_{k_1}, \frac{[N+1]}{(x+1)^m} + \frac{1}{(x+1)^{m-1}}, \overrightarrow{X}_{k_1}, \frac{[N+2]}{(x+1)^m} + \frac{1}{(x+1)^{m-1}}, \ldots].
$$

This completes the proof. □

Theorem 4.13. *Let $q=2$. If $\sum_{i=0}^{(N-1)+\ell} \frac{1}{(x(x+1))^m d_i} =: [0; \overrightarrow{X}_{k_\ell}]$ for $\ell \geq 1$, then*

$$\sum_{i=0}^{(N-1)+\ell+1} \frac{1}{(x(x+1))^m d_i} = [0; \overrightarrow{X}_{k_\ell}, \frac{[N+\ell]+[N]}{(x(x+1))^m}, \overleftarrow{X}_{k_\ell}].$$

In particular,

$$\frac{e}{(x(x+1))^m} = [0; \underbrace{\underbrace{\overrightarrow{X}_{k_1}}, \frac{[N+1]+[N]}{(x(x+1))^m}, \underbrace{\overleftarrow{X}_{k_1}}}, \frac{[N+2]+[N]}{(x(x+1)))^m}, \ldots].$$

Proof. For $\ell \geq 1$, let $\frac{C_{k_\ell}}{D_{k_\ell}} := [0; \overrightarrow{X}_{k_\ell}]$ be the k_ℓ^{th} convergent of the continued fraction expansion of $\sum_{i=0}^{(N-1)+\ell} \frac{1}{(x(x+1))^m d_i}$. Consider

$$\begin{aligned}
\sum_{i=0}^{(N-1)+\ell} \frac{1}{(x(x+1))^m d_i} &= \frac{1}{(x(x+1))^m} + \frac{1}{(x(x+1))^m d_1} + \cdots + \frac{1}{(x(x+1))^m d_{(N-1)+\ell}} \\
&= \frac{d_{(N-1)+\ell} + \frac{d_{(N-1)+\ell}}{d_1} + \frac{d_{(N-1)+\ell}}{d_2} + \cdots + \frac{d_{(N-1)+\ell}}{d_{(N-1)+\ell-1}} + 1}{(x(x+1))^m d_{(N-1)+\ell}}.
\end{aligned}$$

Using Lemma 4.6 and since C_{k_ℓ} and D_{k_ℓ} are relatively prime, $d_{(N-1)+\ell} + \frac{d_{(N-1)+\ell}}{d_1} + \frac{d_{(N-1)+\ell}}{d_2} + \cdots + \frac{d_{(N-1)+\ell}}{d_{(N-1)+\ell-1}} + 1$ and $(x(x+1))^m d_{(N-1)+\ell}$ are monic polynomials over $\mathbb{F}_2$, we get

$$\begin{aligned}
C_{k_\ell} &= d_{(N-1)+\ell} + \frac{d_{(N-1)+\ell}}{d_1} + \frac{d_{(N-1)+\ell}}{d_2} + \cdots + \frac{d_{(N-1)+\ell}}{d_{(N-1)+\ell-1}} + 1 \\
&= d_{(N-1)+\ell}\left(1 + \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_{(N-1)+\ell-1}} + \frac{1}{d_{(N-1)+\ell}}\right) \quad \text{and} \\
D_{k_\ell} &= (x(x+1))^m d_{(N-1)+\ell}.
\end{aligned}$$

We now claim that $D_{k_\ell - 1} = [N]d_{(N-1)+\ell} + C_{k_\ell}$ for all $\ell \geq 1$.
For all $\ell \geq 1$, let $Q = [N]d_{(N-1)+\ell} + C_{k_\ell}$ and $P = \frac{1 + C_{k_\ell}Q}{D_{k_\ell}}$, so we get

$$PD_{k_\ell} - QC_{k_\ell} = \left(\frac{1 + C_{k_\ell}Q}{D_{k_\ell}}\right) D_{k_\ell} - QC_{k_\ell} = 1.$$

We first show that $P \in \mathbb{F}_2[x]$. Note that, for $q = 2$.

- We have

$$x(x+1) \mid [i] \quad \text{for all} \quad i \in \mathbb{N}. \tag{4.6}$$

- From Remark 4.2 (2) and since $x(x+1) \mid [i]$ for all $i \in \mathbb{N}$, we have

$$(x(x+1))^{2^i - 1} \mid d_i \quad \text{for all} \quad i \in \mathbb{N}. \tag{4.7}$$

Now we consider

$$\begin{aligned}
P &= \frac{1 + C_{k_\ell} Q}{D_{k_\ell}} \\
&= \frac{1 + C_{k_\ell}\left([N] d_{(N-1)+\ell} + C_{k_\ell}\right)}{D_{k_\ell}} \\
&= \left\{ 1 + [N] d^2_{(N-1)+\ell} \left(1 + \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_{(N-1)+\ell-1}} + \frac{1}{d_{(N-1)+\ell}}\right) \right. \\
&\quad \left. + d^2_{(N-1)+\ell} \left(1 + \frac{1}{d_1} + \cdots + \frac{1}{d_{(N-1)+\ell-1}} + \frac{1}{d_{(N-1)+\ell}}\right)^2 \right\} / \left(x(x+1)\right)^m d_{(N-1)+\ell} \\
&= \left\{ 1 + [N] d^2_{(N-1)+\ell} \left(1 + \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_{(N-1)+\ell-1}} + \frac{1}{d_{(N-1)+\ell}}\right) \right. \\
&\quad \left. + d^2_{(N-1)+\ell} \left(1 + \frac{1}{d_1^2} + \frac{1}{d_2^2} + \cdots + \frac{1}{d^2_{(N-1)+\ell-1}}\right) + 1 \right\} / \left(x(x+1)\right)^m d_{(N-1)+\ell} \\
&= \left\{ [N] d_{(N-1)+\ell} \left(1 + \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_{(N-1)+\ell-1}} + \frac{1}{d_{(N-1)+\ell}}\right) \right. \\
&\quad \left. + d_{(N-1)+\ell} \left(1 + \frac{1}{d_1^2} + \frac{1}{d_2^2} + \cdots + \frac{1}{d^2_{(N-1)+\ell-1}}\right) \right\} / \left(x(x+1)\right)^m \\
&= \left\{ [N] d_{(N-1)+\ell} \left(1 + \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_{(N-1)+\ell-1}} + \frac{1}{d_{(N-1)+\ell}}\right) \right. \\
&\quad \left. + \left(d_{(N-1)+\ell} + \frac{d_{(N-1)+\ell}}{d_1^2} + \frac{d_{(N-1)+\ell}}{d_2^2} + \cdots + \frac{d_{(N-1)+\ell}}{d^2_{(N-1)+\ell-1}}\right) \right\} / \left(x(x+1)\right)^m .
\end{aligned} \tag{4.8}$$

For a fixed $\ell \geq 1$ and $j \in \{1, 2, \ldots, (N-1)+\ell\}$, we get

$$x^{2^N} + x^{2^j} \equiv 0 \mod \left(x(x+1)\right)^{2^{\min\{N,j\}}}$$

and

$$2^{(N-1)+\ell} - 2^j + 2^{\min\{N,j\}} = \begin{cases} 2^{(N-1)+\ell} & \text{if } \min\{N,j\} = j \\ 2^{(N-1)+\ell} - 2^j + 2^N & \text{if } \min\{N,j\} = N \end{cases}$$
$$\geq 2^N. \tag{4.9}$$

By (4.6), (4.7) and (4.9), it follows that for all $\ell \geq 1$,

$$[N]d_{(N-1)+\ell} \equiv 0 \mod (x(x+1))^{2^{(N-1)+\ell}},$$

and

$$\begin{aligned} \frac{[N]d_{(N-1)+\ell}}{d_j} + \frac{d_{(N-1)+\ell}}{d_{j-1}^2} &= \frac{d_{(N-1)+\ell}}{d_j}([N]+[j]) \\ &= \frac{d_{(N-1)+\ell}}{d_j}\left(\left(x^{2^N}+x\right)+(x^{2^j}+x)\right) \\ &= \frac{d_{(N-1)+\ell}}{d_j}\left(x^{2^N}+x^{2^j}\right) \\ &\equiv 0 \mod (x(x+1))^{2^N} \end{aligned}$$

for all $j \in \{1, 2, \ldots, (N-1)+\ell\}$. Hence we see that $P \in \mathbb{F}_2[x]$.

Now $PD_{k_\ell} - QC_{k_\ell} = 1$ and we have $C_{k_\ell - 1}D_{k_\ell} - D_{k_\ell - 1}C_{k_\ell} = 1$, so

$$\begin{aligned} PD_{k_\ell} - QC_{k_\ell} &= C_{k_\ell-1}D_{k_\ell} - D_{k_\ell-1}C_{k_\ell}, \\ C_{k_\ell}\left(D_{k_\ell-1} - Q\right) &= D_{k_\ell}\left(C_{k_\ell-1} - P\right). \end{aligned}$$

We know C_{k_ℓ} and D_{k_ℓ} are relatively prime and by (4.8)

$$\deg P = \deg d_{(N-1)+\ell} + 2^N - 2m \leq \deg d_{(N-1)+\ell} - 2 < \deg d_{(N-1)+\ell} = \deg C_{k_\ell}.$$

By definition, the degree of $C_{k_\ell-1} - P$ is less than that of C_{k_ℓ}. Thus

$$C_{k_\ell-1} = P \text{ so } D_{k_\ell-1} = Q = [N]d_{(N-1)+\ell} + C_{k_\ell}$$

and the claim is proved.

Next, we show that $\dfrac{[N+\ell]+[N]}{(x(x+1))^m} \in \mathbb{F}_2[x] \setminus \{0\}$ for all $\ell \geq 1$. Consider

$$\begin{aligned}\frac{[N+\ell]+[N]}{(x(x+1))^m} &= \frac{\left(x^{2^{N+\ell}}+x\right)+\left(x^{2^N}+x\right)}{(x(x+1))^m}\\ &= \frac{x^{2^{N+\ell}}+x^{2^N}}{(x(x+1))^m}\\ &= \frac{x^{2^N}\left(x^{2^{N+\ell}-2^N}+1\right)}{(x(x+1))^m}\\ &= \frac{x^{2^N}\left(x^{2^N(2^\ell-1)}+1\right)}{(x(x+1))^m}\\ &= \frac{x^{2^N}(x+1)^{2^N(2^\ell-1)}}{(x(x+1))^m}.\end{aligned}$$

Since $2^\ell - 1 \geq 1$ for all $\ell \geq 1$ and $2^N \geq m$, then $\dfrac{[N+\ell]+[N]}{(x(x+1))^m} \in \mathbb{F}_2[x] \setminus \{0\}$ for all $\ell \geq 1$.

Applying Lemma 4.5 (1), we get

$$\begin{aligned}&[0;\vec{X}_{k_\ell}, \frac{[N+\ell]+[N]}{(x(x+1))^m}, \vec{X}_{k_\ell}]\\ &= \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{1}{D_{k_\ell}^2\left(\left(\dfrac{[N+\ell]+[N]}{(x(x+1))^m}\right)+\left(\dfrac{C_{k_\ell}+D_{k_\ell-1}}{D_{k_\ell}}\right)\right)}\\ &= \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{1}{D_{k_\ell}^2\left(\dfrac{[N+\ell]+[N]}{(x(x+1))^m}+\dfrac{C_{k_\ell}+[N]d_{(N-1)+\ell}+C_{k_\ell}}{D_{k_\ell}}\right)}\\ &= \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{1}{\left((x(x+1))^m d_{(N-1)+\ell}\right)^2\left(\dfrac{[N+\ell]+[N]}{(x(x+1))^m}+\dfrac{[N]d_{(N-1)+\ell}}{\left((x(x+1))^m d_{(N-1)+\ell}\right)}\right)}\\ &= \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{1}{\left((x(x+1))^m d_{(N-1)+\ell}\right)^2\left(\dfrac{[N+\ell]}{(x(x+1))^m}+\dfrac{[N]}{(x(x+1))^m}+\dfrac{[N]}{(x(x+1))^m}\right)}\\ &= \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{1}{(x(x+1))^m d_{(N-1)+\ell}^2[N+\ell]}\\ &= \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{1}{(x(x+1))^m d_{(N-1)+\ell}^2[(N-1)+(\ell+1)]}\end{aligned}$$

$$
\begin{aligned}
&= \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{1}{(x(x+1))^m d_{(N-1)+(\ell+1)}} \\
&= \sum_{i=0}^{(N-1)+\ell} \frac{1}{(x(x+1))^m d_i} + \frac{1}{(x(x+1))^m d_{(N-1)+(\ell+1)}} \\
&= \sum_{i=0}^{(N-1)+\ell+1} \frac{1}{(x(x+1))^m d_i}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\sum_{i=0}^{(N-1)+1} \frac{1}{(x(x+1))^m d_i} &= [0; \overrightarrow{X}_{k_\ell}] \\
\sum_{i=0}^{(N-1)+2} \frac{1}{(x(x+1))^m d_i} &= [0; \overrightarrow{X}_{k_1}, \frac{[N+1]+[N]}{(x(x+1))^m}, \overrightarrow{X}_{k_1}] \\
&\vdots
\end{aligned}
$$

Consequently,

$$
\frac{e}{(x(x+1))^m} = [0; \overrightarrow{X}_{k_1}, \frac{[N+1]+[N]}{(x(x+1))^m}, \overrightarrow{X}_{k_1}, \frac{[N+2]+[N]}{(x(x+1))^m}, \ldots].
$$

This completes the proof. □