CHAPTER V

FOLDED CONTINUED FRACTION EXPANSIONS

From here, our main concern is the field $F((x^{-1}))$ of formal Laurent series over a field F equipped by a degree valuation $|\cdot|_\infty$. In the previous chapter, we construct explicit Ruban continued fraction expansions whose partial quotients are symmetry. The main tool called the Folding Lemma, first appeared in [19], is used in construction. For more discussion of the Folding Lemma see [13], [18], [20], [26], [27], [32], [33] and [34].

In this chapter, we generalize the Folding Lemma for continued fraction expansions with some interesting patterns. We define generalized Folding Lemmas in first two sections. In the last section, some explicit continued fraction expansions for certain series expansions are provided.

5.1 A generalized Folding Lemma

Lemma 4.5 (2) is known as the (classical) Folding Lemma. In this section, we extend this lemma to take care four related possible patterns.

Lemma 5.1. *For* $n \in \mathbb{N}$, *let* $y \in F[x] \setminus \{0\}$ *and* $\dfrac{C_n}{D_n} := [0; a_1, a_2, \ldots, a_n] = [0; \overrightarrow{X}_n]$. *Then*

(1) $[0; \overrightarrow{X}_n, y, \overrightarrow{X}_n] = \dfrac{C_n}{D_n} + \dfrac{(-1)^n}{D_n^2\left(y + \dfrac{C_n + D_{n-1}}{D_n}\right)}$;

(2) $[0; \overrightarrow{X}_n, y, -\overrightarrow{X}_n] = \dfrac{C_n}{D_n} + \dfrac{(-1)^n}{D_n^2\left(y + \dfrac{D_{n-1} - C_n}{D_n}\right)}$;

(3) $[0; \overrightarrow{X}_n, y, \overleftarrow{X}_n] = \dfrac{C_n}{D_n} + \dfrac{(-1)^n}{D_n^2\left(y + \dfrac{2D_{n-1}}{D_n}\right)}$;

(4) $[0; \overrightarrow{X}_n, y, -\overleftarrow{X}_n] = \dfrac{C_n}{D_n} + \dfrac{(-1)^n}{D_n^2 y}$.

Proof. The identities (1) and (4) hold by Lemma 4.5.

Propositions 2.1 and 2.2 yield

$$\begin{aligned}
[0; \overrightarrow{X}_n, y, -\overrightarrow{X}_n] &= [0; a_1, a_2, \ldots, a_n, y, -a_1, -a_2, \ldots, -a_n] \\
&= [0; a_1, a_2, \ldots, a_n, y - \frac{C_n}{D_n}] \\
&= \frac{\left(y - \dfrac{C_n}{D_n}\right) C_n + C_{n-1}}{\left(y - \dfrac{C_n}{D_n}\right) D_n + D_{n-1}} \\
&= \frac{(D_n y - C_n) C_n + D_n C_{n-1}}{(D_n y - C_n) D_n + D_n D_{n-1}} \\
&= \frac{(D_n y - C_n) C_n + C_n D_{n-1} + (-1)^n}{(D_n y - C_n) D_n + D_n D_{n-1}} \\
&= \frac{C_n (D_n y - C_n + D_{n-1}) + (-1)^n}{D_n (D_n y - C_n + D_{n-1})} \\
&= \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2 \left(y + \dfrac{D_{n-1} - C_n}{D_n}\right)},
\end{aligned}$$

and identity (2) holds.

Similarly, by Propositions 4.1, 2.1 and 2.2, we get

$$\begin{aligned}
[0; \overrightarrow{X}_n, y, \overleftarrow{X}_n] &= [0; a_1, a_2, \ldots, a_n, y, a_n, a_{n-1}, \ldots, a_1] \\
&= [0; a_1, a_2, \ldots, a_n, y + \frac{D_{n-1}}{D_n}] \\
&= \frac{\left(y + \dfrac{D_{n-1}}{D_n}\right) C_n + C_{n-1}}{\left(y + \dfrac{D_{n-1}}{D_n}\right) D_n + D_{n-1}} \\
&= \frac{(D_n y + D_{n-1}) C_n + D_n C_{n-1}}{(D_n y + D_{n-1}) D_n + D_n D_{n-1}} \\
&= \frac{(D_n y + D_{n-1}) C_n + C_n D_{n-1} + (-1)^n}{(D_n y + D_{n-1}) D_n + D_n D_{n-1}} \\
&= \frac{C_n (D_n y + 2D_{n-1}) + (-1)^n}{D_n (D_n y + 2D_{n-1})}
\end{aligned}$$

$$= \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2\left(y + \frac{2D_{n-1}}{D_n}\right)},$$

and identity (3) holds. □

We call the four identities in the previous lemma *two-fold continued fraction expansions of types 1 to 4*, respectively.

Theorem 5.2 is an immediate consequence of Lemma 5.1.

Theorem 5.2. *Let $\{\alpha_i\}_{i\geq 1}$ be a sequence of nonconstant polynomial over a field F. For $\ell \in \mathbb{N}$, let $\frac{C_{k_\ell}}{D_{k_\ell}} = [0; \overrightarrow{X}_{k_\ell}]$ be the ${k_\ell}^{th}$ convergent of the continued fraction expansion of $\sum_{i=1}^{\ell} \frac{1}{\alpha_i}$. Then*

(1) $[0; \overrightarrow{X}_{k_\ell}, Y, \overrightarrow{X}_{k_\ell}] = \sum_{i=1}^{\ell+1} \frac{1}{\alpha_i}$, *if*

$$\alpha_{\ell+1} = (-1)^{k_\ell}\left((C_{k_\ell}D_{k_\ell} + D_{k_\ell-1}D_{k_\ell}) + D_{k_\ell}^2 Y\right) \text{ for some } Y \in F[x] \setminus \{0\};$$

(2) $[0; \overrightarrow{X}_{k_\ell}, Y, -\overrightarrow{X}_{k_\ell}] = \sum_{i=1}^{\ell+1} \frac{1}{\alpha_i}$, *if*

$$\alpha_{\ell+1} = (-1)^{k_\ell}\left((D_{k_\ell-1}D_{k_\ell} - C_{k_\ell}D_{k_\ell}) + D_{k_\ell}^2 Y\right) \text{ for some } Y \in F[x] \setminus \{0\};$$

(3) $[0; \overrightarrow{X}_{k_\ell}, Y, \overleftarrow{X}_{k_\ell}] = \sum_{i=1}^{\ell+1} \frac{1}{\alpha_i}$, *if*

$$\alpha_{\ell+1} = (-1)^{k_\ell}\left(2D_{k_\ell-1}D_{k_\ell} + D_{k_\ell}^2 Y\right) \text{ for some } Y \in F[x] \setminus \{0\};$$

(4) $[0; \overrightarrow{X}_{k_\ell}, Y, -\overleftarrow{X}_{k_\ell}] = \sum_{i=1}^{\ell+1} \frac{1}{\alpha_i}$, *if*

$$\alpha_{\ell+1} = (-1)^{k_\ell} D_{k_\ell}^2 Y \text{ for some } Y \in F[x] \setminus \{0\}.$$

Proof. (1) Let $\alpha_{\ell+1} = (-1)^{k_\ell}\left((C_{k_\ell}D_{k_\ell} + D_{k_\ell-1}D_{k_\ell}) + D_{k_\ell}^2 Y\right)$ for some $Y \in F[x] \setminus$

$\{0\}$. By Lemma 5.1 (1), we get

$$
\begin{aligned}
[0; \overrightarrow{X}_{k_\ell}, Y, \overrightarrow{X}_{k_\ell}] &= \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{(-1)^{k_\ell}}{D_{k_\ell}^2\left(Y + \dfrac{C_{k_\ell} + D_{k_\ell - 1}}{D_{k_\ell}}\right)} \\
&= \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{1}{\alpha_{\ell+1}} = \sum_{i=1}^{\ell+1} \frac{1}{\alpha_i}.
\end{aligned}
$$

(2) Let $\alpha_{\ell+1} = (-1)^{k_\ell}\left((D_{k_\ell-1}D_{k_\ell} - C_{k_\ell}D_{k_\ell}) + D_{k_\ell}^2(Y + a_0)\right)$ for some $Y \in F[x] \setminus \{0\}$. By Lemma 5.1 (2), we get

$$
\begin{aligned}
[a_0; \overrightarrow{X}_{k_\ell}, Y, -\overrightarrow{X}_{k_\ell}] &= \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{(-1)^{k_\ell}}{D_{k_\ell}^2\left(Y + \dfrac{D_{k_\ell - 1} - C_{k_\ell}}{D_{k_\ell}}\right)} \\
&= \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{1}{\alpha_{\ell+1}} = \sum_{i=1}^{\ell+1} \frac{1}{\alpha_i}.
\end{aligned}
$$

(3) Let $\alpha_{\ell+1} = (-1)^{k_\ell}\left(2D_{k_\ell-1}D_{k_\ell} + D_{k_\ell}^2 Y\right)$ for some $Y \in F[x] \setminus \{0\}$. By Lemma 5.1 (3), we get

$$
[a_0; \overrightarrow{X}_{k_\ell}, Y, \overleftarrow{X}_{k_\ell}] = \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{(-1)^{k_\ell}}{D_{k_\ell}^2\left(Y + \dfrac{2D_{k_\ell - 1}}{D_{k_\ell}}\right)} = \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{1}{\alpha_{\ell+1}} = \sum_{i=1}^{\ell+1} \frac{1}{\alpha_i}.
$$

(4) Let $\alpha_{\ell+1} = (-1)^{k_\ell} D_{k_\ell}^2 Y$ for some $Y \in F[x] \setminus \{0\}$. By Lemma 5.1 (4), we get

$$
[a_0; \overrightarrow{X}_{k_\ell}, Y, -\overleftarrow{X}_{k_\ell}] = \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{(-1)^{k_\ell}}{D_{k_\ell}^2 Y} = \frac{C_{k_\ell}}{D_{k_\ell}} + \frac{1}{\alpha_{\ell+1}} = \sum_{i=1}^{\ell+1} \frac{1}{\alpha_i}.
$$

This proves our lemma. □

5.2 A generalized 3-tier Folding Lemma

In this section, we derive analogous results of Lemma 5.1 and Theorem 5.2 for *three-fold continued fraction expansions.*

Lemma 5.3. *Let* $y_1, y_2 \in F[x] \setminus \{0\}$ *and* $\dfrac{C_n}{D_n} = [0; \overrightarrow{X}_n]$. *Then*

(1) $$[0; \overrightarrow{X}_n, y_1, \overrightarrow{X}_n, y_2, \overrightarrow{X}_n] = \frac{C_n}{D_n} + \cfrac{(-1)^n}{D_n^2\left(y_1 + \dfrac{C_n + D_{n-1}}{D_n}\right) + \cfrac{(-1)^n}{y_2 + \dfrac{C_n + D_{n-1}}{D_n}}},$$

(2) $$[0; \overrightarrow{X}_n, y_1, \overrightarrow{X}_n, y_2, -\overrightarrow{X}_n] = \frac{C_n}{D_n} + \cfrac{(-1)^n}{D_n^2\left(y_1 + \dfrac{C_n + D_{n-1}}{D_n}\right) + \cfrac{(-1)^n}{y_2 + \dfrac{D_{n-1} - C_n}{D_n}}},$$

(3) $$[0; \overrightarrow{X}_n, y_1, \overrightarrow{X}_n, y_2, \overleftarrow{X}_n] = \frac{C_n}{D_n} + \cfrac{(-1)^n}{D_n^2\left(y_1 + \dfrac{C_n + D_{n-1}}{D_n}\right) + \cfrac{(-1)^n}{y_2 + \dfrac{2D_{n-1}}{D_n}}},$$

(4) $$[0; \overrightarrow{X}_n, y_1, \overrightarrow{X}_n, y_2, -\overleftarrow{X}_n] = \frac{C_n}{D_n} + \cfrac{(-1)^n}{D_n^2\left(y_1 + \dfrac{C_n + D_{n-1}}{D_n}\right) + \dfrac{(-1)^n}{y_2}},$$

(5) $$[0; \overrightarrow{X}_n, y_1, -\overrightarrow{X}_n, y_2, \overrightarrow{X}_n] = \frac{C_n}{D_n} + \cfrac{(-1)^n}{D_n^2\left(y_1 + \dfrac{D_{n-1} - C_n}{D_n}\right) + \cfrac{(-1)^n}{y_2 + \dfrac{C_n - D_{n-1}}{D_n}}},$$

(6) $$[0; \overrightarrow{X}_n, y_1, -\overrightarrow{X}_n, y_2, -\overrightarrow{X}_n] = \frac{C_n}{D_n} + \cfrac{(-1)^n}{D_n^2\left(y_1 + \dfrac{D_{n-1} - C_n}{D_n}\right) + \cfrac{(-1)^n}{y_2 - \dfrac{C_n + D_{n-1}}{D_n}}},$$

(7) $$[0; \overrightarrow{X}_n, y_1, -\overrightarrow{X}_n, y_2, \overleftarrow{X}_n] = \frac{C_n}{D_n} + \cfrac{(-1)^n}{D_n^2\left(y_1 + \dfrac{D_{n-1} - C_n}{D_n}\right) + \dfrac{(-1)^n}{y_2}},$$

(8) $$[0; \overrightarrow{X}_n, y_1, -\overrightarrow{X}_n, y_2, -\overleftarrow{X}_n] = \frac{C_n}{D_n} + \cfrac{(-1)^n}{D_n^2\left(y_1 + \dfrac{D_{n-1} - C_n}{D_n}\right) + \cfrac{(-1)^n}{y_2 - \dfrac{2D_{n-1}}{D_n}}},$$

(9) $$[0; \overrightarrow{X}_n, y_1, \overleftarrow{X}_n, y_2, \overrightarrow{X}_n] = \frac{C_n}{D_n} + \cfrac{(-1)^n}{D_n^2\left(y_1 + \dfrac{2D_{n-1}}{D_n}\right) + \cfrac{(-1)^n}{y_2 + \dfrac{2C_n}{D_n}}},$$

(10) $$[0; \overrightarrow{X}_n, y_1, \overleftarrow{X}_n, y_2, -\overrightarrow{X}_n] = \frac{C_n}{D_n} + \cfrac{(-1)^n}{D_n^2\left(y_1 + \dfrac{2D_{n-1}}{D_n}\right) + \dfrac{(-1)^n}{y_2}},$$

(11) $[0; \overrightarrow{X}_n, y_1, \overleftarrow{X}_n, y_2, \overleftarrow{X}_n] = \dfrac{C_n}{D_n} + \dfrac{(-1)^n}{D_n^2\left(y_1 + \dfrac{2D_{n-1}}{D_n}\right) + \dfrac{(-1)^n}{y_2 + \dfrac{C_n + D_{n-1}}{D_n}}}$,

(12) $[0; \overrightarrow{X}_n, y_1, \overleftarrow{X}_n, y_2, -\overleftarrow{X}_n] = \dfrac{C_n}{D_n} + \dfrac{(-1)^n}{D_n^2\left(y_1 + \dfrac{2D_{n-1}}{D_n}\right) + \dfrac{(-1)^n}{y_2 + \dfrac{C_n - D_{n-1}}{D_n}}}$,

(13) $[0; \overrightarrow{X}_n, y_1, -\overleftarrow{X}_n, y_2, \overrightarrow{X}_n] = \dfrac{C_n}{D_n} + \dfrac{(-1)^n}{D_n^2 y_1 + \dfrac{(-1)^n}{y_2}}$,

(14) $[0; \overrightarrow{X}_n, y_1, -\overleftarrow{X}_n, y_2, -\overrightarrow{X}_n] = \dfrac{C_n}{D_n} + \dfrac{(-1)^n}{D_n^2 y_1 + \dfrac{(-1)^n}{y_2 - \dfrac{2C_n}{D_n}}}$,

(15) $[0; \overrightarrow{X}_n, y_1, -\overleftarrow{X}_n, y_2, \overleftarrow{X}_n] = \dfrac{C_n}{D_n} + \dfrac{(-1)^n}{D_n^2 y_1 + \dfrac{(-1)^n}{y_2 + \dfrac{D_{n-1} - C_n}{D_n}}}$,

(16) $[0; \overrightarrow{X}_n, y_1, -\overleftarrow{X}_n, y_2, -\overleftarrow{X}_n] = \dfrac{C_n}{D_n} + \dfrac{(-1)^n}{D_n^2 y_1 + \dfrac{(-1)^n}{y_2 - \dfrac{C_n + D_{n-1}}{D_n}}}$.

Proof. (1) From the two-fold continued fraction expansion of type 1, we have

$$[0; \overrightarrow{X}_n, y_2, \overrightarrow{X}_n] = \frac{C_n}{D_n} + \frac{(-1)^n}{D_n\left(D_n y_2 + C_n + D_{n-1}\right)},$$

so, by Propositions 2.1 and 2.2, we get

$$\begin{aligned}
&[0; \overrightarrow{X}_n, y_1, \overrightarrow{X}_n, y_2, \overrightarrow{X}_n] \\
&= [0; \overrightarrow{X}_n, y_1 + \frac{C_n}{D_n} + \frac{(-1)^n}{D_n\left(D_n y_2 + C_n + D_{n-1}\right)}] \\
&= \frac{\left(y_1 + \dfrac{C_n}{D_n} + \dfrac{(-1)^n}{D_n\left(D_n y_2 + C_n + D_{n-1}\right)}\right) C_n + C_{n-1}}{\left(y_1 + \dfrac{C_n}{D_n} + \dfrac{(-1)^n}{D_n\left(D_n y_2 + C_n + D_{n-1}\right)}\right) D_n + D_{n-1}}
\end{aligned}$$

$$= \{[y_1 D_n (D_n y_2 + C_n + D_{n-1}) + C_n (D_n y_2 + C_n + D_{n-1}) + (-1)^n] C_n$$
$$+ C_{n-1} D_n (D_n y_2 + C_n + D_{n-1})\} /$$
$$\{[y_1 D_n (D_n y_2 + C_n + D_{n-1}) + C_n (D_n y_2 + C_n + D_{n-1}) + (-1)^n] D_n$$
$$+ D_{n-1} D_n (D_n y_2 + C_n + D_{n-1})\}$$
$$= \frac{C_n}{D_n} + \{C_{n-1} D_n (D_n y_2 + C_n + D_{n-1}) - C_n D_{n-1} (D_n y_2 + C_n + D_{n-1})\} /$$
$$\{[y_1 D_n (D_n y_2 + C_n + D_{n-1}) + C_n (D_n y_2 + C_n + D_{n-1}) + (-1)^n] D_n$$
$$+ D_{n-1} D_n (D_n y_2 + C_n + D_{n-1})\}$$
$$= \frac{C_n}{D_n} + \{(D_n y_2 + C_n + D_{n-1}) (C_{n-1} D_n - C_n D_{n-1})\} /$$
$$\{[y_1 D_n (D_n y_2 + C_n + D_{n-1}) + C_n (D_n y_2 + C_n + D_{n-1}) + (-1)^n] D_n$$
$$+ D_{n-1} D_n (D_n y_2 + C_n + D_{n-1})\}$$
$$= \frac{C_n}{D_n} + \frac{(-1)^n}{\left(y_1 D_n + C_n + \frac{(-1)^n}{D_n y_2 + C_n + D_{n-1}}\right) D_n + D_{n-1} D_n}$$
$$= \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2 \left(y_1 + \frac{C_n + D_{n-1}}{D_n}\right) + \frac{(-1)^n}{y_2 + \frac{C_n + D_{n-1}}{D_n}}}.$$

(2) To prove (2), we recall the two-fold continued fraction expansion of type 2

$$[0; \overrightarrow{X}_n, y_2, -\overrightarrow{X}_n] = \frac{C_n}{D_n} + \frac{(-1)^n}{D_n (D_n y_2 + D_{n-1} - C_n)}.$$

Applying Propositions 2.1 and 2.2, we get

$$[0; \overrightarrow{X}_n, y_1, \overrightarrow{X}_n, y_2, -\overrightarrow{X}_n]$$
$$= [0; \overrightarrow{X}_n, y_1 + \frac{C_n}{D_n} + \frac{(-1)^n}{D_n (D_n y_2 + D_{n-1} - C_n)}]$$
$$= \frac{\left(y_1 + \frac{C_n}{D_n} + \frac{(-1)^n}{D_n (D_n y_2 + D_{n-1} - C_n)}\right) C_n + C_{n-1}}{\left(y_1 + \frac{C_n}{D_n} + \frac{(-1)^n}{D_n (D_n y_2 + D_{n-1} - C_n)}\right) D_n + D_{n-1}}$$
$$= \{[y_1 D_n (D_n y_2 + D_{n-1} - C_n) + C_n (D_n y_2 + D_{n-1} - C_n) + (-1)^n] C_n$$

$$
\begin{aligned}
&\quad +C_{n-1}D_n\left(D_ny_2+D_{n-1}-C_n\right)\}\,/\\
&\quad \{\left[y_1D_n\left(D_ny_2+D_{n-1}-C_n\right)+C_n\left(D_ny_2+D_{n-1}-C_n\right)+(-1)^n\right]D_n\\
&\quad +D_{n-1}D_n\left(D_ny_2+D_{n-1}-C_n\right)\}\\
&=\frac{C_n}{D_n}+\{C_{n-1}D_n\left(D_ny_2+D_{n-1}-C_n\right)-D_{n-1}C_n\left(D_ny_2+D_{n-1}-C_n\right)\}\,/\\
&\qquad \{\left[y_1D_n\left(D_ny_2+D_{n-1}-C_n\right)+C_n\left(D_ny_2+D_{n-1}-C_n\right)+(-1)^n\right]D_n\\
&\qquad +D_{n-1}D_n\left(D_ny_2+D_{n-1}-C_n\right)\}\\
&=\frac{C_n}{D_n}+\{\left(D_ny_2+D_{n-1}-C_n\right)\left(C_{n-1}D_n-D_{n-1}C_n\right)\}\,/\\
&\qquad \{\left[y_1D_n\left(D_ny_2+D_{n-1}-C_n\right)+C_n\left(D_ny_2+D_{n-1}-C_n\right)+(-1)^n\right]D_n\\
&\qquad +D_{n-1}D_n\left(D_ny_2+D_{n-1}-C_n\right)\}\\
&=\frac{C_n}{D_n}+\frac{(-1)^n}{\left(y_1D_n+C_n+\dfrac{(-1)^n}{y_2D_n+D_{n-1}-C_n}\right)D_n+D_{n-1}D_n}\\
&=\frac{C_n}{D_n}+\frac{(-1)^n}{D_n^2\left(y_1+\dfrac{C_n+D_{n-1}}{D_n}\right)+\dfrac{(-1)^n}{y_2+\dfrac{D_{n-1}-C_n}{D_n}}}.
\end{aligned}
$$

(3) We have

$$[0;\overrightarrow{X}_n,y_2,\overleftarrow{X}_n]=\frac{C_n}{D_n}+\frac{(-1)^n}{D_n\left(D_ny_2+2D_{n-1}\right)}.$$

Then

$$
\begin{aligned}
&[0;\overrightarrow{X}_n,y_1,\overrightarrow{X}_n,y_2,\overleftarrow{X}_n]\\
&\quad=[0;\overrightarrow{X}_n,y_1+\frac{C_n}{D_n}+\frac{(-1)^n}{D_n\left(D_ny_2+2D_{n-1}\right)}]\\
&\quad=\frac{\left(y_1+\dfrac{C_n}{D_n}+\dfrac{(-1)^n}{D_n\left(D_ny_2+2D_{n-1}\right)}\right)C_n+C_{n-1}}{\left(y_1+\dfrac{C_n}{D_n}+\dfrac{(-1)^n}{D_n\left(D_ny_2+2D_{n-1}\right)}\right)D_n+D_{n-1}}\\
&\quad=\{\left[y_1D_n\left(D_ny_2+2D_{n-1}\right)+C_n\left(D_ny_2+2D_{n-1}\right)+(-1)^n\right]C_n\\
&\qquad +C_{n-1}D_n\left(D_ny_2+2D_{n-1}\right)\}\,/
\end{aligned}
$$

$$\{[y_1 D_n (D_n y_2 + 2D_{n-1}) + C_n (D_n y_2 + 2D_{n-1}) + (-1)^n] D_n + D_{n-1} D_n (D_n y_2 + 2D_{n-1})\}$$

$$= \frac{C_n}{D_n} + \{C_{n-1} D_n (D_n y_2 + 2D_{n-1}) - D_{n-1} C_n (D_n y_2 + 2D_{n-1})\} / \{[y_1 D_n (D_n y_2 + 2D_{n-1}) + C_n (D_n y_2 + 2D_{n-1}) + (-1)^n] D_n + D_{n-1} D_n (D_n y_2 + 2D_{n-1})\}$$

$$= \frac{C_n}{D_n} + \{(D_n y_2 + 2D_{n-1})(C_{n-1} D_n - D_{n-1} C_n)\} / \{[y_1 D_n (D_n y_2 + 2D_{n-1}) + C_n (D_n y_2 + 2D_{n-1}) + (-1)^n] D_n + D_{n-1} D_n (D_n y_2 + 2D_{n-1})\}$$

$$= \frac{C_n}{D_n} + \frac{(-1)^n}{\left(y_1 D_n + C_n + \dfrac{(-1)^n}{D_n y_2 + 2D_{n-1}}\right) D_n + D_{n-1} D_n}$$

$$= \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2 \left(y_1 + \dfrac{C_n + D_{n-1}}{D_n}\right) + \dfrac{(-1)^n}{y_2 + \dfrac{2D_{n-1}}{D_n}}}.$$

(4) We have

$$[0; \overrightarrow{X}_n, y_2, -\overleftarrow{X}_n] = \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2 y_2},$$

and so

$$[0; \overrightarrow{X}_n, y_1, \overrightarrow{X}_n, y_2, -\overleftarrow{X}_n]$$

$$= [0; \overrightarrow{X}_n, y_1 + \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2 y_2}]$$

$$= \frac{\left(y_1 + \dfrac{C_n}{D_n} + \dfrac{(-1)^n}{D_n^2 y_2}\right) C_n + C_{n-1}}{\left(y_1 + \dfrac{C_n}{D_n} + \dfrac{(-1)^n}{D_n^2 y_2}\right) D_n + D_{n-1}}$$

$$= \frac{(y_1 D_n^2 y_2 + C_n D_n y_2 + (-1)^n) C_n + C_{n-1} D_n^2 y_2}{(y_1 D_n^2 y_2 + C_n D_n y_2 + (-1)^n) D_n + D_{n-1} D_n^2 y_2}$$

$$= \frac{C_n}{D_n} + \frac{C_{n-1} D_n^2 y_2 - D_{n-1} C_n D_n y_2}{(y_1 D_n^2 y_2 + C_n D_n y_2 + (-1)^n) D_n + D_{n-1} D_n^2 y_2}$$

$$= \frac{C_n}{D_n} + \frac{D_n y_2 (C_{n-1} D_n - D_{n-1} C_n)}{(y_1 D_n^2 y_2 + C_n D_n y_2 + (-1)^n) D_n + D_{n-1} D_n^2 y_2}$$

$$= \frac{C_n}{D_n} + \frac{(-1)^n}{\left(y_n D_n + C_n + \dfrac{(-1)^n}{D_n y_2}\right) D_n + D_{n-1} D_n}$$
$$= \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2 \left(y_1 + \dfrac{C_n + D_{n-1}}{D_n}\right) + \dfrac{(-1)^n}{y_2}}.$$

Using the fact that, for any word $\overrightarrow{W}_k$

$$[0; -\overrightarrow{W}_k] = -[0; \overrightarrow{W}_k], \tag{5.1}$$

(5), (6), (7) and (8) follow from the proof of (2), (1), (4) and (3), respectively. We give only the detail of proof for (5).
Using (5.1), we have

$$[0; -\overrightarrow{X}_n, y_2, \overrightarrow{X}_n] = -[0; \overrightarrow{X}_n, -y_2, -\overrightarrow{X}_n].$$

Applying the two-fold continued fraction expansion of type 2 by putting $y = -y_2$, we get

$$[0; -\overrightarrow{X}_n, y_2, \overrightarrow{X}_n] = -\frac{C_n}{D_n} + \frac{(-1)^n}{D_n (D_n y_2 - D_{n-1} + C_n)}.$$

Then the same proof of (2) leads to

$$[0; \overrightarrow{X}_n, y_1, -\overrightarrow{X}_n, y_2, \overrightarrow{X}_n]$$
$$= [0; \overrightarrow{X}_n, y_1 - \frac{C_n}{D_n} + \frac{(-1)^n}{D_n (D_n y_2 - D_{n-1} + C_n)}]$$
$$= \frac{\left(y_1 - \dfrac{C_n}{D_n} + \dfrac{(-1)^n}{D_n (D_n y_2 - D_{n-1} + C_n)}\right) C_n + C_{n-1}}{\left(y_1 - \dfrac{C_n}{D_n} + \dfrac{(-1)^n}{D_n (D_n y_2 - D_{n-1} + C_n)}\right) D_n + D_{n-1}}$$
$$= \{[y_1 D_n (D_n y_2 - D_{n-1} + C_n) - C_n (D_n y_2 - D_{n-1} + C_n) + (-1)^n] C_n$$
$$+ C_{n-1} D_n (D_n y_2 - D_{n-1} + C_n)\} /$$
$$\{[y_1 D_n (D_n y_2 - D_{n-1} + C_n) - C_n (D_n y_2 - D_{n-1} + C_n) + (-1)^n] D_n$$

$$+D_{n-1}D_n(D_ny_2 - D_{n-1} + C_n)\}$$

$$= \frac{C_n}{D_n} + \{C_{n-1}D_n(D_ny_2 - D_{n-1} + C_n) - D_{n-1}C_n(D_ny_2 - D_{n-1} + C_n)\} /$$

$$\{[y_1D_n(D_ny_2 - D_{n-1} + C_n) - C_n(D_ny_2 - D_{n-1} + C_n) + (-1)^n] D_n$$

$$+D_{n-1}D_n(D_ny_2 - D_{n-1} + C_n)\}$$

$$= \frac{C_n}{D_n} + \{(D_ny_2 - D_{n-1} + C_n)(C_{n-1}D_n - C_nD_{n-1})\} /$$

$$\{[y_1D_n(D_ny_2 - D_{n-1} + C_n) - C_n(D_ny_2 - D_{n-1} + C_n) + (-1)^n] D_n$$

$$+D_{n-1}D_n(D_ny_2 - D_{n-1} + C_n)\}$$

$$= \frac{C_n}{D_n} + \frac{(-1)^n}{\left(y_1D_n - C_n + \dfrac{(-1)^n}{D_ny_2 - D_{n-1} + C_n}\right) D_n + D_{n-1}D_n}$$

$$= \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2\left(y_1 + \dfrac{D_{n-1} - C_n}{D_n}\right) + \dfrac{(-1)^n}{y_2 + \dfrac{C_n - D_{n-1}}{D_n}}},$$

which is (5).

If the n^{th} convergent of $[0; \overrightarrow{X}_n]$ is $\dfrac{C_n}{D_n}$, then two consecutive $(n-1)^{\text{th}}$ and n^{th} convergent of $[0; \overleftarrow{X}_n]$ are $\dfrac{C_{n-1}}{C_n}$ and $\dfrac{D_{n-1}}{D_n}$. Substituting them to Lemma 5.1, we obtain a new version of the two-fold continued fraction expansions of types 1 to 4, respectively, as

$$(1)' \quad [0; \overleftarrow{X}_n, y_2, \overleftarrow{X}_n] = \frac{D_{n-1}}{D_n} + \frac{(-1)^n}{D_n^2\left(y_2 + \dfrac{D_{n-1} + C_n}{D_n}\right)},$$

$$(2)' \quad [0; \overleftarrow{X}_n, y_2, -\overleftarrow{X}_n] = \frac{D_{n-1}}{D_n} + \frac{(-1)^n}{D_n^2\left(y_2 + \dfrac{C_n - D_{n-1}}{D_n}\right)},$$

$$(3)' \quad [0; \overleftarrow{X}_n, y_2, \overrightarrow{X}_n] = \frac{D_{n-1}}{D_n} + \frac{(-1)^n}{D_n^2\left(y_2 + \dfrac{2C_n}{D_n}\right)},$$

$$(4)' \quad [0; \overleftarrow{X}_n, y_2, -\overrightarrow{X}_n] = \frac{D_{n-1}}{D_n} + \frac{(-1)^n}{D_n^2 y_2}.$$

Then (9), (10), (11) and (12) are obtained by the same proof of (3), (4), (1) and (2), respectively. We give only the proof of (9).

Using $(3)'$, we have

$$
\begin{aligned}
&[0;\overrightarrow{X}_n, y_1, \overleftarrow{X}_n, y_2, \overrightarrow{X}_n] \\
&= [0; \overrightarrow{X}_n, y_1 + \frac{D_{n-1}}{D_n} + \frac{(-1)^n}{D_n\,(D_n y_2 + 2C_n)}] \\
&= \frac{\left(y_1 + \frac{D_{n-1}}{D_n} + \frac{(-1)^n}{D_n\,(D_n y_2 + 2C_n)}\right) C_n + C_{n-1}}{\left(y_1 + \frac{D_{n-1}}{D_n} + \frac{(-1)^n}{D_n\,(D_n y_2 + 2C_n)}\right) D_n + D_{n-1}} \\
&= \{(y_1 D_n\,(D_n y_2 + 2C_n) + D_{n-1}\,(D_n y_2 + 2C_n) + (-1)^n)\,C_n \\
&\quad + C_{n-1} D_n\,(D_n y_2 + 2C_n)\} \,/ \\
&\quad \{(y_1 D_n\,(D_n y_2 + 2C_n) + D_{n-1}\,(D_n y_2 + 2C_n) + (-1)^n)\,D_n \\
&\quad + D_{n-1} D_n\,(D_n y_2 + 2C_n)\} \\
&= \frac{C_n}{D_n} + \{C_{n-1} D_n\,(D_n y_2 + 2C_n) - D_{n-1} C_n\,(D_n y_2 + 2C_n)\} \,/ \\
&\qquad \{(y_1 D_n\,(D_n y_2 + 2C_n) + D_{n-1}\,(D_n y_2 + 2C_n) + (-1)^n)\,D_n \\
&\qquad + D_{n-1} D_n\,(D_n y_2 + 2C_n)\} \\
&= \frac{C_n}{D_n} + \{(D_n y_2 + 2C_n)\,(C_{n-1} D_n - D_{n-1} C_n)\} \,/ \\
&\qquad \{(y_1 D_n\,(D_n y_2 + 2C_n) + D_{n-1}\,(D_n y_2 + 2C_n) + (-1)^n)\,D_n \\
&\qquad + D_{n-1} D_n\,(D_n y_2 + 2C_n)\} \\
&= \frac{C_n}{D_n} + \frac{(-1)^n}{\left(y_1 D_n + D_{n-1} + \frac{(-1)^n}{D_n y_2 + 2C_n}\right) D_n + D_{n-1} D_n} \\
&= \frac{C_n}{D_n} + \frac{(-1)^n}{D_n^2 \left(y_1 + \frac{2D_{n-1}}{D_n}\right) + \frac{(-1)^n}{y_2 + \frac{2C_n}{D_n}}}.
\end{aligned}
$$

Applying identity (5.1), the remaining cases are obtained similarly.
Using (5.1) and $(4)'$, (13) follows immediately by the same proof of (4).
Using (5.1) and $(3)'$, (14) follows immediately by the same proof of (3).
Using (5.1) and $(2)'$, (15) follows immediately by the same proof of (2).
Using (5.1) and $(1)'$, (16) follows immediately by the same proof of (1). □

Now, we are ready to state an analogue of Theorem 5.2.

Theorem 5.4. *Let* $\{\alpha_i\}_{i\geq 1}$ *be a sequence of nonconstant polynomial over a field* F. *For* $\ell \in \mathbb{N}$, *let* $\frac{C_{k_\ell}}{D_{k_\ell}} = [0; \overrightarrow{X}_{k_\ell}]$ *be the* $k_\ell{}^{th}$ *convergent of the continued fraction expansion of* $\sum_{i=1}^{\ell} \frac{1}{\alpha_i}$. *Then*

(1) $[0; \overrightarrow{X}_{k_\ell}, Y_1, \overrightarrow{X}_{k_\ell}, Y_2, \overrightarrow{X}_{k_\ell}] = \sum_{i=1}^{\ell+1} \frac{1}{\alpha_i}$, *if*

$$\alpha_{\ell+1} = (-1)^{k_\ell} D_n^2 \left(Y_1 + \frac{C_{k_\ell} + D_{k_\ell - 1}}{D_{k_\ell}} \right) + \frac{1}{Y_2 + \frac{C_{k_\ell} + D_{k_\ell - 1}}{D_{k_\ell}}}$$

for some $Y_1, Y_2 \in F[x] \smallsetminus \{0\}$;

(2) $[0; \overrightarrow{X}_{k_\ell}, Y_1, \overrightarrow{X}_{k_\ell}, Y_2, -\overrightarrow{X}_{k_\ell}] = \sum_{i=1}^{\ell+1} \frac{1}{\alpha_i}$, *if*

$$\alpha_{\ell+1} = (-1)^{k_\ell} D_n^2 \left(Y_1 + \frac{C_{k_\ell} + D_{k_\ell - 1}}{D_{k_\ell}} \right) + \frac{1}{Y_2 + \frac{D_{k_\ell - 1} - C_{k_\ell}}{D_{k_\ell}}}$$

for some $Y_1, Y_2 \in F[x] \smallsetminus \{0\}$;

(3) $[0; \overrightarrow{X}_{k_\ell}, Y_1, \overrightarrow{X}_{k_\ell}, Y_2, \overleftarrow{X}_{k_\ell}] = \sum_{i=1}^{\ell+1} \frac{1}{\alpha_i}$, *if*

$$\alpha_{\ell+1} = (-1)^{k_\ell} D_n^2 \left(Y_1 + \frac{C_{k_\ell} + D_{k_\ell - 1}}{D_{k_\ell}} \right) + \frac{1}{Y_2 + \frac{2D_{k_\ell - 1}}{D_{k_\ell}}}$$

for some $Y_1, Y_2 \in F[x] \smallsetminus \{0\}$;

(4) $[0; \overrightarrow{X}_{k_\ell}, Y_1, \overrightarrow{X}_{k_\ell}, Y_2, -\overleftarrow{X}_{k_\ell}] = \sum_{i=1}^{\ell+1} \frac{1}{\alpha_i}$, *if*

$$\alpha_{\ell+1} = (-1)^{k_\ell} D_n^2 \left(Y_1 + \frac{C_{k_\ell} + D_{k_\ell - 1}}{D_{k_\ell}} \right) + \frac{1}{Y_2}$$

for some $Y_1, Y_2 \in F[x] \smallsetminus \{0\}$;

(5) $[0; \overrightarrow{X}_{k_\ell}, Y_1, -\overrightarrow{X}_{k_\ell}, Y_2, \overrightarrow{X}_{k_\ell}] = \sum_{i=1}^{\ell+1} \frac{1}{\alpha_i}$, *if*

$$\alpha_{\ell+1} = (-1)^{k_\ell} D_n^2 \left(Y_1 + \frac{D_{k_\ell - 1} - C_{k_\ell}}{D_{k_\ell}} \right) + \frac{1}{Y_2 + \dfrac{C_{k_\ell} - D_{k_\ell - 1}}{D_{k_\ell}}}$$

for some $Y_1, Y_2 \in F[x] \setminus \{0\}$;

(6) $[0; \overrightarrow{X}_{k_\ell}, Y_1, -\overrightarrow{X}_{k_\ell}, Y_2, -\overrightarrow{X}_{k_\ell}] = \sum_{i=1}^{\ell+1} \frac{1}{\alpha_i}$, *if*

$$\alpha_{\ell+1} = (-1)^{k_\ell} D_n^2 \left(Y_1 + \frac{D_{k_\ell - 1} - C_{k_\ell}}{D_{k_\ell}} \right) + \frac{1}{Y_2 - \dfrac{C_{k_\ell} + D_{k_\ell - 1}}{D_{k_\ell}}}$$

for some $Y_1, Y_2 \in F[x] \setminus \{0\}$;

(7) $[0; \overrightarrow{X}_{k_\ell}, Y_1, -\overrightarrow{X}_{k_\ell}, Y_2, \overleftarrow{X}_{k_\ell}] = \sum_{i=1}^{\ell+1} \frac{1}{\alpha_i}$, *if*

$$\alpha_{\ell+1} = (-1)^{k_\ell} D_n^2 \left(Y_1 + \frac{D_{k_\ell - 1} - C_{k_\ell}}{D_{k_\ell}} \right) + \frac{1}{Y_2}$$

for some $Y_1, Y_2 \in F[x] \setminus \{0\}$;

(8) $[0; \overrightarrow{X}_{k_\ell}, Y_1, -\overrightarrow{X}_{k_\ell}, Y_2, -\overleftarrow{X}_{k_\ell}] = \sum_{i=1}^{\ell+1} \frac{1}{\alpha_i}$, *if*

$$\alpha_{\ell+1} = (-1)^{k_\ell} D_n^2 \left(Y_1 + \frac{D_{k_\ell - 1} - C_{k_\ell}}{D_{k_\ell}} \right) + \frac{1}{Y_2 - \dfrac{2D_{k_\ell - 1}}{D_{k_\ell}}}$$

for some $Y_1, Y_2 \in F[x] \setminus \{0\}$;

(9) $[0; \overrightarrow{X}_{k_\ell}, Y_1, \overleftarrow{X}_{k_\ell}, Y_2, \overrightarrow{X}_{k_\ell}] = \sum_{i=1}^{\ell+1} \frac{1}{\alpha_i}$, *if*

$$\alpha_{\ell+1} = (-1)^{k_\ell} D_n^2 \left(Y_1 + \frac{2D_{k_\ell - 1}}{D_{k_\ell}} \right) + \frac{1}{Y_2 + \dfrac{2C_{k_\ell}}{D_{k_\ell}}}$$

for some $Y_1, Y_2 \in F[x] \setminus \{0\}$;

(10) $[0; \overrightarrow{X}_{k_\ell}, Y_1, \overleftarrow{X}_{k_\ell}, Y_2, -\overrightarrow{X}_{k_\ell}] = \sum_{i=1}^{\ell+1} \frac{1}{\alpha_i}$, *if*

$$\alpha_{\ell+1} = (-1)^{k_\ell} D_n^2 \left(Y_1 + \frac{2D_{k_\ell - 1}}{D_{k_\ell}} \right) + \frac{1}{Y_2}$$

for some $Y_1, Y_2 \in F[x] \setminus \{0\}$;

(11) $[0; \overrightarrow{X}_{k_\ell}, Y_1, \overleftarrow{X}_{k_\ell}, Y_2, \overleftarrow{X}_{k_\ell}] = \sum_{i=1}^{\ell+1} \frac{1}{\alpha_i}$, *if*

$$\alpha_{\ell+1} = (-1)^{k_\ell} D_n^2 \left(Y_1 + \frac{2D_{k_\ell - 1}}{D_{k_\ell}} \right) + \cfrac{1}{Y_2 + \cfrac{C_{k_\ell} + D_{k_\ell - 1}}{D_{k_\ell}}}$$

for some $Y_1, Y_2 \in F[x] \setminus \{0\}$;

(12) $[0; \overrightarrow{X}_{k_\ell}, Y_1, \overleftarrow{X}_{k_\ell}, Y_2, -\overleftarrow{X}_{k_\ell}] = \sum_{i=1}^{\ell+1} \frac{1}{\alpha_i}$, *if*

$$\alpha_{\ell+1} = (-1)^{k_\ell} D_n^2 \left(Y_1 + \frac{2D_{k_\ell - 1}}{D_{k_\ell}} \right) + \cfrac{1}{Y_2 + \cfrac{C_{k_\ell} - D_{k_\ell - 1}}{D_{k_\ell}}}$$

for some $Y_1, Y_2 \in F[x] \setminus \{0\}$;

(13) $[0; \overrightarrow{X}_{k_\ell}, Y_1, -\overleftarrow{X}_{k_\ell}, Y_2, \overrightarrow{X}_{k_\ell}] = \sum_{i=1}^{\ell+1} \frac{1}{\alpha_i}$, *if*

$$\alpha_{\ell+1} = (-1)^{k_\ell} D_n^2 Y_1 + \frac{1}{Y_2}$$

for some $Y_1, Y_2 \in F[x] \setminus \{0\}$;

(14) $[0; \overrightarrow{X}_{k_\ell}, Y_1, -\overleftarrow{X}_{k_\ell}, Y_2, -\overrightarrow{X}_{k_\ell}] = \sum_{i=1}^{\ell+1} \frac{1}{\alpha_i}$, *if*

$$\alpha_{\ell+1} = (-1)^{k_\ell} D_n^2 Y_1 + \cfrac{1}{Y_2 - \cfrac{2C_{k_\ell}}{D_{k_\ell}}}$$

for some $Y_1, Y_2 \in F[x] \setminus \{0\}$;

(15) $[0; \overrightarrow{X}_{k_\ell}, Y_1, -\overleftarrow{X}_{k_\ell}, Y_2, \overleftarrow{X}_{k_\ell}] = \sum_{i=1}^{\ell+1} \frac{1}{\alpha_i}$, *if*

$$\alpha_{\ell+1} = (-1)^{k_\ell} D_n^2 Y_1 + \cfrac{1}{Y_2 + \cfrac{D_{k_\ell - 1} - C_{k_\ell}}{D_{k_\ell}}}$$

for some $Y_1, Y_2 \in F[x] \setminus \{0\}$;

(16) $[0; \overrightarrow{X}_{k_\ell}, Y_1, -\overleftarrow{X}_{k_\ell}, Y_2, -\overleftarrow{X}_{k_\ell}] = \sum_{i=1}^{\ell+1} \frac{1}{\alpha_i}$, *if*

$$\alpha_{\ell+1} = (-1)^{k_\ell} D_n^2 Y_1 + \cfrac{1}{Y_2 - \cfrac{C_{k_\ell} + D_{k_\ell - 1}}{D_{k_\ell}}}$$

for some $Y_1, Y_2 \in F[x] \setminus \{0\}$.

Proof. Using the same proof of Theorem 5.2, but with identities in Lemma 5.3 lead to the desired results. □

5.3 Applications

In this section, some known explicit continued fraction expansions of series expansions are shown as applications of our results. We also construct explicit continued fraction expansions for some interesting series expansions.

5.3.1 Two-fold continued fraction expansion of type 1

It is easily seen that applying the result of Theorem 5.2 (1), by putting $\alpha_\ell = x^m d_{\ell-1}$ for all $\ell \geq 1$ in the case that $F = \mathbb{F}_2$, gives Proposition 4.4.

5.3.2 Two-fold continued fraction expansion of type 3

In 2010, Chaichana [5] has extended and modified Tamuras results, [28], both in the field of real numbers and in the field of formal Laurent series in x^{-1} over a field F of characteristic zero. In this subsection, we apply Theorem 5.2 (3) to prove the main results of Chaichana in the field of formal Laurent series.

We start with definition.

Definition 5.5. For all $n \in \mathbb{N}$, a continued fraction expansion $[a_0, ; a_1, a_2, \ldots, a_n]$ is said to be *palindromic* if the word $a_1, a_2, \ldots, a_n$ is equal to its reversal.

Remark 5.6. *For all $n \in \mathbb{N}$, if a continued fraction expansion $[a_0, ; a_1, a_2, \ldots, a_n] = \dfrac{C_n}{D_n}$ is palindromic, then we have by induction that*

$$C_n = D_{n-1}.$$

Let $f(T) = T(T+2)(T-2)g(T) - T^2 + 2$ be a monic polynomial (in T) in $(F[x])[T]$, for some $g(T) \in (F[x])[T]$. We observe that, $g(T)$ is monic. We consider series expansion of the form, for all $\ell \geq 0$

$$\sum_{n=0}^{\ell} \frac{(-1)^n}{f_0(T)f_1(T)\cdots f_n(T)},$$

where $f_0(T) = T$ and $f_n(T) = f\left(f_{n-1}(T)\right)$ for all $n \geq 1$, which induces $f_n = f_1 \circ f_1 \circ \cdots \circ f_1$ (n composites).

Here, we consider the continued fraction expansion representing an infinite sum of the form

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{f_0(T)f_1(T)\cdots f_n(T)}. \tag{5.2}$$

Let $A_0(T) = 1,\ B_0(T) = f_0(T) = T$ and for $n \geq 1$

$$A_n(T) = (-1)^n + \sum_{m=1}^{n}(-1)^{m+1} f_m(T)f_{m+1}(T)\cdots f_n(T) = (-1)^n + f_n(T)A_{n-1}(T) \tag{5.3}$$

$$B_n(T) = f_0(T)f_1(T)\cdots f_n(T). \tag{5.4}$$

By induction, for all $\ell \geq 0$, it follows that

$$\sum_{n=0}^{\ell} \frac{(-1)^n}{f_0(T)f_1(T)\cdots f_n(T)} = \frac{A_\ell(T)}{B_\ell(T)} \quad \text{and}$$
$$\sum_{n=0}^{\ell+1} \frac{(-1)^n}{f_0(T)f_1(T)\cdots f_n(T)} = \frac{A_\ell(T)}{B_\ell(T)} + \frac{(-1)^{\ell+1}}{f_0(T)f_1(T)\cdots f_{\ell+1}(T)}.$$

Lemma 5.7. *If $f(T) \in (F[x])[T] \setminus \{0\}$, then $A_\ell(f_i(T)) \equiv \pm A_{\ell+i}(T) \mod f_i(T)$ for all $\ell, i \in \mathbb{N} \cup \{0\}$.*

Proof. The case $i = 0$ is trivial.

If $i > 0$ and $\ell = 0$, then the desired result follows from the definition of A_0.

For $\ell, i \geq 1$, we observe that

$$\begin{aligned} A_\ell(f_i(T)) &= (-1)^\ell + \sum_{m=1}^{\ell}(-1)^{m+1} f_m(f_i(T)) \cdots f_\ell(f_i(T)) \\ &= (-1)^\ell + \sum_{m=i+1}^{\ell+i}(-1)^{m+1-i} f_m(T) f_{m+1}(T) \cdots f_{\ell+i}(T). \end{aligned}$$

Consequently,

$$\begin{aligned} &A_{\ell+i}(T) \\ &= (-1)^{\ell+i} + \sum_{m=1}^{i}(-1)^{m+1} f_m(T) \cdots f_{\ell+i}(T) + \sum_{m=i+1}^{\ell+i}(-1)^{m+1} f_m(T) \cdots f_{\ell+i}(T) \\ &= (-1)^i A_\ell(f_i(T)) + \sum_{m=1}^{i}(-1)^{m+1} f_m(T) \cdots f_{\ell+i}(T) \\ &\equiv \pm A_\ell(f_i(T)) \mod f_i(T). \end{aligned}$$

This proves our lemma. □

Next, we substitute T with a nonzero polynomial $Z(x)$ in $F[x]$. For brevity, we will write Z for $Z(x)$.

Lemma 5.8. *If $Z \in F[x] \setminus \{0\}$, then $Z \mid (A_\ell^2(Z) - 1)$ for all $\ell \in \mathbb{N} \cup \{0\}$.*

Proof. Since $f_1(0) = 2$, by induction we have $f_\ell(0) = -2$, for all $\ell \geq 2$. To prove the lemma, it suffices to show that $A_\ell(0) = \pm 1$ for all $\ell \in \mathbb{N} \cup \{0\}$.

Clearly, $A_0(Z) = 1$. Thus,

$$A_1(0) = (-1)^1 + f_1(0) \cdot A_0(0) = -1 + 2 \cdot 1 = 1.$$

By induction, we get $A_\ell(0) = (-1)^{\ell+1}$ for all $\ell \geq 1$, and the desired result follows.

□

Lemma 5.9. *If $Z \in F[x] \setminus F$, then $B_\ell(Z) \neq 0$, $B_\ell(Z) \mid (A_\ell^2(Z) - 1)$ for all $\ell \in \mathbb{N} \cup \{0\}$.*

Proof. For $Z \in F[x] \setminus F$, we have

$$2 \leq |f_0(Z)|_\infty < |f_1(Z)|_\infty < |f_2(Z)|_\infty < \cdots, \tag{5.5}$$

and (5.4) implies that $B_\ell(Z) \neq 0$ for all $\ell \in \mathbb{N} \cup \{0\}$. Now from Lemma 5.8, for all $\ell \geq 0$, we get

$$f_\ell(Z) \mid (A_\ell^2(f_\ell(Z)) - 1). \tag{5.6}$$

We also have from Lemma 5.7 that for $\ell, i \in \mathbb{N} \cup \{0\}$, either

$$A_\ell^2(f_i(Z)) = A_{\ell+i}^2(Z) + 2Df_i(Z)A_{\ell+i}(Z) + D^2 f_i^2(Z),$$

or

$$A_\ell^2(f_i(Z)) = A_{\ell+i}^2(Z) - 2Df_i(Z)A_{\ell+i}(Z) + D^2 f_i^2(Z),$$

for some $D \in F[x]$. By (5.6), for all $\ell, i \in \mathbb{N} \cup \{0\}$, $f_i(Z) \mid (A_{\ell+i}^2(Z) - 1)$. Specifically, for all $i = 0, 1, \ldots \ell$,

$$f_i(Z) \mid (A_{(\ell-i)+i}^2(Z) - 1) = A_\ell^2(Z) - 1. \tag{5.7}$$

It remains to prove that

$$B_\ell(Z) = f_0(Z)f_1(Z)\dots f_\ell(Z) \mid (A_\ell^2(Z) - 1). \tag{5.8}$$

For $j, k \in \mathbb{N} \cup \{0\}$ with $j < k$, since $f_k(Z) = f_{k-j}(f_j(Z)) \equiv f_{k-j}(0) \mod f_j(Z)$, and

$$f_{k-j}(0) = \begin{cases} 2 & \text{for } k = j+1 \\ -2 & \text{for } k > j+1, \end{cases}$$

we deduce that, for all $j \neq k$,

$$\gcd(f_j(Z), f_k(Z)) = \gcd(f_j(Z), 2) \in F, \tag{5.9}$$

i.e., $f_j(Z), f_k(Z)$ are relatively prime. Therefore, (5.8) follows from (5.7) and (5.9). □

An analogoue of Tamuras result in the field of formal Laurent series reads:

Theorem 5.10. *If $Z \in F[x] \setminus F$ is monic (in x), then $\dfrac{1}{f_0(Z)} = [0; Z]$, and for $\ell \geq 1$ if*

$$\sum_{n=0}^{\ell-1} \frac{(-1)^n}{f_0(Z)f_1(Z)\cdots f_n(Z)} = [0; \overrightarrow{X}_{k_\ell}]$$

is a palindromic Ruban continued fraction expansion, then

$$\sum_{n=0}^{\ell} \frac{(-1)^n}{f_0(Z)f_1(Z)\cdots f_n(Z)} = [0; \overrightarrow{X}_{k_\ell}, u_\ell(Z), \overleftarrow{X}_{k_\ell}], \tag{5.10}$$

where

$$u_\ell(Z) = (-1)^{\ell-1}\frac{f_\ell(Z)}{B_{\ell-1}(Z)} - 2\frac{A_{\ell-1}(Z)}{B_{\ell-1}(Z)}, \quad A_{\ell-1}(Z) = C_{k_\ell}, \quad B_{\ell-1}(Z) = D_{k_\ell}$$

with $\dfrac{C_{k_\ell}}{D_{k_\ell}}$ being the ${k_\ell}^{th}$ convergent of $[0; \overrightarrow{X}_{k_\ell}]$.

In particular, the continued fraction expansion representing the infinite sum

(5.2) takes the form

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{f_0(Z)f_1(Z)\cdots f_n(Z)} = [0; Z, u_1(Z), Z, u_2(Z), Z, u_1(Z), Z, u_3(Z), \ldots].$$

Proof. For all $\ell \geq 1$, let $\alpha_\ell = (-1)^{\ell-1} f_0(Z)f_1(Z)\cdots f_{\ell-1}(Z)$ and $\frac{C_{k_\ell}}{D_{k_\ell}} =: [0; \overrightarrow{X}_{k_\ell}]$ be the k_ℓth convergent of the continued fraction expansion of

$$\begin{aligned} \frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \cdots + \frac{1}{\alpha_\ell} &= \frac{1}{f_0(Z)} + \frac{-1}{f_0(Z)f_1(Z)} + \cdots + \frac{(-1)^{\ell-1}}{f_0(Z)f_1(Z)\cdots f_{\ell-1}(Z)} \\ &= \sum_{n=0}^{\ell-1} \frac{(-1)^n}{f_0(Z)f_1(Z)\cdots f_n(Z)}. \end{aligned}$$

Clearly, $\frac{1}{f_0(Z)} = [0; Z] =: \frac{C_{k_1}}{D_{k_1}}$, so k_1 is an odd positive integer.

From Lemma 5.9, we know that $A_{\ell-1}(Z)$ and $B_{\ell-1}(Z)$ are relatively prime. Since $A_{\ell-1}(Z), B_{\ell-1}(Z)$ are monic (in Z) and $B_{\ell-1}(Z), q_{k_\ell}$ are monic (in Z), we infer that

$$B_{\ell-1}(Z) = D_{k_\ell} \tag{5.11}$$

and so $A_{\ell-1}(Z) = C_{k_\ell}$. Since $[0; \overrightarrow{X}_{k_\ell}]$ is palindromic, by Remark 5.6, we have

$$C_{k_\ell} = D_{k_\ell - 1}. \tag{5.12}$$

Next, we show that if $Z \in F[x] \setminus F$, then $u_\ell(Z) \in F[x] \setminus F$ for all $\ell \in \mathbb{N}$. By (5.3), for all $\ell \in \mathbb{N}$, we get

$$A_\ell(Z)^2 = (-1)^{2\ell} + 2(-1)^\ell f_\ell(Z)A_{\ell-1}(Z) + f_\ell(Z)^2 A_{\ell-1}(Z)^2. \tag{5.13}$$

By Lemma 5.9 and (5.13), we get

$$\begin{aligned} &u_\ell(Z) \\ &= (-1)^{\ell-1}(Z)\frac{f_\ell(Z)}{B_{\ell-1}(Z)} - \frac{2A_{\ell-1}(Z)}{B_{\ell-1}(Z)} \\ &= (-1)^\ell \left(-\frac{f_\ell(Z)}{B_{\ell-1}(Z)} + \frac{f_\ell(Z)A_{\ell-1}(Z)^2}{B_{\ell-1}(Z)} - \frac{f_\ell(Z)A_{\ell-1}(Z)^2}{B_{\ell-1}(Z)} \right) - \frac{2A_{\ell-1}(Z)}{B_{\ell-1}(Z)} \end{aligned}$$

$$
\begin{aligned}
&= (-1)^{\ell}\left(\frac{f_\ell(Z)\left(A_{\ell-1}(Z)^2-1\right)}{B_{\ell-1}(Z)}\right) + (-1)^{\ell-1}\left(\frac{f_\ell(Z)A_{\ell-1}(Z)^2 + (-1)^{\ell}2A_{\ell-1}(Z)}{B_{\ell-1}(Z)}\right)\\
&= (-1)^{\ell} f_\ell(Z)\frac{(A_{\ell-1}(Z)^2-1)}{B_{\ell-1}(Z)} + (-1)^{\ell-1}\left(\frac{f_\ell(Z)^2A_{\ell-1}(Z)^2 + (-1)^{\ell}2A_{\ell-1}(Z)f_\ell(Z)}{B_\ell(Z)}\right)\\
&= (-1)^{\ell} f_\ell(Z)\frac{(A_{\ell-1}(Z)^2-1)}{B_{\ell-1}(Z)} + (-1)^{\ell-1}\frac{(A_\ell(Z)^2-1)}{B_\ell(Z)} \in F[x]\setminus F.
\end{aligned}
$$

For all $\ell \in \mathbb{N}$, from (5.11) and (5.12), we get

$$
\begin{aligned}
&-\left(2D_{k_\ell-1}D_{k_\ell} + q_{k_\ell}^2 u_\ell\right)\\
&\quad = -\left(2A_{\ell-1}(Z)B_{\ell-1}(Z) + B_{\ell-1}(Z)^2\left((-1)^{\ell-1}\frac{f_\ell(Z)}{B_{\ell-1}(Z)} - 2\frac{A_{\ell-1}(Z)}{B_{\ell-1}(Z)}\right)\right)\\
&\quad = -\left(2A_{\ell-1}(Z)B_{\ell-1}(Z) + \left((-1)^{\ell-1}f_\ell(Z)B_{\ell-1}(Z) - 2A_{\ell-1}(Z)B_{\ell-1}(Z)\right)\right)\\
&\quad = (-1)^{\ell}B_{\ell-1}(Z)f_\ell(Z)\\
&\quad = (-1)^{\ell}f_0(Z)f_1(Z)\cdots f_\ell(Z)\\
&\quad = \alpha_{\ell+1}.
\end{aligned}
$$

We observe that $\{k_\ell\}_{\ell\geq 1}$ obtained by this process is a sequence of odd positive integers so

$$(-1)^{k_\ell}\left(2D_{k_\ell-1}D_{k_\ell} + q_{k_\ell}^2 u_\ell\right) = \alpha_{\ell+1} \quad \text{for all} \quad \ell \in \mathbb{N}.$$

Using Theorem 5.2 (3), we get

$$[0; \overrightarrow{X}_{k_\ell}, u_\ell(Z), \overleftarrow{X}_{k_\ell}] = \sum_{i=1}^{\ell+1}\frac{1}{\alpha_i} = \sum_{n=0}^{\ell}\frac{(-1)^n}{f_0(Z)f_1(Z)\cdots f_n(Z)}$$

and the proof is complete. □

The proof of the following theorem with some minor changes is also applicable to some other forms of $f(T)$ such as $T(T+2)(T-2)g(T)+T^2-2$.

Theorem 5.11. *If $Z \in F[x]\setminus F$ is monic (in x), then $\frac{1}{f_0(Z)} = [0; Z]$, and for $\ell \geq 1$ if*

$$\sum_{n=0}^{\ell-1}\frac{(-1)^n}{f_0(Z)f_1(Z)\cdots f_n(Z)} = [0; \overrightarrow{X}_{k_\ell}]$$

is a palindromic Ruban continued fraction expansion, then

$$\sum_{n=0}^{\ell} \frac{(-1)^n}{f_0(Z)f_1(Z)\cdots f_n(Z)} = [0; \overrightarrow{X}_{k_\ell}, v_\ell(Z), \overleftarrow{X}_{k_\ell}],$$

where

$$v_\ell(Z) = (-1)^{\ell-1}\frac{f_\ell(Z)}{B_{\ell-1}(Z)} - 2\frac{A_{\ell-1}(Z)}{B_{\ell-1}(Z)}, \quad A_{\ell-1}(Z) = C_{k_\ell}, \quad B_{\ell-1}(Z) = D_{k_\ell}$$

with $\frac{C_{k_\ell}}{D_{k_\ell}}$ *being the* ${k_\ell}^{th}$ *convergent of* $[0; \overrightarrow{X}_{k_\ell}]$.

In particular, the continued fraction expansion representing the infinite sum (5.2) takes the form

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{f_0(Z)f_1(Z)\cdots f_n(Z)} = [0; Z, v_1(Z), Z, v_2(Z), Z, v_1(Z), Z, v_3(Z), \ldots].$$

5.3.3 Two-fold continued fraction expansion of type 4

We can apply the result of Theorem 5.2 (4) to prove Proposition 4.8 as in Section 4.2, by putting $\alpha_{\ell-1} = P_{\ell+1}$ for all $\ell \geq I + 1$ in case of formal series over a finite base field.

5.3.4 Three-fold continued fraction expansion of type 13

Here we are interested in the work of Cohn [8], who considered series expansions of real numbers of the form

$$\sum_{n=0}^{\infty} \frac{1}{f^n(x)},$$

where $f^0(x) = x$ and $f^n(x) = f\left(f^{n-1}(x)\right)$ for all $n \geq 1$, the n^{th} iterate of $f(x)$. Cohn showed that, such a series expansion has explicit Ruban continued fraction expansion if and only if $f(x)$ satisfies one of fourteen congruence conditions.

In this subsection, we consider one of fourteen congruence conditions of Cohn and extend Cohn result in the function field with respect to the degree valuation. The main tool here is Theorem 5.4 (13).

Let $f(T) = T^2(T-1)g(T)+1$ be a monic polynomial (in T) in $(F[x])[T]$, for some $g(T) \in (F[x])[T]$. We observe that, $g(T)$ is a monic polynomial (in T).

We consider series expansions of the form, for all $\ell \geq 0$

$$\sum_{n=0}^{\ell} \frac{1}{f^n(T)},$$

where $f^0(T) = T$ and $f^n(T) = f\left(f^{n-1}(T)\right)$ for all $n \geq 1$, the n^{th} iterate of $f(T)$. Then, for all $n \geq 1$,

$$f^n(T) = f^{n-1}(T)^2\left(f^{n-1}(T)-1\right)g_n(T)+1$$

for some $g_n(T) := g\left(f^{n-1}(T)\right) \in (F[x])[T]$.

Let $A_n(T) = Tf(T)\cdots f^n(T)$ for all $n \geq 0$.

Theorem 5.12. *If $Z \in F[x] \setminus F$ is monic (in x), then*

$$\sum_{n=0}^{1} \frac{1}{f^n(Z)} = [0; Z, -g(Z)(Z-1), -Z+1, -Z-1],$$

and for $\ell \geq 2$, if $\displaystyle\sum_{n=0}^{\ell-1} \frac{1}{f^n(Z)} = [0; \overrightarrow{X}_{k_\ell}]$, *then*

$$\sum_{n=0}^{\ell} \frac{1}{f^n(Z)} = [0; \overrightarrow{X}_{k_\ell}, \frac{g_\ell(Z)g_{\ell-1}(Z)\left(f^{\ell-2}(Z)-1\right)}{A_{\ell-3}(Z)^2}, -\overleftarrow{X}_{k_\ell}, 1, \overrightarrow{X}_{k_\ell}].$$

Proof. For all $\ell \geq 1$, let $\alpha_\ell = f^{\ell-1}(Z)$ and $\dfrac{C_{k_\ell}}{D_{k_\ell}} =: [0; \overrightarrow{X}_{k_\ell}]$ be the k_ℓ^{th} convergent of the continued fraction expansion of

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \cdots + \frac{1}{\alpha_\ell} = \frac{1}{Z} + \frac{1}{f(Z)} + \cdots + \frac{1}{f^{\ell-1}(Z)} = \sum_{n=0}^{\ell-1} \frac{1}{f^n(Z)}.$$

First, we claim that $D_{k_\ell} = A_{\ell-1}(Z)$ for all $\ell \geq 1$. For all $\ell \geq 1$, consider

$$f^\ell(Z) - 1 = f^{\ell-1}(Z)^2\left(f^{\ell-1}(Z)-1\right)g_\ell(Z)$$

$$= f^{\ell-1}(Z)^2 f^{\ell-2}(Z)^2 \left(f^{\ell-2}(Z) - 1\right) g_{\ell-1}(Z) g_\ell(Z)$$

$$\vdots$$

$$= f^{\ell-1}(Z)^2 f^{\ell-2}(Z)^2 \cdots f(Z)^2 \left(f(Z) - 1\right) g_2(Z) \cdots g_{\ell-1}(Z) g_\ell(Z)$$

$$= f^{\ell-1}(Z)^2 f^{\ell-2}(Z)^2 \cdots f(Z)^2 Z^2 \left(Z - 1\right) g(Z) g_2(Z) \cdots g_\ell(Z),$$

and so

$$A_{\ell-1}(Z)^2 \mid \left(f^\ell(Z) - 1\right). \tag{5.14}$$

Then

$$f^\ell(Z) = A_{\ell-1}(Z)^2 B_\ell + 1 \quad \text{for some} \quad B_\ell \in (F[x])[T]. \tag{5.15}$$

For all $\ell \geq 1$, consider

$$\begin{aligned} \frac{C_{k_\ell}}{D_{k_\ell}} &= \frac{1}{Z} + \frac{1}{f(Z)} + \cdots + \frac{1}{f^{\ell-1}(Z)} \\ &= \frac{\left(f(Z) \cdots f^{\ell-1}(Z)\right) + \left(Zf^2(Z) \cdots f^{\ell-1}(Z)\right) + \cdots + \left(Zf(Z) \cdots f^{\ell-2}(Z)\right)}{Zf(Z) \cdots f^{\ell-1}(Z)}. \end{aligned}$$

Suppose that there exists a prime element $p \in F[x]$ such that

$$p \mid \left(\left(f(Z) \cdots f^{\ell-1}(Z)\right) + \left(Zf^2(Z) \cdots f^{\ell-1}(Z)\right) + \cdots + \left(Zf(Z) \cdots f^{\ell-2}(Z)\right)\right)$$

and $p \mid Zf(Z) \cdots f^{\ell-1}(Z)$.
Since $p \mid Zf(Z) \cdots f^{\ell-1}(Z)$, $p \mid f^r(Z)$ for some $0 \leq r \leq \ell - 1$, which implies that $p \mid Zf(Z) \cdots f^{r-1}(Z) f^{r+1}(Z) f^{r+2}(Z) \cdots f^{\ell-1}(Z)$. Then

$$p \mid Zf(Z) \cdots f^{r-1}(Z) \quad \text{or} \quad p \mid f^{r+1}(Z) f^{r+2}(Z) \cdots f^{\ell-1}(Z).$$

If $p \mid Zf(Z) \cdots f^{r-1}(Z)$, using (5.14), we see that $p \mid \left(f^r(Z) - 1\right)$, contradicting $p \mid f^r(Z)$. Then $p \mid f^{r+1}(Z) f^{r+2}(Z) \cdots f^{\ell-1}(Z)$.
By (5.15), we get

$$\begin{aligned} &f^{r+1}(Z) f^{r+2}(Z) \cdots f^{\ell-1}(Z) \\ &\quad = \left(A_r(Z)^2 B_{r+1} + 1\right) \left(A_{r+1}(Z)^2 B_{r+2} + 1\right) \cdots \left(A_{\ell-2}(Z)^2 B_{\ell-1} + 1\right). \end{aligned}$$

Since $f^r(Z) \mid A_j(Z)$ for all $r \leq j$ and $p \mid f^r(Z)$, so we get $p \mid 1$, which is a contradiction.

Then, $\left(f(Z)\cdots f^{\ell-1}(Z)\right) + \left(Zf^2(Z)\cdots f^{\ell-1}(Z)\right) + \cdots + \left(Zf(Z)\cdots f^{\ell-2}(Z)\right)$ and $Zf(Z)\cdots f^{\ell-1}(Z)$ are relatively prime.

Since Z is monic, we get $D_{k_\ell} = Zf(Z)\cdots f^{\ell-1}(Z) = A_{\ell-1}(Z)$ for all $\ell \geq 1$.

Thus, the claim is proved.

Next, we consider

$$\begin{aligned}
\alpha_{\ell+1} &= f^\ell(Z) \\
&= f^{\ell-1}(Z)^2\left(f^{\ell-1}(Z)-1\right)g_\ell(Z)+1 \\
&= f^{\ell-1}(Z)^2 f^{\ell-2}(Z)^2\left(f^{\ell-2}(Z)-1\right)g_{\ell-1}(Z)g_\ell(Z)+1 \\
&= \frac{A_{\ell-1}(Z)^2}{A_{\ell-1}(Z)^2} f^{\ell-1}(Z)^2 f^{\ell-2}(Z)^2\left(f^{\ell-2}(Z)-1\right)g_{\ell-1}(Z)g_\ell(Z)+1 \\
&= A_{\ell-1}(Z)^2\frac{\left(f^{\ell-2}(Z)-1\right)g_{\ell-1}(Z)g_\ell(Z)}{A_{\ell-3}(Z)^2}+1 \\
&= A_{\ell-1}(Z)^2 Y_1 + \frac{1}{Y_2},
\end{aligned}$$

where $Y_1 := \dfrac{\left(f^{\ell-2}(Z)-1\right)g_{\ell-1}(Z)g_\ell(Z)}{A_{\ell-3}(Z)^2}$ and $Y_2 := 1$.

By (5.14), we get $Y_1 \in F[x] \setminus \{0\}$.

Let $\overrightarrow{X}_{k_2}$ be the word $Z, -g(Z)(Z-1), -Z+1, -Z-1$. Then

$$[0; \overrightarrow{X}_{k_2}] = [0; Z, -g(Z)(Z-1), -Z+1, -Z-1] = \frac{1}{Z} + \frac{1}{f(Z)}.$$

We observe that $\{k_\ell\}_{\ell\geq 2}$ obtained by this process is a sequence of even positive integers. Then

$$\alpha_{\ell+1} = (-1)^{k_\ell} D_{k_\ell}^2 Y_1 + \frac{1}{Y_2},$$

where $Y_1, Y_2 = 1 \in F[x] \setminus \{0\}$.

Using Theorem 5.4 (13), we get

$$[0; \overrightarrow{X}_{k_\ell}, \frac{g_\ell(Z)g_{\ell-1}(Z)\left(f^{\ell-2}(Z)-1\right)}{A_{\ell-3}(Z)^2}, -\overleftarrow{X}_{k_\ell}, 1, \overrightarrow{X}_{k_\ell}] = \sum_{i=1}^{\ell+1}\frac{1}{\alpha_i} = \sum_{n=0}^{\ell}\frac{1}{f^n(Z)}.$$

This completes the proof. □