CHAPTER III
COMPLETE CONVERGENCE FOR RANDOM VECTORS IN HILBERT SPACES

Let H be a real separable Hilbert space with the norm $\|\cdot\|$ generated by an inner product $\langle\cdot,\cdot\rangle$ and let $\mathbf{X}$ be an $H-$valued random vector. The inner product $\langle\mathbf{X},\mathbf{e}_j\rangle$ will be denoted by $X^{(j)}$ where $\{\mathbf{e}_j, j\geq 1\}$ is an orthonormal basis in H.

First, we introduce a new dependence concept for a sequence of random vectors taking values in real separable Hilbert spaces called coordinatewise widely orthant dependent random vectors as the following.

Definition 3.1. A sequence $\{\mathbf{X}_n, n\geq 1\}$ of H-valued random vectors is said to be *coordinatewise widely orthant dependent (CWOD)* with dominating coefficient $g(n)$ if for each $j\geq 1$, the sequence $\left\{X_n^{(j)}, n\geq 1\right\}$ of random variables is WOD with dominating coefficient $g(n)$.

Wang et al. [21] showed that NA random variables must be WOD but WOD random variables are not necessarily NA. Therefore, by Definition 3.1, we can see that if a sequence $\{\mathbf{X}_n, n\geq 1\}$ of $H-$valued random vectors is CNA, then it is CWOD but the converse is not true.

The concept of coordinatewise weakly upper bound will be used in this work stated as follows.

Definition 3.2. ([11]) A sequence $\{\mathbf{X}_n, n\geq 1\}$ of $H-$valued random vectors is said to be *coordinatewise weakly upper bounded* by a random vector $\mathbf{X}$ if there exists a positive constant C such that

$$\frac{1}{n}\sum_{i=1}^{n}P\left(\left|X_i^{(j)}\right|>x\right)\leq CP\left(\left|X^{(j)}\right|>x\right)$$

for all $x \geq 0, j \geq 1$ and $n \geq 1$.

In 2014, Huan et al. [11] obtained the complete convergence for a sequence of CNA random vectors. In this chapter, we extend the result of Huan N.V. et al. [11] to the complete convergence for a sequence of CWOD random vectors.

Throughout this work, let $\{\mathbf{X}_n, n \geq 1\}$ be a sequence of $H-$valued CWOD random vectors with dominating coefficients $g_u(n)$ and $g_l(n)$ for all $n \geq 1$. Let $g(n) = \max\{g_u(n), g_l(n)\}$, $\log x = \ln(\max\{x, e\})$ and let $\mathbb{I}(\cdot)$ be the indicator function and C represents positive constant not depending on n which may be different at different places. Our result is stated as follows.

Theorem 3.3. *Let $\{\mathbf{X}_n, n \geq 1\}$ be a sequence of H-valued CWOD random vectors with mean zero and dominating coefficients $g(n)$, with $g(n) = O\left(n^{\alpha\left(1-\frac{r}{2}\right)} \log^{-2} n\right)$ where α and r are positive real numbers such that $\alpha r \geq 1$ and $0 < r < 2$. Assume that $\{\mathbf{X}_n, n \geq 1\}$ is coordinatewise weakly upper bounded by a random vector $\mathbf{X}$ with $\sum_{j=1}^{\infty} E\left|X^{(j)}\right|^2 \log^2\left(1 + \left|X^{(j)}\right|\right) < \infty$. Then for all $\epsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{\alpha r-2} P\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k} \mathbf{X}_i\right\| > \epsilon n^{\alpha}\right) < \infty.$$

3.1 Auxiliary Results

In this section, we give the following lemmas which will be used to prove our main result.

Lemma 3.4. *([8]) Assume that $\{Y_n, n \geq 1\}$ is a sequence of random variables which are coordinatewise weakly upper bounded by a random variable Y. Then for all $n \in \mathbb{N}, \alpha > 0$ and $b > 0$, there exist positive constants C_1 and C_2 such that*

$$\frac{1}{n}\sum_{i=1}^{n} E\left|Y_i\right|^{\alpha} \mathbb{I}\left(\left|Y_i\right| \leq b\right) \leq C_1\left[E\left|Y\right|^{\alpha} \mathbb{I}\left(\left|Y\right| \leq b\right) + b^{\alpha} P\left(\left|Y\right| > b\right)\right]$$

and

$$\frac{1}{n}\sum_{i=1}^{n} E\left|Y_i\right|^{\alpha} \mathbb{I}\left(\left|Y_i\right| > b\right) \leq C_2 E\left|Y\right|^{\alpha} \mathbb{I}\left(\left|Y\right| > b\right).$$

Lemma 3.5. *Let* $\alpha > \frac{1}{2}$ *and let* $\{\mathbf{X}_n, n \geq 1\}$ *be a sequence of H-valued CWOD random vectors with mean zero which is coordinatewise weakly upper bounded by a random vector* $\mathbf{X}$ *with* $\sum_{j=1}^{\infty} E\left|X^{(j)}\right|^2 < \infty$. *For* $n, i, j \geq 1$, *define* $\mathbf{Y}_{ni} = \sum_{j=1}^{\infty} Y_{ni}^{(j)} \mathbf{e}_j$ *where* $Y_{ni}^{(j)} = -n^\alpha \mathbb{I}\left(X_i^{(j)} < -n^\alpha\right) + X_i^{(j)} \mathbb{I}\left(\left|X_i^{(j)}\right| \leq n^\alpha\right) + n^\alpha \mathbb{I}\left(X_i^{(j)} > n^\alpha\right)$. *Then*

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \max_{1\leq k\leq n} \left\| \sum_{i=1}^{k} E\mathbf{Y}_{ni} \right\| = 0.$$

Proof. By the assumption $EX_i^{(j)} = 0$ for $i, j \geq 1$, Definition 3.2 and Lemma 3.4, we obtain that

$$\begin{aligned}
&\frac{1}{n^\alpha} \max_{1\leq k\leq n} \left\| \sum_{i=1}^{k} E\mathbf{Y}_{ni} \right\| \\
&\leq n^{-\alpha} \max_{1\leq k\leq n} \sum_{j=1}^{\infty} \sum_{i=1}^{k} \left| EY_{ni}^{(j)} \right| \\
&= n^{-\alpha} \max_{1\leq k\leq n} \sum_{j=1}^{\infty} \sum_{i=1}^{k} \left| -n^\alpha P\left(X_i^{(j)} < -n^\alpha\right) + EX_i^{(j)} \mathbb{I}\left(\left|X_i^{(j)}\right| \leq n^\alpha\right) + n^\alpha P\left(X_i^{(j)} > n^\alpha\right) \right| \\
&= n^{-\alpha} \max_{1\leq k\leq n} \sum_{j=1}^{\infty} \sum_{i=1}^{k} \left| -n^\alpha P\left(X_i^{(j)} < -n^\alpha\right) - EX_i^{(j)} \mathbb{I}\left(\left|X_i^{(j)}\right| > n^\alpha\right) + n^\alpha P\left(X_i^{(j)} > n^\alpha\right) \right| \\
&\leq n^{-\alpha} \max_{1\leq k\leq n} \sum_{j=1}^{\infty} \sum_{i=1}^{k} \left[n^\alpha P\left(\left|X_i^{(j)}\right| > n^\alpha\right) + E\left|X_i^{(j)}\right| \mathbb{I}\left(\left|X_i^{(j)}\right| > n^\alpha\right) \right] \\
&\leq Cn^{-\alpha} \sum_{j=1}^{\infty} \left[n^{\alpha+1} P\left(\left|X^{(j)}\right| > n^\alpha\right) + nE\left|X^{(j)}\right| \mathbb{I}\left(\left|X^{(j)}\right| > n^\alpha\right) \right] \\
&\leq Cn^{1-\alpha} \sum_{j=1}^{\infty} E\left|X^{(j)}\right| \mathbb{I}\left(\left|X^{(j)}\right| > n^\alpha\right) \\
&\leq Cn^{1-2\alpha} \sum_{j=1}^{\infty} E\left|X^{(j)}\right|^2 \mathbb{I}\left(\left|X^{(j)}\right| > n^\alpha\right).
\end{aligned}$$

Hence $\frac{1}{n^\alpha} \max_{1\leq k\leq n} \left\| \sum_{i=1}^{k} E\mathbf{Y}_{ni} \right\|$ converges to 0 as n goes to infinity. □

By using Lemma 2.45 and the similar argument as the proof of Lemma 2.3 in Ko et al. [14], we can get the following lemma.

Lemma 3.6. *Let* $\{\mathbf{X}_n, n \geq 1\}$ *be a sequence of H-valued CWOD random vectors with dominating coefficient* $g(n)$ *having mean zero and* $E\|\mathbf{X}_n\|^2 < \infty$ *for each* $n \geq 1$. *Then there exists a constant* C *such that*

$$E\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k}\mathbf{X}_i\right\|\right)^2 \leq C(1+g(n))\log^2 n\sum_{i=1}^{n}E\|\mathbf{X}_i\|^2.$$

Proof. From Theorem 2.34, we have

$$\begin{aligned}
E\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k}\mathbf{X}_i\right\|\right)^2 &= E\left(\max_{1\leq k\leq n}\left(\sum_{j=1}^{\infty}\left|\left\langle\sum_{i=1}^{k}\mathbf{X}_i, \mathbf{e}_j\right\rangle\right|^2\right)^{\frac{1}{2}}\right)^2 \\
&= E\left(\max_{1\leq k\leq n}\sum_{j=1}^{\infty}\left|\left\langle\sum_{i=1}^{k}\mathbf{X}_i, \mathbf{e}_j\right\rangle\right|^2\right) \\
&= E\left(\max_{1\leq k\leq n}\sum_{j=1}^{\infty}\left|\sum_{i=1}^{k}X_i^{(j)}\right|^2\right) \\
&\leq \sum_{j=1}^{\infty}E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}X_i^{(j)}\right|\right)^2 \\
&\leq C(1+g(n))\log^2 n\sum_{j=1}^{\infty}\sum_{i=1}^{n}E\left|X_i^{(j)}\right|^2 \\
&= C(1+g(n))\log^2 n\sum_{i=1}^{n}E\|\mathbf{X}_i\|^2,
\end{aligned}$$

where we use Lemma 2.45 to obtain the last inequality. □

3.2 Proof of Main Result

Proof of Theorem 3.3. Let $\epsilon > 0$, from the notation of Y_{ni} in Lemma 3.5, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}n^{\alpha r-2}P\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k}\mathbf{X}_i\right\| > \epsilon n^{\alpha}\right) \\
&= \sum_{n=1}^{\infty}n^{\alpha r-2}P\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k}\sum_{j=1}^{\infty}X_i^{(j)}\mathbf{e}_j\right\| > \epsilon n^{\alpha}\right)
\end{aligned}$$

$$\leq \sum_{n=1}^{\infty} n^{\alpha r-2} P\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k}\sum_{j=1}^{\infty} Y_{ni}^{(j)}\mathbf{e}_j\right\| > \epsilon n^{\alpha}\right) + \sum_{n=1}^{\infty} n^{\alpha r-2}\sum_{j=1}^{\infty}\sum_{i=1}^{n} P\left(\left|X_i^{(j)}\right| > n^{\alpha}\right)$$

$$= \sum_{n=1}^{\infty} n^{\alpha r-2} P\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k} \mathbf{Y}_{ni}\right\| > \epsilon n^{\alpha}\right) + \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} n^{\alpha r-2}\sum_{i=1}^{n} P\left(\left|X_i^{(j)}\right| > n^{\alpha}\right)$$

$$=: A_1 + A_2.$$

To prove the main result, we will show that A_1 and A_2 are finite.

First, we can show that $A_2 < \infty$ as follows. By Definition 3.2,

$$\begin{aligned}
A_2 &=: \sum_{n=1}^{\infty} n^{\alpha r-2}\sum_{j=1}^{\infty}\sum_{i=1}^{n} P\left(\left|X_i^{(j)}\right| > n^{\alpha}\right)\\
&\leq C\sum_{n=1}^{\infty} n^{\alpha r-2\alpha-1}\sum_{j=1}^{\infty} E\left|X^{(j)}\right|^2\\
&\leq C\sum_{n=1}^{\infty} n^{\alpha r-2\alpha-1}\\
&< \infty.
\end{aligned}$$

To prove $A_1 < \infty$, we know from Lemma 3.5 that $\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k} E\mathbf{Y}_{ni}\right\| < \frac{\epsilon n^{\alpha}}{2}$ while n is sufficiently large.

Moreover, from Proposition 2.46, we have $\{Y_{ni}, i \geq 1\}$ is still CWOD.

Hence, by Lemma 3.6, Definition 3.2 and Lemma 3.4,

$$\begin{aligned}
A_1 &\leq \sum_{n=1}^{\infty} n^{\alpha r-2} P\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k}(\mathbf{Y}_{ni} - E\mathbf{Y}_{ni})\right\| + \max_{1\leq k\leq n}\left\|\sum_{i=1}^{k} E\mathbf{Y}_{ni}\right\| > \epsilon n^{\alpha}\right)\\
&\leq \sum_{n=1}^{\infty} n^{\alpha r-2} P\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k}(\mathbf{Y}_{ni} - E\mathbf{Y}_{ni})\right\| > \frac{\epsilon n^{\alpha}}{2}\right)\\
&\leq C\sum_{n=1}^{\infty} n^{\alpha r-2-2\alpha} E\left(\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k}(\mathbf{Y}_{ni} - E\mathbf{Y}_{ni})\right\|^2\right)\\
&\leq C\sum_{n=1}^{\infty}(1+g(n))\, n^{\alpha r-2-2\alpha}\log^2 n\sum_{i=1}^{n} E\left\|\mathbf{Y}_{ni} - E\mathbf{Y}_{ni}\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&= C\sum_{n=1}^{\infty}(1+g(n))\,n^{\alpha r-2-2\alpha}\log^2 n\sum_{j=1}^{\infty}\sum_{i=1}^{n}E\left|Y_{ni}^{(j)}-EY_{ni}^{(j)}\right|^2\\
&\le C\sum_{n=1}^{\infty}(1+g(n))\,n^{\alpha r-2-2\alpha}\log^2 n\sum_{j=1}^{\infty}\sum_{i=1}^{n}E\left|Y_{ni}^{(j)}\right|^2\\
&= C\sum_{n=1}^{\infty}(1+g(n))\,n^{\alpha r-2-2\alpha}\log^2 n\sum_{j=1}^{\infty}\sum_{i=1}^{n}E\Big|-n^{\alpha}\mathbb{I}\left(X_i^{(j)}<-n^{\alpha}\right)\\
&\quad+X_i^{(j)}\mathbb{I}\left(\left|X_i^{(j)}\right|\le n^{\alpha}\right)+n^{\alpha}\mathbb{I}\left(X_i^{(j)}>n^{\alpha}\right)\Big|^2\\
&\le C\sum_{n=1}^{\infty}(1+g(n))\,n^{\alpha r-2-2\alpha}\log^2 n\sum_{j=1}^{\infty}\sum_{i=1}^{n}\Big(n^{2\alpha}P\left(\left|X_i^{(j)}\right|>n^{\alpha}\right)\\
&\quad+E\left|X_i^{(j)}\right|^2\mathbb{I}\left(\left|X_i^{(j)}\right|\le n^{\alpha}\right)\Big)\\
&\le C\sum_{n=1}^{\infty}(1+g(n))\,n^{\alpha r-2-2\alpha}\log^2 n\sum_{j=1}^{\infty}\Big(n^{2\alpha+1}P\left(\left|X^{(j)}\right|>n^{\alpha}\right)\\
&\quad+nE\left|X^{(j)}\right|^2\mathbb{I}\left(\left|X^{(j)}\right|\le n^{\alpha}\right)\Big)\\
&= C\sum_{n=1}^{\infty}n^{\alpha r-1-2\alpha}\log^2 n\sum_{j=1}^{\infty}\Big(n^{2\alpha}P\left(\left|X^{(j)}\right|>n^{\alpha}\right)+E\left|X^{(j)}\right|^2\mathbb{I}\left(\left|X^{(j)}\right|\le n^{\alpha}\right)\Big)\\
&\quad+C\sum_{n=1}^{\infty}g(n)n^{\alpha r-1-2\alpha}\log^2 n\sum_{j=1}^{\infty}\Big(n^{2\alpha}P\left(\left|X^{(j)}\right|>n^{\alpha}\right)+E\left|X^{(j)}\right|^2\mathbb{I}\left(\left|X^{(j)}\right|\le n^{\alpha}\right)\Big)\\
&=: A_{11}+A_{12}.
\end{aligned}
\tag{3.1}
$$

Next, we will show that $A_{11}<\infty$,

$$
\begin{aligned}
A_{11} &= C\sum_{n=1}^{\infty}n^{\alpha r-1}\log^2 n\sum_{j=1}^{\infty}\sum_{k=n}^{\infty}P\left(k^{\alpha}<\left|X^{(j)}\right|\le(k+1)^{\alpha}\right)\\
&\quad+C\sum_{n=1}^{\infty}n^{\alpha r-1-2\alpha}\log^2 n\sum_{j=1}^{\infty}\sum_{k=1}^{n}E\left|X^{(j)}\right|^2\mathbb{I}\left((k-1)^{\alpha}<\left|X^{(j)}\right|\le k^{\alpha}\right)\\
&= C\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}P\left(k^{\alpha}<\left|X^{(j)}\right|\le(k+1)^{\alpha}\right)\sum_{n=1}^{k}n^{\alpha r-1}\log^2 n\\
&\quad+C\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}E\left|X^{(j)}\right|^2\mathbb{I}\left((k-1)^{\alpha}<\left|X^{(j)}\right|\le k^{\alpha}\right)\sum_{n=k}^{\infty}n^{\alpha r-1-2\alpha}\log^2 n
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}k^{\alpha r}\log^{2}kP\left(k^{\alpha}<\left|X^{(j)}\right|\leq(k+1)^{\alpha}\right)\\
&\quad+C\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}k^{\alpha r-2\alpha}\log^{2}kE\left|X^{(j)}\right|^{2}\mathbb{I}\left((k-1)^{\alpha}<\left|X^{(j)}\right|\leq k^{\alpha}\right)\\
&\leq C\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}k^{\alpha r}\log^{2}kE\mathbb{I}\left(k^{\alpha}<\left|X^{(j)}\right|\leq(k+1)^{\alpha}\right)\\
&\quad+C\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}k^{\alpha r}\log^{2}kE\mathbb{I}\left((k-1)^{\alpha}<\left|X^{(j)}\right|\leq k^{\alpha}\right)\\
&\leq C\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}k^{2\alpha}\log^{2}kE\mathbb{I}\left(k^{\alpha}<\left|X^{(j)}\right|\leq(k+1)^{\alpha}\right)\\
&\quad+C\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}k^{2\alpha}\log^{2}kE\mathbb{I}\left((k-1)^{\alpha}<\left|X^{(j)}\right|\leq k^{\alpha}\right)\\
&\leq C\sum_{j=1}^{\infty}E\left|X^{(j)}\right|^{2}\log^{2}\left(1+\left|X^{(j)}\right|\right)\\
&<\infty. \qquad (3.2)
\end{aligned}
$$

Next, we will show that A_{12} is finite. Since $g(n)=O\left(n^{\alpha\left(1-\frac{r}{2}\right)}\log^{-2}n\right)$, we obtain that

$$
\begin{aligned}
A_{12}&\leq C\sum_{n=1}^{\infty}n^{\frac{\alpha r}{2}-1-\alpha}\sum_{j=1}^{\infty}\left(E\left|X^{(j)}\right|^{2}\mathbb{I}\left(\left|X^{(j)}\right|>n^{\alpha}\right)+E\left|X^{(j)}\right|^{2}\mathbb{I}\left(\left|X^{(j)}\right|\leq n^{\alpha}\right)\right)\\
&\leq C\sum_{j=1}^{\infty}E\left|X^{(j)}\right|^{2}\sum_{n=1}^{\infty}n^{\alpha\left(\frac{r}{2}-1\right)-1}\\
&\leq C\sum_{n=1}^{\infty}n^{\alpha\left(\frac{r}{2}-1\right)-1}\\
&<\infty. \qquad (3.3)
\end{aligned}
$$

From (3.1)–(3.3), we have A_1 is finite. Therefore, by the proof is completed. □