CHAPTER IV

COMPLETE MOMENT CONVERGENCE FOR RANDOM VECTORS IN HILBERT SPACES

The concept of complete convergence has been generalized to a more general concept of convergence which is complete moment convergence, introduced in Chow [4]. Later, in 2017, Ding et al. [6] discussed the complete moment convergence for a sequence of WOD random variables with dominating coefficient $g(n)$ such that $g(n) = O\left(n^{\alpha t}\log^{-2} n\right)$ and Liu et al. [18] obtained the complete moment convergence for a sequence of WOD random variables with any dominating coefficient $g(n)$.

In this chapter, we extend the complete moment convergence obtained in Ding et al. [6] and Liu et al. [18] to obtain the complete moment convergence of $H-$valued CWOD random vectors presented in Theorem 4.1 and Theorem 4.2, respectively. Throughout this work, denote $x_+^q = (x_+)^q$ and $x_+ = \max\{x, 0\}$.

Theorem 4.1. *Let $\{\mathbf{X}_n, n \geq 1\}$ be a sequence of $H-$valued CWOD random vectors with mean zero and dominating coefficients $g(n)$ with $g(n) = O\left(n^{\alpha(1-r/2)}\log^{-2} n\right)$ where $\alpha r \geq 1$ and $0 < r < 2$. Assume that $\{\mathbf{X}_n, n \geq 1\}$ is coordinatewise weakly upper bounded by a random vector $\mathbf{X}$ with $\sum_{j=1}^{\infty} E\left|X^{(j)}\right|^2 \log^3\left(1 + \left|X^{(j)}\right|\right) < \infty$. Then, for $0 < q < r$ and $\epsilon > 0$,*

$$\sum_{n=1}^{\infty} n^{\alpha r - \alpha q - 2} E\left[\max_{1\leq k\leq n}\left\|\sum_{i=1}^{k}\mathbf{X}_i\right\| - \epsilon n^{\alpha}\right]_+^q < \infty.$$

Theorem 4.2. *Let $\{\mathbf{X}_n, n \geq 1\}$ be a sequence of $H-$valued CWOD random vectors with mean zero and dominating coefficients $g(n)$. Assume that $\{\mathbf{X}_n, n \geq 1\}$ is coordinatewise weakly upper bounded by a random vector $\mathbf{X}$ with*

$\sum_{j=1}^{\infty} E\left|X^{(j)}\right|^{rp} \log^2\left(1+\left|X^{(j)}\right|\right) < \infty$ *where* $r > 1$ *and* $1 \le p < \frac{2}{r}$. *Then, any* $\epsilon > 0$,

$$\sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}} E\left[\left\|\sum_{i=1}^{n} \mathbf{X}_i\right\| - \epsilon\left(1+g(n)\right) n^{\frac{1}{p}}\right]_+ < \infty.$$

4.1 Proof of Main Results

In this section, the proofs of Theorem 4.1 and Theorem 4.2 will be discussed. To prove Theorem 4.1, the following lemma is needed.

Lemma 4.3. *Let* $\{\mathbf{X}_n, n \ge 1\}$ *be a sequence of H-valued CWOD random vectors with mean zero which is coordinatewise weakly upper bounded by a random vector* $\mathbf{X}$ *with* $\sum_{j=1}^{\infty} E\left|X^{(j)}\right|^2 < \infty$. *Assume* $\alpha > \frac{1}{2}$ *and* $0 < q < 2$. *For* $n, i, j \ge 1$ *and any real number* x *such that* $x \ge n^{\alpha q}$, *define* $\mathbf{Y}_i = \sum_{j=1}^{\infty} Y_i^{(j)} \mathbf{e}_j$ *where*
$Y_i^{(j)} = -x^{\frac{1}{q}} \mathbb{I}\left(X_i^{(j)} < -x^{\frac{1}{q}}\right) + X_i^{(j)} \mathbb{I}\left(\left|X_i^{(j)}\right| \le x^{\frac{1}{q}}\right) + x^{\frac{1}{q}} \mathbb{I}\left(X_i^{(j)} > x^{\frac{1}{q}}\right)$.
Then

$$\lim_{n\to\infty} x^{-\frac{1}{q}} \max_{1\le k\le n} \left\|\sum_{i=1}^{k} E\mathbf{Y}_i\right\| = 0.$$

Proof. From the definition of $\mathbf{Y}_i$, we can show that

$$\begin{aligned}
&x^{-\frac{1}{q}} \max_{1\le k\le n} \left\|\sum_{i=1}^{k} E\mathbf{Y}_i\right\| \\
&\le x^{-\frac{1}{q}} \max_{1\le k\le n} \sum_{j=1}^{\infty}\sum_{i=1}^{k} \left|E\left[-x^{\frac{1}{q}} \mathbb{I}\left(X_i^{(j)} < -x^{\frac{1}{q}}\right) + X_i^{(j)} \mathbb{I}\left(\left|X_i^{(j)}\right| \le x^{\frac{1}{q}}\right) + x^{\frac{1}{q}} \mathbb{I}\left(X_i^{(j)} > x^{\frac{1}{q}}\right)\right]\right| \\
&= x^{-\frac{1}{q}} \max_{1\le k\le n} \sum_{j=1}^{\infty}\sum_{i=1}^{k} \left|E\left[-x^{\frac{1}{q}} \mathbb{I}\left(X_i^{(j)} < -x^{\frac{1}{q}}\right) - X_i^{(j)} \mathbb{I}\left(\left|X_i^{(j)}\right| > x^{\frac{1}{q}}\right) + x^{\frac{1}{q}} \mathbb{I}\left(X_i^{(j)} > x^{\frac{1}{q}}\right)\right]\right| \qquad (4.1) \\
&\le x^{-\frac{1}{q}} \max_{1\le k\le n} \sum_{j=1}^{\infty}\sum_{i=1}^{k} \left[E\left|X_i^{(j)}\right| \mathbb{I}\left(\left|X_i^{(j)}\right| > x^{\frac{1}{q}}\right) + x^{\frac{1}{q}} P\left(\left|X_i^{(j)}\right| > x^{\frac{1}{q}}\right)\right] \\
&\le C x^{-\frac{1}{q}} \sum_{j=1}^{\infty}\sum_{i=1}^{n} \left[E\left|X_i^{(j)}\right| \mathbb{I}\left(\left|X_i^{(j)}\right| > x^{\frac{1}{q}}\right) + x^{\frac{1}{q}} P\left(\left|X_i^{(j)}\right| > x^{\frac{1}{q}}\right)\right]
\end{aligned}$$

$$\leq Cx^{-\frac{1}{q}} \sum_{j=1}^{\infty} \left[nE\left|X^{(j)}\right| \mathbb{I}\left(\left|X^{(j)}\right| > x^{\frac{1}{q}}\right) + nx^{\frac{1}{q}} P\left(\left|X^{(j)}\right| > x^{\frac{1}{q}}\right)\right] \tag{4.2}$$

$$\leq Cn^{1-\alpha} \sum_{j=1}^{\infty} E\left|X^{(j)}\right| \mathbb{I}\left(\left|X^{(j)}\right| > n^{\alpha}\right) \tag{4.3}$$

$$\leq Cn^{1-2\alpha} \sum_{j=1}^{\infty} E\left|X^{(j)}\right|^2 \mathbb{I}\left(\left|X^{(j)}\right| > n^{\alpha}\right), \tag{4.4}$$

where we use the fact that $E\left(X_i^{(j)}\right) = 0$ for $i, j \geq 1$ to obtain (4.1), Lemma 3.4 to obtain (4.2) and $x \geq n^{\alpha q}$ to obtain (4.3).
Since $\alpha > \frac{1}{2}$ and $\sum_{j=1}^{\infty} E\left|X^{(j)}\right|^2 \mathbb{I}\left(\left|X^{(j)}\right| > n^{\alpha}\right) < \infty$, the term in (4.4) converges to 0 as n goes to infinity. □

Proof of Theorem 4.1. For fixed $x > 0$ and $i, j = 1, 2, \ldots$, define

$$\begin{aligned}
Y_i^{(j)} &= -x^{\frac{1}{q}} \mathbb{I}\left(X_i^{(j)} < -x^{\frac{1}{q}}\right) + X_i^{(j)} \mathbb{I}\left(\left|X_i^{(j)}\right| \leq x^{\frac{1}{q}}\right) + x^{\frac{1}{q}} \mathbb{I}\left(X_i^{(j)} > x^{\frac{1}{q}}\right), \\
Z_i^{(j)} &= \left(X_i^{(j)} + x^{\frac{1}{q}}\right) \mathbb{I}\left(X_i^{(j)} < -x^{\frac{1}{q}}\right) + \left(X_i^{(j)} - x^{\frac{1}{q}}\right) \mathbb{I}\left(X_i^{(j)} > x^{\frac{1}{q}}\right), \\
\mathbf{Z}_i &= \sum_{j=1}^{\infty} Z_i^{(j)} \mathbf{e}_j \text{ and } \mathbf{Y}_i = \sum_{j=1}^{\infty} Y_i^{(j)} \mathbf{e}_j.
\end{aligned}$$

Note that $Y_i^{(j)} + Z_i^{(j)} = X_i^{(j)}$ and so

$$\mathbf{X}_i = \sum_{j=1}^{\infty} X_i^{(j)} \mathbf{e}_j = \sum_{j=1}^{\infty} Y_i^{(j)} \mathbf{e}_j + \sum_{j=1}^{\infty} Z_i^{(j)} \mathbf{e}_j = \mathbf{Y}_i + \mathbf{Z}_i.$$

Let $\epsilon > 0$. Then

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha r - \alpha q - 2} E\left[\max_{1 \leq k \leq n} \left\|\sum_{i=1}^{k} \mathbf{X}_i\right\| - \epsilon n^{\alpha}\right]_+^q \\
&= \sum_{n=1}^{\infty} n^{\alpha r - \alpha q - 2} \int_0^{\infty} P\left(\max_{1 \leq k \leq n} \left\|\sum_{i=1}^{k} \mathbf{X}_i\right\| > \epsilon n^{\alpha} + x^{\frac{1}{q}}\right) dx
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} n^{\alpha r-\alpha q-2} \int_{0}^{n^{\alpha q}} P\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k}\mathbf{X}_i\right\| > \epsilon n^{\alpha} + x^{\frac{1}{q}}\right) dx \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha r-\alpha q-2} \int_{n^{\alpha q}}^{\infty} P\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k}\mathbf{X}_i\right\| > \epsilon n^{\alpha} + x^{\frac{1}{q}}\right) dx \\
&\le \sum_{n=1}^{\infty} n^{\alpha r-2} P\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k}\mathbf{X}_i\right\| > \epsilon n^{\alpha}\right) \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha r-\alpha q-2} \int_{n^{\alpha q}}^{\infty} P\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k}\mathbf{X}_i\right\| > x^{\frac{1}{q}}\right) dx \\
&=: I_1 + I_2.
\end{aligned}
$$

We have proved in Theorem 3.3 that $I_1 < \infty$, therefore it remains to show that $I_2 < \infty$.

To prove that $I_2 < \infty$, we first split I_2 into two terms as follows.

$$
\begin{aligned}
I_2 &= \sum_{n=1}^{\infty} n^{\alpha r-\alpha q-2} \int_{n^{\alpha q}}^{\infty} P\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k}(\mathbf{Y}_i + \mathbf{Z}_i)\right\| > x^{\frac{1}{q}}\right) dx \\
&\le \sum_{n=1}^{\infty} n^{\alpha r-\alpha q-2} \int_{n^{\alpha q}}^{\infty} P\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k}\mathbf{Y}_i\right\| + \max_{1\le k\le n}\left\|\sum_{i=1}^{k}\mathbf{Z}_i\right\| > x^{\frac{1}{q}}\right) dx \\
&\le \sum_{n=1}^{\infty} n^{\alpha r-\alpha q-2} \int_{n^{\alpha q}}^{\infty} P\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k}\mathbf{Z}_i\right\| > \frac{x^{\frac{1}{q}}}{2}\right) dx \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha r-\alpha q-2} \int_{n^{\alpha q}}^{\infty} P\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k}\mathbf{Y}_i\right\| > \frac{x^{\frac{1}{q}}}{2}\right) dx \\
&=: I_{21} + I_{22}.
\end{aligned}
$$

We will show that both I_{21} and I_{22} are finite.

For I_{21}, by the same argument of Theorem 2.2 in [6], we have

$$
\begin{aligned}
I_{21} &\le \sum_{n=1}^{\infty} n^{\alpha r-\alpha q-2} \int_{n^{\alpha q}}^{\infty} P\left(\bigcup_{j=1}^{\infty}\bigcup_{i=1}^{n}\left\{\left|X_i^{(j)}\right| > x^{\frac{1}{q}}\right\}\right) dx \\
&\le \sum_{n=1}^{\infty} n^{\alpha r-\alpha q-2} \int_{n^{\alpha q}}^{\infty} \sum_{j=1}^{\infty}\sum_{i=1}^{n} P\left(\left|X_i^{(j)}\right| > x^{\frac{1}{q}}\right) dx
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} n^{\alpha r-\alpha q-2} \sum_{i=1}^{n} \int_{n^{\alpha q}}^{\infty} P\left(\left|X_i^{(j)}\right| > x^{\frac{1}{q}}\right) dx \\
&\le C \sum_{j=1}^{\infty} E\left|X^{(j)}\right|^2 \\
&< \infty.
\end{aligned}
$$

To prove $I_{22} < \infty$, we know from Lemma 4.3 that $\max\limits_{1\le k\le n} \left\|\sum_{i=1}^{k} E\mathbf{Y}_i\right\| < \frac{x^{\frac{1}{q}}}{4}$ while n is sufficiently large.

Moreover, from Proposition 2.46, we have $\{\mathbf{Y}_i, i \ge 1\}$ is still CWOD.

Therefore, from Lemma 4.3, Lemma 3.4 and Lemma 3.6, we have

$$
\begin{aligned}
I_{22} &= \sum_{n=1}^{\infty} n^{\alpha r-\alpha q-2} \int_{n^{\alpha q}}^{\infty} P\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k}(\mathbf{Y}_i - E\mathbf{Y}_i + E\mathbf{Y}_i)\right\| > \frac{x^{\frac{1}{q}}}{2}\right) dx \\
&\le \sum_{n=1}^{\infty} n^{\alpha r-\alpha q-2} \int_{n^{\alpha q}}^{\infty} P\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k}(\mathbf{Y}_i - E\mathbf{Y}_i)\right\| > \frac{x^{\frac{1}{q}}}{4}\right) dx \\
&\le C\sum_{n=1}^{\infty} n^{\alpha r-\alpha q-2} \int_{n^{\alpha q}}^{\infty} x^{-\frac{2}{q}} E\left[\max_{1\le k\le n}\left\|\sum_{i=1}^{k}(\mathbf{Y}_i - E\mathbf{Y}_i)\right\|^2\right] dx \\
&\le C\sum_{n=1}^{\infty} n^{\alpha r-\alpha q-2} \int_{n^{\alpha q}}^{\infty} x^{-\frac{2}{q}} (1+g(n))\log^2 n \sum_{j=1}^{\infty}\sum_{i=1}^{n} E\left|Y_i^{(j)} - EY_i^{(j)}\right|^2 dx \\
&\le C\sum_{n=1}^{\infty} n^{\alpha r-\alpha q-2} \int_{n^{\alpha q}}^{\infty} x^{-\frac{2}{q}} (1+g(n))\log^2 n \sum_{j=1}^{\infty}\sum_{i=1}^{n} E\left|Y_i^{(j)}\right|^2 dx \\
&= C\sum_{n=1}^{\infty} n^{\alpha r-\alpha q-2} \int_{n^{\alpha q}}^{\infty} x^{-\frac{2}{q}} (1+g(n))\log^2 n \sum_{j=1}^{\infty}\sum_{i=1}^{n} E\Big| - x^{\frac{1}{q}}\mathbb{I}\left(X_i^{(j)} < -x^{\frac{1}{q}}\right) \\
&\quad + X_i^{(j)}\mathbb{I}\left(\left|X_i^{(j)}\right| \le x^{\frac{1}{q}}\right) + x^{\frac{1}{q}}\mathbb{I}\left(X_i^{(j)} > x^{\frac{1}{q}}\right)\Big|^2 dx \\
&\le C\sum_{n=1}^{\infty} n^{\alpha r-\alpha q-2} \int_{n^{\alpha q}}^{\infty} x^{-\frac{2}{q}} (1+g(n))\log^2 n \sum_{j=1}^{\infty}\sum_{i=1}^{n} \Big(x^{\frac{2}{q}} P\left(\left|X_i^{(j)}\right| > x^{\frac{1}{q}}\right) \\
&\quad + E\left|X_i^{(j)}\right|^2 \mathbb{I}\left(\left|X_i^{(j)}\right| \le x^{\frac{1}{q}}\right)\Big) dx \\
&\le C\sum_{n=1}^{\infty} n^{\alpha r-\alpha q-2} \int_{n^{\alpha q}}^{\infty} x^{-\frac{2}{q}} (1+g(n))\log^2 n \sum_{j=1}^{\infty} \Big(2n x^{\frac{2}{q}} P\left(\left|X^{(j)}\right| > x^{\frac{1}{q}}\right) \\
&\quad + nE\left|X^{(j)}\right|^2 \mathbb{I}\left(\left|X^{(j)}\right| \le x^{\frac{1}{q}}\right)\Big) dx
\end{aligned}
$$

$$= C\sum_{n=1}^{\infty} n^{\alpha r-\alpha q-1}\log^2 n\int_{n^{\alpha q}}^{\infty} x^{-\frac{2}{q}}\sum_{j=1}^{\infty}\Big(x^{\frac{2}{q}}P\left(\left|X^{(j)}\right|>x^{\frac{1}{q}}\right)$$

$$+E\left|X^{(j)}\right|^2\mathbb{I}\left(\left|X^{(j)}\right|\le x^{\frac{1}{q}}\right)\Big)dx + C\sum_{n=1}^{\infty} n^{\alpha r-\alpha q-1}g(n)\log^2 n$$

$$\times\int_{n^{\alpha q}}^{\infty} x^{-\frac{2}{q}}\sum_{j=1}^{\infty}\Big(x^{\frac{2}{q}}P\left(\left|X^{(j)}\right|>x^{\frac{1}{q}}\right)+E\left|X^{(j)}\right|^2\mathbb{I}\left(\left|X^{(j)}\right|\le x^{\frac{1}{q}}\right)\Big)dx$$

$$=: I_{221}+I_{222}. \tag{4.5}$$

To prove I_{22} is finite, we will prove that I_{221} and I_{222} are finite.

For I_{222},

$$\begin{aligned}
I_{222} &\le C\sum_{n=1}^{\infty} n^{\alpha r-\alpha q-1}g(n)\log^2 n\int_{n^{\alpha q}}^{\infty} x^{-\frac{2}{q}}\sum_{j=1}^{\infty}E\left|X^{(j)}\right|^2 dx\\
&\le C\sum_{n=1}^{\infty} n^{\frac{\alpha r}{2}-\alpha q+\alpha-1}\int_{n^{\alpha q}}^{\infty} x^{-\frac{2}{q}}dx\\
&= C\sum_{n=1}^{\infty} n^{\alpha\left(\frac{r}{2}-1\right)-1}\\
&< \infty.
\end{aligned} \tag{4.6}$$

For I_{221},

$$\begin{aligned}
I_{221} &= C\sum_{n=1}^{\infty} n^{\alpha r-\alpha q-1}\log^2 n\int_{n^{\alpha q}}^{\infty}\sum_{j=1}^{\infty}P\left(\left|X^{(j)}\right|>x^{\frac{1}{q}}\right)dx\\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha r-\alpha q-1}\log^2 n\int_{n^{\alpha q}}^{\infty} x^{-\frac{2}{q}}\sum_{j=1}^{\infty}E\left|X^{(j)}\right|^2\mathbb{I}\left(\left|X^{(j)}\right|\le x^{\frac{1}{q}}\right)dx\\
&=: I_{2211}+I_{2212}.
\end{aligned} \tag{4.7}$$

For I_{2211},

$$\begin{aligned}
I_{2211} &\le C\sum_{n=1}^{\infty} n^{\alpha r-\alpha q-1}\log^2 n\sum_{j=1}^{\infty}E\left|X^{(j)}\right|^q\mathbb{I}\left(\left|X^{(j)}\right|>n^{\alpha}\right)\\
&\le C\sum_{n=1}^{\infty} n^{\alpha r-\alpha q-1+\alpha(q-r)}\log^2 n\sum_{j=1}^{\infty}E\left|X^{(j)}\right|^r\mathbb{I}\left(\left|X^{(j)}\right|>n^{\alpha}\right)
\end{aligned}$$

$$
\begin{aligned}
&= C\sum_{n=1}^{\infty}\frac{\log^2 n}{n}\sum_{j=1}^{\infty}\sum_{k=n}^{\infty}E\left|X^{(j)}\right|^r \mathbb{I}\left(k^\alpha < \left|X^{(j)}\right| \le (k+1)^\alpha\right)\\
&= C\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}E\left|X^{(j)}\right|^r \mathbb{I}\left(k^\alpha < \left|X^{(j)}\right| \le (k+1)^\alpha\right)\sum_{n=1}^{k}\frac{\log^2 n}{n}\\
&\le C\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\log^3 kE\left|X^{(j)}\right|^r \mathbb{I}\left(k^\alpha < \left|X^{(j)}\right| \le (k+1)^\alpha\right)\\
&\le C\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}E\left|X^{(j)}\right|^r \log^3\left(1+\left|X^{(j)}\right|\right)\mathbb{I}\left(k^\alpha < \left|X^{(j)}\right| \le (k+1)^\alpha\right)\\
&\le C\sum_{j=1}^{\infty}E\left|X^{(j)}\right|^2 \log^3\left(1+\left|X^{(j)}\right|\right)\\
&< \infty. \qquad (4.8)
\end{aligned}
$$

For I_{2212},

$$
\begin{aligned}
I_{2212} &= C\sum_{n=1}^{\infty}n^{\alpha r-\alpha q-1}\log^2 n\sum_{j=1}^{\infty}\sum_{k=n}^{\infty}\int_{k^{\alpha q}}^{(k+1)^{\alpha q}}x^{-\frac{2}{q}}E\left|X^{(j)}\right|^2\mathbb{I}\left(\left|X^{(j)}\right|\le x^{\frac{1}{q}}\right)dx\\
&\le C\sum_{n=1}^{\infty}n^{\alpha r-\alpha q-1}\log^2 n\sum_{j=1}^{\infty}\sum_{k=n}^{\infty}k^{\alpha q-2\alpha-1}E\left|X^{(j)}\right|^2\mathbb{I}\left(\left|X^{(j)}\right|\le (k+1)^\alpha\right)\\
&= C\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}k^{\alpha q-2\alpha-1}E\left|X^{(j)}\right|^2\mathbb{I}\left(\left|X^{(j)}\right|\le (k+1)^\alpha\right)\sum_{n=1}^{k}n^{\alpha r-\alpha q-1}\log^2 n\\
&\le C\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}k^{\alpha r-2\alpha-1}\log^2 kE\left|X^{(j)}\right|^2\mathbb{I}\left(\left|X^{(j)}\right|\le (k+1)^\alpha\right)\\
&= C\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}k^{\alpha r-2\alpha-1}\log^2 k\sum_{l=1}^{k+1}E\left|X^{(j)}\right|^2\mathbb{I}\left((l-1)^\alpha < \left|X^{(j)}\right|\le l^\alpha\right)\\
&= C\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}k^{\alpha r-2\alpha-1}\log^2 k\Bigg(E\left|X^{(j)}\right|^2\mathbb{I}\left(0<\left|X^{(j)}\right|\le 1\right)\\
&\quad+\sum_{l=2}^{k+1}E\left|X^{(j)}\right|^2\mathbb{I}\left((l-1)^\alpha < \left|X^{(j)}\right|\le l^\alpha\right)\Bigg)\\
&= C\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}k^{\alpha r-2\alpha-1}\log^2 kE\left|X^{(j)}\right|^2\mathbb{I}\left(0<\left|X^{(j)}\right|\le 1\right)\\
&\quad+C\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}k^{\alpha r-2\alpha-1}\log^2 k\sum_{l=2}^{k+1}E\left|X^{(j)}\right|^2\mathbb{I}\left((l-1)^\alpha < \left|X^{(j)}\right|\le l^\alpha\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq C + C\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} k^{\alpha r-2\alpha-1}\log^2 k \sum_{l=2}^{k+1} E\left|X^{(j)}\right|^2 \mathbb{I}\left((l-1)^{\alpha} < \left|X^{(j)}\right| \leq l^{\alpha}\right) \\
&= C\sum_{j=1}^{\infty}\sum_{l=2}^{\infty} E\left|X^{(j)}\right|^2 \mathbb{I}\left((l-1)^{\alpha} < \left|X^{(j)}\right| \leq l^{\alpha}\right) \sum_{k=l-1}^{\infty} k^{\alpha r-2\alpha-1}\log^2 k \\
&\leq C\sum_{j=1}^{\infty}\sum_{l=2}^{\infty} l^{\alpha r-2\alpha}\log^2 l E\left|X^{(j)}\right|^2 \mathbb{I}\left((l-1)^{\alpha} < \left|X^{(j)}\right| \leq l^{\alpha}\right) \\
&\leq C\sum_{j=1}^{\infty}\sum_{l=2}^{\infty} \log^2 l E\left|X^{(j)}\right|^2 \mathbb{I}\left((l-1)^{\alpha} < \left|X^{(j)}\right| \leq l^{\alpha}\right) \\
&\leq C\sum_{j=1}^{\infty}\sum_{l=2}^{\infty} E\left|X^{(j)}\right|^2 \log^2\left(1+\left|X^{(j)}\right|\right) \mathbb{I}\left((l-1)^{\alpha} < \left|X^{(j)}\right| \leq l^{\alpha}\right) \\
&\leq C\sum_{j=1}^{\infty}\sum_{l=2}^{\infty} E\left|X^{(j)}\right|^2 \log^2\left(1+\left|X^{(j)}\right|\right) \\
&< \infty. \qquad (4.9)
\end{aligned}
$$

From (4.5)–(4.9), we have I_{22} is finite. Therefore, the proof is completed. □

To prove Theorem 4.2, we need the following lemma.

Lemma 4.4. *Let $\{\mathbf{X}_i, 1 \leq i \leq n\}$ and $\{\mathbf{Y}_i, 1 \leq i \leq n\}$ be a sequence of H-valued random vectors. Then for any $q > 1, \epsilon > 0$ and $a > 0$,*

$$
E\left[\left\|\sum_{i=1}^{n}(\mathbf{X}_i + \mathbf{Y}_i)\right\| - \epsilon a\right]_+ \leq \frac{1}{a^{q-1}}\left(\frac{1}{\epsilon^q} + \frac{1}{q-1}\right) E\left\|\sum_{i=1}^{n}\mathbf{X}_i\right\|^q + E\left\|\sum_{i=1}^{n}\mathbf{Y}_i\right\|.
$$

Proof. Note that

$$
\begin{aligned}
E\left[\left\|\sum_{i=1}^{n}\mathbf{X}_i\right\| - \epsilon a\right]_+ &= \int_0^{\infty} P\left(\left\|\sum_{i=1}^{n}\mathbf{X}_i\right\| - \epsilon a > t\right) dt \\
&\leq aP\left(\left\|\sum_{i=1}^{n}\mathbf{X}_i\right\| > \epsilon a\right) + \int_a^{\infty} P\left(\left\|\sum_{i=1}^{n}\mathbf{X}_i\right\| > t\right) dt \\
&\leq \frac{aE\left\|\sum_{i=1}^{n}\mathbf{X}_i\right\|^q}{\epsilon^q a^q} + E\left\|\sum_{i=1}^{n}\mathbf{X}_i\right\|^q \int_a^{\infty}\frac{1}{t^q}dt
\end{aligned}
$$

$$= \frac{E\left\|\sum_{i=1}^{n}\mathbf{X}_i\right\|^q}{\epsilon^q a^{q-1}} + \frac{E\left\|\sum_{i=1}^{n}\mathbf{X}_i\right\|^q}{(q-1)a^{q-1}}$$
$$= \frac{1}{a^{q-1}}\left(\frac{1}{\epsilon^q} + \frac{1}{q-1}\right) E\left\|\sum_{i=1}^{n}\mathbf{X}_i\right\|^q .$$

From the fact that, for any x, y and $z \in H$,

$$(\|x+y\| - \|z\|)_+ \leq (\|x\| - \|z\|)_+ + \|y\| .$$

Hence,

$$E\left[\left\|\sum_{i=1}^{n}(\mathbf{X}_i + \mathbf{Y}_i)\right\| - \epsilon a\right]_+ \leq E\left[\left\|\sum_{i=1}^{n}\mathbf{X}_i\right\| - \epsilon a\right]_+ + E\left\|\sum_{i=1}^{n}\mathbf{Y}_i\right\| .$$

Therefore, the lemma is proved. □

Proof of Theorem 4.2. For $n, i, j \geq 1$, define

$$X_{ni}^{(j)} = -n^{\frac{1}{p}}\mathbb{I}\left(X_i^{(j)} < -n^{\frac{1}{p}}\right) + X_i^{(j)}\mathbb{I}\left(\left|X_i^{(j)}\right| \leq n^{\frac{1}{p}}\right) + n^{\frac{1}{p}}\mathbb{I}\left(X_i^{(j)} > n^{\frac{1}{p}}\right),$$
$$Y_{ni}^{(j)} = X_i^{(j)} - X_{ni}^{(j)} = n^{\frac{1}{p}}\mathbb{I}\left(X_i^{(j)} < -n^{\frac{1}{p}}\right) + X_i^{(j)}\mathbb{I}\left(\left|X_i^{(j)}\right| > n^{\frac{1}{p}}\right) - n^{\frac{1}{p}}\mathbb{I}\left(X_i^{(j)} > n^{\frac{1}{p}}\right),$$
$$Z_{ni}^{(j)} = X_{ni}^{(j)} - EX_{ni}^{(j)}, \mathbf{X}_{ni} = \sum_{j=1}^{\infty} X_{ni}^{(j)}\mathbf{e}_j, \mathbf{Y}_{ni} = \sum_{j=1}^{\infty} Y_{ni}^{(j)}\mathbf{e}_j \text{ and } Z_{ni} = \sum_{j=1}^{\infty} Z_{ni}^{(j)}\mathbf{e}_j.$$

Then $X_i^{(j)} = X_{ni}^{(j)} + Y_{ni}^{(j)} = Z_{ni}^{(j)} + EX_{ni}^{(j)} + Y_{ni}^{(j)}$ and consequently
$\mathbf{X}_i = \mathbf{Z}_{ni} + E\mathbf{X}_{ni} + \mathbf{Y}_{ni}$.

Let $\epsilon > 0$. Then, by Lemma 4.4 with $a = (1 + g(n))n^{\frac{1}{p}}$,

$$\sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}} E\left[\left\|\sum_{i=1}^{n}\mathbf{X}_i\right\| - \epsilon\,(1+g(n))\,n^{\frac{1}{p}}\right]_+$$
$$\leq \sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}} E\left\|\sum_{i=1}^{n}(E\mathbf{X}_{ni} + \mathbf{Y}_{ni})\right\| + C\sum_{n=1}^{\infty} n^{r-2-\frac{2}{p}}\frac{1}{1+g(n)} E\left\|\sum_{i=1}^{n}\mathbf{Z}_{ni}\right\|^2$$

$$\leq \sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}} E \left\| \sum_{i=1}^{n} \mathbf{Y}_{ni} \right\| + \sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}} \left\| \sum_{i=1}^{n} E\mathbf{X}_{ni} \right\| + C \sum_{n=1}^{\infty} n^{r-2-\frac{2}{p}} \frac{1}{1+g(n)} E \left\| \sum_{i=1}^{n} \mathbf{Z}_{ni} \right\|^2$$
$$=: K_1 + K_2 + K_3.$$

To prove the main result, we will show that K_1, K_2 and K_3 are finite.
Note that $\left|Y_{ni}^{(j)}\right| \leq 2\left|X_i^{(j)}\right| \mathbb{I}\left(\left|X_i^{(j)}\right| > n^{\frac{1}{p}}\right)$. We have

$$\begin{aligned}
K_1 &\leq \sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}} \sum_{j=1}^{\infty} \sum_{i=1}^{n} E\left|Y_{ni}^{(j)}\right| \\
&\leq 2\sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}} \sum_{j=1}^{\infty} \sum_{i=1}^{n} E\left|X_i^{(j)}\right| \mathbb{I}\left(\left|X_i^{(j)}\right| > n^{\frac{1}{p}}\right) \\
&\leq C\sum_{n=1}^{\infty} n^{r-1-\frac{1}{p}} \sum_{j=1}^{\infty} E\left|X^{(j)}\right| \mathbb{I}\left(\left|X^{(j)}\right| > n^{\frac{1}{p}}\right) \\
&= C\sum_{n=1}^{\infty} n^{r-1-\frac{1}{p}} \sum_{j=1}^{\infty} \sum_{k=n}^{\infty} E\left|X^{(j)}\right| \mathbb{I}\left(k^{\frac{1}{p}} < \left|X^{(j)}\right| \leq (k+1)^{\frac{1}{p}}\right) \\
&= C\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} E\left|X^{(j)}\right| \mathbb{I}\left(k^{\frac{1}{p}} < \left|X^{(j)}\right| \leq (k+1)^{\frac{1}{p}}\right) \sum_{n=1}^{k} n^{r-1-\frac{1}{p}} \\
&\leq C\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} k^{r-\frac{1}{p}} E\left|X^{(j)}\right| \mathbb{I}\left(k^{\frac{1}{p}} < \left|X^{(j)}\right| \leq (k+1)^{\frac{1}{p}}\right) \\
&\leq C\sum_{j=1}^{\infty} E\left|X^{(j)}\right|^{rp} \\
&< \infty. \qquad (4.10)
\end{aligned}$$

By the fact that $E\left(X_i^{(j)}\right) = 0$ and similar to the proof of (4.10), we can prove that

$$K_2 \leq \sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}} \sum_{j=1}^{\infty} \sum_{i=1}^{n} E\left|X_{ni}^{(j)}\right| \leq C\sum_{j=1}^{\infty} E\left|X^{(j)}\right|^{rp} < \infty.$$

Moreover, from Lemma 2.46, we have $\{\mathbf{Z}_{ni}, i \geq 1\}$ is still CWOD.

For K_3, by Lemma 3.6, we have

$$
\begin{aligned}
K_3 &= C\sum_{n=1}^{\infty} n^{r-2-\frac{2}{p}} \frac{1}{1+g(n)} E\left\|\sum_{i=1}^{n} \mathbf{Z}_{ni}\right\|^2 \\
&\le C\sum_{n=1}^{\infty} n^{r-2-\frac{2}{p}} \frac{1}{1+g(n)} (1+g(n)) \log^2 n \sum_{j=1}^{\infty}\sum_{i=1}^{n} E\left|Z_{ni}^{(j)}\right|^2 \\
&= C\sum_{n=1}^{\infty} n^{r-2-\frac{2}{p}} \log^2 n \sum_{j=1}^{\infty}\sum_{i=1}^{n} E\left|Z_{ni}^{(j)}\right|^2 \\
&= C\sum_{n=1}^{\infty} n^{r-2-\frac{2}{p}} \log^2 n \sum_{j=1}^{\infty}\sum_{i=1}^{n} E\left|X_{ni}^{(j)} - EX_{ni}^{(j)}\right|^2 \\
&\le C\sum_{n=1}^{\infty} n^{r-2-\frac{2}{p}} \log^2 n \sum_{j=1}^{\infty}\sum_{i=1}^{n} E\left|X_{ni}^{(j)}\right|^2 \\
&= C\sum_{n=1}^{\infty} n^{r-2-\frac{2}{p}} \log^2 n \sum_{j=1}^{\infty}\sum_{i=1}^{n} E\Big| -n^{\frac{1}{p}}\mathbb{I}\left(X_i^{(j)} < -n^{\frac{1}{p}}\right) + X_i^{(j)}\mathbb{I}\left(\left|X_i^{(j)}\right| \le n^{\frac{1}{p}}\right) \\
&\quad + n^{\frac{1}{p}}\mathbb{I}\left(X_i^{(j)} > n^{\frac{1}{p}}\right)\Big|^2 \\
&\le C\sum_{n=1}^{\infty} n^{r-2-\frac{2}{p}} \log^2 n \sum_{j=1}^{\infty}\sum_{i=1}^{n} \left[n^{\frac{2}{p}} P\left(\left|X_i^{(j)}\right| > n^{\frac{1}{p}}\right) + E\left|X_i^{(j)}\right|^2 \mathbb{I}\left(\left|X_i^{(j)}\right| \le n^{\frac{1}{p}}\right)\right] \\
&\le C\sum_{n=1}^{\infty} n^{r-1-\frac{2}{p}} \log^2 n \sum_{j=1}^{\infty} \left[n^{\frac{2}{p}} P\left(\left|X^{(j)}\right| > n^{\frac{1}{p}}\right) + E\left|X^{(j)}\right|^2 \mathbb{I}\left(\left|X^{(j)}\right| \le n^{\frac{1}{p}}\right)\right] \\
&\le C\sum_{n=1}^{\infty} n^{r-1-\frac{1}{p}} \log^2 n \sum_{j=1}^{\infty} E\left|X^{(j)}\right| \left(\left|X^{(j)}\right| > n^{\frac{1}{p}}\right) \\
&\quad + C\sum_{n=1}^{\infty} n^{r-1-\frac{2}{p}} \log^2 n \sum_{j=1}^{\infty} E\left|X^{(j)}\right|^2 \mathbb{I}\left(\left|X^{(j)}\right| \le n^{\frac{1}{p}}\right) \\
&=: K_{31} + K_{32}. \qquad (4.11)
\end{aligned}
$$

For K_{31},

$$
\begin{aligned}
K_{31} &= C\sum_{n=1}^{\infty} n^{r-1-\frac{1}{p}} \log^2 n \sum_{j=1}^{\infty}\sum_{k=n}^{\infty} E\left|X^{(j)}\right| \left(k^{\frac{1}{p}} < \left|X^{(j)}\right| \le (k+1)^{\frac{1}{p}}\right) \\
&= C\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} E\left|X^{(j)}\right| \left(k^{\frac{1}{p}} < \left|X^{(j)}\right| \le (k+1)^{\frac{1}{p}}\right) \sum_{n=1}^{k} n^{r-1-\frac{1}{p}} \log^2 n
\end{aligned}
$$

$$
\begin{aligned}
&\leq C \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} k^{r-\frac{1}{p}} \log^2 k E\left|X^{(j)}\right| \left(k^{\frac{1}{p}} < \left|X^{(j)}\right| \leq (k+1)^{\frac{1}{p}}\right) \\
&\leq C \sum_{j=1}^{\infty} E\left|X^{(j)}\right|^{rp} \log^2 \left(1 + \left|X^{(j)}\right|\right) \\
&< \infty. \qquad (4.12)
\end{aligned}
$$

For K_{32},

$$
\begin{aligned}
K_{32} &= C \sum_{n=1}^{\infty} n^{r-1-\frac{2}{p}} \log^2 n \sum_{j=1}^{\infty} \sum_{k=1}^{n} E\left|X^{(j)}\right|^2 \mathbb{I}\left((k-1)^{\frac{1}{p}} < \left|X^{(j)}\right| \leq k^{\frac{1}{p}}\right) \\
&\leq C \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} E\left|X^{(j)}\right|^2 \mathbb{I}\left((k-1)^{\frac{1}{p}} < \left|X^{(j)}\right| \leq k^{\frac{1}{p}}\right) \sum_{n=k}^{\infty} n^{r-1-\frac{2}{p}} \log^2 n \\
&\leq C \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} k^{r-\frac{2}{p}} \log^2 k E\left|X^{(j)}\right|^2 \mathbb{I}\left((k-1)^{\frac{1}{p}} < \left|X^{(j)}\right| \leq k^{\frac{1}{p}}\right) \\
&\leq C \sum_{j=1}^{\infty} E\left|X^{(j)}\right|^{rp} \log^2 \left(1 + \left|X^{(j)}\right|\right) \\
&< \infty. \qquad (4.13)
\end{aligned}
$$

From (4.11)–(4.13), we have K_3 is finite.

Therefore, the proof is completed. □