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Some arithmetic functions and their interesting properties

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Some arithmetic functions and their interesting properties

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ฟังก์ชันเลขคณิต คือ ฟังก์ชันที่มีค่าเป็นจำนวนเชิงซ้อนและมีโดเมนเป็นเซตของจำนวนนับ ในโครงงานนี้เรารวบรวมสมบัติบางประการของฟังก์ชันเลขคณิตที่เป็นที่รู้จักอยู่แล้วและนิยามฟังก์ชันเลขคณิตใหม่โดยใช้แนวคิดของ Atanassov และศึกษาสมบัติของฟังก์ชันที่เราได้นิยามขึ้นมานี้

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An arithmetic function is a complex–valued function whose domain is the set of natural numbers. In this project, we collect some properties of well–known arithmetic functions and define a new arithmetic function based on the idea of Atanassov and study its properties.

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Contents

Chapter 1

Preliminaries

This chapter contains some definitions and theorems in number theory that are used throughout this project, see e.g. [2].

1.1 The greatest common divisor and prime

Definition 1.1.1. Let a, b and c be integers. If $a \mid b$ and $a \mid c$, then we call a a ***common divisor*** of b and c.

Definition 1.1.2. Let $a, b \in \mathbb{N}$. ***The greatest common divisor*** (g.c.d) of a and b is the greatest among their common divisors and is denoted by (a, b).

Example 1.1.3. *Because* 5 *is the highest number such that* $5 \mid 15$ *and* $5 \mid 25$, *then* $(15, 25) = 5$.

Definition 1.1.4. A positive integer $p > 1$ is called a ***prime number*** if all positive divisors of p are 1 and p.

If $n \in \mathbb{N}$ and n is not prime, we call n a ***composite number***.

Theorem 1.1.5. *Let p be a prime number and $a, b \in \mathbb{Z}$. If $p \mid ab$, then $p \mid a$ or $p \mid b$.*

The next theorem is an important theorem that tells us about the unique representation of each number as a product of primes.

Theorem 1.1.6. ***Fundamental theorem of arithmetic****: Every integer $n > 1$ can be represented as a product of prime factors in only one way, apart from the order of the factors.*

1.2 Arithmetic functions

In this section, we begin with the definition of arithmetic functions which plays important role in our study.

Definition 1.2.1. An ***arithmetic function*** is a complex–valued function whose domain is the set of natural numbers.

Some examples of well–known arithmetic functions are shown as follows.

Definition 1.2.2. $\tau(n)$ is an arithmetic function which counts the number of all positive divisors of n.

Example 1.2.3. *Because* $1, 2, 3$ *and* 6 *are all positive divisors of* 6*, then* $\tau(6) = 4$*,*

Theorem 1.2.4. *Let* $n \in \mathbb{N}$*. If* $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$ *where* $p_1, p_2, ..., p_k$ *are distinct primes and* $\alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{N}$*, then*

$$\tau(n) = \prod_{i=1}^{k} (\alpha_i + 1).$$

Note that if p is a prime number, then $\tau(p) = 2$.

Example 1.2.5. *We have* $504 = 2^3 \times 3^2 \times 7^1$*. Then* $\tau(504) = (3 + 1) \times (2 + 1) \times (1 + 1) = 24$.

Definition 1.2.6. $\sigma(n)$ is an arithmetic function which is the sum of positive divisors of n.

Example 1.2.7. *Because* $1, 2, 3$ *and* 6 *are all positive divisors of* 6*, then* $\sigma(6) = 1 + 2 + 3 + 6 = 12$.

Theorem 1.2.8. *Let* $n \in \mathbb{N}$*. If* $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$ *where* $p_1, p_2, ..., p_k$ *are distinct primes and* $\alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{N}$*, then*

$$\sigma(n) = \prod_{i=1}^{k} (\frac{p_i^{\alpha_i + 1} - 1}{p_i - 1}).$$

Note that if p is a prime number, then $\sigma(p) = p + 1$.

Example 1.2.9. *We have* $675 = 5^2 \times 3^3$*. Then*

$$\sigma(675) = (\frac{5^{2+1} - 1}{5 - 1})(\frac{3^{3+1} - 1}{3 - 1}) = 6240.$$

Definition 1.2.10. $\phi(n)$ is an arithmetic function which counts the number of positive integer k less than or equal to n with $(k, n) = 1$.

That is $\phi(n) = |\{k \mid 1 \leq k \leq n \text{ and } (k, n) = 1\}|$.

Theorem 1.2.11. *Let $n \in \mathbb{N}$. If $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$ where $p_1, p_2, ..., p_k$ are distinct primes and $\alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{N}$, then*

$$\phi(n) = \prod_{i=1}^{k} (p_i^{\alpha_i} - p_i^{\alpha_i - 1}) = n \prod_{i=1}^{k} (1 - \frac{1}{p_i}).$$

Note that if p is a prime number, then $\phi(p) = p - 1$.

Example 1.2.12. *We have $968 = 2^3 \times 11^2$. Then*

$$\phi(968) = (2^3 - 2^2)(11^2 - 11) = 440.$$

Definition 1.2.13. $\psi(n)$ is an arithmetic function which is defined by

$$\psi(n) = n \prod_{p|n} (1 + \frac{1}{p}),$$

where p runs over all prime divisors of n.

Example 1.2.14. *We have $48 = 2^4 \times 3^1$. Then*

$$\psi(48) = 48(1 + \frac{1}{2})(1 + \frac{1}{3}) = 96.$$

Definition 1.2.15. $s(n)$ is an arithmetic function which is defined by

$$s(n) = \begin{cases} 1, & \text{if } n = m^2 \text{ for some } m \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Example 1.2.16. *Since 7 is not square, $s(7) = 0$ but $s(49) = 1$ because $49 = 7^2$.*

Definition 1.2.17. $\omega(n)$ is the number of distinct primes dividing n.

Example 1.2.18. *We have $\omega(1) = 0$ and since $100 = 2^2 \times 5^2$, then $\omega(100) = 2$.*

Some properties of arithmetic functions are introduced as follows.

Definition 1.2.19. A function $f : \mathbb{N} \to \mathbb{C}$ is said to be ***multiplicative*** if

$$f(mn) = f(m)f(n),$$

whenever $(m, n) = 1$ and it is said to be ***completely multiplicative*** if the identity holds for all positive integers m and n.

Theorem 1.2.20. *τ, σ and ϕ are multiplicative functions.*

Definition 1.2.21. A function $f : \mathbb{N} \to \mathbb{C}$ is said to be ***additive*** if

$$f(mn) = f(m) + f(n),$$

whenever $(m, n) = 1$.

In this project, we collect some interesting properties of certain well–known arithmetic functions.

1.3 Atanassov's function

In 2016, a new arithmetic function was introduced by Atanassov [1] as follows.

Definition 1.3.1. Let $\downarrow: \mathbb{N} \to \mathbb{C}$ be a function defined by $\downarrow(1) = 1$ and $\downarrow(2) = 1$. For each prime number $p \geq 3$, define $\downarrow(p)$ to be the highest prime number smaller than p. For $n > 2$, if

$$n = \prod_{i=1}^{k} p_i^{\alpha_i},$$

where $k, \alpha_1, \alpha_2, ..., \alpha_k$ are positive integers and $p_1, p_2, ..., p_k$ are distinct primes, define

$$\downarrow(n) = \prod_{i=1}^{k} (\downarrow(p_i))^{\alpha_i}.$$

Throughout this project, we will shorten $\prod_{i=1}^{k}(\downarrow(p_i))^{\alpha_i}$ to $\prod_{i=1}^{k} \downarrow(p_i)^{\alpha_i}$ for our convenience.

Example 1.3.2. $\downarrow(7) = 5$ *and* $\downarrow(36) = \downarrow(2^2 3^2) = 2^2 = 4$.

Then he presented some properties of the function $\downarrow$ as follows.

Theorem 1.3.3. *$\downarrow$ is a multiplicative function.*

Theorem 1.3.4. *For every natural number $n \geq 3$,*

$$\frac{\phi(n)}{n} > \frac{\phi(\downarrow n)}{\downarrow n}.$$

Theorem 1.3.5. *For every natural number $n \geq 2$,*

$$\frac{\sigma(n)}{n} < \frac{\sigma(\downarrow n)}{\downarrow n}.$$

Theorem 1.3.6. *For every natural number $n \geq 2$,*

$$\frac{\psi(n)}{n} < \frac{\psi(\downarrow n)}{\downarrow n}.$$

Theorem 1.3.7. *For every odd number $n \geq 3$,*

$$\phi(n)\sigma(\downarrow n) > \phi(\downarrow n)\sigma(n),$$

$$\phi(n)\psi(\downarrow n) > \phi(\downarrow n)\psi(n).$$

In this project, we will define a new function based on his idea and study its properties.

Chapter 2

Some arithmetic functions and their properties

2.1 Some properties of known arithmetic functions

This section contains some interesting properties of arithmetic functions introduced in Chapter 1.

Theorem 2.1.1. *Let* $n \in \mathbb{N} - \{1\}$ *be such that* $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$ *where* $p_1, p_2, ..., p_k$ *are distinct primes and* $\alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{N}$. *Then*

$$\psi(n) = \prod_{i=1}^{k} (p_i^{\alpha_i} + p^{\alpha_i - 1}).$$

Proof. By Definition 1.2.13, we get

$$\begin{aligned}
\psi(n) &= n \prod_{i=1}^{k} (1 + \frac{1}{p_i}) \\
&= (p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k})(1 + \frac{1}{p_1})(1 + \frac{1}{p_2})...(1 + \frac{1}{p_k}) \\
&= (p_1^{\alpha_1} + p_1^{\alpha_1 - 1})(p_2^{\alpha_2} + p_2^{\alpha_2 - 1})...(p_k^{\alpha_k} + p_k^{\alpha_k - 1}) \\
&= \prod_{i=1}^{k} (p_i^{\alpha_i} + p^{\alpha_i - 1}).
\end{aligned}$$

□

By Theorem 2.1.1, it is easy to see that $\psi(p) = p + 1$ for all prime numbers p.

Theorem 2.1.2. *ψ is a multiplicative function.*

Proof. Let $m, n \in \mathbb{N}$ be such that $(m,n) = 1$.
Case m = 1 or n = 1: Without loss of generality, we may assume that $m = 1$, then

$$\psi(n \cdot 1) = \psi(n) = \psi(n) \cdot 1 = \psi(n)\psi(1).$$

Case m ≠ 1 and n ≠ 1: Assume $m = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$ and $n = q_1^{\beta_1} q_2^{\beta_2} ... q_s^{\beta_s}$. where p_i, q_j are distinct primes and $\alpha_i, \beta_j \in \mathbb{N}$ for all $1 \leq i \leq k$ and $1 \leq j \leq s$. Then

$$mn = (p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k})(q_1^{\beta_1} q_2^{\beta_2} ... q_s^{\beta_s}).$$

Since $(m,n) = 1$, then $p_i \neq q_j$ for all $i \in \{1, 2, ..., k\}$ and $j \in \{1, 2, ..., s\}$. By Theorem 2.1.1, we have

$$\begin{aligned} \psi(mn) &= (p_1^{\alpha_1} + p_1^{\alpha_1 - 1}) ... (p_k^{\alpha_k} + p_k^{\alpha_k - 1})(q_1^{\beta_1} + q_1^{\beta_1 - 1}) ... (q_s^{\beta_s} + q_s^{\alpha_s - 1}) \\ &= \psi(m)\psi(n). \end{aligned}$$

Hence ψ is a multiplicative fuction. □

Note that $\psi(4) = 6$ and $\psi(2) = 3$. Then $\psi(4) \neq \psi(2)\psi(2)$. This implies that ψ is not a completely multiplicative function.

Theorem 2.1.3. *s is a multiplicative function.*

Proof. Let $m, n \in \mathbb{N}$ be such that $(m,n) = 1$.
Case 1: m and n are squares.
Then $s(m) = s(n) = 1$. Moreover, mn is square. Then we have

$$s(mn) = 1 = s(m)s(n).$$

Case 2: m or n is not square.
Without loss of generality, we may assume that m is not square. Then $s(m) = 0$ and there exist a prime p and an odd positive integer k such that

$$p^k \mid m \text{ but } p^{k+1} \nmid m.$$

Since $(m,n) = 1$, then $p \nmid n$. Consequently, we have

$$p^k \mid mn \text{ but } p^{k+1} \nmid mn.$$

Therefore mn is not square and so

$$s(mn) = 0 = s(m)s(n).$$

Hence s is a multiplicative function. □

Theorem 2.1.4. *ω is an additive function.*

Proof. Let $m, n \in \mathbb{N}$ be such that $(m, n) = 1$.
Case 1: $m = 1$ or $n = 1$.
We have $\omega(m) = 0$ or $\omega(n) = 0$. Moreover, we have $\omega(mn) = \omega(n)$ or $\omega(mn) = \omega(m)$. By these two cases, we have

$$\omega(mn) = \omega(m) + \omega(n).$$

Case 2: $m > 1$ and $n > 1$.
We can write $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$ and $m = q_1^{\beta_1} q_2^{\beta_2} ... q_s^{\beta_s}$ where p_i, q_j are distinct primes and $\alpha_i, \beta_j \in \mathbb{N}$ for all $1 \le i \le k$ and $1 \le j \le s$. Since $(m, n) = 1$, $\omega(mn) = k + s$. Therefore

$$\omega(mn) = k + s = \omega(m) + \omega(n).$$

Hence ω is an additive function. □

Some other properties of certain arithmetic functions are shown as follows.

Theorem 2.1.5. *Let p be a prime number. Then*
1) $\tau(p!) = 2\tau((p-1)!)$,
2) $\sigma(p!) = (p+1)\sigma((p-1)!)$,
3) $\phi(p!) = (p-1)\phi((p-1)!)$.

Proof. We have

$$\begin{aligned}\tau(p!) &= \tau(p(p-1)!) \\ &= \tau(p)\tau((p-1)!) \text{ since } (p,(p-1)!) = 1 \\ &= 2\tau((p-1)!).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\sigma(p!) &= \sigma(p(p-1)!) \\ &= \sigma(p)\sigma((p-1)!) \\ &= (p+1)\sigma((p-1)!),\end{aligned}$$

and

$$\begin{aligned}\phi(p!) &= \phi(p(p-1)!) \\ &= \phi(p)\phi((p-1)!) \\ &= (p-1)\phi((p-1)!).\end{aligned}$$

Hence $\tau(p!) = 2\tau((p-1)!)$, $\sigma(p!) = (p+1)\sigma((p-1)!)$ and
$\phi(p!) = (p-1)\phi((p-1)!)$. □

Theorem 2.1.6. *Let p be a prime number. Then $\sigma(p) + \phi(p) = p\tau(p)$.*

Proof. Let p be a prime number. We know that $\sigma(p) = p+1$ and $\phi(p) = p-1$. Then $\sigma(p) + \phi(p) = 2p$. Since $\tau(p) = 2$, we get

$$\sigma(p) + \phi(p) = 2p = p\tau(p).$$

Hence $\sigma(p) + \phi(p) = p\tau(p)$. □

Theorem 2.1.7. *Let $n \in \mathbb{N}$. Then $\tau(n)\phi(n) \geq n$.*

Proof. Let $n \in \mathbb{N}$. If $n = 1$, then

$$\phi(n)\tau(n) = 1 = n.$$

Assume $n \geq 2$. Then we can write $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$ where $p_1, p_2, ..., p_k$ are distinct primes and $\alpha_i \in \mathbb{N}$ for all $1 \leq i \leq k$. By Theorem 1.2.4, we have

$$\tau(n) = \prod_{i=1}^{k} (\alpha_i + 1) \quad \geq \quad 2^k.$$

By Theorem 1.2.11, we have

$$\begin{aligned} \phi(n) = n\prod_{i=1}^{k}(1 - \frac{1}{p_i}) \quad &\geq \quad n\prod_{i=1}^{k}(1 - \frac{1}{2}) \\ &= \quad \frac{n}{2^k}. \end{aligned}$$

Hence $\tau(n)\phi(n) \geq n$. □

Theorem 2.1.8. *Let $n \in \mathbb{N}$. If $n > 2$, then $\phi(n)$ is even.*

Proof. Let $n \in \mathbb{N}$.
Case 1: n is odd.
There is an odd prime factor of n, say p. Write $n = p^k m$ where $k \in \mathbb{N}$ and $(p, m) = 1$. Then

$$\begin{aligned} \phi(n) &= \phi(p^k)\phi(m) \\ &= (p^k - p^{k-1})\phi(m) \\ &= p^{k-1}(p-1)\phi^{(m)}, \end{aligned}$$

where $p - 1$ is even. Therefore $\phi(n)$ is even.
Case 2: n is even.
Write $n = 2^k m$ where $k, m \in \mathbb{N}$ and $(2, m) = 1$. Then

$$\phi(n) \quad = \quad \phi(2^k)\phi(m).$$

By the first case, we get $\phi(m)$ is even. Hence $\phi(n)$ is even. □

2.2 New arithmetic function

First, we define the function $\uparrow$ which is analogous to that defined by Atanassov [1].

Definition 2.2.1. For a prime number p, define $\uparrow p$ to be the smallest prime number higher than p. For a positive number n, if $n = 1$, define $\uparrow (n) = 1$. For $n \geq 2$, write

$$n = \prod_{i=1}^{k} p_i^{\alpha_i},$$

where $k, \alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{N}$ and $p_1, p_2, ..., p_k$ are distinct primes . Define

$$\uparrow (n) = \prod_{i=1}^{k} (\uparrow (p_i))^{\alpha_i}.$$

Throughout this project, we will shorten $\prod_{i=1}^{k} (\uparrow (p_i))^{\alpha_i}$ to $\prod_{i=1}^{k} \uparrow (p_i)^{\alpha_i}$ for our convenience.

Remarks

1. For a positive number n. We have $\uparrow (n) = 1$ if and only if $n = 1$.
2. For $n \geq 2$, we have $\uparrow (n) > n$.
3. For a prime number p, $\uparrow (p)$ is also prime.

Example 2.2.2. $\uparrow (7) = 11$, $\uparrow (8) = \uparrow (2)^3 = 3^3 = 27$.

Next, we will show that $\uparrow$ is a completely multiplicative function.

Theorem 2.2.3. *$\uparrow$ is a completely multiplicative function.*

Proof. Let $m, n \in \mathbb{N}$ be such that $m = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$ and $n = q_1^{\beta_1} q_2^{\beta_2} ... q_s^{\beta_s}$. where p_i, q_j are distinct primes and $\alpha_i, \beta_j \in \mathbb{N}$ for all $1 \leq i \leq k$ and $1 \leq j \leq s$. Then the prime factorization of mn is

$$mn = \prod_{p_i \neq q_j} {p_i}^{\alpha_i} q_j^{\beta_j} \prod_{p_i = q_j} {p_i}^{\alpha_i + \beta_j}.$$

Then

$$\begin{aligned} \uparrow (mn) &= \prod_{p_i \neq q_j} \uparrow (p_i)^{\alpha_i} \uparrow (q_j)^{\beta_j} \prod_{p_i = q_j} \uparrow (p_i)^{\alpha_i + \beta_j} \\ &= \prod_{i=1}^{k} \uparrow (p_i)^{\alpha_i} \prod_{j=1}^{s} \uparrow (q_j)^{\beta_j} \\ &= \uparrow (m) \uparrow (n). \end{aligned}$$

Hence $\uparrow$ is a completely multiplicative function. □

Theorem 2.2.4. *Let $n \in \mathbb{N}$. Then $\downarrow\uparrow(n) = n$, but*

$$\uparrow\downarrow(n) = \begin{cases} n, \text{ if } n \text{ is odd.} \\ \dfrac{n}{2^k}, \text{ if } n = 2^k m \text{ where } m \in \mathbb{N} \text{ is odd and } k \in \mathbb{N}. \end{cases}$$

Proof. Let $n \in \mathbb{N}$.
Case n = 1: $\downarrow\uparrow(1) = \downarrow(1) = 1$ and $\uparrow\downarrow(1) = \uparrow(1) = 1$.
Case n ≥ 2: we can write $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$ where $p_1, p_2, ..., p_k$ are distinct primes and $\alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{N}$. Then

$$\begin{aligned} \downarrow\uparrow(n) &= \downarrow(\uparrow(p_1)^{\alpha_1} \uparrow(p_2)^{\alpha_2} ... \uparrow(p_k)^{\alpha_k}) \\ &= \downarrow(\uparrow(p_1))^{\alpha_1} \downarrow(\uparrow(p_2))^{\alpha_2} ... \downarrow(\uparrow(p_k))^{\alpha_k} \\ &= p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k} \\ &= n. \end{aligned}$$

Now we will consider the composite function $\uparrow\downarrow$.
Case 1: n is odd.
Then all p_i are odd and so $\downarrow(p_i)$ are all primes. By the same argument as the above identity,

$$\uparrow(\downarrow(n)) = n.$$

Case 2: n is even.
There exists $p_i = 2$ for some $i \in \{1, 2, ..., s\}$. Without loss of generality, assume that $p_1 = 2$. Then

$$\begin{aligned} \uparrow(\downarrow(n)) &= \uparrow(\downarrow(p_1)^{\alpha_1} \downarrow(p_2)^{\alpha_2} ... \downarrow(p_k)^{\alpha_k}) \\ &= \uparrow(\downarrow(p_2))^{\alpha_2} ... \uparrow(\downarrow(p_k))^{\alpha_k} \\ &= p_2^{\alpha_2} ... p_k^{\alpha_k} \\ &= \frac{n}{2^{\alpha_1}}. \end{aligned}$$

□

Theorem 2.2.5. *$\uparrow$ is an injection and $ran(\uparrow) \subsetneq \mathbb{N}$.*

Proof. First, we will show $ran(\uparrow) \subsetneq \mathbb{N}$. is not surjective by showing that there is no $m \in \mathbb{N}$ such that $\uparrow(m) = 2$. Let $m \in \mathbb{N}$.
Case m = 1: We have $\uparrow(m) = 1$.
Case m ≥ 2: By Remarks (2), we have $\uparrow(m) > 2$.

Next we will show that $\uparrow$ is injection. Let $m, n \in \mathbb{N}$ be such that

$$\uparrow(n) = \uparrow(m).$$

Then, by Theorem 2.2.4,

$$n = \downarrow(\uparrow(n)) = \downarrow(\uparrow(m)) = m.$$

Hence $\uparrow$ is injection. □

Next, we will study a property related to τ and $\uparrow$.

Theorem 2.2.6. *Let $n \in \mathbb{N}$. Then $\tau(n) = \tau(\uparrow(n))$.*

Proof. Let $n \in \mathbb{N}$.
Case n = 1: $\tau(1) = 1 = \tau(\uparrow(1))$.
Case n ≥ 2: Write $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$ where $p_1, p_2, ..., p_k$ are distinct primes and $\alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{N}$. Then

$$\uparrow(n) = \uparrow(p_1)^{\alpha_1} \uparrow(p_2)^{\alpha_2} ... \uparrow(p_k)^{\alpha_k}.$$

Note that all $\uparrow(p_i)$ are primes. Therefore, by Theorem 1.2.4, we have

$$\tau(n) = \prod_{i=1}^{k} (\alpha_i + 1) = \tau(\uparrow(n)).$$

Hence $\tau(n) = \tau(\uparrow(n))$. □

Next we will study a relationship between ϕ and $\uparrow$.

Lemma 2.2.7. *Let p be a prime number. Then $\dfrac{\phi(\uparrow(p))}{\uparrow(p)} > \dfrac{\phi(p)}{p}$.*

Proof. By Remarks (2), we have

$$\begin{aligned} (p-1)\uparrow(p) &= p\uparrow(p) - \uparrow(p) \\ &< p\uparrow(p) - p \\ &= p(\uparrow(p) - 1). \end{aligned}$$

Then

$$\frac{\uparrow(p) - 1}{\uparrow(p)} > \frac{p-1}{p}.$$

Hence $\dfrac{\phi(\uparrow(p))}{\uparrow(p)} > \dfrac{\phi(p)}{p}$. □

Lemma 2.2.8. *Let p be a prime number and $k \in \mathbb{N}$. Then* $\dfrac{\phi(p^k)}{p^k} = \dfrac{\phi(p)}{p}$.

Proof. By Theorem 1.2.11, we get

$$\begin{aligned} \frac{\phi(p^k)}{p^k} &= \frac{p^{k-1}(p-1)}{p^k} \\ &= \frac{p-1}{p} \\ &= \frac{\phi(p)}{p}. \end{aligned}$$

Hence $\dfrac{\phi(p^k)}{p^k} = \dfrac{\phi(p)}{p}$. □

Finally, we will extend the result in Lemma 2.2.7 for any natural numbers.

Theorem 2.2.9. *For every natural number $n \geq 2$,* $\dfrac{\phi(\uparrow(n))}{\uparrow(n)} > \dfrac{\phi(n)}{n}$.

Proof. Assume that $n \geq 2$. Write $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$ where $p_1, p_2, ..., p_k$ are distinct primes and $\alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{N}$. Therefore, by Lemma 2.2.7 and Lemma 2.2.8, we have

$$\begin{aligned} \frac{\phi(n)}{n} &= \frac{\phi(p_1^{\alpha_1})}{p_1^{\alpha_1}} \frac{\phi(p_2^{\alpha_2})}{p_2^{\alpha_2}} ... \frac{\phi(p_k^{\alpha_k})}{p_k^{\alpha_k}} \\ &= \frac{\phi(p_1)}{p_1} \frac{\phi(p_2)}{p_2} ... \frac{\phi(p_k)}{p_k} \\ &< \frac{\phi(\uparrow(p_1))}{\uparrow(p_1)} \frac{\phi(\uparrow(p_2))}{\uparrow(p_2)} ... \frac{\phi(\uparrow(p_k))}{\uparrow(p_k)} \\ &= \frac{\phi(\uparrow(p_1)^{\alpha_1})}{\uparrow(p_1)^{\alpha_1}} \frac{\phi(\uparrow(p_2)^{\alpha_2})}{\uparrow(p_2)^{\alpha_2}} ... \frac{\phi(\uparrow(p_k)^{\alpha_k})}{\uparrow(p_k)^{\alpha_k}} \\ &= \frac{\phi(\uparrow(n))}{\uparrow(n)}. \end{aligned}$$

□

Next we will study a property between σ and $\uparrow$ for prime numbers.

Lemma 2.2.10. *Let p be a prime number. Then* $\dfrac{\sigma(p)}{p} > \dfrac{\sigma(\uparrow(p))}{\uparrow(p)}$.

Proof. We know that

$$\begin{aligned}(p+1)\uparrow(p) &= p\uparrow(p)+\uparrow(p)\\ &> p\uparrow(p)+p\\ &= p(\uparrow(p)+1).\end{aligned}$$

Then

$$\frac{p+1}{p} > \frac{\uparrow(p)+1}{\uparrow(p)}.$$

Hence $\dfrac{\sigma(p)}{p} > \dfrac{\sigma(\uparrow(p))}{\uparrow(p)}$. □

Lemma 2.2.11. *Let p be a prime number and $k \in \mathbb{N}$. Then* $\dfrac{\sigma(p^k)}{p^k} = \dfrac{\sigma(p)}{p}\left(\dfrac{p^{k+1}-1}{p^{k+1}-p^{k-1}}\right)$.

Proof. By theorem 1.2.8, we have

$$\begin{aligned}\frac{\sigma(p^k)}{p^k} = \frac{p^{k+1}-1}{p^k(p-1)} &= \frac{(p^{k+1}-1)(p+1)}{p^k(p-1)(p+1)}\\ &= \frac{(p+1)(p^{k+1}-1)}{p^k(p^2-1)}\\ &= \left(\frac{p+1}{p}\right)\left(\frac{p^{k+1}-1}{p^{k+1}-p^{k-1}}\right)\\ &= \frac{\sigma(p)}{p}\left(\frac{p^{k+1}-1}{p^{k+1}-p^{k-1}}\right).\end{aligned}$$

□

Lemma 2.2.12. *Let p be a prime number and $k \in \mathbb{N}$. Then* $\dfrac{\sigma(p^k)}{p^k} > \dfrac{\sigma(\uparrow(p)^k)}{\uparrow(p)^k}$.

Proof. By Lemma 2.2.10, the inequality holds for $k = 1$. We may assume that $k > 1$.

Case p = 2: We will show that $\dfrac{\sigma(2^k)}{2^k} > \dfrac{\sigma(3^k)}{3^k}$. Since

$$\begin{aligned}3^k 2^{k+1} - 3^{k+1}2^{k-1} - 3^k + 2^{k-1} &= 3^k(2^{k+1} - 3\cdot 2^{k-1} - 1) + 2^{k-1}\\ &> 2^{k+1} - 3\cdot 2^{k-1} - 1\\ &= 2^{k-1} - 1\\ &> 0,\end{aligned}$$

then

$$3^k 2^{k+1} - 3^k > 3^{k+1} 2^{k-1} - 2^{k-1}$$

and so

$$\frac{2^{k+1}-1}{2^k} > \frac{3^{k+1}-1}{3^k \cdot 2}.$$

Hence, by Lemma 1.2.8, we have

$$\frac{\sigma(2^k)}{2^k} > \frac{\sigma(3^k)}{3^k}.$$

Case p > 2: We will show that $\dfrac{\sigma(p^k)}{p^k} > \dfrac{\sigma(\uparrow(p)^k)}{\uparrow(p)^k}$. By Lemma 2.2.11, this inequality is replaced by

$$\frac{\sigma(p)}{p}\left(\frac{p^{k+1}-1}{p^{k+1}-p^{k-1}}\right) > \frac{\sigma(\uparrow(p))}{\uparrow(p)}\left(\frac{\uparrow(p)^{k+1}-1}{\uparrow(p)^{k+1}-\uparrow(p)^{k-1}}\right).$$

By Lemma 2.2.10, we know that $\dfrac{\sigma(p)}{p} > \dfrac{\sigma(\uparrow(p))}{\uparrow(p)}$.

Hence it remains to show that

$$\left(\frac{p^{k+1}-1}{p^{k+1}-p^{k-1}}\right) > \left(\frac{\uparrow(p)^{k+1}-1}{\uparrow(p)^{k+1}-\uparrow(p)^{k-1}}\right).$$

Consider

$$\begin{aligned} & p^{k-1}\uparrow(p)^{k+1} - p^{k+1}\uparrow(p)^{k-1} - \uparrow(p)^{k+1} + \uparrow(p)^{k-1} + p^{k+1} - p^{k-1} \\ &= \uparrow(p)^{k-1}[p^{k-1}\uparrow(p)^2 - p^{k+1} - \uparrow(p)^2 + 1] + p^{k+1} - p^{k-1} \\ &= \uparrow(p)^{k-1}[\uparrow(p)^2(p^{k-1}-1) - p^{k+1} + 1] + p^{k+1} - p^{k-1}. \end{aligned} \qquad (1)$$

It is easy to see that $\uparrow(p) \geq p+2$ for $p \geq 3$ because $\uparrow(p)$ is the smallest prime number higher than p. Then

$$\begin{aligned} \uparrow(p)^2(p^{k-1}-1) - p^{k+1} &\geq (p+2)^2(p^{k-1}-1) - p^{k+1} \\ &= (p^2+4p+4)(p^{k-1}-1) - p^{k+1} \\ &= 4p^k + 4p^{k-1} - p^2 - 4p - 4 \\ &= (p^k - p^2) + (4p^{k-1} - 4p) + (3p^k - 4) \\ &> 0 \end{aligned}$$

since $k \geq 2$ and $p \geq 3$ which implies that $p^k - p^2 \geq 0$, $4p^{k-1} - 4p \geq 0$ and $3p^k - 4 > 0$. Substituting it to (1), we have

$$p^{k-1}\uparrow(p)^{k+1} - p^{k+1}\uparrow(p)^{k-1} - \uparrow(p)^{k+1} + \uparrow(p)^{k-1} + p^{k+1} - p^{k-1} > 0.$$

Therefore

$$-p^{k+1} \uparrow (p)^{k-1} - \uparrow (p)^{k+1} + \uparrow (p)^{k-1} > -p^{k-1} \uparrow (p)^{k+1} - p^{k+1} + p^{k-1}$$

Adding $(p \uparrow (p))^{k+1}$ to bothsides, it yields

$$(p^{k+1} - 1)(\uparrow (p)^{k+1} - \uparrow (p)^{k-1}) \quad > \quad (\uparrow (p)^{k+1} - 1)(p^{k+1} - p^{k-1}).$$

We then have

$$\frac{p^{k+1} - 1}{p^{k+1} - p^{k-1}} > \frac{\uparrow (p)^{k+1} - 1}{\uparrow (p)^{k+1} - \uparrow (p)^{k-1}}.$$

Hence $\dfrac{\sigma(p^k)}{p^k} > \dfrac{\sigma(\uparrow (p)^k)}{\uparrow (p)^k}$. □

Now, we are ready to state the result for all natural numbers.

Theorem 2.2.13. *For every natural number* $n \geq 2$, $\dfrac{\sigma(n)}{n} > \dfrac{\sigma(\uparrow (n))}{\uparrow (n)}$.

Proof. Assume that $n \geq 2$. Write $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$ where $p_1, p_2, ..., p_k$ are distinct primes and $\alpha_1, ..., \alpha_k \in \mathbb{N}$. Then $\uparrow (n) = \uparrow (p_1)^{\alpha_1} \uparrow (p_2)^{\alpha_2} ... \uparrow (p_k)^{\alpha_k}$ and so

$$\begin{aligned} \frac{\sigma(n)}{n} &= \frac{\sigma(p_1^{\alpha_1})}{p_1^{\alpha_1}} \frac{\sigma(p_2^{\alpha_2})}{p_2^{\alpha_2}} ... \frac{\sigma(p_k^{\alpha_k})}{p_k^{\alpha_k}} \\ &> \frac{\sigma(\uparrow (p_1)^{\alpha_1})}{\uparrow (p_1)^{\alpha_1}} \frac{\sigma(\uparrow (p_2)^{\alpha_2})}{\uparrow (p_2)^{\alpha_2}} ... \frac{\sigma(\uparrow (p_k)^{\alpha_k})}{\uparrow (p_k)^{\alpha_k}} \text{ (by Lemma 2.2.12)} \\ &= \frac{\sigma(\uparrow (n))}{\uparrow (n)}. \end{aligned}$$

□

Next, we will study a property between ψ and $\uparrow$ for prime number.

Lemma 2.2.14. *Let* p *be a prime number. Then* $\dfrac{\psi(p)}{p} > \dfrac{\psi(\uparrow (p))}{\uparrow (p)}$.

Proof. Let p be a prime number. Since

$$\begin{aligned} (p+1) \uparrow (p) &= p \uparrow (p) + \uparrow (p) \\ &> p \uparrow (p) + p \\ &= p(\uparrow (p) + 1). \end{aligned}$$

Then

$$\frac{p+1}{p} > \frac{\uparrow (p) + 1}{\uparrow (p)}.$$

Hence $\dfrac{\psi(p)}{p} > \dfrac{\psi(\uparrow (p))}{\uparrow (p)}$. □

Lemma 2.2.15. *Let p be a prime number and $k \in \mathbb{N}$. Then* $\dfrac{\psi(p^k)}{p^k} = \dfrac{\psi(p)}{p}$.

Proof. By Theorem 2.1.1, we have

$$\begin{aligned}\frac{\psi(p^k)}{p^k} = \frac{p^k + p^{k-1}}{p^k} &= \frac{p^{k-1}(p+1)}{p^k} \\ &= \frac{p+1}{p} \\ &= \frac{\psi(p)}{p}. \qquad \square\end{aligned}$$

Now, we are ready to state a general version of the result.

Theorem 2.2.16. *For every natural number $n \geq 2$,* $\dfrac{\psi(n)}{n} > \dfrac{\psi(\uparrow n)}{\uparrow n}$.

Proof. Assume that $n \geq 2$. Write $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$ where $p_1, p_2, ..., p_k$ are distinct primes and $\alpha_1, ..., \alpha_k \in \mathbb{N}$. Then $\uparrow(n) = \uparrow(p_1)^{\alpha_1} \uparrow(p_2)^{\alpha_2} ... \uparrow(p_k)^{\alpha_k}$. By Theorem 2.1.2, Lemma 2.2.14 and Lemma 2.2.15.

$$\begin{aligned}\frac{\psi(n)}{n} = \frac{\psi(p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k})}{p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}} &= \left(\frac{\psi(p_1^{\alpha_1})}{p_1^{\alpha_1}}\right)\left(\frac{\psi(p_2^{\alpha_2})}{p_2^{\alpha_2}}\right) ... \left(\frac{\psi(p_k^{\alpha_k})}{p_k^{\alpha_k}}\right) \\ &= \left(\frac{\psi(p_1)}{p_1}\right)\left(\frac{\psi(p_2)}{p_2}\right) ... \left(\frac{\psi(p_k)}{p_k}\right) \\ &> \left(\frac{\psi(\uparrow(p_1))}{\uparrow(p_1)}\right)\left(\frac{\psi(\uparrow(p_2))}{\uparrow(p_2)}\right) ... \left(\frac{\psi(\uparrow(p_k))}{\uparrow(p_k)}\right) \\ &= \left(\frac{\psi(\uparrow(p_1)^{\alpha_1})}{\uparrow(p_1)^{\alpha_1}}\right)\left(\frac{\psi(\uparrow(p_2)^{\alpha_2})}{\uparrow(p_2)^{\alpha_2}}\right) ... \left(\frac{\psi(\uparrow(p_k)^{\alpha_k})}{\uparrow(p_k)^{\alpha_k}}\right) \\ &= \frac{\psi(\uparrow(p_1)^{\alpha_1} \uparrow(p_2)^{\alpha_2} ... \uparrow(p_k)^{\alpha_k})}{\uparrow(p_1)^{\alpha_1} \uparrow(p_2)^{\alpha_2} ... \uparrow(p_k)^{\alpha_k}} \\ &= \frac{\psi(\uparrow(n))}{\uparrow(n)}. \qquad \square\end{aligned}$$

Lemma 2.2.17. *Let p be a prime number and $k \in \mathbb{N}$. Then*

$$\phi(\uparrow(p)^k) > \phi(p^k).$$

Proof. Consider

$$\begin{aligned} \phi(\uparrow(p)^k) &= \uparrow(p)^k(\uparrow(p) - 1) \\ &> p^{k-1}(p-1) \\ &= \phi(p^k) \text{ (by Lemma 1.2.11).} \end{aligned}$$

□

Theorem 2.2.18. *For every natural number $n \geq 2$, $\phi(n) < \phi(\uparrow n)$.*

Proof. Assume that $n \geq 2$. Write $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$ where $p_1, p_2, ..., p_k$ are distinct primes and $\alpha_1, ..., \alpha_k \in \mathbb{N}$. Then $\uparrow(n) = \uparrow(p_1)^{\alpha_1} \uparrow(p_2)^{\alpha_2} ... \uparrow(p_k)^{\alpha_k}$ and so

$$\begin{aligned} \phi(n) = \phi(p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}) &= \phi(p_1^{\alpha_1})\phi(p_2^{\alpha_2})...\phi(p_k^{\alpha_k}) \\ &< \phi(\uparrow(p_1)^{\alpha_1})\phi(\uparrow(p_2)^{\alpha_2})...\phi(\uparrow(p_k)^{\alpha_k}) \text{ (by Lemma 2.2.17)} \\ &= \phi(\uparrow(n)). \end{aligned}$$

□

Lemma 2.2.19. *$\psi(\uparrow(p)^k) > \psi(p^k)$ for all prime numbers p.*

Proof. We have

$$\begin{aligned} \psi(\uparrow(p)^k) &= \uparrow(p)^{k-1}(\uparrow(p) + 1) \\ &> p^{k-1}(p+1) \\ &= \psi(p^k) \text{ (by Lemma 2.1.1).} \end{aligned}$$

□

Theorem 2.2.20. *For every natural number $n \geq 2$, $\psi(n) < \psi(\uparrow n)$.*

Proof. Assume that $n \geq 2$. Write $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$ where $p_1, p_2, ..., p_k$ are distinct primes and $\alpha_1, ..., \alpha_k \in \mathbb{N}$. Then $\uparrow(n) = \uparrow(p_1)^{\alpha_1} \uparrow(p_2)^{\alpha_2} ... \uparrow(p_k)^{\alpha_k}$ and so

$$\begin{aligned} \psi(n) = \psi(p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}) &= \psi(p_1^{\alpha_1})\psi(p_2^{\alpha_2})...\psi(p_k^{\alpha_k}) \\ &< \psi(\uparrow(p_1)^{\alpha_1})\psi(\uparrow(p_2)^{\alpha_2})...\psi(\uparrow(p_k)^{\alpha_k}) \text{ (by Lemma 2.2.19)} \\ &= \psi(\uparrow(n)). \end{aligned}$$

□

Before we prove the inequality properties between $\uparrow$ and σ, we first show the following Lemma.

Lemma 2.2.21. $2^{k+2} < 3^{k+1} + 1$ *for all* $k \in \mathbb{N}$.

Proof. We prove this lemma by induction. It is easy to see that $2^{1+2} = 8 < 10 = 3^{1+1} + 1$. Let $b \in \mathbb{N}$ be such that $2^{b+2} < 3^{b+1} + 1$.
Then

$$3 \cdot 2^{b+2} < 3^{b+2} + 3$$

and so

$$2^{b+3} \leq 2^{b+3} + 2^{b+2} - 2 < 3^{b+2} + 1.$$

Hence $2^{k+2} < 3^{k+1} + 1$ for all $k \in \mathbb{N}$.

□

Lemma 2.2.22. $(p-1) \uparrow (p)^k - p^{k+1} > 0$ *for all prime numbers* $p > 2$ *and* $k \in \mathbb{N}$.

Proof. It is easy to see that

$$(p-1) \uparrow (p) - p^2 \geq (p-1)(p+2) - p^2 \quad = \quad p - 2 > 0.$$

Let $b \in \mathbb{N}$ be such that $(p-1) \uparrow (p)^b - p^{b+1} > 0$. Then, by the induction hypothesis,

$$(p-1) \uparrow (p)^{b+1} - p^{b+2} > p(p-1) \uparrow (p)^b - p^{b+2} = p((p-1) \uparrow (p)^b - p^{b+1}) > 0$$

Hence $(p-1) \uparrow (p)^k - p^{k+1} > 0$ for all $k \in \mathbb{N}$. □

Lemma 2.2.23. $\sigma(p^k) < \sigma(\uparrow (p)^k)$ *for all prime numbers* p *and* $k \in \mathbb{N}$.

Proof. **Case p = 2:** We know that $2^{k+2} < 3^{k+1} + 1$ (by Lemma 2.2.21).
Then

$$2^{k+2} - 2 < 3^{k+1} - 1$$

and so

$$2^{k+1} - 1 < \frac{3^{k+1} - 1}{2}.$$

Therefore

$$\sigma(2^k) < \sigma(3^k).$$

Case p ≥ 3: Consider

$$\begin{aligned} &p \uparrow (p)^{k+1} - \uparrow (p)^{k+1} - p^{k+1} \uparrow (p) + \uparrow (p) - p + p^{k+1} \\ &= \uparrow (p)[(p-1) \uparrow (p)^k - p^{k+1}] + \uparrow (p) - p + p^{k+1} \\ &> 0 \text{ (by Lemma 2.2.22).} \end{aligned}$$

Therefore

$$p \uparrow (p)^{k+1} - p - \uparrow (p)^{k+1} > p^{k+1} \uparrow (p) - \uparrow (p) - p^{k+1}.$$

Thus

$$\frac{\uparrow (p)^{k+1} - 1}{\uparrow (p) - 1} > \frac{p^{k+1} - 1}{p - 1}$$

and so $\sigma(\uparrow (p)^k) > \sigma(p^k)$. □

Then we will prove the inequality properties between $\uparrow$ and σ for natural numbers.

Theorem 2.2.24. *For every natural number $n \geq 2$, $\sigma(n) < \sigma(\uparrow n)$.*

Proof. Assume that $n \geq 2$. Write $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$ where $p_1, p_2, ..., p_k$ are distinct primes and $\alpha_1, ..., \alpha_k \in \mathbb{N}$. Then $\uparrow (n) = \uparrow (p_1)^{\alpha_1} \uparrow (p_2)^{\alpha_2} ... \uparrow (p_k)^{\alpha_k}$ and so

$$\begin{aligned} \sigma(n) = \sigma(p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}) &= \sigma(p_1^{\alpha_1}) \sigma(p_2^{\alpha_2}) ... \sigma(p_k^{\alpha_k}) \\ &< \sigma(\uparrow (p_1)^{\alpha_1}) \sigma(\uparrow (p_2)^{\alpha_2}) ... \sigma(\uparrow (p_k)^{\alpha_k}) \text{ (by Lemma 2.2.23)} \\ &= \sigma(\uparrow (n)). \end{aligned}$$

□

Finally, we will prove the inequality properties between $\uparrow$ and known arithmetic fuctions more than 1 function.

Theorem 2.2.25. *For every natural number $n \geq 2$,*

$$\phi(\uparrow (n)) \psi(n) > \phi(n) \psi(\uparrow (n)).$$

Proof. Assume that $n \geq 2$. By Theorem 2.2.9 and Theorem 2.2.16, we obtain

$$\frac{\phi(\uparrow (n))}{\uparrow (n)} \cdot \frac{\psi(n)}{n} > \frac{\phi(n)}{n} \cdot \frac{\psi(\uparrow (n))}{\uparrow (n)}.$$

Consequently, we have $\phi(\uparrow (n)) \psi(n) > \phi(n) \psi(\uparrow (n))$. □

Theorem 2.2.26. *For every natural number $n \geq 2$,*

$$\sigma(n) \phi(\uparrow (n)) > \sigma(\uparrow (n)) \phi(n).$$

Proof. The Theorem holds by using Theorem 2.2.9, Theorem 2.2.24 and the proof is similar as in Theorem 2.2.25. □

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Appendix

Appendix

Project Tittle (Thai)	ฟังก์ชันเลขคณิตบางชนิดและสมบัติที่น่าสนใจ
Project Tittle (English)	Some arithmetic functions and their interesting properties.
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Background and Rationale

An *arithmetic function* $\boldsymbol{f}: \mathbb{N} \to \mathbb{C}$ is a complex–valued function whose domain is the set of natural numbers $\mathbb{N}$. Some examples of these particular functions are :

(1) the function $\boldsymbol{\tau(n)}$ counting the number of all positive divisors of $\boldsymbol{n}$;

(2) the function $\boldsymbol{\sigma(n)}$ adding the positive divisors of $\boldsymbol{n}$;

(3) the function $\boldsymbol{\emptyset(n)}$ counting the number of positive integer $\boldsymbol{k}$ less than or equal to $\boldsymbol{n}$ with $\boldsymbol{(k,n)=1}$.

An important property of arithmetic functions is multiplicativity. A function $\boldsymbol{f}: \mathbb{N} \to \mathbb{C}$ is said to be *multiplicative* if

$$\boldsymbol{f(mn) = f(m)f(n)}$$

whenever $\boldsymbol{(m,n)=1}$ and it is said to be *completely multiplicative* if the identity holds for all positive integers $\boldsymbol{m}$ and $\boldsymbol{n}$.

It is known that the function $\boldsymbol{\tau}, \boldsymbol{\sigma}$ and $\boldsymbol{\emptyset}$ are multiplicative, see [2]. Moreover, there are some interesting properties of these functions. In [2], some inequalities of these functions were determined. Examples of inequalities include the following :

(1) $\emptyset(n) \geq \sqrt{n}$ for all n with $n \neq 2$ and $n \neq 6$;

(2) $n\tau(n) \geq \sigma(n) + \emptyset(n)$ for all $n \geq 2$;

(3) $\emptyset(n)\sigma(n) \geq n$ for all n.

The first objective of this project is to investigate possible properties of some known arithmetic functions.

In 2016, a new arithmetic function was introduced by Atanassov [1] as follows. Define ↓(1) = 1 and ↓(2) = 1. For each prime number $p \geq 3$, define $\downarrow(p)$ to be the highest prime number smaller than p. For n > 2, write

$$n = \prod_{i=1}^{k} p_i^{\alpha_i}$$

where $\boldsymbol{k}$, $\boldsymbol{\alpha_1},\ldots,\boldsymbol{\alpha_k} \in \mathbb{N}$ and $\boldsymbol{p_1},\ldots,\boldsymbol{p_k}$ are distinct primes and define

$$\downarrow(\boldsymbol{n}) = \prod_{i=1}^{k} \downarrow(\boldsymbol{p_i})^{\alpha_i}$$

He proved that the function ↓ is multiplicative. Some of its properties were also presented. The second objective of this project is to define a new function based on Atanassov's idea and study its properties.

Objectives

1. Investigate possible properties of some known an arithmetic functions.

2. Define a new arithmetic function based on the idea of Atanassov [2] and study its properties.

Scope

The results of the second objective are based on the idea of Atanassov [2].

Project Activities

1. Study research papers on arithmetic functions and related topics.
2. Present a proposal of the project.
3. Investigate possible properties of some known arithmetic functions.
4. Define a new arithmetic function based on the work of Atanassov[2] and study its properties.
5. Write the report.

Periods.

Description	2019					2020			
	08	09	10	11	12	01	02	03	04
1. Study research papers on arithmetic funtions and related topics.	■	■	■	■					
2. Present a proposal of the project.				■					
3. Investigate possible properties of some known arithmetic functions.				■	■	■			
4. Define a new arithmetic function based on the work of Atanassov [2] and study its properties.						■	■	■	
5. Write the report.								■	■

Benefits

1. Obtain interesting properties of some arithmetic functions.
2. Obtain the technique of doing mathematical reseach.
3. Obtain computer typing skills.

Equipment

1. Computer
2. Microsoft Word 2013
3. Latex

ประวัติผู้เขียน

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