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Minimal surfaces and harmonic mappings

ชื่อนิสิต นายภูมิ เลิศภิญโญวงศ์ **เลขประจำตัว** 5933539723

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์
สาขาวิชา คณิตศาสตร์

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คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

การศึกษาพื้นผิวมินิมอลและการส่งแบบฮาร์โมนิก

นายภูมิ เลิศภิญโญวงศ์

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คณะวิทยาศาสตร์จุฬาลงกรณ์มหาวิทยาลัย

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Minimal surfaces and harmonic mappings

Mr. Poom Lertpinyowong

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สาขาวิชา คณิตศาสตร์
อาจารย์ที่ปรึกษาโครงงาน รองศาสตราจารย์ ดร. ณัฐพันธ์ กิติสิน

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และวิทยาการคอมพิวเตอร์
(ศาสตราจารย์ ดร.กฤษณะ เนียมมณี)

คณะกรรมการสอบโครงงาน

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.... กรรมการ
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.... กรรมการ
(ผู้ช่วยศาสตราจารย์ ดร.พงษ์เดช มนทกานติรัตน์)

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ในโครงงานนี้เราศึกษาการส่งแบบฮาร์โมนิก $f_{n,m}$ ที่สร้างจากวิธีการแปลงของคลูนี่และไชล์-สมอลล์กับฟังก์ชันเชิงวงรี $F(z,1)=\int_0^z \frac{d\zeta}{(1-\zeta^2)}$ ด้วยไดเลชัน $\omega(z)=m^{2n}z^{2n}$ เมื่อ n เป็นจำนวนนับใดๆ และจำนวนเชิงซ้อน m ซึ่ง $|m|\leq 1$ ในกรณีที่ $|m|=1$ เราพบว่าภาพของ $\mathbb{D}$ ภายใต้ $f_{n,m}$ จะเป็นรูปสี่เหลี่ยมด้านขนาน จากนั้นเราใช้สูตรของไวแยร์สทราสส์เพื่อสร้างพื้นผิวมินิมอลเหนือรูปสี่เหลี่ยมด้านขนานดังกล่าว รวมถึงแสดงว่าพื้นผิวดังกล่าวนั้นเป็นพื้นผิวเจเอส กล่าวคือเป็นพื้นผิวมินิมอลเหนือรูปหลายเหลี่ยมซึ่งมีขนาดของความสูงเป็นอนันต์ที่ขอบของรูปหลายเหลี่ยมนั้น

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต ณัฐพันธ์ กิติสิน

สาขาวิชา คณิตศาสตร์ ลายมือชื่อ อ.ที่ปรึกษาโครงงาน ภูมิ เลิศภิญโญวงศ์

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POOM LERTPINYOWONG: MINIMAL SURFACES AND HARMONIC MAPPINGS. ADVISOR: ASSOC. PROF. NATAPHAN KITISIN, PH.D., 91 PP.

In this project, we construct harmonic shear $f_{n,m}$ of elliptic integral $F(z,1) = \int_0^z \frac{d\zeta}{(1-\zeta^2)}$ with dilation $\omega(z) = m^{2n}z^{2n}$ where n is an arbitrary natural number and m is a complex number such that $|m| \leqslant 1$. In particular case, we found that the image of $\mathbb{D}$ under $f_{n,m}$ is a parallelogram when $|m| = 1$. We then use the Weierstrass representation to construct a family of minimal graphs and prove that minimal graphs in this family are JS surfaces, minimal graphs over polygonal domains that become infinite in magnitude at the domain boundary.

Department	Mathematics and Computer Science	Student's Signature	ณัฐพันธ์ กิตติสิน
Field of Study	Mathematics	Advisor's Signature	ภูมิ เลิศภิญโญวงศ์
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Poom Lertpinyowong
March 2020

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List of Notations

$\mathbb{N}$	Set of all natural numbers
$\mathbb{Z}$	Set of all integers
$\mathbb{Q}$	Set of all rational numbers
$\mathbb{Q}^c$	Set of all irrational numbers
$\mathbb{R}$	Set of all real numbers
$\mathbb{C}$	Set of all complex numbers
$\mathbb{D}$	Unit disk in complex plane
$\partial\Omega$	Boundary of a set Ω
$\Re(z)$	Real part of complex number z
$\Im(z)$	Imaginary part of complex number z
$\bar{z}$	Complex conjugate of complex number z

Chapter 1

Introduction

A *minimal surface*, intuitively, is a surface which for each sufficiently small portion of it has the minimum area among all surfaces with the same boundary. Minimal surface can physically be interpreted as a soap film that spans a wire frame when it is dipped in soap solution. Some standard examples of minimal surfaces in Euclidean space are the plane, the catenoid, and the helicoid. History of minimal surfaces can be traced back to 1744 when Euler first described the catenoid surface. Since then minimal surface has become one of the most interesting geometric object which challenges a lot of great mathematicians.

Theory of minimal surfaces involves many branches in mathematics, including differential geometry, partial differential equation and complex analysis. One of the most important techniques to construct new minimal surfaces is called the *Weierstrass representation*.

Our work will focus on special class of minimal surfaces called *minimal graphs*. A minimal graph is a minimal surface which is lifted from its domain. One well-known example of minimal graph is Scherk's doubly periodic surface which has a square as its domain. There are several papers that study this kind of surfaces such as [5], [7], and [8]. Constructions of minimal graphs in those papers are involving constructing *harmonic univalent mapping of the unit disc* $\mathbb{D}$, which is defined as an one to one function from $\mathbb{D}$ to $\mathbb{C}$ whose real part and imaginary part satisfy Laplace equation, and then use *modified Weierstrass representation* to construct minimal graphs. Two main techniques that use to construct harmonic univalent mapping are *Clunie and Sheil-Small shearing method* [3] (see example [7], and [8]) and *Radó-Kneser-Choquet theorem* [9, Chapter 3 and 4] (see example [5], and [10]).

In this project, we will try to apply those methods mentioned above to obtain certain harmonic univalent mapping of the unit disc $\mathbb{D}$ and use it to construct minimal graphs over certain domains.

Chapter 2

Preliminaries

Our approach to construct minimal surfaces involve various branches in mathematics, including differential geometry, complex analysis and harmonic mapping theory, so we dedicate this chapter to provide all facts that we will use in later chapters. We divide this chapter into five sections. First, we give some basic background of differential geometry in section 2.1, just sufficient to rigorously define minimal surfaces and some special classes of them in section 2.2. In section 2.3, we provide some facts in complex analysis and harmonic mapping theory that are necessary for our method to construct minimal surfaces. Then we introduce the Weierstrass representation, the formula to construct minimal surfaces, in section 2.4. Finally, in section 2.5, we review some research papers that we have studied and also describe the main goal of our work.

All of theorems in this chapter will be stated without proof, but most of them can be found on [6], [9] and [12].

2.1 Background in differential geometry

The main objective of our project is to construct some minimal surfaces, which we have already described in the introduction that a minimal surface is "a surface which each sufficiently small portion of it has the minimum area among all surfaces with the same boundary". But in order to rigorously define this kind of surfaces, we need some essential background in differential geometry in $\mathbb{R}^3$. So we begin this chapter by providing some definitions and facts in differential geometry.

Two main objects that we study in differential geometry in $\mathbb{R}^3$ are curves and surfaces. Roughly speaking, a curve is an one dimensional subset of $\mathbb{R}^3$ while a surface is a two dimensional subset of $\mathbb{R}^3$.

Definition 2.1.1. *A* ***(parametrized) curve*** *in* $\mathbb{R}^3$ *is a map* $\gamma : I \to \mathbb{R}^3$, *for some open interval* $I \subseteq \mathbb{R}$. *Moreover, the curve* γ *is called a* ***unit speed curve*** *or* ***arclenght-parametrized curve*** *if* $|\gamma'(t)| = 1$ *for all* $t \in I$.

Definition 2.1.2. *The* ***curvature*** $\kappa(t)$ *of an unit speed curve* γ *at* t *is defined by* $\kappa(t) = |\gamma''(t)|$.

Examples 2.1.3. *Here are some examples of curve:*

1. *A unit speed line, parametrized by* $\gamma(t) = (x_0 + at, y_0 + bt, z_0 + ct)$ *for all* $t \in \mathbb{R}$ *and fixed real numbers* x_0, y_0, z_0, a, b, c *such that* $a^2 + b^2 + c^2 = 1$, *is an unit speed curve in* $\mathbb{R}^3$ *with zero curvature at every point.*

2. *A unit circle, parametrized by* $\gamma(t) = (\cos 2t, \sin 2t, z_0)$ *for all* $t \in [0, \pi)$ *and fixed real number* z_0, *is a in* $\mathbb{R}^3$ *which is not an unit speed curve because* $\gamma'(t) = 2$ *for all* $t \in [0, \pi)$.

Definition 2.1.4. *A connected subset* $S \subseteq \mathbb{R}^3$ *is called a* ***(parametrized) surface*** *if each point* p *in* S *has a neighborhood* U, *a domain* $\Omega \subseteq \mathbb{R}^2$, *and* C^2 *one-to-one function* $\mathbf{x} : \Omega \to U$ *where* $\mathbf{x}(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v))$. *The function* $\mathbf{x}(u, v)$ *is called a* ***parametrization*** *of a surface* S, *and a pair* $(\Omega, \mathbf{x})$ *is called a* ***chart*** *at* p. *The collection of charts that covers* S *is called an* ***atlas*** *of* S.

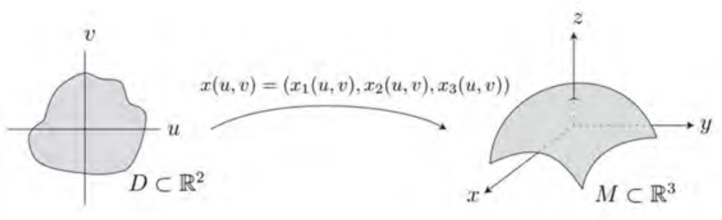


Figure 2.1: Parametrized surface.
([6], Figure 2.7, p. 106)

Definition 2.1.5. *The* ***unit normal vector*** *of a surface* S *at a point* $p = \mathbf{x}(a, b)$ *is* $\mathbf{N}(a, b) = \frac{\mathbf{x}_u(a,b) \times \mathbf{x}_v(a,b)}{|\mathbf{x}_u(a,b) \times \mathbf{x}_v(a,b)|}$. *A* ***tangent vector*** *at a point* p *is a vector* v *started at* p *which orthogonal to* $\mathbf{N}(a, b)$, *i.e.* $v \cdot N(a, b) = 0$.

Note that not every surface has a well-defined unit normal vector, which we called ***non-orientable surface***, *Möbius band* for instance, but in this project, we consider only on surface which has well-defined unit normal vector. We called such a surface, an ***orientable surface***.

The unit normal vector can also be considered as a map from Ω to 2-dimensional sphere $\mathbb{S}^2$. This map is called the ***Gauss map***. And if we compose this map by stereographic projection with respect to $(0,0,1)$, we get the map G which maps from $\Omega \subseteq \mathbb{R}^2 \backsimeq \mathbb{C}$ to $\mathbb{C}$. We also called G, the (steroegraphic projection of the) Gauss map.

Definition 2.1.6. *The* ***tangent plane*** *of a surface S at a point p is*

$$T_pS = \{v|v \text{ is tangent to } S \text{ at } p\}.$$

The geometric interpretation of T_pS is the plane which perpendicular to the unit normal vector and attached to S at p (see Figure 2.2) and it can be proved that $T_pS = \operatorname{span}\{x_u, x_v\}$.

Next we will define the way to measure curvature of a surface S at a

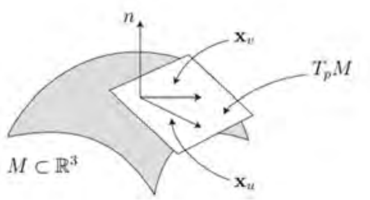


Figure 2.2: A tangent plane and unit normal vector.
([6], Figure 2.13, p. 112)

point p. Surely, the definition of curvature of a surface should be related to a curve. After we defined a curve and a surface, we can also talk about a ***curve on a surface*** which is a curve $\gamma : I \subseteq \mathbb{R} \to \mathbb{R}^3$ such that its image $\gamma(I)$ is in S. We say that the curve γ *passing through* a point p in S if p is in an image of γ.

Definition 2.1.7. *Let S be a surface parametrized by $\mathbf{x}(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v))$, $p = \mathbf{x}(a,b)$ be a point on S, $\mathbf{w}$ be a unit vector in T_pS, and γ be a unit speed curve on S such that $\gamma(t) = p$ and $\gamma'(t) = \mathbf{w}$. The* ***normal curvature*** *of S at p in the $\mathbf{w}$ direction is $\kappa(\mathbf{w}) = \gamma''(t) \cdot N(a,b)$.*

We can think of the normal curvature of S at p in the $\mathbf{w}$ direction as a curvature (with a sign) of the curve obtained by intersecting S with a plane span by $\mathbf{w}$ and $N(a,b)$ (see Figure 2.3).

If we rotate $\mathbf{w}$ around the point p and compute normal curvature in every direction around p, in the case that κ is non-constant, we can find one direction with the maximal value of κ, denoted by $\kappa_1(p)$,and one direction with the minimal value of κ, denoted by $\kappa_2(p)$. These two directions are

Figure 2.3: Normal curvatures in different directions.
([11], Figure 2.2, p. 45)

called ***the principal direction*** at a point p and $\kappa_2(p), \kappa_2(p)$ are called ***the principal curvature*** at p.

It can be proved that principal directions of S at a point p are always perpendicular to each other.

The next definition is the main key to define minimal surface, which we will explain in the next section.

Definition 2.1.8. *The mean curvature (i.e., average curvature) of a surface S at p is $H = \frac{\kappa_1+\kappa_2}{2}$.*

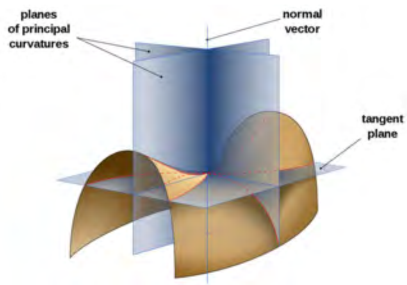


Figure 2.4: Principal curvatures and principal direction.
(Gaba, Eric. (June 2006). **Minimal surface curvature planes-en.svg.**
Available at: https://commons.wikimedia.org/wiki/
File:Minimal_surface_curvature_planes-en.svg [Accessed 23 Feb 2020])

It turns out that, according to the definition, the mean curvature is very hard to compute because we have to find the maximal and minimal curvatures at that point. The next theorem, which we state without a proof, gives us a practical way to compute it.

Theorem 2.1.9. *The mean curvature can be expressed in the form*

$$H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)}, \tag{2.1}$$

where $E = \mathbf{x}_u \cdot \mathbf{x}_u, F = \mathbf{x}_u \cdot \mathbf{x}_v, G = \mathbf{x}_v \mathbf{x}_v$*, the coefficients of the first fundamental form, and* $e = \mathbf{N} \cdot \mathbf{x}_{uu}, f = \mathbf{N} \cdot \mathbf{x}_{uv}, g = \mathbf{N} \cdot \mathbf{x}_{vv}$*, the coefficients of the second fundamental form.*

Examples 2.1.10. *Here are some examples of surfaces:*

1. *The* ***plane****, parametrized by* $\mathbf{x}(u, v) = (u, v, 0); \forall (u, v) \in \Omega \subseteq \mathbb{R}^2$*, is a surface in* $\mathbb{R}^3$ *with zero curvature at every point,* $\kappa_1 = \kappa_2 = 0$*. This implies that its mean curvature* $H = \frac{\kappa_1 + \kappa_2}{2} = 0$ *at every point. See Figure 2.5.*

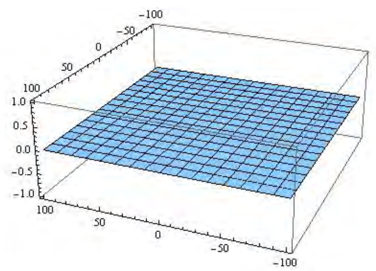


Figure 2.5: The plane.

2. *A* ***sphere*** *with given radius* $r > 0$*, parametrized by* $\mathbf{x}(u, v) = (r \cos u \sin v, r \sin u \sin v, r \cos v)$ *for all* $u \in (0, 2\pi)$ *and* $v \in (0, \pi)$ *is a surface in* $\mathbb{R}^3$*.By some computation, we found that* $\kappa_1 = \kappa_2 = -\frac{1}{r}$ *at every point on the sphere, so* $H = \frac{\kappa_1 + \kappa_2}{2} = -\frac{1}{r}$ *at every point.*

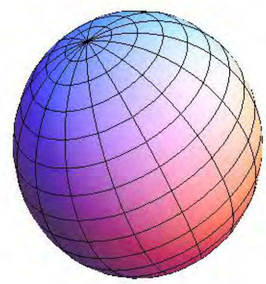


Figure 2.6: Sphere with radius 1.

3. *A* ***cylinder*** *with given radius $r > 0$, parametrized by $\mathbf{x}(u, v) = (r\cos v, r\sin v, u)$ for all $u \in \mathbb{R}$ and $v \in (0, 2\pi)$, is a surface in $\mathbb{R}^3$. By some computation, we found that $\kappa_1 = 0$ and $\kappa_2 = -\frac{1}{r}$ at every point on the cylinder, so $H = \frac{\kappa_1+\kappa_2}{2} = -\frac{1}{2r}$ at every point.*

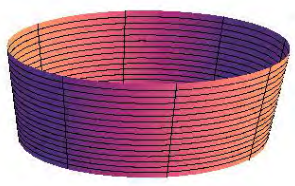


Figure 2.7: A cylinder with radius 1.

4. *A* ***helicoid****, parametrized by $x(u, v) = (u\cos v, u\sin v, bv); \exists a, b > 0, \forall u \in [0, a)$ and $\forall v \in \mathbb{R}$, is a surface. By Equation (2.1) and some calculation, we found that it has zero mean curvature at every point. (see Figure 2.8 for a helicoid with $a = 1, b = 0.2$).*

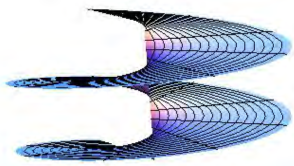


Figure 2.8: A helicoid.

5. *A* ***catenoid****, parametrized by $x(u, v) = (a\cosh v\cos u, a\cosh v\sin u, av), \forall u \in [0, 2\pi)$ and $\forall v \in \mathbb{R}$, is a surface. By Equation (2.1) and some calculation, we found that it has zero mean curvature at every point. (see Figure 2.9 for a catenoid with $a = 1$).*

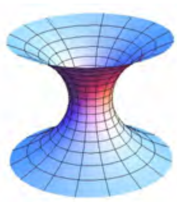


Figure 2.9: A catenoid.

6. *The* ***Enneper's surface****, parametrized by* $x(u,v) = (u - \frac{1}{3}u^3 + uv^2, v - \frac{1}{3}v^3 + u^2v, u^2 - v^2)$ *for all* $u \in \mathbb{R}$ *and* $v \in (0, 2\pi)$*, is a surface. By Equation (2.1) and some calculation, we found that it has zero mean curvature at every point. (see Figure 2.10).*

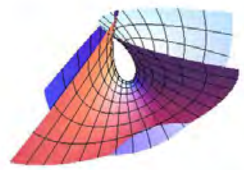


Figure 2.10: The Enneper's surface.

7. *The* ***Scherk's doubly periodic surface****, parametrized by* $x(u,v) = (u, v, \ln\left(\frac{\cos u}{\cos v}\right))$*for all* $u \in (-\pi/2, \pi/2)$ *and* $v \in (-\pi/2, \pi/2)$*, is a surface. By Equation (2.1) and some calculation, we found that it has zero mean curvature at every point. (see Figure 2.11).*

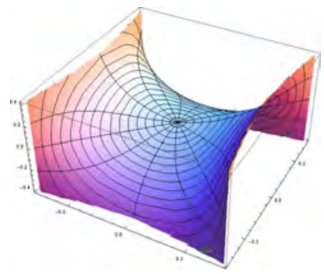


Figure 2.11: The Scherk's doubly periodic surface.

2.2 Minimal surfaces, minimal graphs and JS surfaces

Now, we are ready to define a minimal surface.

Definition 2.2.1. *A* ***minimal surface*** *in* $\mathbb{R}^3$ *is a surface* S *such that its mean curvature* $H = \frac{\kappa_1 + \kappa_2}{2}$ *is zero at every points on* S*.*

One way to understand this definition is by imagining a surface that every point on this surface is saddle point, i.e. a point that look like a peak in one direction, but look like a bottom in other direction (look at the middle point of a surface shown in Figure 2.12 for an example). This is because mean curvature of a minimal surface is vanish everywhere, which mean that at every point, two principle directions of its are curved equally but just in opposite direction ($\kappa_1 = -\kappa_2$).

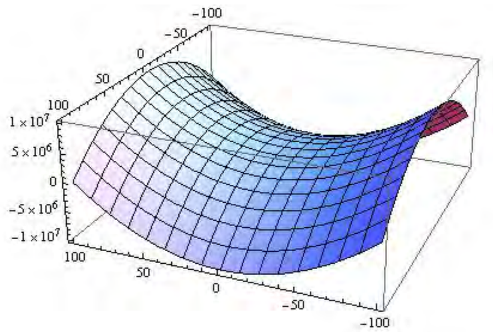


Figure 2.12: Saddle point.

Note that, according to this definition, it isn't obvious that a minimal surface is a surface, as we described in the introduction, with minimum area among all surfaces with the same boundary, but it can be proved that these two definitions are equivalent. In fact, there is a lot of equivalent ways to define minimal surface, but these are beyond our scope, so we do not mention them here.

In this research project, we focus on special class of minimal surfaces called *minimal graphs* which is defined as follow:

Definition 2.2.2. *A* ***minimal graph*** *is a minimal surface such that can be parametrized in the form* $(u, v, \mathcal{F}(u, v))$ *where* $\mathcal{F}(u, v)$ *is function from* $\Omega \subseteq \mathbb{R}^2$ *to* $\mathbb{R}$. *Shortly, a minimal graph is a minimal surface which is also a graph over some domain in* $\mathbb{R}^2$.

Remark 2.2.3. *Equivalently, we can define minimal graph as a surface* S *parametrized by* $(u, v, \mathcal{F}(u, v))$ *where* $\mathcal{F}(u, v)$ *is function from* $\Omega \subseteq \mathbb{R}^2$ *to* $\mathbb{R}$*which satisfies minimal surface equation:*

$$\left(1 - \mathcal{F}_u^2\right) \mathcal{F}_{uu} - 2\mathcal{F}_u \mathcal{F}_v \mathcal{F}_{uv} + \left(1 - \mathcal{F}_v^2\right) \mathcal{F}_{vv} = 0.$$

One special kind of minimal graphs which we focus on is *JS surface*. Roughly speaking, JS surface, which named after Jenkins and Serrin, is a minimal graph over simple bounded polygonal domains where, on approaching each edge bounding the domain, the graph becomes either positively or negatively infinite.

Definition 2.2.4. *Let P be a polygonal domain with finitely many bounding edges partitioned into sets $\{A_i\}$ and $\{B_i\}$. The minimal graph $(u, v, \mathcal{F}(u, v))$ for $(u, v) \in P$ is a* ***JS surface*** *if it satisfies boundary values:*

$$\mathcal{F}(u,v) \to +\infty \quad as\,(u,v) \to \mathrm{int}A_i,$$

$$\mathcal{F}(u,v) \to -\infty \quad as\,(u,v) \to \mathrm{int}B_i,$$

for all $(u, v) \in P$.

Examples 2.2.5. *Here are some examples of minimal surfaces:*

1. *The plane. As shown in Example 2.1.10, mean curvature of the plane is zero everywhere, so it is a minimal surface and hence a minimal graph. (See Figure 2.5).*

2. *A helicoid. As shown in Example 2.1.10, mean curvature of helicoid is zero everywhere, so it is a minimal surface. Otherwise, helicoid is not a minimal graph because it can not be parametrized as $(u, v, \mathcal{F}(u, v))$. (see Figure 2.8).*

3. *A catenoid. As shown in Example 2.1.10, mean curvature of catenoid is zero everywhere, so it is a minimal surface. Otherwise, catenoid is not a minimal graph because it can not be parametrized as $(u, v, \mathcal{F}(u, v))$. (see Figure 2.9).*

4. *The Enneper's surface. As shown in Example 2.1.10, mean curvature of Enneper's surface is zero everywhere, so it is a minimal surface. Otherwise, Enneper's surface is not a minimal graph because it can not be parametrized as $(u, v, \mathcal{F}(u, v))$. (see Figure 2.10).*

5. *The Scherk's doubly periodic surface. As shown in Example 2.1.10, mean curvature of Scherk's doubly periodic surface is zero everywhere, so it is a minimal surface and hence a minimal graph over a square (see Figure 2.11). From its parametrization $x(u, v) = (u, v, \ln\left(\frac{\cos u}{\cos v}\right))$for all $u \in (-\pi/2, \pi/2)$ and $v \in (-\pi/2, \pi/2)$, we found that as u approaches $\pi/2$ or $-\pi/2$, $\ln\left(\frac{\cos u}{\cos v}\right)$ will go to negative infinity while as v approaches $\pi/2$ or $-\pi/2$, $\ln\left(\frac{\cos u}{\cos v}\right)$ will go to positive infinity. Hence Scherk's doubly periodic surface is JS surface.*

Following from Example 2.1.10, we found that sphere and cylinder are not minimal surfaces.

It is interesting to ask that which polygonal domain can be a domain for a JS surface. The theorem , following from [5], stated below is the result of Jenkins and Serrin which completely answers this question.

Theorem 2.2.6. *Let P be a polygonal domain with finitely many bounding edges partitioned into sets A_i and B_i. Let Π be a connected polygonal subset of P whose boundary is the union of some segments from A_i and B_i, possibly including additional line segments contained in P whose endpoints are vertices of P. Let $|\Pi|$ be the length of the boundary of Π. Then there exists a JS surface $(u, v, F(u, v)) : (u, v) \in P$ if and only if*

(a) *no two edges of A_i nor of B_i meet at a convex vertex,*

(b) $2\sum_{A_i\in\Pi} |A_i| < |\Pi|$ *and* $2\sum_{B_i\in\Pi} |B_i| < |\Pi|$ *for each such* Π, $\Pi \neq P$,

(c) $\sum |A_i| = \sum |B_i|$ *when* $\Pi = P$.

If the JS surface exists, it is unique up to translation.

It is not hard to see that a square satisfies all the criterion given above, hence there exist a JS surface over a square, which is, in fact, Scherk's doubly periodic surface introduced in Example 2.2.5.

2.3 Background in complex analysis and harmonic mapping theory

As mentioned above, our construction of minimals graphs is involved theory of harmonic univalent mapping on unit disk $\mathbb{D}$. In this section, we will briefly introduce some definitions and facts about this kind of map. We also introduce two main methods to construct them, which called *Clunie and Sheil-Small shearing method* and *Radó-Kneser-Choquet theorem.*

Definition 2.3.1. *A* ***holomorphic function*** *is a complex-valued function of complex variable that is, at every point of its domain, complex differentiable in a neighbourhood of the point. A* ***meromorphic function*** *is a function that is holomorphic on all of domain except for a set of isolated points, which are poles of the function.*

Definition 2.3.2. *A complex-valued of complex variable function $f(z)$ is called a* ***conformal mapping*** *if and only if it preserves local angles.*

Theorem 2.3.3. *An holomorphic function is conformal at any point where it has a nonzero derivative.*

Examples 2.3.4. *Here are some examples of complex-valued of complex variable function:*

1. $f(z) = e^{i\theta}z$, *for some* $\theta \in \mathbb{R}$, *is a holomorphic function and a conformal mapping from* $\mathbb{C}$ *to* $\mathbb{C}$,

2. *$f(z) = 1/z$ is a meromorphic function on $\mathbb{C}$ with only one pole at $z = 0$ and it is a conformal mapping on $\mathbb{C} \setminus \{0\}$,*

3. *$f(z) = \ln(z)$ is a holomorphic function on $\mathbb{C} \setminus (-\infty, 0]$ which is not a meromorphic function on $\mathbb{C}$ because it cannot be defined on the whole complex plane while only excluding a set of isolated points,*

4. *$f(z) = 1/\sin(z)$ is a meromorphic function on $\mathbb{C}$ with countable poles at $z = 2n\pi$ for all $n \in \mathbb{N}$.*

Theorem 2.3.5 (Cauchy-Riemann equations)**.** *Let $f = u + iv$ be a complex-valued function where u and v are real-valued functions. If f is complex diffrentiable at $z_0 = x_0 + y_0$, then u and v satisfy the Cauchy-Riemann equation:*

$$\tfrac{\partial u}{\partial x}(x_0, y_0) = \tfrac{\partial v}{\partial y}(x_0, y_0) \text{ and } \tfrac{\partial u}{\partial y}(x_0, y_0) = -\tfrac{\partial v}{\partial x}(x_0, y_0).$$

A pair of functions (u, v) that satisfies the Cauchy–Riemann equations is said to be a ***conjugate pair****, and v is called the* ***harmonic conjugate*** *of u.*

Definition 2.3.6. *A rational function*

$$B(z) = c \prod_{k=1}^{n} \frac{z - a_k}{1 - \bar{a}_k z}$$

is called ***finite Blaschke product of order*** *n for the unit disc where $|c| = 1$ and $|a_k| < 1, 1 \leq k \leq n$.*

Remark 2.3.7. *A finite Blaschke product of order n is holomorphic on unit disk $\mathbb{D}$. A finite Blaschke product of order 1 is called* ***Möbius transformation*** *which is an automorphism on $\mathbb{D}$.*

Definition 2.3.8. *Let Ω be a domain in $\mathbb{R}^2$. A function $u : \Omega \to \mathbb{R}$ is* ***(real) harmonic*** *if it satisfies Laplace's equation:*

$$\Delta u(x, y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy} = 0.$$

Examples 2.3.9. *By using little calculation, it's easy to show that these following functions are real harmonic:*

1. *$u(x, y) = xy$*

2. *$u(x, y) = x^3 - 3xy^2$ and $v(x, y) = -3x^2y + y^3$*

3. $u(x,y) = \ln(x^2+y^2)$.

Definition 2.3.10. *A* ***(planar) harmonic mapping*** *is a complex-valued function $f(z) = u(z) + iv(z)$ defined on some domain $\Omega \subseteq \mathbb{C}$ such that $u(z) = u(x+iy) = u(x,y)$ and $v(z) = v(x+iy) = v(x,y)$ are real harmonic function.*

Definition 2.3.11. *A harmonic mapping is said to be* ***univalent*** *if and only if it is one-to-one mapping. A harmonic mapping $f = u+iv$ is said to be* ***locally univalent*** *at z in its domain Ω if and only if there is a neighborhood U of z in Ω such that $f|_U$ is univalent. Moreover, if f is univalent and its Jacobian $J_f(z) := u_x v_y - u_y v_x > 0$, we said that f is* ***sense-preserving*** *(or orientation-preserving). On other hand, if f is univalent and its Jacobian $J_f(z) := u_x v_y - u_y v_x < 0$, we said that f is* ***sense-reversing****.*

Theorem 2.3.12. *Let f be harmonic mapping defined on domain $\Omega \subseteq \mathbb{C}$. If Ω is a simply-connected domain then there exist holomorphic functions h and g defined on Ω such that $f = h + \bar{g}$. Moreover, a pair of h and g is unique up to an additive constant.*

Remark 2.3.13. *The harmonic function $f = h + \bar{g}$ can also be written in the form $f(z) = \Re(h(z)+g(z)) + i\Im(h(z)-g(z))$.*

Definition 2.3.14. *The* ***dilation*** *of $f = h+\bar{g}$, where h and g are holomorphic functions on domain $\Omega \subseteq \mathbb{C}$ of f is defined by $\omega(z) = \frac{g'(z)}{h'(z)}$.*

Examples 2.3.15. *These are some examples of harmonic mappings:*

1. *Any holomorphic function $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$. Consider f as $u+iv$ where u and v are real-valued functions. By Cauchy-Riemann equation, $\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y)$ and $\frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y)$ for all $(x,y) \in \mathbb{C}$. By differentiating these two equations, we get $\frac{\partial^2 u}{\partial x^2}(x,y) = \frac{\partial^2 v}{\partial x \partial y}(x,y) = \frac{\partial^2 v}{\partial x \partial y}(y,x) = -\frac{\partial^2 u}{\partial y^2}(x,y)$ and $\frac{\partial^2 v}{\partial y^2}(x,y) = \frac{\partial^2 u}{\partial y \partial x}(x,y) = \frac{\partial^2 u}{\partial x \partial y}(x,y) = -\frac{\partial^2 v}{\partial x^2}(x,y)$. So u and v satisfy Laplace's equation $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$, which imply f is harmonic mapping. It also easy to see that f can be expressed as $h + \bar{g}$ where g is a constant function and $h = f - \bar{g}$ is a holomorphic function. Thus we can conclude that dilation of f is $\omega(z) = \frac{g'(z)}{h'(z)} = 0$*

2. *$f(z) = f(x,y) = x^3 - 3xy^2 + i(-3x^2y + y^3)$ defined on $\mathbb{D}$. From example 2.3.9, we have known that $u(x,y) = x^3 - 3xy^2$ and $v(x,y) = -3x^2y + y^3$ are real harmonic, therefore f is harmonic mapping. We can express $f(z)$ as $h(z) + \overline{g(z)} = z + \frac{1}{3}\bar{z}^3$ and its dilation $\omega(z) = z^2$.*

Theorem 2.3.16. *A harmonic mapping $f = h + \bar{g}$ defined on $\mathbb{D}$ is locally univalent and sense-preserving if and only if $|\omega(z)| < 1$ for all z in $\mathbb{D}$.*

Definition 2.3.17. *A domain Ω is* ***convex in the direction*** $e^{i\varphi}$ *if, for every $a \in \mathbb{C}$, the set*

$$\Omega \cap \left\{a + te^{i\varphi} | t \in \mathbb{R}\right\}$$

is either connected or empty. In particular, a domain Ω is ***convex in the direction of the real axis*** *or* ***convex in horizontal direction****, CHD for short, if every line parallel to the real axis has a connected intersection with Ω (see figure 2.13 for an example of CHD domain).*

The next two theorems are criterion for a function that maps a disk $\mathbb{D}$univalently onto a domain convex in one direction. The first one is for a holomorphic function, which proved by W.C.Royster and M. Ziegler in 1976, but unfortunately we can not find their original publication, titled *"Univalent functions convex in one direction"*, so we follow the statement from [4]. The second one is for a harmonic function, proved by Clunie and Sheil-Small in 1984, which is one of the most important theorem for our project.

Theorem 2.3.18 (W.C.Royster and M. Ziegler)**.** *Let F be a non-constant holomorphic function in $\mathbb{D}$. The function F maps $\mathbb{D}$ univalently onto a domain convex in the direction of φ if and only if there are numbers μ and ν where $0 \leq \mu < 2\pi$ and $0 \leq \nu \leq \pi$ so that*

$$\Re[e^{i(\mu-\varphi)}(1 - 2\cos\nu e^{-i\mu}z + e^{-2i\mu}z^2)F'(z)] \geq 0. \tag{2.2}$$

Theorem 2.3.19 (Clunie and Sheil-Small,[3])**.** *Let $\varphi \in [0, \pi)$. A harmonic $f = h + \bar{g}$ locally univalent in $\mathbb{D}$ is a univalent mapping of $\mathbb{D}$ onto a domain convex in the direction of φ if and only if $h - e^{2i\varphi}g$ is a conformal univalent mapping of $\mathbb{D}$ onto a domain convex in the direction of φ.*

Remark 2.3.20. *In particular, A harmonic $f = h + \bar{g}$ locally univalent in $\mathbb{D}$ is a univalent mapping of $\mathbb{D}$ onto a CHD domain if and only if $h - g$ is a conformal univalent mapping of $\mathbb{D}$ onto a CHD domain.*

This theorem gives us a method to construct harmonic univalent mappings onto certain CHD domain from given dilation $\omega = g'/h'$ which $|\omega(z)| < 1$ for all z in $\mathbb{D}$ (according to Theorem 2.3.16, this condition required to make sure that a harmonic mapping constructed from this method is locally univalent on $\mathbb{D}$, and hence univalent on $\mathbb{D}$) and a conformal univalent mapping $F = h - g$. By differentiating the relation $F = h - g$, we get $F' = h' - g' = h'(1 - \omega) = g'\left(\frac{1-\omega}{\omega}\right)$. So we can find h and g by integration,

$$h(z) = \int_0^z \frac{F'(\zeta)}{1 - \omega(\zeta)} d\zeta \quad and \quad g(z) = \int_0^z \frac{F'(\zeta)\omega(\zeta)}{1 - \omega(\zeta)} d\zeta. \tag{2.3}$$

This method is called **Clunie and Sheil-Small shearing method** or **shearing method** for short.

Examples 2.3.21. *These are some examples of constructon of harmonic univalent mappings by using shearing method:*

1. *Let $F(z) = z - \frac{1}{3}z^3$, which is a conformal univalent mapping of $\mathbb{D}$ onto a CHD domain (see Figure 2.13), and $\omega(z) = z^2$. By equation (2.3) , we get $h(z) = z$ and $g(z) = \frac{1}{3}z^3$. We rediscover the harmonic map mentioned in Example 2.3.15, $f(z) = h(z) + \bar{g(z)}(z) = z + \frac{1}{3}\bar{z}^3$, we can conclude that it is a harmonic univalent mapping of $\mathbb{D}$ onto a CHD domain, as shown in figure 2.14.*

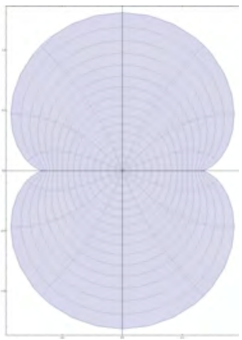


Figure 2.13: Image of $\mathbb{D}$ under $F(z) = z - \frac{1}{3}z^3$.

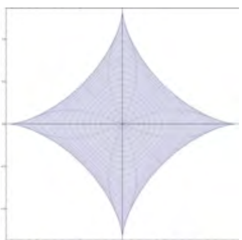


Figure 2.14: Image of $\mathbb{D}$ under $f(z) = z + \frac{1}{3}\bar{z}^3$.

2. *Let $F(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$, which is a conformal univalent mapping of $\mathbb{D}$ onto a horizontal strip convex in the direction of the real axis (see figure 2.15), and $\omega(z) = -z^2$. By equation (2.3) , we get $h(z) = \frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{i}{4}\log\left(\frac{i+z}{i-z}\right)$ and $g(z) = -\frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{i}{4}\log\left(\frac{i+z}{i-z}\right)$. Hence $f(z) = \Re\frac{i}{2}\log\left(\frac{i+z}{i-z}\right) + i\Im\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$, we can conclude that it is a harmonic univalent mapping of $\mathbb{D}$ onto a CHD domain. In fact, the image of $\mathbb{D}$ under f is a square as shown in figure 2.16.*

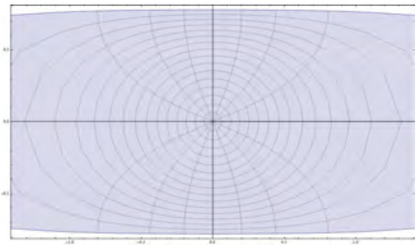


Figure 2.15: Image of $\mathbb{D}$ under $F(z) = \frac{1}{2}\log(\frac{1+z}{1-z})$.

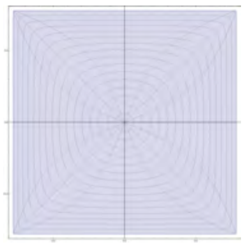


Figure 2.16: Image of $\mathbb{D}$ under $f(z) = \Re\frac{i}{2}\log\left(\frac{i+z}{i-z}\right) + i\Im\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$.

See [6] and [12] for more examples.

The next theorem, followed statement from [9], is one of the important theorem that give us another method to construct a harmonic univalent mapping of $\mathbb{D}$ onto a fixed bounded convex domain.

Theorem 2.3.22 (Radó-Kneser-Choquet theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ be a bounded convex domain whose boundary is a simple closed curve Γ. Let φ map $\partial\mathbb{D}$ continuously onto Γ and suppose that $\varphi(e^{it})$ runs once around Γ monotonically as e^{it} runs around $\partial\mathbb{D}$. Then the harmonic extension*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2}\varphi(e^{it})dt \tag{2.4}$$

is univalent in $\mathbb{D}$ and defines a harmonic mapping of $\mathbb{D}$ onto Ω.

In fact, according to [9], the theorem remains true even if φ has points of discontinuity, provided that $\varphi(\partial\mathbb{D})$ does not lie on a line and the values of φ go monotonically once around Γ. The harmonic extension f will then map $\mathbb{D}$ univalently onto the interior of the convex hull of $\varphi(\partial\mathbb{D})$. For instance, if φ is piecewise constant and monotonic, and its values are not collinear, then f maps $\mathbb{D}$ univalently onto the interior of the convex polygon whose vertices are the values of φ.

In [10], P. Duren, J. McDougall and L. Schaubroeck generalized this result for general polygonal domain.

Suppose that Ω is a general polygon with vertices $c_1, c_2, ..., c_m$, taken in counterclockwise order on the boundary $\Gamma = \partial\mathbb{D}$. For

$$0 \leq t_0 < t_1 < ... < t_m = t_0 + 2\pi, \tag{2.5}$$

let the points

$$b_k = e^{it_k}, k = 0, 1, ..., m, \tag{2.6}$$

determine an arbitrary partition of the unit circle into m subarcs. Note that $b_m = b_0$. Given the boundary correspondence

$$f(e^{it}) = c_k \text{ for } e^{it} \in (b_{k-1}, b_k), k = 1, 2, ..., m,$$

construct the harmonic extension

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2}\varphi(e^{it})dt$$

which, in this case, can be expressed as

$$f(z) = \frac{1}{\pi}\sum_{k=1}^{m} c_k \arg\left\{\frac{z-b_k}{z-b_{k-1}}\right\} - \hat{c}, z \in \mathbb{D} \tag{2.7}$$

where $\hat{c} = \frac{1}{2\pi}\sum_{k=1}^{m} c_k \arg\left\{\frac{b_k}{b_{k-1}}\right\}$. It can be proved that the dilatation of any function f of the form (2.7) is a Blaschke product with at most $m-2$ factors of the form $\varphi_\zeta(z) = \frac{\zeta - z}{1-\bar{\zeta}z}, |\zeta| \neq 1$. Some zeros ζ of the dilatation may be situated outside $\mathbb{D}$.

P. Duren, J. McDougall and L. Schaubroeck also gave a criterion for function of this form to be univalent as follow:

Theorem 2.3.23. *[10] Let f be a harmonic function of the form (2.7), constructed as above from a piecewise constant boundary function with values on the m vertices of a polygonal region Ω, so that the dilatation ω of f is a Blaschke product with at most $m-2$ factors. Then f is univalent in $\mathbb{D}$ if and only if all zeros of ω lie in $\mathbb{D}$. In this case, f is a harmonic mapping of $\mathbb{D}$ onto Ω.*

Example 2.3.24. *Consider Ω to be a square with vertices $c_1 = \frac{\pi}{\sqrt{2}}e^{i\pi/4} = \frac{\pi}{4} + \frac{i\pi}{4}, c_2 = \frac{\pi}{\sqrt{2}}e^{3i\pi/4} = -\frac{\pi}{4} + \frac{i\pi}{4}, c_3 = \frac{\pi}{\sqrt{2}}e^{5i\pi/4} = -\frac{\pi}{4\pi} - \frac{i\pi}{4}, c_4 = \frac{\pi}{\sqrt{2}}e^{7i\pi/4} = \frac{\pi}{4} - \frac{i\pi}{4}$, taken in counterclockwise order on the boundary $\Gamma = \partial\mathbb{D}$. Let the points $b_1 = 1, b_2 = i, b_3 = -1, b_4 = -i$. Apply to Equation (2.7) with some computation, we get $f(z) = \frac{1}{2}\left(\arg\left\{\frac{z-i}{z+i}\right\} + i\arg\left\{\frac{z+1}{z-1}\right\}\right) = h(z) + \bar{g(z)}$ where $h(z) = \frac{1}{4}\left(\log\left(\frac{z+1}{z-1}\right) - i\log\left(\frac{z-i}{z+i}\right)\right)$ and $g(z) = \frac{1}{4}\left(-\log\left(\frac{z+1}{z-1}\right) - i\log\left(\frac{z-i}{z+i}\right)\right)$. Hence we can compute its dilation $\omega(z) = -z^2$ which is a Blaschke product with at most 2 factors and all zeros are lied in $\mathbb{D}$. Therefore this map f is a harmonic univalent mapping of $\mathbb{D}$ onto the square Ω.*

Remark 2.3.25. *By using similar argument, we can construct a harmonic univalent mapping of* $\mathbb{D}$ *onto a regular* $n-$*gon and prove that its dilation is* $\omega(z) = -z^{n-2}$. *(See [9], Chapter 4).*

Although Clunie and Sheil-Small shearing method and Radó-Kneser-Choquet theorem gave us two different ways to construct a harmonic univalent mapping of $\mathbb{D}$, these two methods have different advantages and disadvantages. Even though we can fix dilation of a map if we construct it by using shearing method, we can't fixed its image. In contrast, for Radó-Kneser-Choquet theorem, we can fix its image, but can't fix its dilation. In the next section, we will show that why this is a big disadvantage for using Radó-Kneser-Choquet theorem.

2.4 Connection between harmonic univalent mapping and minimal graph

So far, theory of minimal surface and harmonic mapping are constructed separately, but surprisingly these two subjects have a lot of connections. In this section, we will introduce some of these connections which will be important for our construction of minimal surfaces.

Definition 2.4.1. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be isothermal parametrizations (i.e.* $E = \mathbf{x}_u \cdot \mathbf{x}_u = \mathbf{x}_v \cdot \mathbf{x}_v = G$ *and* $F = \mathbf{x}_u \cdot \mathbf{x}_v = 0$*) of minimal surfaces such that their component functions are pairwise harmonic conjugates. That is,*

$$\mathbf{x}_u = \mathbf{y}_v \text{ and } \mathbf{x}_v = -\mathbf{y}_u.$$

In such a case, $\mathbf{x}$ *and* $\mathbf{y}$ *are called* ***conjugate minimal surfaces****.*

Theorem 2.4.2. *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are conjugate minimal surfaces then*

$$\mathbf{z} = (\cos t)\mathbf{x} + (\sin t)\mathbf{y}$$

where $t \in \mathbb{R}$ *is also a minimal surface. Note that when* $t = 0$ *we have the minimal surface parametrized by* $\mathbf{x}$*, and when* $t = \frac{\pi}{2}$ *we have the minimal surface parametrized by* $\mathbf{y}$*. So for* $0 \leq t \leq \frac{\pi}{2}$*, we have a continuous parameter of minimal surfaces known as* ***associated surfaces****.*

Example 2.4.3. *The helicoid and the catenoid are conjugate surfaces (see Figure 2.17).*

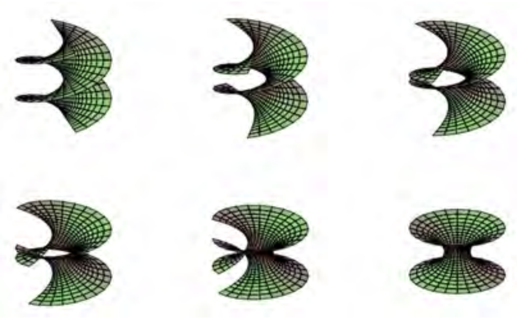


Figure 2.17: Associate surfaces of the helicoid and the catenoid with various value of t. ([6], Figure 2.25, p. 132)

Lemma 2.4.4. *Let Φ_1, Φ_2, Φ_3 be holomorphic functions on the simply connected domain Ω such that $\Phi_1^2 + \Phi_2^2 + \Phi_3^2 = 0$ and $0 < |\Phi_1^2| + |\Phi_2^2| + |\Phi_3^2| < \infty$ and is finite. Let $z_0 \in \Omega$ Then the formula*

$$\mathbf{x}(z) = \left(\Re \int_{z_0}^{z} \Phi_1(w)dw, \Re \int_{z_0}^{z} \Phi_2(w)dw, \Re \int_{z_0}^{z} \Phi_3(w)dw \right), z \in \Omega$$

parameterizes a minimal surface on Ω.

Consider a holomorphic function p and a meromorphic function q in some domain $\Omega \subseteq \mathbb{C}$ having the property that at each point where q has a pole of order m, p has a zero of order at least $2m$. Let $\Phi_1 = p(1+q^2), \Phi_2 = -ip(1-q^2)$ and $\Phi_3 = -2ipq$. These functions satisfy the requirement of the lemma mentioned above. Hence we get:

Theorem 2.4.5 ((General) Weierstrass representation)**.** *Let p be a holomorphic function and q a meromorphic function in some domain $\Omega \subseteq \mathbb{C}$ having the property that at each point where q has a pole of order m, p has a zero of order at least $2m$. Then every minimal surface has a local isothermal parametric representation (i.e. $E = G$ and $F = 0$) of the form*

$$\mathbf{x}(z) = \left(\Re \left\{ \int_{z_0}^{z} p(1+q^2)d\zeta \right\}, \Re \left\{ \int_{z_0}^{z} -ip(1-q^2)d\zeta \right\}, \Re \left\{ \int_{z_0}^{z} -2ipqd\zeta \right\} \right) \tag{2.8}$$

where z_0 is a constant in Ω

Example 2.4.6. *Using $p(z) = 1$ and $q(z) = iz$, the Weierstrass representation yields*

$$\mathbf{x}(z) = \left(\Re \left\{ z - \frac{1}{3}z^3 \right\}, \Re \left\{ -i \left(z + \frac{1}{3}z^3 \right) \right\}, \Re(z^2) \right).$$

Letting $z = u + iv$, this yields $\mathbf{x}(u, v) = (u - \frac{1}{3}u^3 + uv^2, v - \frac{1}{3}v^3 + u^2v, u^2 - v^2)$ which is the Enneper surface.

The representation above give us the method to construct minimal surfaces, but our approach is more specific. As mentioned before, we want to construct minimal graphs. The next theorem, though less general than the previous one, gives us the easier way to construct minimal graphs and it connects theory of harmonic mapping to theory of minimal surface.

Let $f = h + \bar{g}$ be a harmonic univalent mapping of $\mathbb{D}$ onto domain Ω which its dilation can be expressed as q^2 where q is analytic (holomorphic). Consider $\Phi_1 = h' + g', \Phi_2 = -i(h' - g')$ and $\Phi_3 = -2i\sqrt{h'g'} = -2ih'\sqrt{w}$. These functions satisfy the requirement of Lemma 2.4.4, hence we get:

Theorem 2.4.7 ((Modified) Weierstrass representation for minimal graph)**.** *If $f = h + \bar{g}$ is a sense-preserving harmonic univalent mapping of $\mathbb{D}$ onto some domain Ω with dilatation $\omega = q^2$ for some analytic function q in $\mathbb{D}$, then the formulas*

$$\begin{aligned} u &= \Re\left\{h(z) + g(z)\right\} = \Re\left\{f(z)\right\}, \\ v &= \Im\left\{h(z) - g(z)\right\} = \Im\left\{f(z)\right\}, \\ \mathcal{F}(u, v) = \mathcal{F}(z) &= 2\Im\left\{\int_0^z \sqrt{h'(\zeta)g'(\zeta)}d\zeta\right\}. \end{aligned} \tag{2.9}$$

define a minimal graph whose projection onto the complex plane i $f(\mathbb{D})$. If $\mathbf{x}^*(u, v) = (x_1^*(u, v), x_2^*(u, v), x_3^*(u, v))$ is a parametrization of conjugate surface defined above, we get

$$\begin{aligned} x_1^*(z) &= \Re\left\{h(z) - g(z)\right\}, \\ x_2^*(z) &= \Im\left\{h(z) + g(z)\right\}, \\ x_3^*(z) &= 2\Re\left\{\int_0^z \sqrt{h'(\zeta)g'(\zeta)}d\zeta\right\}. \end{aligned} \tag{2.10}$$

Theorem 2.4.8. *If $(u, v, \mathcal{F}(u, v))$ is a parametrization of the minimal graph defined by (2.9), then the (stereographic projection of the) Gauss map G of the surface, where we take the surface normal to be upward so that $|G(z)| > 1$, is related to the dilation ω of a harmonic map f by the relation*

$$\omega(z) = -\frac{1}{G^2(z)} \tag{2.11}$$

Examples 2.4.9. *Here are some examples:*

1. *Apply modified Weierstrass reprentation to any holomorphic function $f : \mathbb{D} \to \mathbb{C}$, then we get a plane.*

2. *Consider f as in the second example in Examples 2.3.21 which is a harmonic univalent mapping which maps $\mathbb{D}$ to a square. Since $\omega(z) = g'(z)/h'(z) = -z^2$ is a square of an analytic function in $\mathbb{D}$, we can apply modified Weierstrass reprentation. This yields*

$$\mathbf{x}(z) = \left(\Re\left[\frac{i}{2}\log\left(\frac{i+z}{i-z}\right)\right], \Im\left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right], 2\Im\left[i\log\left(\frac{1+z^2}{1-z^2}\right)\right]\right)$$

which is a parametrization of Scherk's doubly-periodic surface.

The next two theorems are the results of D. Bshouty, and A. Weitsman in [2]. The first one makes us know when $\mathcal{F}(u, v)$ of JS surface over polygonal domain Ω will change its sign around the boundary $\partial\Omega$. While the second one gives us a criterion for a minimal graph to be a JS surface.

Let S be a JS surface over a polygonal domain Ω parametrized by (2.9). Now, f is the Poisson integral of a step function having values $z_1, z_2, ..., z_n$ which are vertices of Ω. If

$$f(e^{i\varphi}) = \begin{cases} z_1 & \text{when } \varphi \in (t_1, t_2) \\ z_2 & \text{when } \varphi \in (t_2, t_3) \\ \vdots & \\ z_n & \text{when } \varphi \in (t_n, t_{n+1}) \end{cases}$$

where $t_1 < t_2 < t_3 < ... < t_{n+1} = t_0 + 2\pi$. Let $C_j = \{e^{i\varphi} | \varphi \in (t_j, t_{j+1})\}$ and let $\zeta_j = e^{it_j}$ for $j = 1, 2, 3, ..., n+1$. If $0 < \alpha_j < 2\pi$ is the interior angle at z_j and we take a continuous branch of $\arg(\omega(z))$ on $\overline{C_j}$, then

$$\frac{1}{2}\left(\arg(\omega(\zeta_{j+1})) - \arg(\omega(\zeta_j))\right) = \alpha_j \tag{2.12}$$

or

$$\frac{1}{2}\left(\arg(\omega(\zeta_{j+1})) - \arg(\omega(\zeta_j))\right) = \alpha_j - \pi \tag{2.13}$$

In case $0 < \alpha_j < \pi$, then z_j is a ***point of convexity*** and (2.12) must hold. However, if $\alpha_j \geq \pi$ and (2.12) still holds, we call z_j a ***full resting point***.

Theorem 2.4.10. *[2] If z_j is a full resting point for the JS surface S given by $\mathcal{F}(u, v)$, then $\mathcal{F}$ changes sign on the sides adjacent to z_j . If z_j is neither a point of convexity nor full resting point then $\mathcal{F}$ does not change sign at z_j.*

Theorem 2.4.11 (Criterion for JS surface, [2])**.** *Let S be a minimal graph over a polygonal domain D having k sides. If the Gauss map G for S in the parametrization (2.4.7) has the form $c/B(z)$ where c is a constant of modulus 1 and $B(z)$ is a Blaschke product of order n, then S is a JS surface, and $k \geq 2n + 2$.*

2.5 Previous results and our objectives

Our work is mainly followed from research paper titled *"Harmonic shears of elliptic integrals"* [8] written by M. Dorff and J. Szynal in 2005. They applied shearing method to shear elliptic integral of the first and second kind, which can be represented in the following forms, respectively:

$$F(z,k) = \int_0^z \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}};$$
$$E(z,k) = \int_0^z \sqrt{\frac{1-k^2\zeta^2}{1-\zeta^2}}d\zeta,$$

where $k, m \in \overline{\mathbb{D}}$.

M. Dorff and J. Szynal showed that for any fixed complex number $k \in \bar{\mathbb{D}}$, $F(z,k)$ maps $\mathbb{D}$ univalently onto a convex region. Let m be a complex number with modulus less than or equal to 1. By applying Theorem 2.3.19 to $F(z,k)$ with dilation $\omega(z) = m^2z^2$, they constructed a collection of harmonic univalent mappings of $\mathbb{D}$ onto a CHD domain. Then they consider special case when $k = 1$ which the corresponding harmonic mapping is $f = h + \bar{g}$ where

$$h(z) = \frac{1}{2(1-m^2)}\log\left(\frac{1+z}{1-z}\right) + \frac{m}{2(m^2-1)}\log\left(\frac{1+mz}{1-mz}\right); \tag{2.14}$$

$$g(z) = \frac{m^2}{2(1-m^2)}\log\left(\frac{1+z}{1-z}\right) + \frac{m}{2(m^2-1)}\log\left(\frac{1+mz}{1-mz}\right). \tag{2.15}$$

They found that, in this case, if $m = e^{i\theta}$, the image of the map f is a parallelogram with four vertices $(\pi(1-\cos\theta)/4\sin\theta, \pi/4)$, $(-\pi(1+\cos\theta)/4\sin\theta, \pi/4)$, $(-\pi(1-\cos\theta)/4\sin\theta, -\pi/4)$, and $(\pi(1+\cos\theta)/4\sin\theta, -\pi/4)$. Although the authors didn't mention, it can be easy to prove that this parallelogram is, in fact, a rhombus with angle θ and length $\pi/2\sin\theta$, see Figure 2.18.

For elliptic integral of the second kind, let θ be any fixed real number in $[0, 2\pi)$, $E(z, e^{i\theta})$ maps $\mathbb{D}$ univalently onto a domain convex in the direction $e^{i\pi/2}$, i.e. convex the direction of the imaginary axis. By applying Theorem 2.3.19 to $E(z, e^{i\theta})$ with dilation $\omega(z) = m^2z^2$, they constructed a collection of harmonic univalent mappings of $\mathbb{D}$ onto a domain convex in the direction of the imaginary axis. Despite the fact that the authors discovered interesting special case for image of $\mathbb{D}$ under harmonic shear of $F(z,k)$, they didn't know much for $E(z,k)$, except for its convexity in the direction of the imaginary

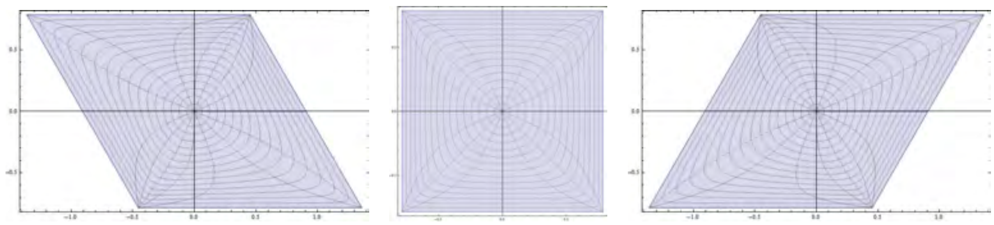


Figure 2.18: Image of $\mathbb{D}$ under $f(z)$ when $m = e^{\pi/3}$, $m = e^{\pi/2}$ and $m = e^{2\pi/3}$ respectively.

axis, see some figures from [8] for examples.

Since all harmonic univalent maps above recived from shearing method with dilation $\omega(z) = m^2z^2$ which has analytic square root, we can use modified Weierstrass representation (Theorem 2.4.7) to construct a minimal graph over image of $\mathbb{D}$ under such harmonic univalent mapping. The authors mainly focus on harmonic shear $f(z)$ of $F(z, 1)$ with dilation $\omega(z) = e^{2i\theta}z^2$, which mentioned above that $f(\mathbb{D})$ is a rhombus. They found that if $\theta = \pi/2$, $f(\mathbb{D})$ is a square and a minimal graph over $f(\mathbb{D})$ is Scherk's doubly periodic surface and by varying value of θ, they found a family of minimal graphs over rhombus which they called *slanted Scherk surfaces*, which in fact are JS surfaces, see Figure 2.19. They also proved that as θ approching to zero, surfaces will approch to a helicoid. Moreover, they investigated their conjugate surfaces, see Figure 2.20, which has catenoid as a limit surface when θ approching to zero.

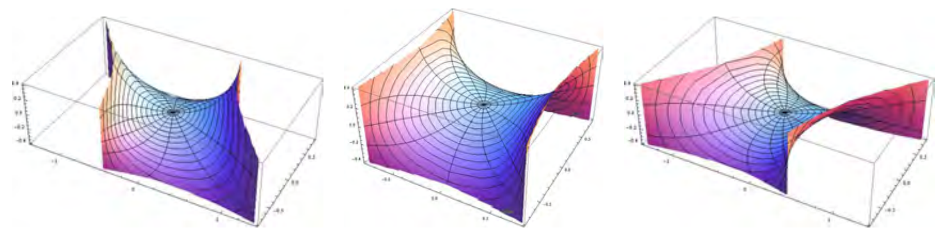


Figure 2.19: Minimal graphs corresponding to $f(z)$ when $m = e^{\pi/3}$, $m = e^{\pi/2}$ and $m = e^{2\pi/3}$ respectively.

In 2008, J. McDougall and L. Schaubroeck investigated a new family of minimal graphs in their publication *"Minimal surfaces over stars"*[5]. The authors applied modified Weierstrass representation to harmonic univalent mappings of $\mathbb{D}$ onto star domains, which they already constructed in [10] by using Radó-Kneser-Choquet theorem. They also proved that minimal graphs over stars that they constructed are JS surfaces (See Figure 2.21).

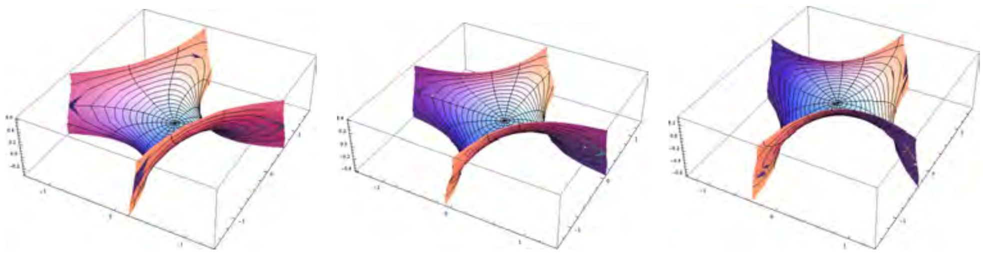


Figure 2.20: Conjugate minimal graphs corresponding to $f(z)$ when $m = e^{\pi/3}$, $m = e^{\pi/2}$ and $m = e^{2\pi/3}$ respectively.

By including this family to the list of previous JS surfaces mentioned in the third section of their paper, we get:

1. Scherk's doubly periodic surface (see Example 2.1.10), discovered in 1834 by H. Scherk.

2. The JS surface over a regular hexagon, studied by Hermann Schwarz.

3. The JS surface over a regular $2n$-gon , where $n \geq 3$, studied by Karcher in the context of constructing saddle towers.

4. The JS surface over a rhombus (see Figure 2.19), as mentioned above, studied by M.Dorf and J. Szynal in 2005 [8].

5. The JS surface over a star.

Figure 2.21: A minimal graph over a star domain.
([5], Fig. 3., p. 731)

Our objectives are to use shearing method or Radó-Kneser-Choquet theorem to construct a harmonic univalent mapping of $\mathbb{D}$ onto certain domain and then use modified Weierstrass represntation to lift such domain to a minimal graph over it. We divided our results into two chapters as follow:

Chapter 3: **Construction of harmonic univalent mapping.** Since minimal graph over rhombus was already constructed in [8] and it is a generalization of Scherk's doubly periodic surface, at first we try to generalize to arbitrary parallelogram. Hence we try to construct a harmonic mapping onto parallelogram domain. As we discussed in the last paragraph of section 2.3, Radó-Kneser-Choquet theorem is a better choice of method to construct this kind of map because we want to construct a harmonic map onto a fixed domain. Unfortunately, dilation of harmonic map that we construction in this approach turns out to has non analytic square root (except for the case of rhombus which the map concise with construction via shearing method in [8]), which doesn't match the requirement of modified Weierstrass representation. So we can't construct minimal graph corresponding to this map.

To avoid this problem, we change our approach to shearing method. In [8], M. Dorff and M. Szynal sheared the elliptic integral $F(z, 1)$ with dilation $\omega(z) = m^2z^2$ where $|m| \leq 1$ and found that in the case $|m| = 1$, the map will map $\mathbb{D}$ to a rhombus. We try to follow this construction and generalize this by changing the dilation to be $\omega(z) = m^{2n}z^{2n}$, which has analytic square root, where $|m| \leq 1$ and $n \in \mathbb{N}$. It turns out that, by the same argument given in [8], we can prove that the map will map $\mathbb{D}$ to a parallelogram, which is not a rhombus for $n \geq 2$.

Chapter 4: **Minimal graphs over parallelograms and their conjugate surfaces.** In this chapter, we apply modified Weierstrass representation to the maps constructed in Chapter 3 to get minimal graphs over parallelograms. We can also prove that these minimal graphs are JS surfaces and can conclude interesting trigonometric identity as a corollary.

Chapter 3

Construction of harmonic univalent mappings

3.1 Using Radó-Kneser-Choquet theorem to construct harmonic mapping onto parallelogram domain

In this section, we going to construct harmonic univalent mapping onto arbitrary parallelogram domain by using Radó-Kneser-Choquet theorem. Given arbitrary positive real numbers a, b and $\theta, \beta \in (0, \pi)$. Consider the convex quadrilateral with vertices $c_1 = (\frac{b+a\cos\theta}{2}, \frac{a\sin\theta}{2}), c_2 = (\frac{-b+a\cos\theta}{2}, \frac{a\sin\theta}{2}), c_3 = (\frac{-b-a\cos\theta}{2}, -\frac{a\sin\theta}{2})$ and $c_4 = (\frac{b-a\cos\theta}{2}, -\frac{a\sin\theta}{2})$ taken in counterclockwise order. It is easy to see that this quadrilateral is a parallelogram with $c_1c_2 = c_3c_4 = b, c_2c_3 = c_4c_1 = a$ and $c_4\hat{c_1}c_2 = c_2\hat{c_3}c_4 = \theta, c_1\hat{c_2}c_3 = c_3\hat{c_4}c_1 = \pi - \theta$, as shown in Figure 3.1. Let $t_0 = 0, t_1 = \pi - \beta, t_2 = \pi, t_3 = 2\pi - \beta$ and $t_4 = 2\pi$.

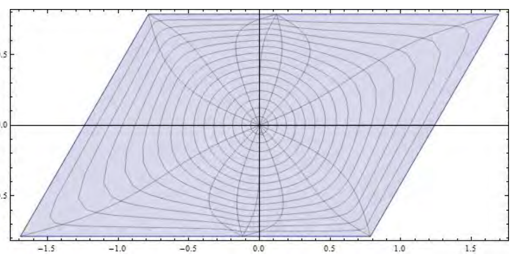


Figure 3.1: A parallelogram.

It is obvious that $0 \le t_0 < t_1 < t_2 < t_3 < t_4 = t_0 + 2\pi$. For $k = 0, 1, 2, 3, 4$, let $b_k = e^{it_k}$ be points on the unit circle. Define a step function on the unit circle.

$$f(e^{it}) = \begin{cases} c_1, & t \in (t_0, t_1) \\ c_2, & t \in (t_1, t_2) \\ c_3, & t \in (t_2, t_3) \\ c_4, & t \in (t_3, t_4) \end{cases} \tag{3.1}$$

Then, by (2.7) the harmonic extension to $\mathbb{D}$ is

$$f(z) = \frac{1}{\pi} \sum_{k=1}^{4} c_k \arg\left\{ \frac{z - b_k}{z - b_{k-1}} \right\} - \hat{c}, z \in \mathbb{D} \tag{3.2}$$

where $\hat{c} = \frac{1}{2\pi} \sum_{k=1}^{4} c_k \arg\left\{ \frac{b_k}{b_{k-1}} \right\}$

$$\begin{aligned} &= \frac{1}{2\pi}\left(\frac{b + a\cos\theta}{2} + \frac{ia\sin\theta}{2}\right)(\pi - \beta) + \frac{1}{2\pi}\left(\frac{-b + a\cos\theta}{2} + \frac{ia\sin\theta}{2}\right)\beta \\ &+ \frac{1}{2\pi}\left(\frac{-b - a\cos\theta}{2} - \frac{ia\sin\theta}{2}\right)(\pi - \beta) + \frac{1}{2\pi}\left(\frac{b - a\cos\theta}{2} - \frac{ia\sin\theta}{2}\right)\beta \\ &= 0. \end{aligned}$$

Hence

$$\begin{aligned} f(z) &= \frac{1}{\pi} \sum_{k=1}^{4} c_k \arg\left\{ \frac{z - b_k}{z - b_{k-1}} \right\} \\ &= \frac{1}{\pi}\left(\frac{b + a\cos\theta}{2} + \frac{ia\sin\theta}{2}\right)\arg\left\{\frac{z - e^{i(\pi-\beta)}}{z - 1}\right\} \\ &+ \frac{1}{\pi}\left(\frac{-b + a\cos\theta}{2} + \frac{ia\sin\theta}{2}\right)\arg\left\{\frac{z + 1}{z - e^{i(\pi-\beta)}}\right\} \\ &+ \frac{1}{\pi}\left(\frac{-b - a\cos\theta}{2} - \frac{ia\sin\theta}{2}\right)\arg\left\{\frac{z - e^{i(2\pi-\beta)}}{z + 1}\right\} \\ &+ \frac{1}{\pi}\left(\frac{b - a\cos\theta}{2} - \frac{ia\sin\theta}{2}\right)\arg\left\{\frac{z - 1}{z - e^{i(2\pi-\beta)}}\right\} \\ &= \frac{1}{\pi}\left[b \arg\left\{\frac{z - e^{i(\pi-\beta)}}{z - e^{i(2\pi-\beta)}}\right\} + a(\cos\theta + i\sin\theta)\arg\left\{\frac{z+1}{z-1}\right\}\right]. \end{aligned}$$

And f can be expressed in the form $h + \bar{g}$ where

$$h(z) = -\frac{1}{2\pi}\left[bi \log\left(\frac{e^{i\beta}z + 1}{e^{i\beta}z - 1}\right) + a(i\cos\theta - \sin\theta)\log\left(\frac{z+1}{z-1}\right)\right]$$

and $g(z) = -\dfrac{1}{2\pi}\left[bi\log\left(\dfrac{e^{i\beta}z+1}{e^{i\beta}z-1}\right) + a(i\cos\theta+\sin\theta)\log\left(\dfrac{z+1}{z-1}\right)\right]$.

We get

$$\begin{aligned}
h'(z) &= -\frac{1}{2\pi}\left[bi\left(\frac{e^{i\beta}}{e^{i\beta}+1} - \frac{e^{i\beta}}{e^{i\beta}-1}\right) + a(i\cos\theta-\sin\theta)\left(\frac{1}{z+1}-\frac{1}{z-1}\right)\right]\\
&= \frac{1}{\pi}\left[\frac{bie^{i\beta}}{e^{2i\beta}z^2-1} + \frac{ai(\cos\theta+i\sin\theta)}{z^2-1}\right]
\end{aligned}$$

and

$$\begin{aligned}
g'(z) &= -\frac{1}{2\pi}\left[bi\left(\frac{e^{i\beta}}{e^{i\beta}+1} - \frac{e^{i\beta}}{e^{i\beta}-1}\right) + a(i\cos\theta+\sin\theta)\left(\frac{1}{z+1}-\frac{1}{z-1}\right)\right]\\
&= \frac{1}{\pi}\left[\frac{bie^{i\beta}}{e^{2i\beta}z^2-1} + \frac{ai(\cos\theta-i\sin\theta)}{z^2-1}\right].
\end{aligned}$$

So we can compute the dilation

$$\begin{aligned}
\omega(z) = \frac{g'(z)}{h'(z)} &= \frac{(be^{i\beta}+ae^{i(2\beta-\theta)})z^2 - (be^{i\beta}+ae^{-i\theta})}{(be^{i\beta}+ae^{i(2\beta+\theta)})z^2 - (be^{i\beta}+ae^{i\theta})}\\
&= -\frac{be^{i\beta}+ae^{i(2\beta-\theta)}}{be^{i\beta}+ae^{i\theta}}\left(\frac{z^2-A}{1-\bar{A}z^2}\right)
\end{aligned}$$

where $A = \frac{be^{i\beta}+ae^{-i\theta}}{be^{i\beta}+ae^{i(2\beta-\theta)}}$. Suppose $A = r(\cos\alpha + i\sin\alpha)$ where $r \geq 0$ and $\alpha \in [0, 2\pi)$ and let $B = \sqrt{r}(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2})$. Hence $\omega(z) = -\frac{be^{i\beta}+ae^{i(2\beta-\theta)}}{be^{i\beta}+ae^{i\theta}}\left(\frac{z-B}{1-\bar{B}z}\right)\left(\frac{z-(-B)}{1-(-\bar{B})z}\right)$ which is a Blaschke product of order 2. Theorem 2.3.23 tells us that this map is univalent in $\mathbb{D}$ if and only if B lie in $\mathbb{D}$ which means that $|A| = r < 1$.

Since $\theta, \beta \in (0, \pi)$, $\sin\theta > 0$ and $\sin\beta > 0$. Hence $\sin\theta\sin\beta > 0$ which implies that $\cos(\theta+\beta) < \cos(\theta-\beta)$. And since

$$\begin{aligned}
&|A| = \left|\frac{be^{i\beta}+ae^{-i\theta}}{be^{i\beta}+ae^{i(2\beta-\theta)}}\right| < 1\\
&\iff |be^{i\beta}+ae^{-i\theta}| < |be^{i\beta}+ae^{i(2\beta-\theta)}|\\
&\iff (be^{i\beta}+ae^{-i\theta})(be^{-i\beta}+ae^{i\theta}) < (be^{i\beta}+ae^{i(2\beta-\theta)})(be^{-i\beta}+ae^{i(\theta-2\beta)})\\
&\iff e^{i(\beta+\theta)} + e^{-i(\beta+\theta)} < e^{i(\theta-\beta)} + e^{i(\beta-\theta)}\\
&\iff \cos(\theta+\beta) < \cos(\theta-\beta),
\end{aligned}$$

we can conclude that $|A| = r < 1$ and hence the map f is univalent in $\mathbb{D}$ and it maps $\mathbb{D}$ onto the parallelogram mentioned above.

Because we want to use this map to construct minimal surface over parallelogram, we should consider when its dilation ω is analytic square root. Since ω has two roots, B and $-B$ in $\mathbb{D}$, if $B \neq 0$, B and $-B$ are single roots of ω, square root of ω should have a branch cut emerge from B and $-B$ which means that ω is not a square of an analytic function.

If $B = 0$, $\frac{be^{i\beta}+ae^{-i\theta}}{be^{i\beta}+ae^{i(2\beta-\theta)}} = A = 0$. Therefore $be^{i\beta} + ae^{-i\theta} = 0$. Since $a, b > 0$ and $\theta, \beta \in (0, \pi)$, $a = b$ and $\theta = \pi - \beta$. So ω is analytic square root if and only if $a = b$ and $\theta = \pi - \beta$ and the image of $\mathbb{D}$ under f is a rhombus. We can conclude this theorem.

Theorem 3.1.1. *The harmonic map f which obtained from the construction above has analytic square root dilation if and only if $a = b$ and $\theta = \pi - \beta$ and the image of $\mathbb{D}$ under f is a rhombus.*

In fact, in this case, if we give $a = \frac{\pi}{2\sin\theta}$ and $m = e^{i\beta}$, this map will concise with the map constructed in [8] which we mentioned in (2.14) and (2.15).

So we can't construct minimal graph over a parallelogram domain which is not a rhombus via these collection of maps and in the case of rhombus, its corresponding minimal graph is already known. In the next section, we will try to use other approach, the shearing method, to construct the map.

3.2 Harmonic shear of $F(z, 1)$ with dilation $\omega(z) = m^{2n}z^{2n}$ where $|m| \leq 1$ and natural number n

In [8], M. Dorff and M. Szynal sheared the elliptic integral $F(z, 1)$ with dilation $\omega(z) = m^2 z^2$ where $|m| \leq 1$. We try to follow this construction and generalize this by changing the dilation to be $\omega(z) = m^{2n}z^{2n}$, which has analytic square root, where $|m| \leq 1$ and $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and $m \in \mathbb{C}$ such that $|m| \leq 1$. Let $f_{n,m}(z) = h_{n,m}(z) + \overline{g_{n,m}(z)}$, where $h_{n,m}$ and $g_{n,m}$ are holomorphic functions on $\mathbb{D}$, be a harmonic shear of $F(z, 1) = \int_0^z \frac{d\zeta}{(1-\zeta^2)}$ with dilation $\omega(z) = m^{2n}z^{2n}$. By Equation (2.3), we get

$$h_{n,m}(z) = \int_0^z \frac{F'(\zeta)}{1-\omega(\zeta)} d\zeta = \int_0^z \frac{1}{(1-\zeta^2)(1-m^{2n}\zeta^{2n})} d\zeta \qquad (3.3)$$

$$g_{n,m}(z) = \int_0^z \frac{F'(\zeta)\omega(\zeta)}{1-\omega(\zeta)} d\zeta = \int_0^z \frac{m^{2n}\zeta^{2n}}{(1-\zeta^2)(1-m^{2n}\zeta^{2n})} d\zeta. \qquad (3.4)$$

First, we will find $h_{n,m}(z)$ by computing the partial fraction decomposition of

$$\frac{1}{(1-\zeta^2)(1-m^{2n}\zeta^{2n})} = \frac{1}{(1-\zeta^2)(1-m^2\zeta^2)(1-m^2\omega_{2n}^2\zeta^2)...(1-m^2\omega_{2n}^{2(n-1)}\zeta^2)}$$

where $\omega_{2n} = e^{i\pi/n}$ is the primitive $2n$-th root of unity. Suppose that the above fraction can be written in the form

$$\frac{A_1}{(1-m^2\zeta^2)} + \frac{A_2}{(1-m^2\omega_{2n}^2\zeta^2)} + ... + \frac{A_n}{(1-m^2\omega_{2n}^{2(n-1)}\zeta^2)} + \frac{A_{n+1}}{(1-\zeta^2)}$$

where $A_j \in \mathbb{C}$ for all $j = 1, 2, 3, ..., n+1$. We get

$$\begin{aligned} 1 = A_1 &\left(\frac{(1-\zeta^2)(1-m^{2n}\zeta^{2n})}{(1-m^2\zeta^2)}\right) + A_2 \left(\frac{(1-\zeta^2)(1-m^{2n}\zeta^{2n})}{(1-m^2\omega_{2n}^2\zeta^2)}\right) + ... \\ &+ A_n \left(\frac{(1-\zeta^2)(1-m^{2n}\zeta^{2n})}{(1-m^2\omega_{2n}^{2(n-1)}\zeta^2)}\right) + A_{n+1}(1-m^{2n}\zeta^{2n}). \end{aligned}$$

For $p = 0, 1, 2, ..., n-1$, consider $\left(\frac{(1-\zeta^2)(1-m^{2n}\zeta^{2n})}{(1-m^2\omega_{2n}^{2p}\zeta^2)}\right)$. Since

$$1-m^{2n}\zeta^{2n} = (1-m^2\omega_{2n}^{2p}\zeta^2)(1+m^2\omega_{2n}^{2p}\zeta^2+m^4\omega_{2n}^{4p}\zeta^4+...+m^{2(n-1)}\omega_{2n}^{2p(n-1)}\zeta^{2(n-1)}),$$

we get

$$\begin{aligned} \frac{(1-\zeta^2)(1-m^{2n}\zeta^{2n})}{(1-m^2\omega_{2n}^{2p}\zeta^2)} &= (1-\zeta^2)(1+m^2\omega_{2n}^{2p}\zeta^2+m^4\omega_{2n}^{4p}\zeta^4+...+m^{2(n-1)}\omega_{2n}^{2p(n-1)}\zeta^{2(n-1)}) \\ &= \sum_{j=0}^{n-1}(m^{2j}\omega_{2n}^{2pj}\zeta^{2j} - m^{2j}\omega_{2n}^{2pj}\zeta^{2(j+1)}) \\ &= \left(\sum_{j=0}^{n-1} m^{2j}\omega_{2n}^{2pj}\zeta^{2j}\right) - \left(\sum_{j=0}^{n-1} m^{2j}\omega_{2n}^{2pj}\zeta^{2(j+1)}\right) \\ &= 1 - m^{2(n-1)}\omega_{2n}^{2p(n-1)}\zeta^{2n} + \sum_{j=1}^{n-1} m^{2(j-1)}\omega_{2n}^{2pj}\zeta^{2j}(m^2-\omega_{2n}^{-2p}). \end{aligned}$$

Therefore

$$\begin{aligned}
1 &= \sum_{p=0}^{n-1} A_{p+1}\left(1 - m^{2(n-1)}\omega_{2n}^{2p(n-1)}\zeta^{2n} + \sum_{j=1}^{n-1} m^{2(j-1)}\omega_{2n}^{2pj}\zeta^{2j}(m^2 - \omega_{2n}^{-2p})\right)\\
&\quad + A_{n+1}(1 - m^{2n}\zeta^{2n})\\
&= (A_1 + A_2 + A_3 + ... + A_{n+1}) - m^{2(n-1)}\zeta^{2n}\left(m^2 A_{n+1} + \sum_{p=0}^{n-1}(\omega_{2n}^{2p(n-1)}A_{p+1})\right)\\
&\quad + \sum_{j=1}^{n-1}\left(m^{2(j-1)}\zeta^{2j}\left(\sum_{p=0}^{n-1}(\omega_{2n}^{2pj}(m^2 - \omega_{2n}^{-2p})A_{p+1})\right)\right).
\end{aligned}$$

Comparing the coefficient of ζ^{2j} for $j = 0, 1, 2, ..., n$ and get

$$\begin{aligned}
j = 0;\ & A_1 + A_2 + ... + A_{n+1} = 1\\
j \in 1, 2, 3, ..., n-1;\ & m^{2(j-1)}\left(\sum_{p=0}^{n-1}(\omega_{2n}^{2pj}(m^2 - \omega_{2n}^{-2p})A_{p+1})\right) = 0\\
j = n;\ & -m^{2(n-1)}\left(m^2 A_{n+1} + \sum_{p=0}^{n-1}(\omega_{2n}^{2p(n-1)}A_{p+1})\right) = 0.
\end{aligned}$$

Since $m \neq 0$, we can divide equations by m and get

$$\begin{aligned}
&(0);\ A_1 + A_2 + ... + A_{n+1} = 1\\
&(j);\ \sum_{p=0}^{n-1}(\omega_{2n}^{2pj}(m^2 - \omega_{2n}^{-2p})A_{p+1}) = 0 \text{ for } j = 1, 2, 3, ..., n-1\\
&(*);\ m^2 A_{n+1} + \sum_{p=0}^{n-1}(\omega_{2n}^{2p(n-1)}A_{p+1}) = 0.
\end{aligned}$$

By $m^2(0) - (*)$, we get

$$(n); \sum_{p=0}^{n-1}\omega_{2n}^{2pn}(m^2 - \omega_{2n}^{-2p})A_{p+1} = m^2.$$

For $r = 0, 1, 2, ..., n-1$, consider $\sum_{j=1}^{n}\omega_{2n}^{2rj}(j)$ and get

$$\sum_{j=1}^{n}\sum_{p=0}^{n-1}\left(\omega_{2n}^{2(p+r)j}\left(m^2 - \omega_{2n}^{-2p}\right)A_{p+1}\right) = m^2$$

$$\sum_{p=0}^{n-1}\left(\left(m^2 - \omega_{2n}^{-2p}\right)A_{p+1}\sum_{j=1}^{n}\left(\omega_{2n}^{2(p+r)j}\right)\right) = m^2$$

Since for $p+r \equiv 1,2,3,...,n-1(\mathrm{mod}\ n)$, $\omega_{2n}^{2(p+r)}$ are roots of $z+z^2+z^3+...+z^n=0$, we get

$$\sum_{j=1}^{n} \omega_{2n}^{2(p+r)j} = 0 \text{ for all } p+r \equiv 1,2,3,...,n-1(\mathrm{mod}\ n).$$

Therefore

$$\begin{aligned} nA_{n-r+1}(m^2 - \omega_{2n}^{-2(n-r)}) &= m^2 \\ A_{n-r+1} &= \frac{m^2}{n(m^2 - \omega_{2n}^{-2(n-r)})}. \end{aligned}$$

Which means $A_j = \frac{m^2}{n(m^2-\omega_{2n}^{2(1-j)})}$ for $j = 1,2,3,...,n$. Since $\frac{1}{(1-\zeta^2)(1-m^{2n}\zeta^{2n})} = \frac{A_1}{(1-m^2\zeta^2)} + \frac{A_2}{(1-m^2\omega_{2n}^2\zeta^2)} + ... + \frac{A_n}{(1-m^2\omega_{2n}^{2(n-1)}\zeta^2)} + \frac{A_{n+1}}{(1-\zeta^2)}$, we get

$$\begin{aligned} \frac{1}{(1-\zeta^2)(1-m^{2n}\zeta^{2n})} &= \sum_{j=1}^{n} \left[\left(\frac{m^2}{n(m^2-\omega_{2n}^{2(1-j)})} \right) \left(\frac{1}{1-m^2\omega_{2n}^{2(j-1)}\zeta^2} \right) \right] + \frac{A_{n+1}}{(1-\zeta^2)} \\ &= \sum_{j=0}^{n-1} \left[\left(\frac{m^2}{n(m^2-\omega_{2n}^{-2j})} \right) \left(\frac{1}{1-m^2\omega_{2n}^{2j}\zeta^2} \right) \right] + \frac{A_{n+1}}{(1-\zeta^2)}. \end{aligned}$$

And since $A_1 + A_2 + ... + A_{n+1} = 1$, we get

$$\begin{aligned} A_{n+1} &= 1 - \sum_{j=1}^{n} A_j \\ &= 1 - \sum_{j=1}^{n} \frac{m^2}{n} \left(\frac{1}{n(m^2-\omega_{2n}^{2(1-j)})} \right) \\ &= 1 - \frac{m^2}{n} \left(\sum_{j=0}^{n-1} \frac{1}{n(m^2-\omega_{2n}^{2j})} \right) \\ &= 1 - \frac{m^2}{n} \left(\frac{\sum_{j=0}^{n-1} \left(\frac{m^{2n}-\omega_{2n}^{2jn}}{m^{2n}-\omega_{2n}^{2j}} \right)}{m^{2n}-1} \right) \\ &= 1 - \frac{m^2}{n} \left(\frac{\sum_{j=0}^{n-1} \sum_{p=0}^{n-1} m^{2(n-1-p)} \omega_{2n}^{2jp}}{m^{2n}-1} \right) \\ &= 1 - \frac{m^2}{n} \left(\frac{\sum_{p=0}^{n-1} m^{2(n-1-p)} \left(\sum_{j=0}^{n-1} \omega_{2n}^{2jp} \right)}{m^{2n}-1} \right). \end{aligned}$$

Since $\sum_{j=0}^{n-1} \omega_{2n}^{2jp} = 0$ for all $p = 1, 2, 3, ..., n-1$, we get

$$\begin{aligned} A_{n+1} &= 1 - \frac{m^2}{n}\left(\frac{nm^{2(n-1)}}{m^{2n}-1}\right) \\ &= 1 - \frac{m^{2n}}{m^{2n}-1} \\ &= -\frac{1}{m^{2n}-1}. \end{aligned}$$

Hence

$$\frac{1}{(1-\zeta^2)(1-m^{2n}\zeta^{2n})} = \sum_{j=0}^{n-1}\left[\left(\frac{m^2}{n(m^2-\omega_{2n}^{-2j})}\right)\left(\frac{1}{1-m^2\omega_{2n}^{2j}\zeta^2}\right)\right] - \frac{1}{(m^{2n}-1)(1-\zeta^2)}.$$

For $j = 0, 1, 2, ..., n-1$, consider

$$\begin{aligned} \int_0^z \frac{1}{1-m^2\omega_{2n}^{2j}\zeta^2}d\zeta &= \frac{1}{2}\int_0^z\left(\frac{1}{1-m\omega_{2n}^j\zeta} + \frac{1}{1+m\omega_{2n}^j\zeta}\right)d\zeta \\ &= \frac{1}{2m\omega_{2n}^j}\int_0^z \frac{1}{1+m\omega_{2n}^j\zeta}d(1+m\omega_{2n}^j\zeta) \\ &\quad - \frac{1}{2m\omega_{2n}^j}\int_0^z \frac{1}{1-m\omega_{2n}^j\zeta}d(1-m\omega_{2n}^j\zeta) \\ &= \left(\frac{1}{2m\omega_{2n}^j}\right)\log\left(\frac{1+m\omega_{2n}^j\zeta}{1-m\omega_{2n}^j\zeta}\right). \end{aligned}$$

And $\int_0^z \frac{1}{1-\zeta^2}d\zeta = \frac{1}{2}\int_0^z\left(\frac{1}{1-\zeta}+\frac{1}{1+\zeta}\right)d\zeta = \frac{1}{2}\left[\int_0^z \frac{1}{1+\zeta}d(1+\zeta) - \int_0^z \frac{1}{1-\zeta}d(1-\zeta)\right] = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$, hence

$$\begin{aligned} h_{n,m}(z) &= \int_0^z \frac{1}{(1-\zeta^2)(1-m^{2n}\zeta^{2n})}d\zeta \\ &= \sum_{j=0}^{n-1}\left[\left(\frac{m}{2n(m^2-\omega_{2n}^{-2j})\omega_{2n}^j}\right)\log\left(\frac{1+m\omega_{2n}^j z}{1-m\omega_{2n}^j z}\right)\right] + \left(\frac{1}{2(1-m^{2n})}\right)\log\left(\frac{1+z}{1-z}\right). \end{aligned}$$

Next, we will find $g_{n,m}(z)$ by computing the partial fraction decomposition of

$$\frac{m^{2n}\zeta^{2n}}{(1-\zeta^2)(1-m^{2n}\zeta^{2n})} = \frac{m^{2n}\zeta^{2n}}{(1-\zeta^2)(1-m^2\zeta^2)(1-m^2\omega_{2n}^2\zeta^2)...(1-m^2\omega_{2n}^{2(n-1)}\zeta^2)}$$

where $\omega_{2n} = e^{i\pi/n}$ is the primitive $2n$-th root of unity. Suppose that the above fraction can be written in the form

$$\frac{B_1}{(1-m^2\zeta^2)} + \frac{B_2}{(1-m^2\omega_{2n}^2\zeta^2)} + ... + \frac{B_n}{(1-m^2\omega_{2n}^{2(n-1)}\zeta^2)} + \frac{B_{n+1}}{(1-\zeta^2)}$$

where $B_j \in \mathbb{C}$ for all $j = 1, 2, 3, ..., n+1$. We get

$$\begin{aligned} m^{2n}\zeta^{2n} = B_1 &\left(\frac{(1-\zeta^2)(1-m^{2n}\zeta^{2n})}{(1-m^2\zeta^2)}\right) + B_2\left(\frac{(1-\zeta^2)(1-m^{2n}\zeta^{2n})}{(1-m^2\omega_{2n}^2\zeta^2)}\right) + ... \\ &+ B_n\left(\frac{(1-\zeta^2)(1-m^{2n}\zeta^{2n})}{(1-m^2\omega_{2n}^{2(n-1)}\zeta^2)}\right) + B_{n+1}(1-m^{2n}\zeta^{2n}). \end{aligned}$$

By the same argument above, we get

$$\begin{aligned} j = 0;\ & B_1 + B_2 + ... + B_{n+1} = 0 \\ j \in 1, 2, 3, ..., n-1;\ & m^{2(j-1)}\left(\sum_{p=0}^{n-1}(\omega_{2n}^{2pj}(m^2 - \omega_{2n}^{-2p})B_{p+1})\right) = 0 \\ j = n;\ & -m^{2(n-1)}\left(m^2 A_{n+1} + \sum_{p=0}^{n-1}(\omega_{2n}^{2p(n-1)}B_{p+1})\right) = m^{2n}. \end{aligned}$$

Since $m \neq 0$, we can divide equations by m and get

$$\begin{aligned} &(0);\ B_1 + B_2 + ... + B_{n+1} = 0 \\ &(j);\ \sum_{p=0}^{n-1}(\omega_{2n}^{2pj}(m^2 - \omega_{2n}^{-2p})B_{p+1}) = 0 \text{ for } j = 1, 2, 3, ..., n-1 \\ &(*);\ m^2 B_{n+1} + \sum_{p=0}^{n-1}(\omega_{2n}^{2p(n-1)}B_{p+1}) = -m^2. \end{aligned}$$

By $m^2(0) - (*)$, we get

$$(n); \sum_{p=0}^{n-1}\omega_{2n}^{2pn}(m^2 - \omega_{2n}^{-2p})B_{p+1} = m^2.$$

We can solve this system of equations $(0) - (n)$ and get $B_j = \frac{m^2}{n(m^2 - \omega_{2n}^{2(1-j)})}$ for $j = 1, 2, 3, ..., n$ and

$$B_{n+1} = -\sum_{j=0}^{n} B_j = -\frac{m^{2n}}{m^{2n}-1}.$$

Hence

$$\frac{m^{2n}\zeta^{2n}}{(1-\zeta^2)(1-m^{2n}\zeta^{2n})} = \sum_{j=0}^{n-1}\left[\left(\frac{m^2}{n(m^2-\omega_{2n}^{-2j})}\right)\left(\frac{1}{1-m^2\omega_{2n}^{2j}\zeta^2}\right)\right] - \frac{m^{2n}}{(m^{2n}-1)(1-\zeta^2)}.$$

We get

$$\begin{aligned} g_{n,m}(z) &= \int_0^z \frac{m^{2n}\zeta^{2n}}{(1-\zeta^2)(1-m^{2n}\zeta^{2n})}d\zeta \\ &= \sum_{j=0}^{n-1}\left[\left(\frac{m}{2n(m^2-\omega_{2n}^{-2j})\omega_{2n}^j}\right)\log\left(\frac{1+m\omega_{2n}^j z}{1-m\omega_{2n}^j z}\right)\right] + \left(\frac{m^{2n}}{2(1-m^{2n})}\right)\log\left(\frac{1+z}{1-z}\right). \end{aligned}$$

By Clunie and Sheil-Small shearing method, we can conclude that $f_{n,m}(z) = h_{n,m}(z) + \overline{g_{n,m}(z)}$ where

$$h_{n,m}(z) = \sum_{j=0}^{n-1}\left[\left(\frac{m}{2n(m^2-\omega_{2n}^{-2j})\omega_{2n}^j}\right)\log\left(\frac{1+m\omega_{2n}^j z}{1-m\omega_{2n}^j z}\right)\right] + \left(\frac{1}{2(1-m^{2n})}\right)\log\left(\frac{1+z}{1-z}\right)$$

and

$$g_{n,m}(z) = \sum_{j=0}^{n-1}\left[\left(\frac{m}{2n(m^2-\omega_{2n}^{-2j})\omega_{2n}^j}\right)\log\left(\frac{1+m\omega_{2n}^j z}{1-m\omega_{2n}^j z}\right)\right] + \left(\frac{m^{2n}}{2(1-m^{2n})}\right)\log\left(\frac{1+z}{1-z}\right)$$

is a harmonic univalent mapping of $\mathbb{D}$ onto a CHD domain with dilation $\omega(z) = m^{2n}z^{2n}$.

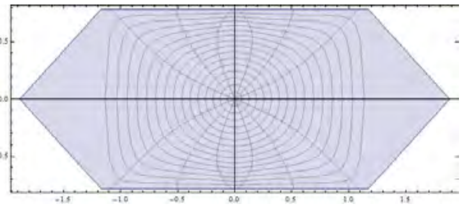
In the case that $n = 1$, this mapping is exactly the same as a map constructed by M. Dorff and M. Szynal in [8] which is $f_{1,m}(z) = h_{1,m}(z) + \overline{g_{1,m}(z)}$ where

$$h_{1,m}(z) = \left(\frac{1}{2(1-m^2)}\right)\log\left(\frac{1+z}{1-z}\right) + \frac{1}{2(1-m^2)}\log\left(\frac{1+mz}{1-mz}\right)$$

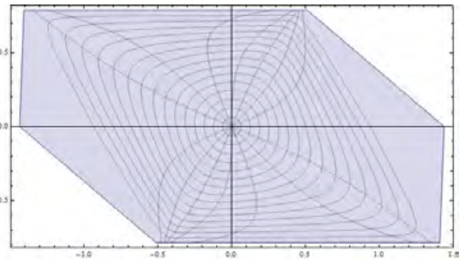
and

$$g_{1,m}(z) = \left(\frac{m^2}{2(1-m^{2n})}\right)\log\left(\frac{1+z}{1-z}\right) + \frac{m^2}{2(1-m^2)}\log\left(\frac{1+mz}{1-mz}\right).$$

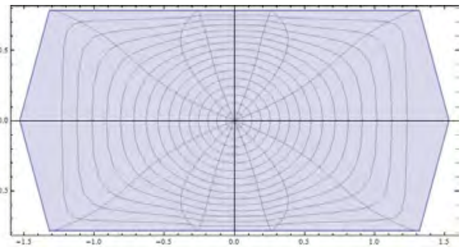
Although we know that $f_{n,m}(\mathbb{D})$ is convex in direction of the real axis, we do not know much about it, see Figure 3.2 which show images of $\mathbb{D}$ under $f_{n,m}$ for various values of n and m. However, in the next section, we will analyse the image of $\mathbb{D}$ under $f_{n,m}$ where $|m| = 1$.



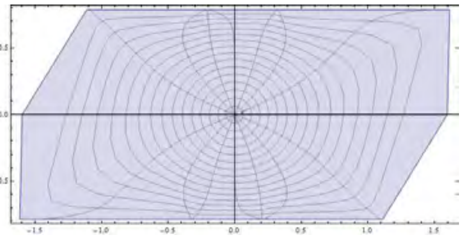
(a) $n = 2$ and $m = 0.95e^{\frac{i\pi}{4}}$



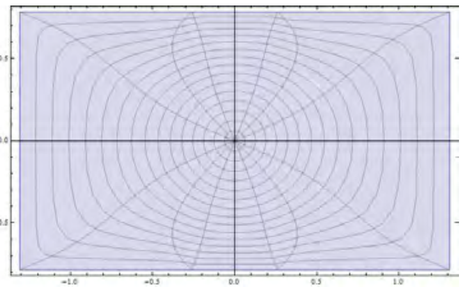
(b) $n = 1$ and $m = 0.95e^{\frac{i\pi}{3}}$



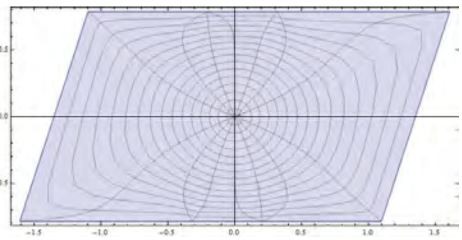
(c) $n = 3$ and $m = 0.99e^{\frac{i\pi}{6}}$



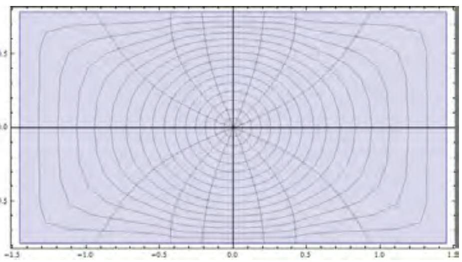
(d) $n = 3$ and $m = 0.99e^{\frac{i\pi}{5}}$



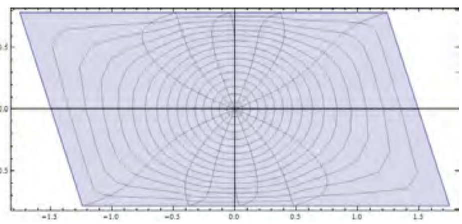
(e) $n = 3$ and $m = e^{\frac{i\pi}{6}}$



(f) $n = 3$ and $m = e^{\frac{i\pi}{5}}$



(g) $n = 4$ and $m = e^{\frac{i\pi}{8}}$



(h) $n = 4$ and $m = e^{\frac{i\pi}{10}}$

Figure 3.2: images of $\mathbb{D}$ under $f_{n,m}$ for various values of n and m

3.3 Images of $\mathbb{D}$ under $f_{n,m}$ where $|m| = 1$ and natural number n

In [8], M. Dorff and M. Szynal have proven that the image of $\mathbb{D}$ under $f_{1,m}$ where $|m| = 1$, in fact this should except the cases that $m = 1, -1$ to avoid zero denominator, is a parallelogram which in fact is a rhombus. We will prove the same result for $f_{n,m}$ where $|m| = 1$ such that $m^{2n} \neq 1$ and an arbitrary natural number n. The restriction of value of m is to avoid zero denominator in formula of $f_{n,m}$. Then we will find the proportion of non-parallel sides of the parallelogram and conclude that for the case $n > 1$ the image of $\mathbb{D}$ under $f_{n,m}$ is a parallelogram which is not a rhombus.

Firstly, we will prove some propositions.

Proposition 3.3.1. *For any complex number z, $\Re(iz) = -\Im(z)$.*

Proof. Let z be a complex number. Then $iz = i\left(\Re(z) + i\Im(z)\right) = i\Re(z) - \Im(z)$. Since $\Re(z)$ and $-\Im(z)$ are real numbers, we get $\Re(iz) = -\Im(z)$ as desired. □

Proposition 3.3.2. *For any non-zero complex number z, $\Im\left(\log(z)\right) = \arg(z)$.*

Proof. Let $z = re^{i\theta}$ be a non-zero complex number where r is a positive real number and $\theta \in [0, 2\pi)$. Therefore $\Im(\log(z)) = \Im(log(r) + i\theta)$. Since $\log(r)$ and θ are real numbers, we get $\Im\left(\log(z)\right) = \theta = \arg(z)$. □

Proposition 3.3.3. *For a complex number $z = e^{i\varphi}$, z and $z^2 - 1$ are perpendicular and $\frac{z}{z^2-1} = -\frac{i}{2\sin\varphi}$.*

Proof. Let $z = e^{i\varphi}$ be a complex number, then $|z|^2 z\bar{z} = 1$ and $\frac{z}{z^2-1} = \frac{z\bar{z}}{(z^2-1)\bar{z}} = \frac{1}{z-\bar{z}} = \frac{1}{2i\sin\varphi} = -\frac{i}{\sin\varphi}$. This implies that $\frac{z}{z^2-1} = \left(-\frac{1}{\sin\varphi}\right)e^{i\pi/2}$ where $-\frac{1}{\sin\varphi}$ is a real number. This means z and $z^2 - 1$ are perpendicular, as shown in Figure 3.3. □

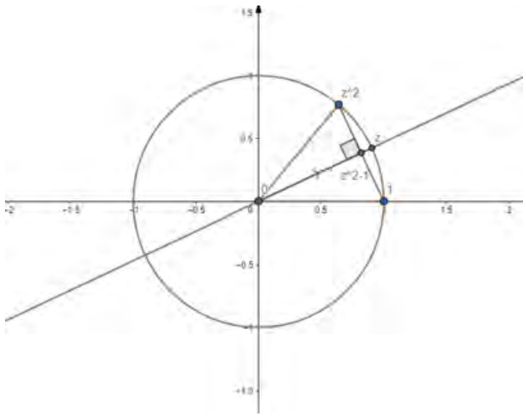


Figure 3.3: Geometric interpretation of Proposition 3.3.3 when $\varphi \in (0, \pi/2)$.

Proposition 3.3.4. *For a complex number $z = e^{i\varphi}$, $z+1$ and $z-1$ are perpendicular and $\frac{1+z}{1-z} = i\cot\frac{\varphi}{2}$.*

Proof. Let $z = e^{i\varphi}$ be a complex number, then $|z|^2 z\bar{z} = 1$ and $\frac{1+z}{1-z} = \frac{(1+z)(1+\bar{z})}{(1-z)(1+\bar{z})} = \frac{2+z+\bar{z}}{-z+\bar{z}} = \frac{2+2\cos\varphi}{-2i\sin\varphi} = \frac{i(1+\cos\varphi)}{\sin\varphi}$. Since $\cot\left(\frac{\varphi}{2}\right) = \frac{\cos\left(\frac{\varphi}{2}\right)}{\sin\left(\frac{\varphi}{2}\right)} = \frac{2\cos^2\left(\frac{\varphi}{2}\right)}{2\sin\left(\frac{\varphi}{2}\right)\cos\left(\frac{\varphi}{2}\right)} = \frac{(1+\cos\varphi)}{\sin\varphi}$, we get $\frac{1+z}{1-z} = i\cot\frac{\varphi}{2}$. This implies that $\frac{1+z}{1-z} = \cot\left(\frac{\varphi}{2}\right)e^{i\pi/2}$ where $\cot\left(\frac{\varphi}{2}\right)$ is a real number. This means $1+z$ and $1-z$ are perpendicular. Geometrically, $1+z$ and $1-z$ can interpret as diagonals of a rhombus which each side has 1 unit length, as shown in Figure 3.4. $\square$

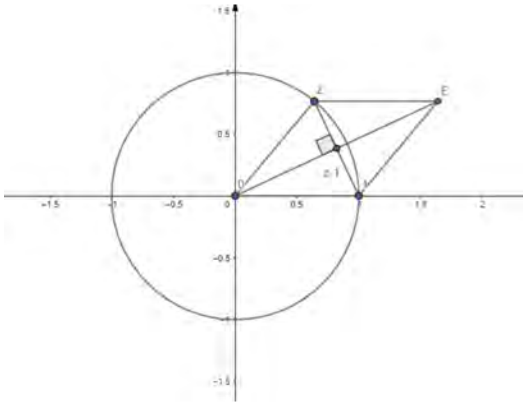


Figure 3.4: Geometric interpretation of Proposition 3.3.4 when $\varphi \in (0, \pi/2)$.

From now, we let n be an arbitrary natural number and $m = e^{i\theta}$ be a complex number where $0 < \theta < \frac{\pi}{n}$. The reason we focus only on $0 < \theta < \frac{\pi}{n}$ is that the image of $\mathbb{D}$ under $f_{n,e^{i\theta}}$ is periodic with period $\frac{\pi}{n}$ respect to θ. We will consider the value of $f_{n,m}$ at the boundary of $\mathbb{D}$ which is $\mathbb{S}^1 = \{z \in \mathbb{C} \| z| = 1\}$. Let $z = e^{i\varphi}$ where $0 < \varphi < 2\pi$ and $\varphi \neq \pi, -\theta + \frac{k\pi}{n}$ where $k = 1, 2, 3, ..., 2n$. This restriction is to avoid zero denominator and zero argument of logarithm in formula of $f_{n,m}$. Consider $f_{n,m}(z) = h_{n,m}(z) + \overline{g_{n,m}(z)} = \Re\left(h_{n,m}(z) + g_{n,m}(z)\right) + \Im\left(h_{n,m}(z) - g_{n,m}(z)\right)$, we will find real part and imaginary part of $f_{n,m}$. From formulas of $h_{n,m}$, $g_{n,m}$, Proposition 3.3.2

and Proposition 3.3.4, we get

$$\begin{aligned}\Im(f_{n,m}) &= \Im(h_{n,m}(z) - g_{n,m}(z)) \\ &= \Im\left(\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right) \\ &= \frac{1}{2}\arg\left(\frac{1+z}{1-z}\right) \\ &= \frac{1}{2}\arg\left(i\cot\frac{\varphi}{2}\right).\end{aligned}$$

Since $\cot\frac{\varphi}{2}$ is a real number and $i = e^{i\pi/2}$, the value of $\arg\left(i\cot\frac{\varphi}{2}\right)$ is $\frac{\pi}{2}$ or $-\frac{\pi}{2}$ depending only on sign of $\cot\frac{\varphi}{2}$. Since $0 < \frac{\varphi}{2} < \pi$, $\cot\frac{\varphi}{2}$ is positive when $0 < \frac{\varphi}{2} < \frac{\pi}{2}$ and negative when $\frac{\pi}{2} < \frac{\varphi}{2} < \pi$. We get

$$\Im(f_{n,m}) = \frac{1}{2}\arg\left(i\cot\frac{\varphi}{2}\right) = \begin{cases}\frac{\pi}{4} & \text{if } 0 < \varphi < \pi \\ -\frac{\pi}{4} & \text{if } \pi < \varphi < 2\pi\end{cases}$$

Next we will find $\Re(f_{n,m}) = \Re(h_{n,m}(z) + g_{n,m}(z))$ which is more complicated.

$$\begin{aligned}\Re(f_{n,m}) &= \Re(h_{n,m}(z) + g_{n,m}(z)) \\ &= \Re\left(\sum_{j=0}^{n-1}\left[\left(\frac{m}{n(m^2-\omega_{2n}^{-2j})\omega_{2n}^{j}}\right)\log\left(\frac{1+m\omega_{2n}^{j}z}{1-m\omega_{2n}^{j}z}\right)\right] + \left(\frac{1+m^{2n}}{2(1-m^{2n})}\right)\log\left(\frac{1+z}{1-z}\right)\right).\end{aligned}$$

For $j = 0, 1, 2, ..., n-1$, consider $\frac{m}{n(m^2-\omega_{2n}^{-2j})\omega_{2n}^{j}} = \frac{m}{n(m^2\omega_{2n}^{2j}-1)\omega_{2n}^{-j}} = \frac{m\omega_{2n}^{j}}{n(m^2\omega_{2n}^{2j}-1)}$.
Since $m\omega_{2n}^{j} = e^{i\left(\theta+\frac{j\pi}{n}\right)}$ and by Proposition 3.3.3, we get $\frac{m}{n(m^2-\omega_{2n}^{-2j})\omega_{2n}^{j}} = -\frac{i}{2n\sin\left(\theta+\frac{j\pi}{n}\right)}$. Since $m^{2n} = e^{2ni\theta}$ and by Proposition 3.3.4, we get $\frac{1+m^{2n}}{2(1-m^{2n})} = \frac{i}{2}\cot(n\theta)$. Similarly, we get $\frac{1+z}{1-z} = i\cot(\frac{\varphi}{2})$ and $\frac{1+m\omega_{2n}^{j}z}{1-m\omega_{2n}^{j}z} = i\cot\left(\frac{\theta+\varphi}{2}+\frac{j\pi}{2n}\right)$ for all $j = 0, 1, 2, ..., n-1$. So

$$\begin{aligned}\Re(f_{n,m}) = \sum_{j=0}^{n-1}\Re&\left[\left(-\frac{i}{2n\sin\left(\theta+\frac{j\pi}{n}\right)}\right)\log\left(i\cot\left(\frac{\theta+\varphi}{2}+\frac{j\pi}{2n}\right)\right)\right] \\ &+\Re\left(\left(\frac{i}{2}\cot(n\theta)\right)\log\left(i\cot\left(\frac{\varphi}{2}\right)\right)\right).\end{aligned}$$

By Proposition 3.3.1 and Proposition 3.3.2, we get

$$\begin{aligned}\Re(f_{n,m}) = \sum_{j=0}^{n-1}&\left(\frac{1}{2n\sin\left(\theta+\frac{j\pi}{n}\right)}\right)\arg\left(i\cot\left(\frac{\theta+\varphi}{2}+\frac{j\pi}{2n}\right)\right) \\ &-\left(\frac{1}{2}\cot(n\theta)\right)\arg\left(i\cot\left(\frac{\varphi}{2}\right)\right).\end{aligned}$$

For $k = 1, 2, 3, ..., 2n$, we will consider value of $\arg\left(i\cot\left(\frac{\theta+\varphi}{2} + \frac{j\pi}{2n}\right)\right)$ where $\varphi \in (\frac{k\pi}{n} - \theta, \frac{(k+1)\pi}{n} - \theta)$ and $j = 0, 1, 2, ..., n-1$. Since $\frac{k\pi}{n} - \theta < \varphi < \frac{(k+1)\pi}{n} - \theta$, we get $\frac{(k+j)\pi}{2n} < \frac{\theta+\varphi}{2} + \frac{j\pi}{2n} < \frac{(k+j+1)\pi}{2n}$. Hence

$$\arg\left(i\cot\left(\frac{\theta+\varphi}{2} + \frac{j\pi}{2n}\right)\right) = \begin{cases} \frac{\pi}{2} & \text{if } j \le n-k-1 \text{ or } j \ge 2n-k \\ -\frac{\pi}{2} & \text{if } n-k \le j \le 2n-k-1 \end{cases}$$

For $1 \le k \le n-1$, we get

$$\sum_{j=0}^{n-1}\left(\frac{1}{2n\sin\left(\theta + \frac{j\pi}{n}\right)}\right)\arg\left(i\cot\left(\frac{\theta+\varphi}{2} + \frac{j\pi}{2n}\right)\right)$$

$$= \sum_{j=0}^{n-k-1}\left(\frac{\pi}{4n\sin\left(\theta + \frac{j\pi}{n}\right)}\right) - \sum_{j=n-k}^{n-1}\left(\frac{\pi}{4n\sin\left(\theta + \frac{j\pi}{n}\right)}\right)$$

For $k = n$, we get

$$\sum_{j=0}^{n-1}\left(\frac{1}{2n\sin\left(\theta + \frac{j\pi}{n}\right)}\right)\arg\left(i\cot\left(\frac{\theta+\varphi}{2} + \frac{j\pi}{2n}\right)\right) = -\sum_{j=0}^{n-1}\left(\frac{\pi}{4n\sin\left(\theta + \frac{j\pi}{n}\right)}\right)$$

For $n+1 \le k \le 2n-1$, we get

$$\sum_{j=0}^{n-1}\left(\frac{1}{2n\sin\left(\theta + \frac{j\pi}{n}\right)}\right)\arg\left(i\cot\left(\frac{\theta+\varphi}{2} + \frac{j\pi}{2n}\right)\right)$$

$$= -\sum_{j=0}^{2n-k-1}\left(\frac{\pi}{4n\sin\left(\theta + \frac{j\pi}{n}\right)}\right) + \sum_{j=2n-k}^{n-1}\left(\frac{\pi}{4n\sin\left(\theta + \frac{j\pi}{n}\right)}\right)$$

Finally, for $k = 2n$ which, in fact, equivalent to $k = 0$, we get

$$\sum_{j=0}^{n-1}\left(\frac{1}{2n\sin\left(\theta + \frac{j\pi}{n}\right)}\right)\arg\left(i\cot\left(\frac{\theta+\varphi}{2} + \frac{j\pi}{2n}\right)\right) = \sum_{j=0}^{n-1}\left(\frac{\pi}{4n\sin\left(\theta + \frac{j\pi}{n}\right)}\right)$$

And since

$$\arg\left(i\cot\frac{\varphi}{2}\right) = \begin{cases} \frac{\pi}{2} & \text{if } 0 < \varphi < \pi \\ -\frac{\pi}{2} & \text{if } \pi < \varphi < 2\pi \end{cases},$$

we can conclude that

$\Re\left(f_{n,m}\right) =$

$$
\begin{cases}
\frac{\pi}{4n}\sum_{j=0}^{n-1}\left(\frac{1}{\sin\left(\theta+\frac{j\pi}{n}\right)}\right)-\frac{\pi}{4}\cot(n\theta) & \text{if } 0<\varphi<\frac{\pi}{n}-\theta\\
\frac{\pi}{4n}\sum_{j=0}^{n-2}\left(\frac{1}{\sin\left(\theta+\frac{j\pi}{n}\right)}\right)-\left(\frac{\pi}{4n\sin\left(\theta+\frac{(n-1)\pi}{n}\right)}\right)-\frac{\pi}{4}\cot(n\theta) & \text{if } \frac{\pi}{n}-\theta<\varphi<\frac{2\pi}{n}-\theta\\
\frac{\pi}{4n}\sum_{j=0}^{n-3}\left(\frac{1}{\sin\left(\theta+\frac{j\pi}{n}\right)}\right)-\sum_{j=n-2}^{n-1}\left(\frac{\pi}{4n\sin\left(\theta+\frac{(n-1)\pi}{n}\right)}\right)-\frac{\pi}{4}\cot(n\theta) & \text{if } \frac{2\pi}{n}-\theta<\varphi<\frac{3\pi}{n}-\theta\\
\vdots & \\
-\sum_{j=0}^{n-1}\left(\frac{\pi}{4n\sin\left(\theta+\frac{j\pi}{n}\right)}\right)-\frac{\pi}{4}\cot(n\theta) & \text{if } \pi-\theta<\varphi<\pi\\
-\sum_{j=0}^{n-1}\left(\frac{\pi}{4n\sin\left(\theta+\frac{j\pi}{n}\right)}\right)+\frac{\pi}{4}\cot(n\theta) & \text{if } \pi<\varphi<\frac{(n+1)\pi}{n}-\theta\\
-\sum_{j=0}^{n-2}\left(\frac{\pi}{4n\sin\left(\theta+\frac{j\pi}{n}\right)}\right)+\left(\frac{\pi}{4n\sin\left(\theta+\frac{(n-1)\pi}{n}\right)}\right)+\frac{\pi}{4}\cot(n\theta) & \text{if } \frac{(n+1)\pi}{n}-\theta<\varphi<\frac{(n+2)\pi}{n}-\theta\\
-\sum_{j=0}^{n-3}\left(\frac{\pi}{4n\sin\left(\theta+\frac{j\pi}{n}\right)}\right)+\sum_{j=n-2}^{n-1}\left(\frac{\pi}{4n\sin\left(\theta+\frac{(n-1)\pi}{n}\right)}\right)+\frac{\pi}{4}\cot(n\theta) & \text{if } \frac{(n+2)\pi}{n}-\theta<\varphi<\frac{(n+3)\pi}{n}-\theta\\
\vdots & \\
\frac{\pi}{4n}\sum_{j=0}^{n-1}\left(\frac{1}{\sin\left(\theta+\frac{j\pi}{n}\right)}\right)+\frac{\pi}{4}\cot(n\theta) & \text{if } 2\pi-\theta<\varphi<2\pi
\end{cases}
$$

Combining $\Re\left(f_{n,m}\right)$ with $\Im\left(f_{n,m}\right)$, we can conclude that the restriction of the map $f_{n,m}$ on $\partial\mathbb{D}$ except for the points 1,-1 and $e^{i\left(\frac{k\pi}{n}-\theta\right)}$ where $k = 1, 2, ..., 2n-1$ is a piecewise step function to $2n+2$ points. Now, let

$$b_1 = \left(\frac{\pi}{4n}\sum_{j=0}^{n-1}\left(\frac{1}{\sin\left(\theta+\frac{j\pi}{n}\right)}\right)-\frac{\pi}{4}\cot(n\theta)\right)+i\left(\frac{\pi}{4}\right)$$

$$b_2 = \left(\frac{\pi}{4n}\sum_{j=0}^{n-2}\left(\frac{1}{\sin\left(\theta+\frac{j\pi}{n}\right)}\right)-\left(\frac{\pi}{4n\sin\left(\theta+\frac{(n-1)\pi}{n}\right)}\right)-\frac{\pi}{4}\cot(n\theta)\right)+i\left(\frac{\pi}{4}\right)$$

$\vdots$

$$b_{n+1} = \left(-\sum_{j=0}^{n-1}\left(\frac{\pi}{4n\sin\left(\theta+\frac{j\pi}{n}\right)}\right)-\frac{\pi}{4}\cot(n\theta)\right)+i\left(\frac{\pi}{4}\right)$$

$$b_{n+2} = \left(-\sum_{j=0}^{n-1}\left(\frac{\pi}{4n\sin\left(\theta+\frac{j\pi}{n}\right)}\right)+\frac{\pi}{4}\cot(n\theta)\right)-i\left(\frac{\pi}{4}\right)$$

$$b_{n+3} = \left(-\sum_{j=0}^{n-2} \left(\frac{\pi}{4n \sin\left(\theta + \frac{j\pi}{n}\right)} \right) + \left(\frac{\pi}{4n \sin\left(\theta + \frac{(n-1)\pi}{n}\right)} \right) + \frac{\pi}{4}\cot(n\theta) \right) - i\left(\frac{\pi}{4}\right)$$

$\vdots$

$$b_{2n+2} = \left(\frac{\pi}{4n} \sum_{j=0}^{n-1} \left(\frac{1}{\sin\left(\theta + \frac{j\pi}{n}\right)} \right) + \frac{\pi}{4}\cot(n\theta) \right) - i\left(\frac{\pi}{4}\right).$$

We can express $f_{n,m}$ by

$$f_{n,m}(e^{i\varphi}) = \begin{cases} b_1 & \text{when } \varphi \in (0, \frac{\pi}{n} - \theta) \\ b_k & \text{when } \varphi \in (\frac{(k-1)\pi}{n} - \theta, \frac{k\pi}{n} - \theta) \text{ where } k = 2, 3, 4, ..., n \\ b_{n+1} & \text{when } \varphi \in (\pi - \theta, \pi) \\ b_{n+2} & \text{when } \varphi \in (\pi, \frac{(n+1)\pi}{n} - \theta) \\ b_k & \text{when } \varphi \in (\frac{(k-2)\pi}{n} - \theta, \frac{(k-1)\pi}{n} - \theta) \text{ where } k = n+3, ..., 2n+1 \\ b_{2n+2} & \text{when } \varphi \in (2\pi - \theta, 2\pi). \end{cases}$$

It's easy to see that $b_1, b_2, ..., b_{n+1}$ lie on the line $\Im(\zeta) = \frac{\pi}{4}$ in the complex plane, while $b_{n+2}, b_{n+3}, ..., b_{2n+2}$ lie on the line $\Im(\zeta) = -\frac{\pi}{4}$ and these two lines are parallel. Because of the fact that $\left(\frac{1}{\sin\left(\theta + \frac{j\pi}{n}\right)}\right) > 0$ for all $j = 0, 1, 2, ..., n-1$, we can conclude that $b_2, b_3, ..., b_n$ lie between b_1 and b_{n+1}, while $b_{n+3}, b_{n+4}, ..., b_{2n+1}$ lie between b_{n+2} and b_{2n+2}. Moreover the line through b_1 and b_{2n+2} has slope $-\tan(n\theta)$ as same as the line through b_{n+1} and b_{n+2}. So $b_1, b_2, ..., b_{2n+2}$ all lie counterclockwise on the boundary of a parallelogram whose vertices are b_1, b_{n+1}, b_{n+2} and b_{2n+2}.

Now, we can conclude the lemma:

Lemma 3.3.5. *The restriction of the map $f_{n,m}$ where $m = e^{i\theta}$ which $0 < \theta < \frac{\pi}{n}$ on $\partial\mathbb{D}$ except for the points 1,-1 and $e^{i\left(\frac{k\pi}{n} - \theta\right)}$ where $k = 1, 2, ..., 2n-1$ is a piecewise step function and it maps $\partial\mathbb{D}$ counterclockwise to $2n+2$ points on the boundary of a parallelogram whose vertices are b_1, b_{n+1}, b_{n+2} and b_{2n+2}.*

Next, we will prove that the map $f_{n,m}$ map $\mathbb{D}$ to the interior of this parallelogram. We will start by proving some lemma.

Lemma 3.3.6. *Let $\varphi \in (0, \pi)$ and Φ be a complex variable function defined on $\mathbb{D}$ by*

$$\Phi(z) = ie^{-i\varphi}\left(\frac{1 - e^{2i\varphi}z}{(1-z)}\right).$$

Then Φ maps $\mathbb{D}$ to the right half plane $\mathbb{C}_{\Re>0} = \{\zeta \in \mathbb{C} | \Re(\zeta) > 0\}$.

Proof. Let $\varphi \in (0, \pi)$ and Φ defined as above. We get

$$\begin{aligned}
\Re(\Phi(z)) &= \frac{1}{2}\left(\Phi(z) + \overline{\Phi(z)}\right) \\
&= \frac{1}{2}\left(ie^{-i\varphi}\left(\frac{1-e^{2i\varphi}z}{(1-z)}\right) - ie^{i\varphi}\left(\frac{1-e^{-2i\varphi}\bar{z}}{(1-\bar{z})}\right)\right) \\
&= \frac{1}{2}\left(\frac{-i(e^{i\varphi}-e^{-i\varphi}) + i|z|^2(e^{i\varphi}-e^{-i\varphi})}{|1-z|^2}\right) \\
&= \frac{1}{2}\left(\frac{i(|z|^2-1)(2i\sin\varphi)}{|1-z|^2}\right) \\
&= \frac{(1-|z|^2)(\sin\varphi)}{|1-z|^2}.
\end{aligned}$$

Since $|z| < 1$ for all $z \in \mathbb{D}$ and $\sin\varphi > 0$, we get $\Re(\Phi(z)) > 0$ for all $z \in \mathbb{D}$. This means that Φ maps $\mathbb{D}$ to the right half plane $\mathbb{C}_{\Re>0}$. □

Proposition 3.3.7. *For $m = e^{i\theta}$ which $0 < \theta < \frac{\pi}{n}$, the map $f_{n,m}$ maps $\mathbb{D}$ onto a parallelogram whose vertices are b_1, b_{n+1}, b_{n+2} and b_{2n+2}.*

Proof. First we will show that the image of $\mathbb{D}$ is convex. By Theorem 2.3.19, it is equivalent to prove that the function

$$\begin{aligned}
F(z) &= h_{n,m}(z) - e^{2i\varphi}g_{n,m}(z) \\
&= \sum_{j=0}^{n-1}\left(\frac{m^2(1-e^{2i\varphi})}{2n(m^2-\omega_{2n}^{-2j})\omega_{2n}^{j}}\log\left(\frac{1+m\omega_{2n}^{j}z}{1-m\omega_{2n}^{j}z}\right)\right) \\
&\quad + \left(\frac{1-e^{2i\varphi}m^{2n}}{2(1-m^{2n})}\right)\log\left(\frac{1+z}{1-z}\right)
\end{aligned}$$

is convex in the direction $e^{i\varphi}$ for all $\varphi \in [0, \pi)$. From the construction, we already known that this holds for $\varphi = 0$. So we will prove in the case that $\varphi \in (0, \pi)$. According to Theorem 2.3.18, this can be done by showing that

$$\Re[e^{i(\mu-\varphi)}(1 - 2\cos\nu e^{-i\mu}z + e^{-2i\mu}z^2)F'(z)] \geq 0 \tag{3.5}$$

for some real numbers μ and ν where $0 \leq \mu < 2\pi$ and $0 \leq \nu \leq \pi$.

We choose $\mu = \frac{\pi}{2}$ and $\nu = \frac{\pi}{2}$. Let

$$\begin{aligned}
H(z) &= e^{i(\mu-\varphi)}(1 - 2\cos\nu e^{-i\mu}z + e^{-2i\mu}z^2)F'(z) \\
&= e^{i(\frac{\pi}{2}-\varphi)}(1-z^2)F'(z).
\end{aligned}$$

Since

$$\begin{aligned}F'(z) &= h'_{n,m}(z) - e^{2i\varphi} g'_{n,m}(z) \\ &= \frac{1}{(1-z^2)(1-m^{2n}z^{2n})} - \frac{e^{2i\varphi}m^{2n}z^{2n}}{(1-z^2)(1-m^{2n}z^{2n})} \\ &= \frac{1-e^{2i\varphi}m^{2n}z^{2n}}{(1-z^2)(1-m^{2n}z^{2n})},\end{aligned}$$

we get

$$H(z) = ie^{-i\varphi)}\left(\frac{1-e^{2i\varphi}m^{2n}z^{2n}}{(1-m^{2n}z^{2n})}\right).$$

Let $\Phi(z) = ie^{-i\varphi}\left(\frac{1-e^{2i\varphi}z}{(1-z)}\right)$ and $\sigma(z) = m^{2n}z^{2n}$. Hence $H(z) = (\Phi \circ \sigma)(z)$. By Proposition 3.3.6, Φ maps $\mathbb{D}$ to the right half plane $\mathbb{C}_{\Re>0}$, while σ is a conformal mapping on $\mathbb{D}$. So H maps $\mathbb{D}$ to $\mathbb{C}_{\Re>0}$ which means $\Re(H(z)) > 0$ for all $z \in \mathbb{D}$. Hence the image of $\mathbb{D}$ under $f_{n,m}$ is convex in every direction $\varphi \in [0, \pi)$. This implies the image of $\mathbb{D}$ under $f_{n,m}$ is convex.

Lemma 3.3.5 shows that the boundary of the $\mathbb{D}$ gets mapped to $2n+2$ points, b_1, b_{n+1}, b_{n+2} and b_{2n+2}, under $f_{n,m}$. According to Remark 3.4 of [1], Bshouty and Hengartner note that in this case the image of $\mathbb{D}$ under $f_{n,m}$ must be the convex polygon. That is, the image of $\mathbb{D}$ under $f_{n,m}$ is a parallelogram bounded by b_1, b_{n+1}, b_{n+2} and b_{2n+2}. □

Next, we will analyze some geometric properties of this parallelogram. For a natural number n and a complex number $m = e^{i\theta}$ where $0 < \theta < \frac{\pi}{n}$, let $P_{n,m}$ be the image of D under $f_{n,m}$. We known that $P_{n,m}$ is a parallelogram whose vertices are b_1, b_{n+1}, b_{n+2} and b_{2n+2}. For $k = 1, 2, 3, ..., 2n+2$, let α_k be the interior angle of the $P_{n,m}$ at the point b_k. Since $b_2, b_3, b_4, ...b_n, b_{n+3}, b_{n+4}, ..., b_{2n+1}$ are on the sides of $P_{n,m}$, we can conclude that $\alpha_2, \alpha_3, ..., \alpha_n, \alpha_{n+3}, \alpha_{n+4}, ..., \alpha_{2n+1} = \pi$. And since the line through b_1 and b_{2n+2} has slope $-\tan(n\theta) = \tan(\pi - n\theta)$ as same as the line through b_{n+1} and b_{n+2}, we can conclude that $\alpha_{n+1} = \alpha_{2n+2} = n\theta$ and $\alpha_1 = \alpha_{n+2} = \pi - n\theta$. This also implies that $P_{n,m}$ is a rectangle if and only if $m = e^{\frac{i\pi}{2n}}$ which is the middle of the interval $(0, \frac{\pi}{n})$.

For points A and B in the complex plane, denote the length of line segment $\overline{AB}$ by $l(A, B)$.

According to the formula b_k, it is easy to see that for $k = 1, 2, 3, ..., n$,

$$l(b_k, b_{k+1}) = \frac{\pi}{2n \sin\left(\theta + \frac{(n-k)\pi}{n}\right)}.$$

And for $k = n+2, n+3...2n+1$,

$$l(b_k, b_{k+1}) = \frac{\pi}{2n \sin\left(\theta + \frac{(2n+1-k)\pi}{n}\right)}.$$

And hence

$$l(b_1, b_{n+1}) = l(b_{n+2}, b_{2n+2}) = \frac{\pi}{2n} \sum_{j=0}^{n-1} \left(\frac{1}{\sin(\theta + \frac{j\pi}{n})} \right). \tag{3.6}$$

By Pythagorean theorem, we can compute

$$\begin{aligned} l(b_{2n+2}, b_1) = l(b_{n+1}, b_{n+2}) &= \sqrt{\left(\frac{\pi}{2}\cot(n\theta)\right)^2 + \left(\frac{\pi}{2}\right)^2} \\ &= \frac{\pi}{2\sin(n\theta)}. \end{aligned} \tag{3.7}$$

Notice that $l(b_{2n+2}, b_1), l(b_{n+1}, b_{n+2})$ don't depend on m. By dividing (3.6) by (3.7), we get the ratio of length of non parallel sides of the parallelogram $P_{n,m}$.

$$\frac{l(b_1, b_{n+1})}{l(b_{2n+2}, b_1)} = \frac{1}{n} \sum_{j=0}^{n-1} \left(\frac{\sin(n\theta)}{\sin(\theta + \frac{j\pi}{n})} \right). \tag{3.8}$$

For a fixed natural number n, let $r_n : (0, \frac{\pi}{n}) \longrightarrow \mathbb{R}$ be a function defined by

$$r_n(\theta) = \frac{1}{n} \sum_{j=0}^{n-1} \left(\frac{\sin(n\theta)}{\sin(\theta + \frac{j\pi}{n})} \right) = \frac{1}{n} \sum_{j=0}^{n-1} \left(\frac{(-1)^j \sin(n(\theta + \frac{j\pi}{n}))}{\sin(\theta + \frac{j\pi}{n})} \right).$$

We will investigate some properties of r_n. Firstly, it's obvious that r_n is smooth on $(0, \frac{\pi}{n})$. And

$$\begin{aligned} r_n\left(\frac{\pi}{n} - \theta\right) &= \frac{1}{n} \sum_{j=0}^{n-1} \left(\frac{\sin(n(\frac{\pi}{n} - \theta))}{\sin(\frac{\pi}{n} - \theta + \frac{j\pi}{n})} \right) \\ &= \frac{1}{n} \sum_{j=0}^{n-1} \left(\frac{\sin(\pi - n\theta)}{\sin(\frac{(j+1)\pi}{n} - \theta)} \right) \\ &= \frac{1}{n} \sum_{j=0}^{n-1} \left(\frac{\sin(n\theta)}{\sin(\frac{(n-j-1)\pi}{n} + \theta)} \right) \\ &= \frac{1}{n} \sum_{j=0}^{n-1} \left(\frac{\sin(n\theta)}{\sin(\theta + \frac{j\pi}{n})} \right) = r_n(\theta) \end{aligned}$$

for all $\theta \in (0, \frac{\pi}{n})$. Next, we will consider asymtotic behavior of r_n. If θ converges to 0, we get

$$r_n(0^+) = \lim_{\theta \longrightarrow 0^+} \frac{\sin(n\theta)}{n \sin\theta} = \lim_{\theta \longrightarrow 0^+} \left(\frac{\sin(n\theta)}{n\theta} \right) \left(\frac{\theta}{\sin\theta} \right) = 1.$$

Combine with the above relation, we get

$$r_n\left(\frac{\pi}{n}^-\right) = 1.$$

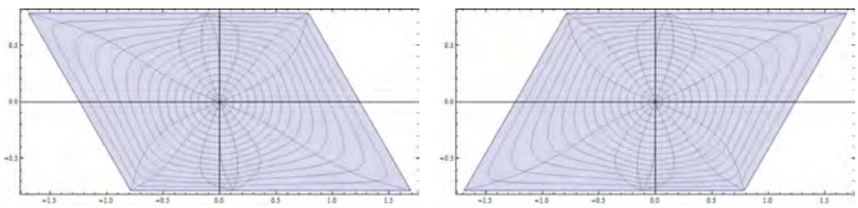


Figure 3.5: For fixed n, the Image when $m = e^{i\theta}$ (left) and when $m = e^{i\left(\frac{\pi}{n}-\theta\right)}$ (right) is a reflection of each other. This figure illustrates this fact for the case $n = 2$ and $\theta = \frac{\pi}{6}$.

For example, in the case that $n = 1$, we get $r_n(\theta) = 1$ for all $\theta \in (0, \frac{\pi}{n})$ which implies that $P_{1,m}$ is a rhombus, as we mentioned in Section 2.5. Let look at the case $n = 2$. We get $r_2(\theta) = \frac{1}{2}\left(\frac{\sin(2\theta)}{\sin(\theta)} + \frac{\sin(2\theta)}{\sin(\theta+\frac{\pi}{2})}\right) = \sin\theta + \cos\theta$. In this case, we can see that $r_2'(\theta) = \cos\theta - \sin\theta$ which is positive when $0 < \theta < \frac{\pi}{4}$ and negative when $\frac{\pi}{4} < \theta < \frac{\pi}{2}$. So $r_2(\theta)$ is increasing when $0 < \theta < \frac{\pi}{4}$, decreasing when $\frac{\pi}{4} < \theta < \frac{\pi}{2}$ and maximum when $\theta = \frac{\pi}{4}$ which its value is $r_n\left(\frac{\pi}{4}\right) = \sqrt{2}$. And in that case ($\theta = \frac{\pi}{4}$), the image $P_{2,m}$ is a rectangle. And since $1 < r_2(\theta) = \sin\theta + \cos\theta \leq \sqrt{2}$ when $0 < \theta < \frac{\pi}{2}$, we can conclude that $P_{2,m}$ is not a rhombus for all $0 < \theta < \frac{\pi}{2}$. We will prove the same result for an arbitrary natural number $n \geq 2$.

We will use the following facts:

Proposition 3.3.8. *For real numbers a, d and a natural number k,*

$$\sum_{j=0}^{k-1} \cos(a + jd) = \frac{\sin\left(\frac{kd}{2}\right)}{\sin\frac{d}{2}} \cos\left(a + \frac{(k-1)d}{2}\right).$$

Proof. Let a, d be real numbers and k be a natural number. Let $\omega = e^{id}$, we get

$$\begin{aligned} e^{ia}\sum_{j=0}^{k-1}\omega^j &= \left(\frac{\omega^k - 1}{\omega - 1}\right) e^{ia} \\ &= \left(\frac{e^{ikd} - 1}{e^{id} - 1}\right) e^{ia} \\ &= \left(\frac{e^{ikd} - 1}{e^{ikd/2}}\right)\left(\frac{e^{id/2}}{e^{id} - 1}\right) e^{i\left(\frac{(k-1)d}{2}+a\right)}. \end{aligned}$$

From Propositon 3.3.3, we get

$$\begin{aligned} e^{ia}\sum_{j=0}^{k-1}\omega^j &= \left(\frac{2\sin\left(\frac{kd}{2}\right)}{-i}\right)\left(\frac{-i}{2\sin\frac{d}{2}}\right)e^{i\left(a+\frac{(k-1)d}{2}\right)} \\ &= \left(\frac{2\sin\left(\frac{kd}{2}\right)}{2\sin\frac{d}{2}}\right)e^{i\left(a+\frac{(k-1)d}{2}\right)}. \end{aligned}$$

By comparing the real parts of two sides of the equation, we get this proposition. □

Proposition 3.3.9. *For a natural number k and a real number α,*

$$\frac{\sin(k\alpha)}{\sin\alpha} = 1 + 2\sum_{\substack{r=1 \\ odd\ r}}^{k-2}\cos((k-r)\alpha) \text{ if } k \text{ is odd}$$

and

$$\frac{\sin(k\alpha)}{\sin\alpha} = 2\sum_{\substack{r=1 \\ odd\ r}}^{k-1}\cos((k-r)\alpha) \text{ if } k \text{ is even.}$$

Proof. Let k be a natural number and α be a real number. We get

$$\begin{aligned} \sin(k\alpha) &= \sin((k-1)\alpha)\cos\alpha + \cos((k-1)\alpha)\sin\alpha \\ &= \{\sin((k-2)\alpha)\cos\alpha + \cos((k-2)\alpha)\sin\alpha\}\cos\alpha + \cos((k-1)\alpha)\sin\alpha \\ \frac{\sin(k\alpha)}{\sin\alpha} &= \frac{\sin((k-2)\alpha)}{\sin\alpha}\cos^2\alpha + \cos((k-2)\alpha)\cos\alpha + \cos((k-1)\alpha) \\ &= \frac{\sin((k-2)\alpha)}{\sin\alpha}(1-\sin^2\alpha) + \cos((k-2)\alpha)\cos\alpha + \cos((k-1)\alpha) \\ &= \frac{\sin((k-2)\alpha)}{\sin\alpha} - \sin((k-2)\alpha)\sin\alpha + \cos((k-2)\alpha)\cos\alpha \\ &\quad + \cos((k-1)\alpha) \\ &= \frac{\sin((k-2)\alpha)}{\sin\alpha} + 2\cos((k-1)\alpha). \end{aligned}$$

By mathematical induction and this identity, we can conclude this proposition. □

First, let consider when n is even. From Proposition 3.3.9, we get

$$\begin{aligned} \frac{\sin(n(\theta+\frac{j\pi}{n}))}{\sin(\theta+\frac{j\pi}{n})} &= 2\sum_{\substack{k=1 \\ \text{odd } k}}^{n-1}\cos\left((n-k)\left(\theta+\frac{j\pi}{n}\right)\right) \\ &= 2\sum_{\substack{k=1 \\ \text{odd } k}}^{n-1}\cos\left(k\left(\theta+\frac{j\pi}{n}\right)\right). \end{aligned}$$

Hence

$$
\begin{aligned}
r_n(\theta) &= \frac{1}{n}\sum_{j=0}^{n-1}\left(\frac{(-1)^j \sin(n(\theta+\frac{j\pi}{n}))}{\sin(\theta+\frac{j\pi}{n})}\right) \\
&= \frac{2}{n}\sum_{j=0}^{n-1}\left((-1)^j \sum_{\substack{k=1\\ \text{odd } k}}^{n-1} \cos\left(k\left(\theta+\frac{j\pi}{n}\right)\right)\right) \\
&= \frac{2}{n}\sum_{\substack{j=0\\ \text{even } j}}^{n-1}\sum_{\substack{k=1\\ \text{odd } k}}^{n-1} \cos\left(k\left(\theta+\frac{j\pi}{n}\right)\right) - \frac{2}{n}\sum_{\substack{j=0\\ \text{odd } j}}^{n-1}\sum_{\substack{k=1\\ \text{odd } k}}^{n-1} \cos\left(k\left(\theta+\frac{j\pi}{n}\right)\right) \\
&= \frac{2}{n}\sum_{\substack{k=1\\ \text{odd } k}}^{n-1}\sum_{\substack{j=0\\ \text{even } j}}^{n-1} \cos\left(k\left(\theta+\frac{j\pi}{n}\right)\right) - \frac{2}{n}\sum_{\substack{k=1\\ \text{odd } k}}^{n-1}\sum_{\substack{j=0\\ \text{odd } j}}^{n-1} \cos\left(k\left(\theta+\frac{j\pi}{n}\right)\right) \\
&= \frac{2}{n}\sum_{\substack{k=1\\ \text{odd } k}}^{n-1}\sum_{r=0}^{\frac{n-1}{2}} \cos\left(k\left(\theta+\frac{2r\pi}{n}\right)\right) - \frac{2}{n}\sum_{\substack{k=1\\ \text{odd } k}}^{n-1}\sum_{r=0}^{\frac{n-1}{2}} \cos\left(k\left(\theta+\frac{(2r+1)\pi}{n}\right)\right)
\end{aligned}
$$

By Proposition 3.3.8, we get

$$
\sum_{r=0}^{\frac{n-1}{2}} \cos\left(k\left(\theta+\frac{2r\pi}{n}\right)\right) = \left(\frac{\sin\left(\frac{k\pi}{2}\right)}{\sin\left(\frac{k\pi}{n}\right)}\right)\cos\left(k\theta+\frac{(n-2)\pi k}{2n}\right)
$$

and

$$
\begin{aligned}
\sum_{r=0}^{\frac{n-1}{2}} \cos\left(k\left(\theta+\frac{(2r+1)\pi}{n}\right)\right) &= \left(\frac{\sin\left(\frac{k\pi}{2}\right)}{\sin\left(\frac{k\pi}{n}\right)}\right)\cos\left(k\left(\theta+\frac{\pi}{n}\right)+\frac{(n-2)\pi k}{2n}\right) \\
&= \left(\frac{\sin\left(\frac{k\pi}{2}\right)}{\sin\left(\frac{k\pi}{n}\right)}\right)\cos\left(k\left(\theta+\frac{\pi}{2}\right)\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
r_n(\theta) &= \frac{2}{n}\sum_{\substack{k=1\\ \text{odd } k}}^{n-1}\left(\frac{\sin\left(\frac{k\pi}{2}\right)}{\sin\left(\frac{k\pi}{n}\right)}\right)\left[\cos\left(k\theta+\frac{(n-2)\pi k}{2n}\right)-\cos\left(k\left(\theta+\frac{\pi}{2}\right)\right)\right] \\
&= \frac{2}{n}\sum_{\substack{k=1\\ \text{odd } k}}^{n-1}\left[\left(\frac{\sin\left(\frac{k\pi}{2}\right)}{2\sin\left(\frac{k\pi}{2n}\right)\cos\left(\frac{k\pi}{2n}\right)}\right)\left(2\sin\left(k\theta+\frac{(n-2)\pi k}{2n}\right)\sin\left(\frac{\pi k}{2n}\right)\right)\right] \\
&= \frac{1}{n}\sum_{\substack{k=1\\ \text{odd } k}}^{n-1}\left(\frac{2\sin\left(\frac{k\pi}{2}\right)\sin\left(k\theta+\frac{(n-2)\pi k}{2n}\right)}{\cos\left(\frac{k\pi}{2n}\right)}\right)
\end{aligned}
$$

$$= \frac{1}{n}\sum_{\substack{k=1\\ \text{odd } k}}^{n-1}\left(\frac{\cos\left(k\theta-\frac{\pi k}{2n}\right)-\cos\left(\pi k+k\theta-\frac{\pi k}{2n}\right)}{\cos\left(\frac{k\pi}{2n}\right)}\right)$$

$$= \frac{2}{n}\sum_{\substack{k=1\\ \text{odd } k}}^{n-1}\left(\frac{\cos\left(k\theta-\frac{\pi k}{2n}\right)}{\cos\left(\frac{k\pi}{2n}\right)}\right)$$

$$= \frac{2}{n}\sum_{\substack{k=1\\ \text{odd } k}}^{n-1}\left(\cos(k\theta)+\sin(k\theta)\tan\left(\frac{\pi k}{2n}\right)\right)$$

Differentiating this equation two times, we get

$$r_n''(\theta) = -\frac{2}{n}\sum_{\substack{k=1\\ \text{odd } k}}^{n-1}\left(k^2\cos(k\theta)+k^2\sin(k\theta)\tan\left(\frac{\pi k}{2n}\right)\right)$$

For $\theta \in \left(0, \frac{\pi}{2n}\right]$, $0 < k\theta < \frac{\pi}{2}$ for all $k = 1, 3, ..., n-1$. Hence $\cos(k\theta) > 0$, $\sin(k\theta) > 0$ and $\tan\left(\frac{\pi k}{2n}\right) > 0$ for all $k = 1, 3, ..., n-1$. This implies that $r_n''(\theta) < 0$ for all $\theta \in \left(0, \frac{\pi}{2n}\right]$. And since $r_n(\frac{\pi}{n}) = r_n(\theta)$, we can conclude that $r_n''(\frac{\pi}{n}) = r_n''(\theta)$ for all $\theta \in \left(0, \frac{\pi}{n}\right)$. So $r_n''(\theta) < 0$ for all $\theta \in \left(0, \frac{\pi}{n}\right)$ which means $r_n'(\theta)$ is decreasing on $\left(0, \frac{\pi}{n}\right)$.

Since $r_n(\frac{\pi}{n} - \theta) = r_n(\theta)$, we can conclude that $r_n'(\frac{\pi}{n} - \theta) = -r_n'(\theta)$ for all $\theta \in \left(0, \frac{\pi}{n}\right)$. This implies that $r_n'(\frac{\pi}{2n}) = 0$. And since $r_n'(\theta)$ is decreasing on $\left(0, \frac{\pi}{n}\right)$, we can conclude that $r_n'(\theta) > 0$ on $\left(0, \frac{\pi}{2n}\right)$ and $r_n'(\theta) < 0$ on $\left(\frac{\pi}{2n}, \frac{\pi}{n}\right)$. Hence $r_n(\theta)$ is increasing when $0 < \theta < \frac{\pi}{2n}$, decreasing when $\frac{\pi}{2n} < \theta < \frac{\pi}{n}$ and maximum when $\theta = \frac{\pi}{2n}$, which has value

$$r_n\left(\frac{\pi}{2n}\right) = \frac{2}{n}\left(\sum_{\substack{k=1\\ \text{odd } k}}^{n-1}\sec\left(\frac{k\pi}{2n}\right)\right).$$

And in that case ($\theta = \frac{\pi}{2n}$), the image $P_{n,m}$ is a rectangle. Moreover, since $r_n(\theta)$ is increasing when $0 < \theta < \frac{\pi}{2n}$ and $r_n(0^+) = 1$, $1 < r_n(\theta)$ when $0 < \theta < \frac{\pi}{2n}$. Similarly, since $r_n(\theta)$ is decreasing when $\frac{\pi}{2n} < \theta < \frac{\pi}{n}$, $1 < r_n(\theta)$ when $0 < \theta < \frac{\pi}{2n}$. So $1 < r_n(\theta)$ when $0 < \theta < \frac{\pi}{n}$. Now, we can conclude that $P_{n,m}$ is not a rhombus for all $0 < \theta < \frac{\pi}{n}$.

Next, let consider when n is odd. From Proposition 3.3.9, we get

$$\frac{\sin(n(\theta+\frac{j\pi}{n}))}{\sin(\theta+\frac{j\pi}{n})} = 1 + 2\sum_{\substack{k=1\\ \text{odd } k}}^{n-1}\cos\left((n-k)\left(\theta+\frac{j\pi}{n}\right)\right)$$

$$= 1 + 2\sum_{\substack{k=1\\ \text{even } k}}^{n-1}\cos\left(k\left(\theta+\frac{j\pi}{n}\right)\right).$$

Hence

$$
\begin{aligned}
r_n(\theta) &= \frac{1}{n}\sum_{j=0}^{n-1}\left(\frac{(-1)^j \sin(n(\theta+\frac{j\pi}{n}))}{\sin(\theta+\frac{j\pi}{n})}\right) \\
&= \frac{1}{n}\left(1+2\sum_{j=0}^{n-1}(-1)^j \sum_{\substack{k=1 \\ \text{even } k}}^{n-1} \cos\left(k\left(\theta+\frac{j\pi}{n}\right)\right)\right) \\
&= \frac{1}{n}\left(1+2\sum_{\substack{j=0 \\ \text{even } j}}^{n-1}\sum_{\substack{k=1 \\ \text{even } k}}^{n-1} \cos\left(k\left(\theta+\frac{j\pi}{n}\right)\right) - 2\sum_{\substack{j=0 \\ \text{odd } j}}^{n-1}\sum_{\substack{k=1 \\ \text{even } k}}^{n-1} \cos\left(k\left(\theta+\frac{j\pi}{n}\right)\right)\right) \\
&= \frac{1}{n}\left(1+2\sum_{r=0}^{\frac{n-1}{2}}\sum_{\substack{k=1 \\ \text{even } k}}^{n-1} \cos\left(k\left(\theta+\frac{2r\pi}{n}\right)\right) - 2\sum_{r=0}^{\frac{n-1}{2}}\sum_{\substack{k=1 \\ \text{even } k}}^{n-1} \cos\left(k\left(\theta+\frac{(2r+1)\pi}{n}\right)\right)\right) \\
&= \frac{1}{n}\left(1+2\sum_{\substack{k=1 \\ \text{even } k}}^{n-1}\sum_{r=0}^{\frac{n-1}{2}} \cos\left(k\left(\theta+\frac{2r\pi}{n}\right)\right) - 2\sum_{\substack{k=1 \\ \text{even } k}}^{n-1}\sum_{r=0}^{\frac{n-1}{2}} \cos\left(k\left(\theta+\frac{(2r+1)\pi}{n}\right)\right)\right)
\end{aligned}
$$

By Proposition 3.3.8, we get

$$
\sum_{r=0}^{\frac{n-1}{2}} \cos\left(k\left(\theta+\frac{2r\pi}{n}\right)\right) = \left(\frac{\sin\left(\frac{k(n+1)\pi}{2n}\right)}{\sin\left(\frac{k\pi}{n}\right)}\right)\cos\left(k\theta+\frac{(n-1)\pi k}{2n}\right)
$$

and

$$
\begin{aligned}
\sum_{r=0}^{\frac{n-1}{2}} \cos\left(k\left(\theta+\frac{(2r+1)\pi}{n}\right)\right) &= \left(\frac{\sin\left(\frac{k(n-1)\pi}{2n}\right)}{\sin\left(\frac{k\pi}{n}\right)}\right)\cos\left(k\left(\theta+\frac{\pi}{n}\right)+\frac{(n-3)\pi k}{2n}\right) \\
&= \left(\frac{\sin\left(\frac{k(n-1)\pi}{2n}\right)}{\sin\left(\frac{k\pi}{n}\right)}\right)\cos\left(k\theta+\frac{(n-1)\pi k}{2n}\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
r_n(\theta) &= \frac{1}{n}\left(1+2\sum_{\substack{k=1 \\ \text{even } k}}^{n-1}\left(\frac{\cos\left(k\theta+\frac{(n-1)\pi k}{2n}\right)}{\sin\left(\frac{k\pi}{n}\right)}\right)\left[\sin\left(\frac{k(n+1)\pi}{2n}\right)-\sin\left(\frac{k(n-1)\pi}{2n}\right)\right]\right) \\
&= \frac{1}{n}\left(1+2\sum_{\substack{k=1 \\ \text{even } k}}^{n-1}\left(\frac{\cos\left(k\theta+\frac{(n-1)\pi k}{2n}\right)}{2\sin\left(\frac{k\pi}{2n}\right)\cos\left(\frac{k\pi}{2n}\right)}\right)\left(2\cos\left(\frac{k\pi}{2}\right)\sin\left(\frac{k\pi}{2n}\right)\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{n}\left(1 + 2\sum_{\substack{k=1\\ \text{even } k}}^{n-1}\left(\frac{\cos\left(k\theta + \frac{(n-1)\pi k}{2n}\right)\cos\left(\frac{k\pi}{2}\right)}{\cos\left(\frac{k\pi}{2n}\right)}\right)\right)\\
&= \frac{1}{n}\left(1 + \sum_{\substack{k=1\\ \text{even } k}}^{n-1}\left(\frac{\cos\left(k\theta + \frac{(2n-1)\pi k}{2n}\right) + \cos\left(k\theta - \frac{k\pi}{2n}\right)}{\cos\left(\frac{k\pi}{2n}\right)}\right)\right)\\
&= \frac{1}{n}\left(1 + \sum_{\substack{k=1\\ \text{even } k}}^{n-1}\left(\frac{2\cos\left(k\theta - \frac{k\pi}{2n}\right)}{\cos\left(\frac{k\pi}{2n}\right)}\right)\right)\\
&= \frac{1}{n} + \frac{2}{n}\sum_{\substack{k=1\\ \text{even } k}}^{n-1}\left(\frac{\cos k\theta\cos\left(\frac{k\pi}{2n}\right) + \sin k\theta\sin\left(\frac{k\pi}{2n}\right)}{\cos\left(\frac{k\pi}{2n}\right)}\right)\\
&= \frac{1}{n} + \frac{2}{n}\sum_{\substack{k=1\\ \text{even } k}}^{n-1}\left(\cos(k\theta) + \sin(k\theta)\tan\left(\frac{k\pi}{2n}\right)\right)
\end{aligned}
$$

Differentiating this equation two times, we get

$$r_n''(\theta) = -\frac{2}{n}\sum_{\substack{k=1\\ \text{even } k}}^{n-1}\left(k^2\cos(k\theta) + k^2\sin(k\theta)\tan\left(\frac{\pi k}{2n}\right)\right)$$

For $\theta \in \left(0, \frac{\pi}{2n}\right]$, $0 < k\theta < \frac{\pi}{2}$ for all $k = 2, 4, ..., n-1$. Hence $\cos(k\theta) > 0$, $\sin(k\theta) > 0$ and $\tan\left(\frac{\pi k}{2n}\right) > 0$ for all $k = 2, 4, ..., n-1$. This implies that $r_n''(\theta) < 0$ for all $\theta \in \left(0, \frac{\pi}{2n}\right]$. And since $r_n(\frac{\pi}{n}) = r_n(\theta)$, we can conclude that $r_n''(\frac{\pi}{n}) = r_n''(\theta)$ for all $\theta \in \left(0, \frac{\pi}{n}\right)$. So $r_n''(\theta) < 0$ for all $\theta \in \left(0, \frac{\pi}{n}\right)$ which means $r_n'(\theta)$ is decreasing on $\left(0, \frac{\pi}{n}\right)$.

Since $r_n(\frac{\pi}{n} - \theta) = r_n(\theta)$, we can conclude that $r_n'(\frac{\pi}{n} - \theta) = -r_n'(\theta)$ for all $\theta \in \left(0, \frac{\pi}{n}\right)$. This implies that $r_n'(\frac{\pi}{2n}) = 0$. And since $r_n'(\theta)$ is decreasing on $\left(0, \frac{\pi}{n}\right)$, we can conclude that $r_n'(\theta) > 0$ on $\left(0, \frac{\pi}{2n}\right)$ and $r_n'(\theta) < 0$ on $\left(\frac{\pi}{2n}, \frac{\pi}{n}\right)$. Hence $r_n(\theta)$ is increasing when $0 < \theta < \frac{\pi}{2n}$, decreasing when $\frac{\pi}{2n} < \theta < \frac{\pi}{n}$ and maximum when $\theta = \frac{\pi}{2n}$, which has value

$$r_n\left(\frac{\pi}{2n}\right) = \frac{1}{n} + \frac{2}{n}\left(\sum_{\substack{k=1\\ \text{even } k}}^{n-1}\sec\left(\frac{k\pi}{2n}\right)\right).$$

And in that case ($\theta = \frac{\pi}{2n}$), the image $P_{n,m}$ is a rectangle. Moreover, since $r_n(\theta)$ is increasing when $0 < \theta < \frac{\pi}{2n}$ and $r_n(0^+) = 1$, $1 < r_n(\theta)$ when $0 < \theta < \frac{\pi}{2n}$. Similarly, since $r_n(\theta)$ is decreasing when $\frac{\pi}{2n} < \theta < \frac{\pi}{n}$, $1 < r_n(\theta)$

when $0 < \theta < \frac{\pi}{2n}$. So $1 < r_n(\theta)$ when $0 < \theta < \frac{\pi}{n}$. Now, we can conclude that $P_{n,m}$ is not a rhombus for all $0 < \theta < \frac{\pi}{n}$.

Finally, we will prove that for a given real number $\lambda > 1$, there exist a natural number n and a complex number $m = e^{i\theta}$ where $0 < \theta < \frac{\pi}{n}$ such that the ratio of lengths of the non parallel sides of $P_{n,m}$ is equal to λ, equivalently $r_n(\theta) = \lambda$.

Theorem 3.3.10. *Let $\lambda > 1$, then there exist a natural number n and a complex number $m = e^{i\theta}$ where $0 < \theta < \frac{\pi}{n}$ such that $r_n(\theta) = \lambda$.*

Proof. Let q be an odd positive integer. Let $l \in \{1, 3, 5, ..., q\}$. Since

$$\begin{aligned}
\lim_{n\longrightarrow\infty} \frac{2}{n}\left(\sec\left(\frac{(n-l)\pi}{2n}\right)\right) &= \lim_{n\longrightarrow\infty} \frac{2}{n}\left(\frac{1}{\cos\left(\frac{(n-l)\pi}{2n}\right)}\right)\\
&= \lim_{n\longrightarrow\infty} \frac{2}{n}\left(\frac{1}{\cos\left(\frac{\pi}{2}-\frac{l\pi}{2n}\right)}\right)\\
&= \lim_{n\longrightarrow\infty} \frac{4}{l\pi}\left(\frac{\frac{l\pi}{2n}}{\sin\left(\frac{l\pi}{2n}\right)}\right)\\
&= \frac{4}{l\pi},
\end{aligned}$$

there exists a positive integer N_l such that for all $n \geq N_l$,

$$\frac{4}{l\pi} - \frac{8}{q(q+1)\pi} < \frac{2}{n}\left(\sec\left(\frac{(n-l)\pi}{2n}\right)\right) < \frac{4}{l\pi} + \frac{8}{q(q+1)\pi}$$

Let $N = max\{N_1, N_3, N_5, ..., N_q, q\}$. Let n be an even positive integer greater that N. We get

$$\frac{4}{l\pi} - \frac{8}{q(q+1)\pi} < \frac{2}{n}\left(\sec\left(\frac{(n-l)\pi}{2n}\right)\right) < \frac{4}{l\pi} + \frac{8}{q(q+1)\pi} \text{ for all } l = 1, 3, 5, ..., q.$$

Consider

$$r_n\left(\frac{\pi}{2n}\right) = \frac{2}{n}\left(\sum_{\substack{k=1\\ \text{odd } k}}^{n-1} \sec\left(\frac{k\pi}{2n}\right)\right).$$

Since $0 < \frac{k\pi}{2n} < \frac{\pi}{2}$ for all $k = 1, 3, 5, ..., n-1$, $\sec\left(\frac{k\pi}{2n}\right) > 0$ for all $k =$

$1, 3, 5, ..., n-1$. Hence

$$\begin{aligned}
r_n\left(\frac{\pi}{2n}\right) &= \frac{2}{n}\left(\sum_{\substack{k=1\\ \text{odd } k}}^{n-1} \sec\left(\frac{k\pi}{2n}\right)\right) \\
&\geq \frac{2}{n}\left(\sum_{\substack{k=n-q\\ \text{odd } k}}^{n-1} \sec\left(\frac{k\pi}{2n}\right)\right) \\
&= \sum_{\substack{l=1\\ \text{odd } l}}^{q} \frac{2}{n}\left(sec\left(\frac{(n-l)\pi}{2n}\right)\right) \\
&> \left(\sum_{\substack{l=1\\ \text{odd } l}}^{q} \frac{4}{l\pi}\right) - \frac{4}{q\pi} \\
&= \frac{4}{\pi}\left(\sum_{\substack{l=1\\ \text{odd } l}}^{q-1} \frac{1}{l}\right).
\end{aligned}$$

Therefore for each positive odd integer q there exists a natural number N such that for all positive even number n greater than N,

$$r_n\left(\frac{\pi}{2n}\right) > \frac{4}{\pi}\left(\sum_{\substack{l=1\\ \text{odd } l}}^{q-1} \frac{1}{l}\right).$$

Let λ be any positive real number greater than 1. Since

$$\lim_{q\longrightarrow\infty} \frac{4}{\pi}\left(\sum_{\substack{l=1\\ \text{odd } l}}^{q-1} \frac{1}{l}\right) = \infty,$$

there exists $Q \in \mathbb{N}$ such that

$$\frac{4}{\pi}\left(\sum_{\substack{l=1\\ \text{odd } l}}^{Q-1} \frac{1}{l}\right) > \lambda$$

Hence there exists a natural number N_Q such that for all positive even numbers n greater than N_Q,

$$r_n\left(\frac{\pi}{2n}\right) > \frac{4}{\pi}\left(\sum_{\substack{l=1\\ \text{odd } l}}^{Q-1}\frac{1}{l}\right) > \lambda.$$

Since $r_n(0^+) = 1 < \lambda < r_n\left(\frac{\pi}{2n}\right)$ and r_n is continuous on $(0, \frac{\pi}{2n})$, we can conclude by Intermediate Value Theorem that there exists $\theta \in (0, \frac{\pi}{2n})$ such that $r_n(\theta) = \lambda = r_n(\frac{\pi}{n} - \theta)$. $\square$

Although this theorem implies that we can find a harmonic map $f_{n,m}$ which maps $\mathbb{D}$ onto a parallelogram that has the ratio of lengths of the non parallel sides equals to a given number $\lambda > 1$, it doesn't imply that for any given parallelogram (up to similarity) there exists a harmonic map $f_{n,m}$ which maps $\mathbb{D}$ onto it. In fact, a parallelogram (up to similarity) can be characterized by only one of its angel α together with its ratio of lengths of the non parallel sides λ. It is easy to see that if we fix n and α, we should choose $m = e^{i\theta}$ where $\theta = \frac{\alpha}{n}$ or $\theta = \frac{\pi-\alpha}{n}$ to get the parallelogram which has one of its angle equals to α. But this means that n and α control the value of $\lambda = r_n(\theta)$. So we may can not find a harmonic map $f_{n,m}$ which maps $\mathbb{D}$ onto a parallelogram with fixed α and λ.

In the next chapter, we will construct a minimal graph corresponding to the map $f_{n,m}$

Chapter 4

Minimal graphs over parallelograms and their conjugate surfaces

In this chapter, we will construct minimal graphs over parallelograms by applying modified Weierstrass representation (Theorem 2.4.7) to the map $f_{n,m}$ constructed in Chapter 3. We also construct their conjugate surfaces.

4.1 Construction

Applying Equation (2.9) to harmonic mapping $f_{n,m}$, we get the parametrize of minimal graph $(u(z), v(z), \mathcal{F}_{n,m}(z)) = (\Re\{f_{n,m}(z)\}, \Im\{f_{n,m}(z)\}, \mathcal{F}_{n,m}(z))$ where

$$
\begin{aligned}
\mathcal{F}_{n,m}(z) &= 2\Im\left\{\int_0^z \sqrt{h'_{n,m}(\zeta)g'_{n,m}(\zeta)}d\zeta\right\} \\
&= 2\Im\left\{\int_0^z h'_{n,m}(\zeta)\sqrt{\omega(\zeta)}d\zeta\right\} \\
&= 2\Im\left\{\int_0^z m^n\zeta^n h'_{n,m}(\zeta)d\zeta\right\}.
\end{aligned}
$$

From Equation (3.3), we get $h'_{n,m}(z) = \frac{1}{(1-z^2)(1-m^{2n}z^{2n})}$. Hence

$$
\mathcal{F}_{n,m}(z) = 2\Im\left\{\int_0^z \frac{m^n\zeta^n}{(1-\zeta^2)(1-m^{2n}\zeta^{2n})}d\zeta\right\}.
$$

From now, we will consider for $n \geq 2$ because the case that $n = 1$ is already done by M.Dorff and J.Szynal and the method that we use here is not compatible with that case. find $\mathcal{F}_{n,m}(z)$ by computing the partial fraction decomposition of

$$\frac{m^n\zeta^n}{(1-\zeta^2)(1-m^{2n}\zeta^{2n})} = \frac{m^n\zeta^n}{(1-m\omega_{2n}\zeta)(1-m\omega_{2n}^2\zeta)...(1-m\omega_{2n}^{2n}\zeta)(1-\zeta)(1+\zeta)}$$

where $\omega_{2n} = e^{i\pi/n}$ is the primitive $2n$-th root of unity. Suppose that the above fraction can be written in the form

$$\frac{C_1}{(1-m\omega_{2n}\zeta)} + \frac{C_2}{(1-m\omega_{2n}^2\zeta)} + ... + \frac{C_{2n}}{(1-m\omega_{2n}^{2n}\zeta)} + \frac{C_{2n+1}}{(1-\zeta)} + \frac{C_{2n+2}}{(1+\zeta)}$$

where $C_j \in \mathbb{C}$ for all $j = 1, 2, 3, ..., 2n+2$. We get

$$\begin{aligned} m^n\zeta^n = C_1 &\left(\frac{(1-\zeta^2)(1-m^{2n}\zeta^{2n})}{(1-m\omega_{2n}\zeta)}\right) + C_2\left(\frac{(1-\zeta^2)(1-m^{2n}\zeta^{2n})}{(1-m\omega_{2n}^2\zeta)}\right) + ... \\ &+ C_{2n}\left(\frac{(1-\zeta^2)(1-m^{2n}\zeta^{2n})}{(1-m\omega_{2n}^{2n}\zeta)}\right) + C_{2n+1}(1+\zeta)(1-m^{2n}\zeta^{2n}) \\ &+ C_{2n+2}(1-\zeta)(1-m^{2n}\zeta^{2n}). \end{aligned}$$

For $p = 1, 2, ..., 2n$, consider $\left(\frac{(1-\zeta^2)(1-m^{2n}\zeta^{2n})}{(1-m\omega_{2n}^p\zeta)}\right)$. Since

$$1 - m^{2n}\zeta^{2n} = (1-m\omega_{2n}^p\zeta)(1+m\omega_{2n}^p\zeta + m^2\omega_{2n}^{2p}\zeta^2 + ... + m^{2n-1}\omega_{2n}^{p(2n-1)}\zeta^{2n-1}),$$

we get

$$\begin{aligned} \frac{(1-\zeta^2)(1-m^{2n}\zeta^{2n})}{(1-m\omega_{2n}^p\zeta)} &= (1-\zeta^2)(1+m\omega_{2n}^p\zeta + m^2\omega_{2n}^{2p}\zeta^2 + ... + m^{2n-1}\omega_{2n}^{p(2n-1)}\zeta^{2n-1}) \\ &= \sum_{j=0}^{2n-1}(m^j\omega_{2n}^{pj}\zeta^j - m^j\omega_{2n}^{pj}\zeta^{j+2}) \\ &= \left(\sum_{j=0}^{2n-1} m^j\omega_{2n}^{pj}\zeta^j\right) - \left(\sum_{j=0}^{2n-1} m^j\omega_{2n}^{pj}\zeta^{j+2}\right) \\ &= \left(\sum_{j=0}^{2n-1} m^j\omega_{2n}^{pj}\zeta^j\right) - \left(\sum_{j=2}^{2n+1} m^{j-2}\omega_{2n}^{p(j-2)}\zeta^j\right) \\ &= 1 + m\omega_{2n}^p\zeta - m^{2n-2}\omega_{2n}^{p(2n-2)}\zeta^{2n} - m^{2n-1}\omega_{2n}^{p(2n-1)}\zeta^{2n+1} \\ &\quad + \sum_{j=2}^{2n-1}\left(m^j\omega_{2n}^{pj}\zeta^j - m^{j-2}\omega_{2n}^{p(j-2)}\zeta^j\right) \\ &= 1 + m\omega_{2n}^p\zeta - m^{2n-2}\omega_{2n}^{p(2n-2)}\zeta^{2n} - m^{2n-1}\omega_{2n}^{p(2n-1)}\zeta^{2n+1} \\ &\quad + \sum_{j=2}^{2n-1}\left(m^{j-2}\omega_{2n}^{p(j-2)}(m^2\omega_{2n}^{2p}-1)\zeta^j\right). \end{aligned}$$

Therefore

$$\begin{aligned}
m^n\zeta^n &= \sum_{p=1}^{2n} C_p\left(1+m\omega_{2n}^p\zeta - m^{2n-2}\omega_{2n}^{p(2n-2)}\zeta^{2n} - m^{2n-1}\omega_{2n}^{p(2n-1)}\zeta^{2n+1}\right)\\
&\quad+\sum_{p=1}^{2n} C_p\left(\sum_{j=2}^{2n-1}\left(m^{j-2}\omega_{2n}^{p(j-2)}(m^2\omega_{2n}^{2p}-1)\zeta^j\right)\right)\\
&\quad+C_{2n+1}(1+\zeta)(1-m^{2n}\zeta^{2n})+C_{2n+2}(1-\zeta)(1-m^{2n}\zeta^{2n})\\
&=\sum_{p=1}^{2n+2} C_p+\left(\left(\sum_{p=1}^{2n} C_p m\omega_{2n}^p\right)+C_{2n+1}-C_{2n+2}\right)\zeta\\
&\quad+\sum_{j=2}^{2n-1}\left(\sum_{p=1}^{2n} C_p m^{j-2}\omega_{2n}^{p(j-2)}(m^2\omega_{2n}^{2p}-1)\right)\zeta^j\\
&\quad-\left[\left(\sum_{p=1}^{2n} C_j m^{2n-2}\omega_{2n}^{p(2n-2)}\right)+\left(C_{2n+1}+C_{2n+2}\right)m^{2n}\right]\zeta^{2n}\\
&\quad-\left[\left(\sum_{p=1}^{2n} C_p m^{2n-1}\omega_{2n}^{p(2n-1)}\right)+\left(C_{2n+1}-C_{2n+2}\right)m^{2n}\right]\zeta^{2n+1}.
\end{aligned}$$

Comparing the coefficient of ζ^j for $j=0,1,2,...,2n+1$, we get

$$\begin{aligned}
j=0;&\ C_1+C_2+...+C_{2n+2}=0\\
j=1;&\ \left(\sum_{p=1}^{2n} C_p m\omega_{2n}^p\right)+C_{2n+1}-C_{2n+2}=0\\
j\in\{2,3,4,...,2n-1\}\setminus\{n\};&\ \sum_{p=1}^{2n} C_p m^{j-2}\omega_{2n}^{p(j-2)}(m^2\omega_{2n}^{2p}-1)=0\\
j=n;&\ \sum_{p=1}^{2n} C_p m^{n-2}\omega_{2n}^{p(n-2)}(m^2\omega_{2n}^{2p}-1)=m^n\\
j=2n;&\ \left(\sum_{p=1}^{2n} C_j m^{2n-2}\omega_{2n}^{p(2n-2)}\right)+\left(C_{2n+1}+C_{2n+2}\right)m^{2n}=0\\
j=2n+1;&\ \left(\sum_{p=1}^{2n} C_j m^{2n-1}\omega_{2n}^{p(2n-1)}\right)+\left(C_{2n+1}-C_{2n+2}\right)m^{2n}=0.
\end{aligned}$$

Since $m \neq 0$, we can divide the equations by m and get

$$(0);\ C_1 + C_2 + ... + C_{2n+2} = 0$$

$$(1);\ \left(\sum_{p=1}^{2n} C_p m \omega_{2n}^{p}\right) + C_{2n+1} - C_{2n+2} = 0$$

$$(j);\ \sum_{p=1}^{2n} C_p \omega_{2n}^{p(j-2)} (m^2 \omega_{2n}^{2p} - 1) = 0 \text{ for } j = 2, 3, 4, ..., n-1, n+1, ..., 2n-1$$

$$(n);\ \sum_{p=1}^{2n} C_p \omega_{2n}^{p(n-2)} (m^2 \omega_{2n}^{2p} - 1) = m^2$$

$$(*);\ \left(\sum_{p=1}^{2n} C_p \omega_{2n}^{p(2n-2)}\right) + (C_{2n+1} + C_{2n+2})\, m^2 = 0$$

$$(**);\ \left(\sum_{p=1}^{2n} C_p \omega_{2n}^{p(2n-1)}\right) + (C_{2n+1} - C_{2n+2})\, m = 0.$$

By $m^2(0) - (*)$, we get

$$(2n); \sum_{p=1}^{2n} C_p \omega_{2n}^{p(2n-2)} (m^2 \omega_{2n}^{2p} - 1) = 0.$$

By $m(1) - (**)$, we get

$$(2n+1); \sum_{p=1}^{2n} C_p \omega_{2n}^{p(2n-1)} (m^2 \omega_{2n}^{2p} - 1) = 0.$$

For $r = 1, 2, 3, ..., 2n$, consider $\sum_{j=2}^{2n+1} \omega_{2n}^{-rj}(j)$ and get

$$\sum_{j=2}^{2n+1} \sum_{p=1}^{2n} C_p \omega_{2n}^{p(j-2)-rj} (m^2 \omega_{2n}^{2p} - 1) = \omega_{2n}^{-rn} m^2$$

$$\sum_{p=1}^{2n} \left(C_p (m^2 \omega_{2n}^{2p} - 1) \omega_{2n}^{-2p} \left(\sum_{j=2}^{2n+1} \omega_{2n}^{(p-r)j} \right) \right) = \omega_{2n}^{-rn} m^2$$

Since for $p - r \equiv 1, 2, 3, ..., 2n-1 (\text{mod } 2n)$, ω_{2n}^{p-r} are roots of $z^2 + z^+ z^4 + ... + z^{2n+1} = 0$, we get

$$\sum_{j=2}^{2n+1} \omega_{2n}^{(p-r)j} = 0 \text{ for all } p - r \equiv 1, 2, 3, ..., 2n-1 (\text{mod } 2n).$$

Therefore

$$\begin{aligned} 2nC_r(m^2\omega_{2n}^{2r}-1)\omega_{2n}^{-2r} &= \omega_{2n}^{-rn}m^2 \\ C_r &= \frac{\omega_{2n}^{-r(n-2)}m^2}{2n(m^2\omega_{2n}^{2r}-1)} \end{aligned}$$

for all $r = 1, 2, 3, ..., 2n$.

From $(*)$, we get

$$C_{2n+1} + C_{2n+2} = -\frac{1}{m^2}\sum_{p=1}^{2n} C_p\omega_{2n}^{-2p} = -\frac{1}{2n}\sum_{p=1}^{2n}\frac{(-1)^p}{m^2\omega_{2n}^{2p}-1}. \tag{4.1}$$

From $(**)$, we get

$$C_{2n+1} - C_{2n+2} = -\frac{1}{m}\sum_{p=1}^{2n} C_p\omega_{2n}^{-p} = -\frac{m}{2n}\sum_{p=1}^{2n}\frac{(-1)^p\omega_{2n}^{p}}{m^2\omega_{2n}^{2p}-1}. \tag{4.2}$$

Solving this system of equations, we get

$$\frac{1}{2}\left((4.1)+(4.2)\right); C_{2n+1} = -\frac{1}{4n}\sum_{p=1}^{2n}\frac{(-1)^p}{m\omega_{2n}^{p}-1}$$

$$\frac{1}{2}\left((4.1)-(4.2)\right); C_{2n+2} = \frac{1}{4n}\sum_{p=1}^{2n}\frac{(-1)^p}{m\omega_{2n}^{p}+1}.$$

We get

$$\begin{aligned} C_{2n+1} &= -\frac{1}{4n}\left(\frac{\sum_{p=1}^{2n}\left((-1)^p\sum_{j=0}^{2n-1} m^j\omega_{2n}^{pj}\right)}{m^{2n}-1}\right) \\ &= -\frac{1}{4n}\left(\frac{\sum_{p=1}^{2n}\left(\sum_{j=0}^{2n-1} m^j\omega_{2n}^{(j+n)p}\right)}{m^{2n}-1}\right) \\ &= -\frac{1}{4n}\left(\frac{\sum_{j=0}^{2n-1} m^j\left(\sum_{p=1}^{2n}\omega_{2n}^{(j+n)p}\right)}{m^{2n}-1}\right) \\ &= -\frac{1}{4n}\left(\frac{2nm^n}{m^{2n}-1}\right) \\ &= -\frac{m^n}{2(m^{2n}-1)}, \end{aligned}$$

and

$$\begin{aligned}
C_{2n+2} &= \frac{1}{4n}\sum_{p=1}^{2n}\frac{(-1)^p}{m\omega_{2n}^{2n+p}+1} \\
&= \frac{1}{4n}\left(\sum_{p=1}^{n}\frac{(-1)^p}{m\omega_{2n}^{2n+p}-\omega_{2n}^{n}}+\sum_{p=n+1}^{2n}\frac{(-1)^p}{m\omega_{2n}^{2n+p}-\omega_{2n}^{n}}\right) \\
&= \frac{1}{4n}\left(\sum_{p=1}^{n}\frac{(-1)^p}{(m\omega_{2n}^{n+p}-1)\omega_{2n}^{n}}+\sum_{p=n+1}^{2n}\frac{(-1)^p}{(m\omega_{2n}^{n+p}-1)\omega_{2n}^{n}}\right) \\
&= \frac{1}{4n}\left(\sum_{p=1}^{n}\frac{(-1)^{p+1}}{m\omega_{2n}^{n+p}-1}+\sum_{p=n+1}^{2n}\frac{(-1)^{p+1}}{m\omega_{2n}^{n+p}-1}\right) \\
&= \frac{1}{4n}\left(\sum_{p=n+1}^{2n}\frac{(-1)^{p-n+1}}{m\omega_{2n}^{p}-1}+\sum_{p=1}^{n}\frac{(-1)^{p+n+1}}{m\omega_{2n}^{2n+p}-1}\right) \\
&= \frac{1}{4n}\left(\sum_{p=n+1}^{2n}\frac{(-1)^{p+n+1}}{m\omega_{2n}^{p}-1}+\sum_{p=1}^{n}\frac{(-1)^{p+n+1}}{m\omega_{2n}^{p}-1}\right) \\
&= \frac{1}{4n}\left(\sum_{p=1}^{2n}\frac{(-1)^{p+n+1}}{m\omega_{2n}^{p}-1}\right) \\
&= \frac{(-1)^{n+1}}{4n}\left(\sum_{p=1}^{2n}\frac{(-1)^{p}}{m\omega_{2n}^{p}-1}\right) \\
&= \frac{(-1)^{n+1}}{4n}\left(\frac{2nm^n}{m^{2n}-1}\right) \\
&= \frac{(-1)^{n+1}m^n}{2(m^{2n}-1)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{m^n\zeta^n}{(1-\zeta^2)(1-m^{2n}\zeta^{2n})} &= \sum_{j=1}^{2n}\left[\left(\frac{\omega_{2n}^{-j(n-2)}m^2}{2n(m^2\omega_{2n}^{2j}-1)}\right)\left(\frac{1}{1-m\omega_{2n}^{j}\zeta}\right)\right] \\
&\quad -\frac{m^n}{2(m^{2n}-1)(1-\zeta)}+\frac{(-1)^{n+1}m^n}{2(m^{2n}-1)(1+\zeta)}.
\end{aligned}$$

Despite the fact that this method is not compatible with the case $n = 1$, the result above also holds for that case.

$$\begin{aligned}
\frac{m\zeta}{(1-\zeta^2)(1-m^{2n}\zeta^{2n})} &= m\zeta\left(\frac{1}{(1-\zeta^2)(1-m^{2n}\zeta^{2n})}\right)\\
&= m\zeta\left(\frac{1}{1-m^2}\left(\frac{1}{1-\zeta^2}\right)-\frac{m^2}{1-m^2}\left(\frac{1}{1-m^2\zeta^2}\right)\right)\\
&= \frac{m}{1-m^2}\left(\frac{\zeta}{1-\zeta^2}\right)-\frac{m^2}{1-m^2}\left(\frac{m\zeta}{1-m^2\zeta^2}\right)\\
&= \frac{m}{2(1-m^2)}\left(\frac{1}{1-\zeta}-\frac{1}{1+\zeta}\right)-\frac{m^2}{2(1-m^2)}\left(\frac{1}{1-m\zeta}-\frac{1}{1-m\zeta}\right)\\
&= \frac{m^2}{2(m^2-1)}\left(\frac{1}{1-m\zeta}\right)-\frac{m^2}{2(m^2-1)}\left(\frac{1}{1+m\zeta}\right)\\
&\quad-\frac{m}{2(m^2-1)}\left(\frac{1}{1-\zeta}\right)+\frac{m}{2(m^2-1)}\left(\frac{1}{1+\zeta}\right)\\
&= \sum_{j=1}^{2n}\left[\left(\frac{\omega_{2n}^{-j(n-2)}m^2}{2n(m^2\omega_{2n}^{2j}-1)}\right)\left(\frac{1}{1-m\omega_{2n}^{j}\zeta}\right)\right]\\
&\quad-\frac{m^n}{2(m^{2n}-1)(1-\zeta)}+\frac{(-1)^{n+1}m^n}{2(m^{2n}-1)(1+\zeta)}
\end{aligned}$$

where $n=1$. So from now we let n be an arbitrary natural number. We will find $\int_0^z\left(\frac{m^n\zeta^n}{(1-\zeta^2)(1-m^{2n}\zeta^{2n})}\right)d\zeta$

For $j=1,2,3,...,2n$, consider

$$\begin{aligned}
\int_0^z\frac{1}{1-m\omega_{2n}^{j}\zeta}d\zeta &= -\frac{1}{m\omega_{2n}^{j}}\int_0^z\frac{1}{1-m\omega_{2n}^{j}\zeta}d(1-m\omega_{2n}^{j}\zeta)\\
&= -\left(\frac{1}{m\omega_{2n}^{j}}\right)\log\left(1-m\omega_{2n}^{j}z\right),
\end{aligned}$$

and $\int_0^z\frac{1}{1-\zeta}d\zeta=-\int_0^z\frac{1}{1-\zeta}d(1-\zeta)=-\log(1-z)$, $\int_0^z\frac{1}{1+\zeta}d\zeta=\int_0^z\frac{1}{1+\zeta}d(1+\zeta)=\log(1+z)$. Hence,

$$\begin{aligned}
\int_0^z\left(\frac{m^n\zeta^n}{(1-\zeta^2)(1-m^{2n}\zeta^{2n})}\right)d\zeta &= \sum_{j=1}^{2n}\left[-\left(\frac{\omega_{2n}^{-j(n-1)}m}{2n(m^2\omega_{2n}^{2j}-1)}\right)\log\left(1-m\omega_{2n}^{j}z\right)\right]\\
&\quad+\left(\frac{m^n}{2(m^{2n}-1)}\right)\log(1-z)+\frac{(-1)^{n+1}m^n}{2(m^{2n}-1)}\log(1+z)\\
&= \sum_{j=1}^{2n}\left[\left(\frac{(-1)^{j+1}m\omega_{2n}^{j}}{2n(m^2\omega_{2n}^{2j}-1)}\right)\log\left(1-m\omega_{2n}^{j}z\right)\right]\\
&\quad+\left(\frac{m^n}{2(m^{2n}-1)}\right)\log(1-z)+\frac{(-1)^{n+1}m^n}{2(m^{2n}-1)}\log(1+z).
\end{aligned}$$

From $\mathcal{F}_{n,m}(z) = 2\Im\left\{\int_0^z \frac{m^n\zeta^n}{(1-\zeta^2)(1-m^{2n}\zeta^{2n})}d\zeta\right\}$, we get

$$\begin{aligned}\mathcal{F}_{n,m}(z) = &\sum_{j=1}^{2n} \Im\left\{\left(\frac{(-1)^{j+1}m\omega_{2n}^j}{n(m^2\omega_{2n}^{2j}-1)}\right)\log\left(1-m\omega_{2n}^j z\right)\right\}\\ &+\Im\left\{\left(\frac{m^n}{m^{2n}-1}\right)\log(1-z)\right\}+\Im\left\{\left(\frac{(-1)^{n+1}m^n}{m^{2n}-1}\right)\log(1+z)\right\}.\end{aligned}$$

Next, we will construct the conjugate surface $S^*_{n,m}$ of $S_{n,m}$. Let $\mathbf{x}^*(z) = (x_1^*(z), x_2^*(z), x_3^*(z))$ be a parametrization of $S^*_{n,m}$. By Equation (2.10), we get

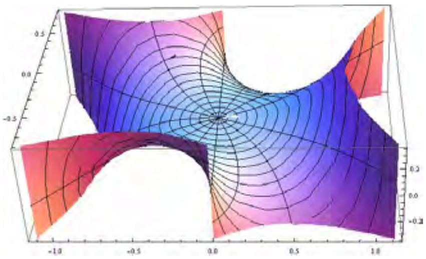
$$\begin{aligned}x_1^*(z) &= \Re\left\{h_{n,m}(z) - g_{n,m}(z)\right\},\\ x_2^*(z) &= \Im\left\{h_{n,m}(z) + g_{n,m}(z)\right\},\\ x_3^*(z) &= 2\Re\left\{\int_0^z \sqrt{h'_{n,m}(\zeta)g'_{n,m}(\zeta)}d\zeta\right\}.\end{aligned}$$

Hence,

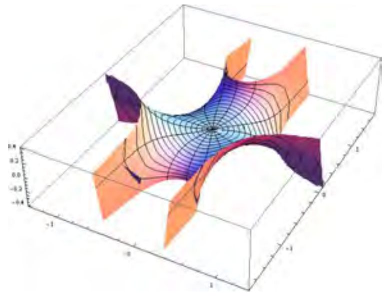
$$\begin{aligned}x_1^*(z) &= \frac{1}{2}\Re\left(\log\left(\frac{1+z}{1-z}\right)\right)\\ x_2^*(z) &= \Im\left(\sum_{j=0}^{n-1}\left[\left(\frac{m}{n(m^2-\omega_{2n}^{-2j})\omega_{2n}^j}\right)\log\left(\frac{1+m\omega_{2n}^j z}{1-m\omega_{2n}^j z}\right)\right]+\left(\frac{1+m^{2n}}{2(1-m^{2n})}\right)\log\left(\frac{1+z}{1-z}\right)\right)\\ x_3^*(z) &= \sum_{j=1}^{2n}\Re\left\{\left(\frac{(-1)^{j+1}m\omega_{2n}^j}{n(m^2\omega_{2n}^{2j}-1)}\right)\log\left(1-m\omega_{2n}^j z\right)\right\}\\ &\quad+\Re\left\{\left(\frac{m^n}{m^{2n}-1}\right)\log(1-z)\right\}+\Re\left\{\left(\frac{(-1)^{n+1}m^n}{m^{2n}-1}\right)\log(1+z)\right\}.\end{aligned}$$

Figure 4.1 shows some figures of $S_{n,m}$ and $S^*_{n,m}$.

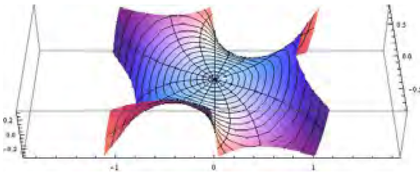
In the next section, we will prove that if $m = e^{i\theta}$ where $0 < \theta < \frac{\pi}{n}$, $S_{n,m}$ is a JS surface over parallelogram domain and then conclude some interesting corollaries.



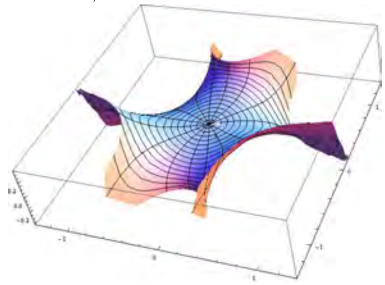
(a) $S_{n,m}$; $n = 2$ and $m = e^{\frac{i\pi}{4}}$



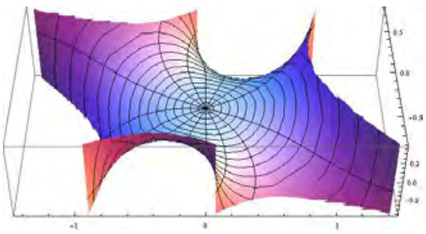
(b) $S^*_{n,m}$; $n = 2$ and $m = e^{\frac{i\pi}{4}}$



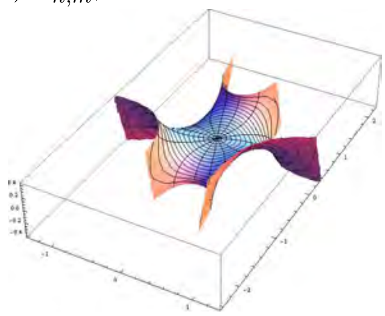
(c) $S_{n,m}$; $n = 2$ and $m = 0.95e^{\frac{i\pi}{4}}$



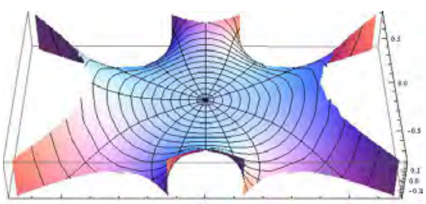
(d) $S^*_{n,m}$; $n = 2$ and $m = 0.95e^{\frac{i\pi}{4}}$



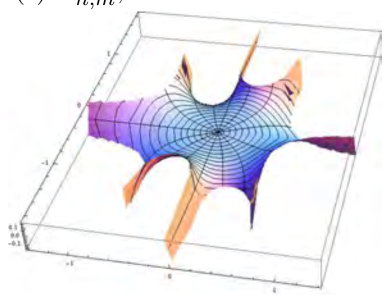
(e) $S_{n,m}$; $n = 2$ and $m = e^{\frac{i\pi}{5}}$



(f) $S^*_{n,m}$; $n = 2$ and $m = e^{\frac{i\pi}{5}}$



(g) $S_{n,m}$; $n = 3$ and $m = e^{\frac{i\pi}{6}}$



(h) $S^*_{n,m}$ where $n = 3$ and $m = e^{\frac{i\pi}{6}}$

Figure 4.1: Minimal graphs over $f_{n,m}(\mathbb{D})$ and their conjugate surfaces for various values of n and m

4.2 Some corollaries

For the case that $m = e^{i\theta}$ where $0 < \theta < \frac{\pi}{n}$, the parametrization $(u(z), v(z), \mathcal{F}_{n,m}(z))$ is a minimal graph $S_{n,m}$ over a parallelogram domain with $2n+2$, $b_1, b_2, ..., b_{2n+2}$, and hence $2n+2$ sides (see Section 3.3). Since this parametrization obtained from harmonic mapping $f_{n,m}$ which has dilation $\omega(z) = m^{2n}z^{2n}$, the Equation (2.11) tell us that (the stereographic projection of) the Gauss map $G(z)$ of $S_{n,m}$ is

$$G(z) = \frac{i}{\sqrt{\omega(z)}} = \frac{i}{m^n z^n}$$

which is in the form $c/B(z)$ where c is a constant of modulus 1 and $B(z)$ is a Blaschke product of order n. By Theorem 2.4.11, we can conclude that $S_{n,m}$ is a JS surface.

For $j = 1, 2, 3, ..., 2n$, let α_j be the interior angle of the parallelogram at the vertex b_j. According to Section 3.3, we get $\alpha_1 = \alpha_{n+2} = \pi - n\theta$ and $\alpha_{n+1} = \alpha_{2n+2} = n\theta$, while the rests are π. So b_1, b_{n+1}, b_{n+2} and b_{2n+2} are points of convexity and Theorem 2.4.10 tells us that $\mathcal{F}_{n,m}(z)$ must change signs on the sides adjacent to each of these four points.

For $j = 2, 3, 4, ..., n$, we known that $f_{n,m}(e^{i\varphi}) = b_j$ when $\varphi \in (\frac{(j-1)\pi}{n} - \theta, \frac{j\pi}{n} - \theta)$. Consider

$$\begin{aligned}
&\frac{1}{2}\left(\arg(\omega(e^{\frac{j\pi i}{n}-\theta})) - \arg(\omega(e^{\frac{(j-1)\pi}{n}-\theta}))\right) \\
&= \frac{1}{2}\left(2n\left(\theta + \left(\frac{j\pi i}{n} - \theta\right)\right) - 2n\left(\theta + \left(\frac{(j-1)\pi}{n} - \theta\right)\right)\right) \\
&= j\pi - (j-1)\pi \\
&= \pi = \alpha_j.
\end{aligned}$$

For $j = n+3, n+4, n+5, ..., 2n+1$, we known that $f_{n,m}(e^{i\varphi}) = b_j$ when $\varphi \in (\frac{(j-2)\pi}{n} - \theta, \frac{(j-1)\pi}{n} - \theta)$. Consider

$$\begin{aligned}
&\frac{1}{2}\left(\arg(\omega(e^{\frac{(j-1)\pi i}{n}-\theta})) - \arg(\omega(e^{\frac{(j-2)\pi}{n}-\theta}))\right) \\
&= \frac{1}{2}\left(2n\left(\theta + \left(\frac{(j-1)\pi i}{n} - \theta\right)\right) - 2n\left(\theta + \left(\frac{(j-2)\pi}{n} - \theta\right)\right)\right) \\
&= (j-1)\pi - (j-2)\pi \\
&= \pi = \alpha_j
\end{aligned}$$

So for $j = 2, 3, 4, ..., n, n+3, n+4, n+5, ..., 2n+1$, b_j is a full resting point. We can conclude by Theorem 2.4.10 that $\mathcal{F}_{n,m}(z)$ must also change signs on the sides adjacent to each of these points.

Now, we can get the corollary:

Corollary 4.2.1. *For a natural number n and a complex number $m = e^{i\theta}$ where $0 < \theta < \frac{\pi}{n}$, minimal graph $S_{n,m}$ parametrized by $(u(z), v(z), \mathcal{F}_{n,m}(z))$ is a JS surface over parallelogram domain with $2n+2$ vertices, $b_1, b_2, ..., b_{2n+2}$. Moreover, $\mathcal{F}_{n,m}(z)$ changes signs on the sides adjacent to b_j every $j = 1, 2, 3, ..., 2n+2$.*

Since we can construct JS surface over parallelogram $P_{n,m}$ with vertices $b_1, b_2, b_3, ..., b_{2n+2}$, Theorem 2.2.6 tells us that this parallelogram must satisfy

(a) no two edges of A_i nor of B_i meet at a convex vertex,

(b) $2\sum_{A_i\in\Pi}|A_i| < |\Pi|$ and $2\sum_{B_i\in\Pi}|B_i| < |\Pi|$ for each such Π, $\Pi \neq P$,

(c) $\sum|A_i| = \sum|B_i|$ when $\Pi = P$.

where $P_{n,m}$ has finitely many bounding edges partitioned into sets A_i and B_i. Π is a connected polygonal subset of $P_{n,m}$ whose boundary is the union of some segments from A_i and B_i, possibly including additional line segments contained in $P_{n,m}$ whose endpoints are vertices of $P_{n,m}$. $|\Pi|$ is the length of the boundary of Π.

We will focus only on condition (c).

In the case that n is even, $\mathcal{F}$ has the same sign when it approaches the sides $b_1b_2, b_3b_4, ..., b_{n+1}b_{n+2}, ..., b_{2n+1}b_{2n+2}$ which have opposite sign to the sides $b_2b_3, b_4b_5, ..., b_nb_{n+1}, ..., b_{2n}b_{2n+1}, b_{2n+2}b_1$. From Section 3.3, we have known that

$$l(b_{2n+2}, b_1) = \frac{\pi}{2\sin(n\theta)} = l(b_{n+1}, b_{n+2}),$$

while $\mathcal{F}$ has opposite sign when it approaches the sides $b_{n+1}b_{n+2}$ and $b_{2n+2}b_1$. And for $k = 1, 2, 3, ..., n+1$

$$\begin{aligned} l(b_k, b_{k+1}) &= \frac{\pi}{2n\sin\left(\theta + \frac{(n-k)\pi}{n}\right)} \\ &= \frac{\pi}{2n\sin\left(\theta + \frac{(2n+1-(k+n+1))\pi}{n}\right)} \\ &= l(b_{k+n+1}, b_{k+n+2}). \end{aligned}$$

Since k and $k+n+1$ have different parity, $\mathcal{F}$ also has opposite sign when it approaches the sides b_kb_{k+1} and $b_{k+n+1}b_{k+n+2}$. So we can pair the sides with opposite sign which has the same length bijectively. This obviously implies the condition (c).

It is much more interesting for the case that n is odd. In this case, we can not pair the sides as for the previous case. Since n is odd, $\mathcal{F}$ has the same sign when it approaches the sides $b_1b_2, b_3b_4, ..., b_nb_{n+1}, ..., b_{2n+1}b_{2n+2}$ which are opposite sign to the sides $b_2b_3, b_4b_5, ..., b_{n+1}b_{n+2}, ..., b_{2n}b_{2n+1}, b_{2n+2}b_1$. According to the condition (c), this implies that the sum of the lengths of the sides $b_1b_2, b_3b_4, ..., b_nb_{n+1}, ..., b_{2n+1}b_{2n+2}$ must be equal to the sum of the lengths of the sides $b_2b_3, b_4b_5, ..., b_{n+1}b_{n+2}, ..., b_{2n}b_{2n+1}, b_{2n+2}b_1$. According to Section 3.3, we can conclude an interesting trigonometric identity as follow:

Corollary 4.2.2. *For odd n,*

$$\sum_{k=0}^{n-1}\left(\frac{\sin\left(n\left(\theta+\frac{k\pi}{n}\right)\right)}{\sin\left(\theta+\frac{k\pi}{n}\right)}\right)=n.$$

Proof. From the argument above, we get

$$\sum_{\substack{k=1\\ \text{odd } k}}^{2n+1} l(b_k,b_{k+1})=\sum_{\substack{k=2\\ \text{even } k}}^{2n} l(b_k,b_{k+1})+l(b_{2n+2},b_1). \tag{4.3}$$

Consider the left hand side,

$$\begin{aligned}\sum_{\substack{k=1\\ \text{odd } k}}^{2n+1} l(b_k,b_{k+1}) &= \sum_{\substack{k=1\\ \text{odd } k}}^{n} l(b_k,b_{k+1})+\sum_{\substack{k=n+2\\ \text{odd } k}}^{2n+1} l(b_k,b_{k+1})\\ &= \sum_{\substack{k=1\\ \text{odd } k}}^{n}\left(\frac{\pi}{2n\sin\left(\theta+\frac{(n-k)\pi}{n}\right)}\right)+\sum_{\substack{k=n+2\\ \text{odd } k}}^{2n+1}\left(\frac{\pi}{2n\sin\left(\theta+\frac{(2n+1-k)\pi}{n}\right)}\right)\\ &= \sum_{\substack{k=1\\ \text{odd } k}}^{n}\left(\frac{\pi}{2n\sin\left(\theta+\frac{(n-k)\pi}{n}\right)}\right)+\sum_{\substack{k=1\\ \text{odd } k}}^{n}\left(\frac{\pi}{2n\sin\left(\theta+\frac{(n-k)\pi}{n}\right)}\right)\\ &= \sum_{\substack{k=1\\ \text{odd } k}}^{n}\left(\frac{\pi}{n\sin\left(\theta+\frac{(n-k)\pi}{n}\right)}\right).\end{aligned}$$

Similarly for the right hand side,

$$\begin{aligned}\sum_{\substack{k=2\\ \text{even } k}}^{2n} l(b_k,b_{k+1})+l(b_{2n+2},b_1) &= \sum_{\substack{k=2\\ \text{even } k}}^{n-1} l(b_k,b_{k+1})+\sum_{\substack{k=n+3\\ \text{even } k}}^{2n} l(b_k,b_{k+1})\\ &\quad+l(b_{n+1},b_{n+2})+l(b_{2n+2},b_1)\\ &= \sum_{\substack{k=2\\ \text{even } k}}^{n-1}\left(\frac{\pi}{n\sin\left(\theta+\frac{(n-k)\pi}{n}\right)}\right)+\frac{\pi}{\sin(n\theta)}.\end{aligned}$$

From (4.3), we get

$$\sum_{\substack{k=1\\ \text{odd } k}}^{n}\left(\frac{\pi}{n\sin\left(\theta+\frac{(n-k)\pi}{n}\right)}\right)=\sum_{\substack{k=2\\ \text{even } k}}^{n-1}\left(\frac{\pi}{n\sin\left(\theta+\frac{(n-k)\pi}{n}\right)}\right)+\frac{\pi}{\sin(n\theta)}.$$

$$\sum_{\substack{k=1\\ \text{odd } k}}^{n}\left(\frac{\pi}{n\sin\left(\theta+\frac{(n-k)\pi}{n}\right)}\right)-\sum_{\substack{k=2\\ \text{even } k}}^{n-1}\left(\frac{\pi}{n\sin\left(\theta+\frac{(n-k)\pi}{n}\right)}\right)=\frac{\pi}{\sin(n\theta)}.$$

$$\sum_{k=1}^{n}\left(\frac{(-1)^{k+1}}{n\sin\left(\theta+\frac{(n-k)\pi}{n}\right)}\right)=\frac{1}{\sin(n\theta)}.$$

$$\sum_{k=1}^{n}\left(\frac{(-1)^{k+1}\sin(n\theta)}{\sin\left(\theta+\frac{(n-k)\pi}{n}\right)}\right)=n.$$

Hence

$$\begin{aligned}\sum_{k=1}^{n}\left(\frac{\sin\left(n\left(\theta+\frac{k\pi}{n}\right)\right)}{\sin\left(\theta+\frac{k\pi}{n}\right)}\right)&=n\\ \sum_{k=1}^{n-1}\left(\frac{\sin\left(n\left(\theta+\frac{k\pi}{n}\right)\right)}{\sin\left(\theta+\frac{k\pi}{n}\right)}\right)+\frac{\sin\left(n\theta+n\pi\right)}{\sin\left(\theta+\pi\right)}&=n\\ \sum_{k=1}^{n-1}\left(\frac{\sin\left(n\left(\theta+\frac{k\pi}{n}\right)\right)}{\sin\left(\theta+\frac{k\pi}{n}\right)}\right)+\frac{-\sin\left(\theta\right)}{-\sin\left(\theta\right)}&=n\\ \sum_{k=1}^{n-1}\left(\frac{\sin\left(n\left(\theta+\frac{k\pi}{n}\right)\right)}{\sin\left(\theta+\frac{k\pi}{n}\right)}\right)+\frac{\sin\left(n\theta\right)}{\sin\left(\theta\right)}&=n\\ \sum_{k=0}^{n-1}\left(\frac{\sin\left(n\left(\theta+\frac{k\pi}{n}\right)\right)}{\sin\left(\theta+\frac{k\pi}{n}\right)}\right)&=n\end{aligned}$$

as desired. $\square$

Remark 4.2.3. *In fact, this corollary can be proven purely algebraically, as follow:*

Proof. (Algebraic proof of Cor. 4.2.2) By Prop 3.3.9, for $k=0,1,2,...,n-1$,

$$\frac{\sin\left(n\left(\theta+\frac{k\pi}{n}\right)\right)}{\sin\left(\theta+\frac{k\pi}{n}\right)}=1+2\sum_{\substack{r=1\\ \text{odd } r}}^{n-2}\cos\left((n-r)\left(\theta+\frac{k\pi}{n}\right)\right).$$

Therefore,

$$\sum_{k=0}^{n-1}\left(\frac{\sin\left(n\left(\theta+\frac{k\pi}{n}\right)\right)}{\sin\left(\theta+\frac{k\pi}{n}\right)}\right) = n + 2\sum_{k=0}^{n-1}\sum_{\substack{r=1\\ \text{odd } r}}^{n-2}\cos\left((n-r)\left(\theta+\frac{k\pi}{n}\right)\right)$$
$$= n + 2\sum_{\substack{r=1\\ \text{odd } r}}^{n-2}\sum_{k=0}^{n-1}\cos\left((n-r)\left(\theta+\frac{k\pi}{n}\right)\right).$$

For fixed $r \in \{1, 3, ..., n-2\}$, by Prop 3.3.8, we get

$$\sum_{k=0}^{n-1}\cos\left((n-r)\left(\theta+\frac{k\pi}{n}\right)\right) = \sum_{k=0}^{n-1}\cos\left((n-r)\theta + k\left(\frac{(n-r)\pi}{n}\right)\right)$$
$$= \frac{\sin\left(\frac{((n-r)\pi)}{2}\right)}{\sin\left(\frac{((n-r)\pi)}{2n}\right)}\cos\left((n-r)\theta + \frac{(n-1)\left((n-r)\pi\right)}{2n}\right).$$

Since n, r are odds, $n-r$ is even. Hence $\frac{n-r}{2}$ is an integer. So $\sin\left(\frac{((n-r)\pi)}{2}\right) = 0$ which implies

$$\sum_{k=0}^{n-1}\cos\left((n-r)\left(\theta+\frac{k\pi}{n}\right)\right) = 0.$$

Therefore

$$\sum_{k=0}^{n-1}\left(\frac{\sin\left(n\left(\theta+\frac{k\pi}{n}\right)\right)}{\sin\left(\theta+\frac{k\pi}{n}\right)}\right) = n$$

as desired. $\square$

Chapter 5

Conclusion

5.1 Our Results

We obtained some harmonic mappings onto an arbitrary parallelogram domain via Radó-Kneser-Choquet theorem but we can prove that this kind of maps has analytic square root dilation only when it maps $\mathbb{D}$ onto a rhombus, see Theorem 3.1.1. We then constructed harmonic shear $f_{n,m}$ of elliptic integral $F(z,1)=\int_0^z \frac{d\zeta}{(1-\zeta^2)}$ with dilation $\omega(z)=m^{2n}z^{2n}$ where n is an arbitrary natural number and m is a complex number such that $|m|\leqslant 1$. In the case that $|m|=1$, we found that the image of $\mathbb{D}$ under $f_{n,m}$ is a parallelogram, see Proposition 3.3.7. We applied the Weierstrass representation to the harmonic map $f_{n,m}$ where $|m|=1$ and received a family of minimal graphs which are JS surfaces over parallelogram domains, see Corollary 4.2.1. In fact, the result in the case $n=1$ is the same as result of M.Dorff and J.Szynal in [8] which is a minimal graph over a rhombus, but when $n\geq 2$ the minimal graph that we construct is a minimal graph over a parallelogram which is not a rhombus. We also rediscover an interesting trigonometry identity, see Corollary 4.2.2.

5.2 Problems for further investigation

Although there is a lot of researches on minimal surfaces, there are still a lot of problems to investigate. Here are some interesting questions arising from our work:

1. Describe the image of $\mathbb{D}$ under $f_{n,m}$ where n is an arbitrary natural number and m is a complex number where $|m|<1$ and its associated minimal graph. Figure 3.2 shows some image of $\mathbb{D}$ under $f_{n,m}$ where

$|m| < 1$. We can make a interesting claim that for this case the image of $\mathbb{D}$ under $f_{n,m}$ is a hexagon but we haven't prove it yet.

2. Construct minimal graph over an arbitrary parallelogram domain and determine whether it is JS surface or not. In Chapter 3, we have constructed a harmonic map which maps $\mathbb{D}$ onto an arbitrary parallelogram domain via Radó-Kneser-Choquet theorem but unfortunately its dilation has non analytic square root, see Theorem 3.1.1. However, there is a lot of possible way to construct a harmonic map which maps $\mathbb{D}$ onto a parallelogram domain via Radó-Kneser-Choquet theorem so that there still a hope to find some of these maps with analytic square root dilation.

3. Apply shearing method to elliptic integral $F(z,k)$ and $E(z,k)$ by using other dilations. Investigate the image of $\mathbb{D}$ under this map and construct minimal graph over it.

4. Construct minimal graphs over other polygonal domains and examine whether it is a JS surface or not.

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Appendix: Project Proposal

กลุ่มที่ 28

เอกสารนี้ได้รับการอนุมัติจากอาจารย์ที่ปรึกษาโครงงานแล้ว

ลงชื่อ ....

(วันที่ ....)

The Project Proposal of Course 2301399 Project Proposal Academic Year 2019

Project Tittle (Thai) การศึกษาพื้นผิวมินิมอลและการส่งแบบฮาร์โมนิก

Project Tittle (English) Minimal surfaces and harmonic mappings

Project Advisor Assoc. Prof. Dr. Nataphan Kitisin

By Mr. Poom Lertpinyowong Student ID 5933539723

Mathematics, Department of Mathematics and Computer Science,
Faculty of Science, Chulalongkorn University

Background and Rationale

A *minimal surface*, intuitively, is a surface which for each sufficiently small portion of it has the minimum area among all surfaces with the same boundary. Minimal surface can physically be interpreted as a soap film that spans a wire frame when it is dipped in soap solution. Some standard examples of minimal surfaces in Euclidean space are the plane, the catenoid, and the helicoid. History of minimal surfaces can be traced back to 1744 when Euler first described the catenoid surface. Since then minimal surface has become one of the most interesting geometric object which challenges a lot of great mathematicians.

Theory of minimal surfaces involves many branches in mathematics, including differential geometry, partial differential equation and complex analysis. One of the most important techniques to construct new minimal surfaces is called the *Weierstrass representation*.

Our work will focus on special class of minimal surfaces called *minimal graphs*. A minimal graph is a minimal surface which is lifted from its domain. One general example of minimal graph is Scherk's doubly periodic surface which has a square as its domain in the complex plane. There are several papers that study this kind of surfaces such as [3], [5], [6], and [8]. Constructions of minimal graphs in those papers are involving constructing *harmonic univalent mapping of the unit disc* $\mathbb{D}$, which is defined as an one to one function from $\mathbb{D}$ to $\mathbb{C}$ whose real part and imaginary part are twice differentiable and satisfy Laplace equation, and then use *modified Weierstrass representation* to construct minimal graphs. Two main techniques that use to construct harmonic univalent mapping are *Clunie and Sheil-Small shearing method*

[2] (see example [5], and [6]) and *Radó-Kneser-Choquet theorem* [7, Chapter 3&4] (see example [3], and [8]).

In this project, we will try to apply those methods mentioned above to obtain certian harmonic univalent mapping of the unit disc $\mathbb{D}$ and use it to construct minimal graphs over certain domains.

Objectives

To construct minimal graphs over certain domains by using harmonic univalent mapping of the unit disc $\mathbb{D}$, given by Clunie and Sheil-Small shearing method or Radó-Kneser-Choquet theorem.

Scope

In this project, we focus only on minimal graphs over certain domains.

Project Activities

A. Processes

1. Reviewing basic knowledges about minimal surfaces and harmonic univalent mapping from [2], [7], and [9].
2. Reading related research papers from [1], [3], [4], [5], [6], and [8].
3. Presenting project proposal.
4. Constructing minimal graphs over certain domains.
5. Writing the project report and project presenting.

B. Duration

Processes	2019					2020			
	Aug.	Sep.	Oct.	Nov.	Dec.	Jan.	Feb.	Mar.	Apr.
1. Reviewing basic knowledges about minimal surfaces and harmonic univalent mapping from [2], [7], and [9].									
2. Reading related research papers from [1], [3], [4], [5], [6], and [8].									
3. Presenting project proposal.									
4. Constructing minimal graphs over certain domains.									
5. Writing the project report and project presenting									

Benefits

A. The benefits for students who implement this project

1. To improve mathematical thinking and proving skill.
2. To improve academic research skill.
3. To review some knowledges in mathematics, especially in differential geometry and complex analysis.
4. To learn some new knowledges in mathematics, especially in differential geometry and complex analysis.
5. To understand properties of minimal graphs and univalent harmonic mappings.

B. The benefits for users of the project

1. To obtain minimal graphs over certain domains.

2. To understand more about properties of minimal graphs and univalent harmonic mappings.

Equipment

A. Hardware

1. Computer
2. Printer
3. Textbooks
4. Stationery

B. Software

1. Microsoft Window 10
2. Microsoft Office 2010
3. Wolfram Mathemaica 9
4. Complex Tool 6.0
5. MinSurfTool 3.0
6. LaTeX

Budget

1. Textbooks	4,500	Baht
2. Stationery	500	Baht

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Author's Profile

Mr. Poom Lertpinyowong
ID 5933539723
Department of Mathematics and Computer Science
Faculty of Science Chulalongkorn University