สูตรรูปแบบปิดสำหรับโมเมนต์แบบมีเงื่อนไขของกระบวนการค็อกซ์-อินเกอร์ซอล-รอสส์แบบทั่วไป

นายพีรพัฒน์ สุทธิมาศ

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CLOSED-FORM FORMULAS FOR CONDITIONAL MOMENTS OF GENERALIZED COX–INGERSOLL–ROSS PROCESSES

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A Dissertation Submitted in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy Program in Applied Mathematics and Computational Science

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กระบวนการค็อกซ์-อินเกอร์ซอล์-รอสส์ เริ่มรู้จักในปี 1985 เป็นกระบวนการแบบ 1 ตัวแปร สำหรับการอธิบายการวิวัฒน์ของอัตราดอกเบี้ยและการกำหนดราคาของอนุพันธ์ทางการเงิน ซึ่งภายหลัง กระบวนการนี้ได้มีการขยายให้ค่าพารามิเตอร์ต่าง ๆ ขึ้นกับเวลา และเรียกว่า กระบวนการค็อกซ์-อินเกอร์ซอล์-รอสส์แบบขยาย ซึ่งได้มีการศึกษากันอย่างกว้างขวางมากขึ้นและนำไปประยุกต์ใช้กับงานวิจัยที่หลากหลาย อีกทั้งได้ศึกษาเกี่ยวกับรูปแบบทั่วไปของกระบวนการค็อกซ์-อินเกอร์ซอล์-รอสส์ เพื่อการประยุกต์ทางด้านการเงินได้มากขึ้น อย่างไรก็ตาม การประยุกต์ได้ต้องใช้ความรู้และสมบัติของค่าคาดหมายและค่าโมเมนต์แบบมีเงื่อนไข ซึ่งส่วนมากยังไม่ได้มีการศึกษาหรือพัฒนาออกมาได้ในรูปแบบที่สมบูรณ์ ในงานนี้ ได้นำเสนอสูตรแบบปิดที่ได้มาจากการใช้การแทนของไฟน์แมน-แคค สำหรับกระบวนการค็อกซ์-อินเกอร์ซอล์-รอสส์แบบทั่วไปสองกระบวนการ ได้แก่ การแพร่ความยืดหยุ่นคงตัวของความแปรปรวนแบบดริฟท์ไม่เชิงเส้น และการแพร่แบบเพียร์สัน ซึ่งเป็นการพัฒนามาจากกระบวนการค็อกซ์-อินเกอร์ซอล์-รอสส์แบบขยาย รวมถึงมีการศึกษาสมบัติอื่น ๆ เช่น ความแปรปรวนและโมเมนต์ผสมแบบมีเงื่อนไขเพิ่มเติมด้วย นอกจากนี้ ยังได้ทำการขยายผลการศึกษาค้นคว้าเกี่ยวกับกระบวนการค็อกซ์-อินเกอร์ซอล์-รอสส์แบบขยาย เพื่อนำไปประยุกต์ใช้กับการแลกเปลี่ยนอัตราดอกเบี้ยอีกด้วย สูตรรูปแบบปิดต่าง ๆ ที่ได้พัฒนามีการตรวจสอบและยืนยันความถูกต้องโดยการทดสอบด้วยวิธีการเชิงตัวเลขโดยใช้การจำลองของมอนติคาร์โล

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Cox–Ingersoll–Ross (CIR) process, introduced in 1985, is a one factor model used to describe the evolution of interest rate and pricing the financial derivatives. It was later extended to have time-dependent parameters called the extended CIR (ECIR) process, which is more widely studied and used in a variety of applications. The generalized versions of CIR process are also studied and investigated for more applications in finance. However, most of these applications rely on the knowledge and properties of conditional expectations and moments, which most of them are not yet fully developed into closed form. In this work, we propose closed-form formulas derived from applying the Feynman–Kac representation for the two generalized CIR processes: the nonlinear drift constant elasticity of variance and the Pearson diffusions which are developed from the ECIR process, as well as further study of their properties such as variance and conditional mixed moments. In addition, we also extending the result of ECIR process by applying it for valuation of interest rate swaps. The formulas derived in this work are numerically verified and validated based on the Monte–Carlo simulations.

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CHAPTER I

INTRODUCTION

1.1 Motivation and literature surveys

Cox–Ingersoll–Ross (CIR) process [16] is one of the most popular processes in financial mathematics particularly to describe the evolution of interest rates and price the derivatives. Actually, all parameters of this process are constants. Thus, this CIR process has been extended to the time-dependent parameters called the extended CIR (ECIR) process [39]. The ECIR process is one of the most widely used processes in financial mathematics, which was first considered in 1990 by Hull and White [39] to generalize models constructed by Vasicek in 1977, see [76]. The ECIR process is usually applied to price financial derivatives, such as zero-coupon bond, ex-coupon, interest rate swaps (IRSs) and options, which often involves evaluation of conditional expectations, see e.g. [5, 35, 56]. Moreover, the process is a continuous-time Markov process that posses some useful properties including mean reversion and analytical formulas for its expectation and variance, in which there are a number of methods readily available for the calibration of the ECIR process parameters, see more details in [80]. Thus, mathematical properties of the ECIR process are challenging topics for observing and applying in financial applications.

In the context of the ECIR process, r_t, the drift factor of the ECIR process is identical to that of the extended Vasicek process [39]. The only difference between the ECIR and the extended Vasicek process is the diffusion term which prevents r_t from being negative [44]. In contrast to the extended Vasicek process, the ECIR process has r_t approaching zero as well as the diffusion term. This characteristic of the ECIR process makes r_t to have non-negative value, which is a main reason that the process becomes famous for the study of the behavior of interest rates.

However, there are some evidences to support that the nonlinearity in the drift factor may be suitable for studying of financial derivatives described by dynamics of interest rates, for details see [11, 41, 53]. In particular, Chan et al. [10] provided the empirical analysis that gives a number of interesting results between a financial derivative and the drift and diffusion terms of the process. Some extensions of the CIR process are required.

The generalized CIR process is one of the most widely used processes in financial mathematics particularly to describe the evolution of interest rates and price their derivatives which can be separated into many types of process. In literature, the generalized CIR processes are considered in two important names: the nonlinear drift constant elasticity of variance (NLD-CEV) diffusion and the Pearson diffusion processes. They are widely utilized to describe various real-world applications, e.g., in derivative pricing of interest rate swaps [56] and variance swaps [9, 29, 67].

The moments and conditional moments play significant roles in many real-world applications and are especially beneficial for estimating parameters, pricing financial derivatives, etc. In fact, these moments can be directly calculated by applying the transition probability density function (PDF), which is often unknown or unavailable in closed form. The formulas for the conditional moments of the stochastic differential equation (SDE) may be unavailable in closed form as well. Based on solving the partial differential equation (PDE) corresponding to the Feynman–Kac representation, the closed-form formulas for the moments and conditional moments are obtained. The approach does not require any knowledge of eigenfunctions or the transition PDF.

The main objective of this dissertation is to obtain closed-form formulas of conditional moments of both generalized CIR processes, namely, the NLD-CEV diffusion and the Pearson diffusion. Basic knowledge and details of these processes are given as the followings.

1.1.1 The NLD-CEV processes

An extended case of the constant elasticity of variance (CEV) process was first proposed by Marsh and Rosenfeld [58] which becomes the CIR process when $\beta = 1$. In this study, we extend their CEV process by replacing the constant parameters with time dependent functions. We subsequently call this process the NLD-CEV process which is of the form

$$dR_t = \kappa(t)\left(\theta(t)R_t^{-(1-\beta)} - R_t\right)dt + \sigma(t)R_t^{\beta/2}dW_t, \qquad \beta \in [0,2) \cup (2,\infty), \tag{1.1}$$

with a positive initial value R_{t_0} and satisfies some sufficient conditions that make $R_t > 0$ for all $t \in [t_0, T]$, where $\theta(t)$, $\kappa(t)$ and $\sigma(t)$ are time dependent functions and W_t is the standard Brownian motion. Comparing with (1.1), the diffusion factor $\sigma(t)R_t^{\beta/2}$ is identical to that of the standard CEV process studied by Cox [16] and Black [7]. While, the difference between (1.1) and the standard CEV process is the nonlinear drift factor $\kappa(t)\left(\theta(t)R_t^{-(1-\beta)} - R_t\right)$. Moreover, when considered as mean reversion processes, the NLD-CEV process becomes an ECIR process [39] when $\beta = 1$, becomes a lognormal process studied by Merton [59] as $\beta \to 2$, and behaves like Ornstein–Uhlenbeck (OU) process [76] when $\beta = 0$. In extension, for $\beta \in (2, \infty)$, the NLD-CEV process exhibits the mean-reverting feature, e.g., an inverse Feller (IF) process or 3/2-stochastic volatility model (SVM) when $\beta = 3$. However, a number of empirical evidences are presented to support that the mean-reverting drift is not necessary and the nonlinearity in the drift factor may be suitable for the financial derivative described by the dynamics of interest rates, see [11,41,53]. In particular, Chan et al. [10] provided the empirical analysis that gives a number of interesting results between a financial derivative and its characteristic parameters β of the model. Moreover, they also concluded that $\beta > 2$ is more suitable to capture the dynamics of the short-term rate better than those of $\beta < 2$.

In this work, we provide two novel forms of the process (1.1) on two different cases with respect to β. First, when $0 \leq \beta < 2$ with $\beta = \frac{2\alpha-1}{\alpha}$, (1.1) can be written as

$$dR_t = \kappa(t)\left(\theta(t)R_t^{\frac{\alpha-1}{\alpha}} - R_t\right)dt + \sigma(t)R_t^{\frac{2\alpha-1}{2\alpha}}dW_t, \qquad \alpha \geq \tfrac{1}{2}. \tag{1.2}$$

Second, when $\beta > 2$ with $\beta = \frac{2\alpha+1}{\alpha}$, (1.1) can be written as

$$dR_t = \kappa(t)\left(\theta(t)R_t^{\frac{\alpha+1}{\alpha}} - R_t\right)dt + \sigma(t)R_t^{\frac{2\alpha+1}{2\alpha}}dW_t, \qquad \alpha > 0. \tag{1.3}$$

Note that, from (1.1), (1.2) and (1.3), $\alpha \to \infty$ if and only if $\beta \to 2$.

Under the probability measure $\mathcal{P}$ and σ-field $\mathcal{F}_t$, in this work we propose closed-from formulas for conditional moments based on the NLD-CEV process,

$$\mathbf{E}\left[R_T^{\frac{\gamma}{\alpha}} \mid \mathcal{F}_t\right] = \mathbf{E}\left[R_T^{\frac{\gamma}{\alpha}} \mid R_t = R\right], \qquad 0 \leq t \leq T, \tag{1.4}$$

where $R > 0$ for any order $\gamma \in \mathbb{R}$ of $\alpha \geq \frac{1}{2}$ for (1.2) and $\alpha > 0$ for (1.3).

The formulas would be advantageous for market practitioners who require closed-form formulas for pricing a derivative in which the NLD-CEV model is adopted to describe the dynamics of volatility or interest rates. For instance, in 1999, Ahn and Gao [1] derived the conditional ν-moments of the process (1.1) in case 3/2-SVM (also known as the IF process, when $\beta = 3$ which corresponds to the case of $\alpha = 1$ in (1.3)) for studying the distribution of their model with term-structure data. Their obtained formula involves both the Kummer's and Gamma functions in the integral forms which cannot be exactly evaluated and the formula is unavailable in a close form. In 2003, Zhou [81] needed the conditional moments vector of the process (1.1) in the case of $\beta \in [0, 2)$, which is equivalent to our constructed process (1.2), to estimate parameters by using the generalized method of moments (GMM). Because the needed vector has no closed-form, Zhou merely approximated the first and second moments through a diffusion process by exploiting the Itô's lemma. In 2011, in order to make the Heston hybrid model affine, Lech and Oosterlee [36] used the first-order Taylor expansion to approximate the conditional $\frac{1}{2}$-moment for the CIR process or the process (1.2) with $\alpha = 1$ and constant parameter functions $\kappa(t)$, $\theta(t)$ and $\sigma(t)$. In 2014, Rujivan and Zhu [67] needed to calculate the first and second conditional moments in order to obtain a closed-form solution for a pricing discretely-sampled variance swap based on ECIR process in the Heston model. Moreover, pricing and hedging by using variance and higher-order moment swaps are challenging problems. The skewness and kurtosis swaps are nowadays traded in practice. These two special types of moment swaps have been widely studied and more useful. Recently, Chumpong et al. [14] proposed analytical formulas for pricing discretely-sampled skewness and kurtosis swaps for comodities.

The study of the NLD-CEV process in this research is presented in Chapter 2, which is published in *Applied Mathematics and Computation*, see [73].

1.1.2 The Pearson diffusion processes

A class of Pearson diffusions is defined by linear drift as the CIR process, but having a quadratic squared diffusion coefficient which extends the CIR process in this respect. It satisfies an SDE in the form

$$dX_t = \theta\,(\mu - X_t)\,dt + \sqrt{2\theta\,(aX_t^2 + bX_t + c)}\,dW_t, \tag{1.5}$$

when X_t is in the state space, where $\theta > 0$ and a, b, c are real constants such that the quadratic squared diffusion term in (1.5) is well-defined. The well-known term W_t is a Wiener process. The parameters contained in (1.5) are often referred as follows: the parameter θ corresponds to the speed of adjustment to the mean of the invariant distribution, and the parameters a, b, c determine the state space of the diffusion as well as the shape of the invariant distribution. Well-known instances for the class of stationary distributions reduced from the full Pearson diffusion processes are OU, CIR, Jacobi processes which have been well investigated and applied in practice. In contrast, the processes with heavy-tailed distributions such as Fisher–Snedecor, reciprocal gamma and Student processes are poor studied and hard to work with. In particular, using the criteria based on the characteristic property of the polynomial $d(x)$ yields the six cases as follows.

1. OU diffusion: $\deg(d) = 0,$
2. CIR diffusion: $\deg(d) = 1,$
3. Jacobi diffusion: $\deg(d) = 2, \quad \Delta(d) > 0, \quad a < 0,$
4. Fisher–Snedecor diffusion: $\deg(d) = 2, \quad \Delta(d) > 0, \quad a > 0,$
5. Reciprocal gamma diffusion: $\deg(d) = 2, \quad \Delta(d) = 0, \quad a > 0,$
6. Student diffusion: $\deg(d) = 2, \quad \Delta(d) < 0, \quad a > 0,$

where $d(x) = a\theta x^2 + b\theta x + c\theta$ and the discriminant $\Delta(d) := b^2 - 4ac$.

Even though the Pearson diffusion processes (1.5) have very general forms consisting of the OU diffusion, CIR diffusion (also known as squared diffusion), Jacobi diffusion, Fisher-Snedecor diffusion, reciprocal gamma diffusion and Student diffusion processes, they have constant parameters which are not suitable for describing time-varying data. A number of strong empirical evidences which have been found that an extreme movement in finance-based practices tends to be followed by time, see [37, 39, 55]. Therefore, the dynamics of the diffusion processes might be governed by time-varying parameter functions as follows,

$$dX_t = \theta(t)\left(\mu(t) - X_t\right)dt + \sqrt{2\theta(t)\left(a(t)X_t^2 + b(t)X_t + c(t)\right)}\,dW_t, \tag{1.6}$$

where $0 \le t \le T$ and parameters θ, μ, a, b, c are changed to be time-dependent functions. Well-known instances deduced by the process (1.6) are the extended Ornstein–Uhlenbeck (EOU) and the ECIR processes, see Hull and White [39].

In 2003, the explicit formulas for conditional polynomial moments of a subclass of the Pearson diffusions are first applied for GMM estimation by Zhou [81]. Since the needed vector has no closed-form expression, Zhou merely approximated the first and second moments through the diffusions by utilizing the Itô's lemma. However, in 2005, the statistical inference of the most of the Pearson diffusions was investigated by Bibby et al. [6], who derived a closed-form formula of the conditional first moment and the correlation for the diffusion-type processes with the given marginal distribution. It should be noted that the closed-form formula given by Bibby is only for the first moment and only on Student diffusion. Note that most of the conditional moments based on the Pearson diffusion processes were studied in 2008 by Forman et al. [32] under some sufficient conditions that the conditional moments (1.7) hold, see Kessler [43]. Those moments were presented as recurrence relations involving eigenvalues and eigenfunctions. Actually, the eigenvalues and eigenfunctions are solved from Fokker–Planck equation which is complicated for obtaining the conditional moments. Thus, one may say that any closed-form formula of the moments for Pearson diffusion processes has not yet been satisfactorily achieved.

In this work, under the probability measure $\mathcal{P}$ and σ-field $\mathcal{F}_t$, we propose the integral form formula for conditional moments of the extended Pearson diffusion (1.6) in the form

$$\mathbf{E}\left[X_T^{\gamma} \mid \mathcal{F}_t\right] = \mathbf{E}\left[X_T^{\gamma} \mid X_t = X\right], \qquad 0 \le t \le T, \tag{1.7}$$

for real order γ and $X > 0$. In this work, the closed-form formulas for conditional moments of the extended Pearson diffusion processes (1.5) are provided. Furthermore, some properties for each of the classification are given in the concise forms, e.g., conditional variance, central moment, covariance and correlation. The study of this part of the research is described in Chapter 3, which is published in *Communications in Nonlinear Science and Numerical Simulation*, see [71].

1.1.3 Interest rate swap pricing by using ECIR process

An ECIR process [39] is one of the most widely used processes in financial mathematics. If the parameters are constant, then the process becomes the well-known CIR process [18]. The ECIR process is usually applied to price financial derivatives, such as IRSs.

A swap is a derivative contract for two parties involving the exchange of a series of cash flows. In this section, we consider a fixed-to-floating IRS where a buyer agrees to pay a floating interest rate on a predetermined principle, called a notional principle P, in order to receive a fixed one from a seller over a specified period of time $[t, T]$; see more details in [56]. The IRSs are the most traded swaps at present and have many potential uses in practice, such as in hedging, portfolio management, and speculation.

In this part of dissertation, we provide an extension of the recent results given by Sutthimat et al. [72] and propose an analytical formula for a conditional expectation of a path-dependent product of polynomial and exponential functions described in the form

$$\mathbf{E}^{\mathcal{P}}\left[\left(\sum_{j=0}^{n}\lambda_j^{(l)} r_{t_l}^{j}\right) e^{\sum_{k=1}^{m}\alpha_k^{(l)} r_{t_k}} \mid r_{t_1}=r\right], \tag{1.8}$$

where $n, m \in \mathbb{N}$, $l = 1, 2, 3, \ldots, m$, $0 \leq t_1 < t_2 < t_3 < \cdots < t_m = T < \infty$, $\lambda_j^{(l)}, \alpha_k^{(l)} \in \mathbb{R}$ and $\{r_t\}_{0 \leq t \leq T}$ is assumed to follow the ECIR process. We consider primarily on the case that the exponential term depends on the values $r_{t_1}, r_{t_2}, r_{t_3}, \ldots, r_{t_m}$ at a fixed times $t_1, t_2, t_3, \ldots, t_m$, as described in (1.8), for valuation of IRS described above or some other financial products having similar behaviour as IRS. The study of this work is described in Chapter 4 which is published in *Research in the Mathematical Sciences*, see [74].

1.2 Research objective and dissertation overview

The aims of this dissertation are the followings: (i) to derive the closed-form formulas and their properties for conditional moments of the generalized NLD-CEV and inhomogeneous Pearson diffusion processes, (ii) to extend the result of ECIR process for application of IRS.

In the dissertation, we collect all three research articles related to the topic of the dissertation as follows: (i) the result of the NLD-CEV process is presented in Chapter 2, (ii) the result of the Pearson diffusion process is presented in Chapter 3, and (iii) the extension result of ECIR process with application for IRS is given in Chapter 4. Moreover, the conclusion of the dissertation is provided in Chapter 5.

CHAPTER II

CLOSED-FORM FORMULA FOR CONDITIONAL MOMENTS OF GENERALIZED NONLINEAR DRIFT CEV PROCESS

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Abstract

This paper studied a generalized case of the constant elasticity of variance diffusion (CEV) process whereas the drift term is substantially nonlinear in the short rate. Well-known instances deduced by this process are the extended Cox–Ingersoll–Ross (ECIR) process and the extended inverse Feller (EIF) process or 3/2-stochastic volatility model (SVM). We found particular sufficient conditions of existence and uniqueness of a positive pathwise strong solution for time-dependent parameter functions, and obtained closed-form formulas for conditional moments based on Feynman–Kac theorem. The accuracy and validity of the formulas were further investigated based on Monte Carlo simulations.

Keywords: nonlinear drift CEV process, ECIR process, 3/2-SVM, conditional moment, closed-form formula

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2.1 Introduction

The constant elasticity of variance diffusion (CEV) process introduced in 1975 by Cox [16] is considered as an extension of the Ornstein–Uhlenbeck (OU) process [76] for applications in finance, which are subsequently studied, extended and generalized for applications in many areas. Recently, Araneda et al. [3] provided a study of sub–fractional CEV model and Cao et al. [9] priced variance swaps under hybrid CEV and stochastic volatility. The first generalized case of Cox's CEV process was proposed by Marsh and Rosenfeld [58] by considering time dependent parameters with nonlinear drift term. We subsequently refer to this process as the nonlinear drift CEV (NLD-CEV) process, which is written in the form of

$$dR_t = \kappa(t)\left(\theta(t)R_t^{-(1-\beta)} - R_t\right)dt + \sigma(t)R_t^{\beta/2}dW_t, \quad \beta \in [0,2) \cup (2,\infty), \tag{2.1}$$

where $\theta(t)$, $\kappa(t)$, and $\sigma(t)$ are time dependent parameters for $t \in [t_0, T]$ with an initial value $R_{t_0} > 0$. Note that the diffusion factor $\sigma(t)R_t^{\beta/2}$ in (2.1) is identical to those of the standard Cox's CEV process [16] and Black [7], while the difference is the nonlinear drift factor $\kappa(t)(\theta(t)R_t^{-(1-\beta)} - R_t)$. By considering different values of the parameter β, one can see the followings: for $\beta = 1$, the NLD-CEV process becomes an extended Cox–Ingersoll–Ross (ECIR) process [39] as a mean reversion processes; for $\beta = 0$, it behaves like OU process [76]; and for $\beta \to 2$, it is a lognormal process studied by Merton [59]. In extension for $\beta \in (0,\infty)$, the NLD-CEV process also exhibits the mean-reverting feature, e.g., when $\beta = 3$ it becomes an inverse Feller (IF) process (or 3/2-stochastic volatility model, SVM).

There are some evidences to support that the nonlinearity in the drift factor may be suitable for studying of financial derivatives described by dynamics of interest rates, for details see [11, 41, 53]. In particular, Chan et al. [10] provided the empirical analysis that gives a number of interesting results between a financial derivative and its characteristic parameter β of the model, and concluded that $\beta > 2$ is more suitable to capture the dynamics of the short-term rate than those of $\beta < 2$.

In this study, we consider two novel forms of the process (2.1) in two different cases according to β. For the first case for $0 \leq \beta < 2$, we set $\beta = \frac{2\alpha-1}{\alpha}$, (2.1) can be written as

$$dR_t = \kappa(t) \left(\theta(t) R_t^{\frac{\alpha-1}{\alpha}} - R_t\right) dt + \sigma(t) R_t^{\frac{2\alpha-1}{2\alpha}} dW_t, \quad \alpha \geq \tfrac{1}{2}. \tag{2.2}$$

For the second case for $\beta > 2$, we set $\beta = \frac{2\alpha+1}{\alpha}$, (2.1) can be written as

$$dR_t = \kappa(t) \left(\theta(t) R_t^{\frac{\alpha+1}{\alpha}} - R_t\right) dt + \sigma(t) R_t^{\frac{2\alpha+1}{2\alpha}} dW_t, \quad \alpha > 0. \tag{2.3}$$

Note from (2.1) that $\beta \to 2$ is equivalent to $\alpha \to \infty$ for both (2.2) and (2.3).

In this work we propose closed-form formulas for conditional moments based on the NLD-CEV process under the probability measure $\mathcal{P}$ and σ-field $\mathcal{F}_t$ in the form of

$$\mathbf{E}\left[R_T^{\frac{\gamma}{\alpha}} \mid \mathcal{F}_t\right] = \mathbf{E}\left[R_T^{\frac{\gamma}{\alpha}} \mid R_t = R\right], \quad 0 \leq t \leq T, \tag{2.4}$$

for order $\gamma \in \mathbb{R}$ and $R > 0$. The obtained analytical formulas would benefit market practitioners who require formulas for pricing financial derivatives in which the NLD-CEV model is adopted to describe the dynamics of volatility or interest rates. For instance, in 1999 Ahn and Gao [1] derived the conditional ν^{th} moments of the process 3/2-SVM (corresponding to (2.1) with $\beta = 3$ or (2.3) with $\alpha = 1$) to study the distribution of their model with term-structure data; this process is also known as the inverse Feller (IF). In their work, the formula involves the integral form of the Kummer's and Gamma functions which is not available in closed-form. In 2003, Zhou [81] required the conditional moments of the process (2.1) in the case of $\beta \in [0, 2)$ to estimate parameters using the generalized method of moments (GMM). Because the required moment has no closed-form, Zhou merely approximated the first and second moments through a diffusion process by exploiting Itô's lemma. In order to make the Heston hybrid model affine, in 2011 Lech and Oosterlee [36] applied the first-order Taylor expansion to approximate the conditional 1/2-moment for the CIR process (corresponding to process (2.2) when $\alpha = 1$ with constant parameters). In 2014, Rujivan and Zhu [67] calculated the first and second conditional moments to obtain a closed-form value of discretely-sampled variance swap based on ECIR process in the Heston model. Pricing and hedging by using variance and higher-order moment swaps are also challenging, the skewness and kurtosis swaps have been widely studied, e.g., Chumpong et al. [15] proposed analytical formulas for pricing discretely-sampled skewness and kurtosis swaps for commodities.

The rest of paper is organized as follows. In Section 2.2, we provide the sufficient conditions required for the study of NLD-CEV process. In Section 2.3, the formulas for conditional moments are technically derived based on the solution of PDE for the NLD-CEV process, and some special cases of the formulas are observed according to parameters. In this section,we also provide the closed-form formulas for unconditional moments for the case of constant parameters. The analysis of convergences is also mentioned here. Experimental validation of the accuracy and efficiency of the proposed formulas are presented in Section 2.4 based on Monte Carlo simulations, and the case that formulas having infinite series forms are also discussed some examples. Section 2.5 provides the discussion and conclusion of the study.

2.2 Conditions for existence and uniqueness

An analysis of the distribution of the process R_t in (2.1) has been also studied by Marsh and Rosenfeld [58]. Their idea is utilizing the Itô's lemma together with the transformation $V_t = R_t^{2-\beta}$ in the process (2.1) to yield the following ECIR process

$$dV_t = A(t)\left(B(t) - V_t\right)dt + C(t)\sqrt{V_t}dW_t, \tag{2.5}$$

where $A(t) = (2-\beta)\kappa(t)$, $B(t) = \theta(t) + \frac{(1-\beta)\sigma^2(t)}{2\kappa(t)}$ and $C(t) = (2-\beta)\sigma(t)$. In this paper, we apply their idea combining with the parameters $\beta = \frac{2\alpha-1}{\alpha}$ and $\beta = \frac{2\alpha+1}{\alpha}$ in the process (2.5), which is corresponding to the processes (2.2) and (2.3), respectively.

In order to perform with the process V_t in (2.5) for some cases of β, we provide some sufficient assumptions. In 1987, Rogers and Williams [63] proposed the theoretical study of the ECIR process (2.5) using the Yamada–Watanabe theorem to guarantee the existence of a unique pathwise strong solution for ECIR process (2.5) under Assumption 2.1.

Assumption 2.1. *The functions $A(t), B(t)$ and $C(t)$ in ECIR process* (2.5) *are strictly positive and smooth functions depending on the temporal variable $t \in [0, T]$. Moreover, $A(t)/C^2(t)$ is locally bounded on $[0, T]$.*

Assumption 2.2. *For the process* (2.5)*, the following inequality holds*

$$\lim_{V\to 0}\left(\mu(V,t) - \frac{1}{2}\frac{\partial\delta^2}{\partial V}(V,t)\right) \geq 0,$$

where $\mu(V,t) = A(t)\left(B(t) - V\right)$ and $\delta(V,t) = C(t)\sqrt{V}$.

Recently, Ekström et al. [24] mentioned that if the Assumption 2.2 holds for ECIR process (2.5), the boundary at the zero is non-attainable for the process V_t. In the other word, $\mathbb{P}(V_t > 0, \text{ for all } t > 0) = 1$ for each the initial value $V > 0$. However, in this study, the analysis of the boundary conditions is focused on two different cases; $0 \leq \beta < 2$ and $\beta > 2$.

2.2.1 Sufficient conditions for $0 \leq \beta < 2$

It is not difficult to observe that the Assumptions 2.3 and 2.4 given below significantly imply the Assumptions 2.1 and 2.2, respectively.

Assumption 2.3. *The functions $\kappa(t), \theta(t)$ and $\sigma(t)$ in the NLD-CEV process* (2.1) *are strictly positive and smooth functions depending on the temporal variable $t \in [0, T]$. Moreover, $\kappa(t)/\sigma^2(t)$ is locally bounded on $[0, T]$.*

Assumption 2.4. *The process R_t in* (2.1) *holds the inequality, $2\kappa(t)\theta(t) \geq \sigma(t)^2$.*

Technically speaking, for the Assumption 2.3, if the parameters $\kappa(t), \theta(t)$ and $\sigma(t)$ in (2.1) hold, the parameters $A(t), B(t)$ and $C(t)$ in (2.5) also hold for the Assumption 2.1 from the definitions of A, B, C in (2.5). Next, since $2 - \beta > 0$, the process R_t in (2.1) holds with the Assumption 2.4 for the functions $\kappa(t)$, $\theta(t)$ and $\sigma(t)$, we have that

$$2A(t)B(t) = (2-\beta)\left(2\kappa(t)\theta(t) + (1-\beta)\sigma^2(t)\right) \geq (2-\beta)\left(\sigma^2(t) + (1-\beta)\sigma^2(t)\right) = C^2(t).$$

It is easy to see that $\frac{\partial \delta^2}{\partial V}(V, t) = C^2(t)$ as $V \to 0$. This makes that the parameters $A(t), B(t)$ and $C(t)$ in (2.5) hold for the Assumption 2.2. Afterward, in 2015, Alfonsi [2] further presented Assumption 2.2 which is sufficient to guarantee that the process V_t in (2.5) avoids zero almost surely with respect to a probability measure $\mathcal{P}$ for all $t \in [0, T]$. From the transformation $V_t = R_t^{2-\beta}$, the held properties in the process V_t also imply directly to the process R_t in (2.1). Therefore, the process R_t has a pathwise unique strong solution that almost surely avoids zero regarding to a probability measure $\mathcal{P}$ for all $t \in [0, T]$.

2.2.2 Sufficient conditions for $\beta > 2$

In the case of the process (2.3), under Assumption 2.5 given below, it is easy to see that all parameter functions $A(t)$, $B(t)$, $C(t)$ in the process (2.5) satisfy Assumption 2.1. Next, considering Assumption 2.2 yields $2\kappa(t)\theta(t) \leq \sigma^2(t)$ for all $t \in [0, T]$. Since $2 - \beta < 0$, we also have

$$2A(t)B(t) = (2-\beta)\left(2\kappa(t)\theta(t) + (1-\beta)\sigma^2(t)\right) \geq (2-\beta)\left(\sigma^2(t) + (1-\beta)\sigma^2(t)\right) = C^2(t),$$

for all $t \in [0, T]$. Thus, Assumption 2.5 implies that all parameters in ECIR process (2.5) correspond to Assumptions 2.1 and 2.2.

Assumption 2.5. *The functions* $-\kappa(t), \theta(t)$ *and* $-\sigma(t)$ *in the NLD-CEV process* (2.1) *are strictly positive and smooth functions depending on the temporal variable* $t \in [0, T]$*. Moreover,* $\kappa(t)/\sigma^2(t)$ *is locally bounded on* $[0, T]$.

2.3 Main results

In this section, we derive the closed-form formulas of the conditional moments of the process (2.1) according to the two cases depending on the range of β, and provide some investigation of the results. Firstly, the formulas are derived as integral-forms for the processes (2.2) and (2.3), then applied to the cases of constant parameters to get closed-forms. The key idea relies on the Feynman–Kac formula by expressing solution of the corresponding partial differential equation (PDE) as an infinite series and solving the coefficients to receive a closed-form. The motivation of the expressing form for the conditional moments is similar to that presented in [13, 32, 61, 71, 74].

2.3.1 The conditional $\frac{\gamma}{\alpha}$-moments of the process when $0 \le \beta < 2$

The strategy for constructing a closed-form formula of all conditional moments in the process (2.2) is provided. The concept of Feynman–Kac formula is applied to solve the explicit formula of (2.4) which is assumed in term of the infinitely sum (2.6) as shown in the following theorem.

Theorem 2.1. *Suppose* R_t *follows the NLD-CEV process* (2.2) *and the Assumptions 2.3 and 2.4 are assumed for all* $0 < t \le T$*. The conditional* $\frac{\gamma}{\alpha}$*-moment for* $\gamma \in \mathbb{R}$*,* $R > 0$ *and* $\tau = T - t \ge 0$ *is defined by*

$$U_\alpha^{\langle\gamma\rangle}(R, \tau) := E\left[R_T^{\frac{\gamma}{\alpha}} \mid R_t = R\right] = \sum_{k=0}^{\infty} A_\alpha^{\langle k\rangle}(\tau) R^{\frac{\gamma-k}{\alpha}}, \tag{2.6}$$

for all $(R, \tau) \in D_\alpha^{\langle\gamma\rangle} \subset (0, \infty) \times [0, \infty)$ *which the infinite series in* (2.6) *uniformly converges on* $D_\alpha^{\langle\gamma\rangle}$*. Then, the coefficients in* (2.6) *can be expressed by*

$$\begin{cases} A_\alpha^{\langle 0\rangle}(\tau) = e^{-\frac{\gamma}{\alpha}\int_0^\tau \kappa(T-\xi)d\xi}, \\ A_\alpha^{\langle k\rangle}(\tau) = \displaystyle\int_0^\tau e^{-\frac{\gamma-k}{\alpha}\int_\eta^\tau \kappa(T-\xi)d\xi} B_\alpha^{\langle k-1\rangle}(T-\eta) A_\alpha^{\langle k-1\rangle}(\eta) d\eta, \end{cases} \tag{2.7}$$

for $k \in \mathbb{Z}^+$, where

$$B_\alpha^{\langle j\rangle}(\tau) = \left(\frac{\gamma - j}{\alpha}\right)\left(\frac{1}{2}\left(\frac{\gamma - j}{\alpha} - 1\right)\sigma^2(\tau) + \kappa(\tau)\theta(\tau)\right). \tag{2.8}$$

Proof. First, the Feynman–Kac formula is applied to solve $U_\alpha^{\langle\gamma\rangle}(R,\tau) := U$ in (2.6) which satisfies the following PDE

$$U_\tau - \frac{1}{2}\sigma^2(T-\tau)R^{\frac{2\alpha-1}{\alpha}}U_{RR} - \kappa(T-\tau)\left(\theta(T-\tau)R^{\frac{\alpha-1}{\alpha}} - R\right)U_R = 0 \tag{2.9}$$

for all $R > 0$ and $0 < \tau \leq T$, subject to the initial condition

$$U_\alpha^{\langle\gamma\rangle}(R,0) = E\left[R_T^{\frac{\gamma}{\alpha}} \mid R_T = R\right] = R^{\frac{\gamma}{\alpha}}. \tag{2.10}$$

Next, comparing the coefficients between (2.6) and (2.10), we obtain the following conditions $A_\alpha^{\langle 0\rangle}(0) = 1$ and $A_\alpha^{\langle k\rangle}(0) = 0$ for $k \in \mathbb{Z}^+$. After that, we compute (2.9) using (2.6) to find the partial derivatives U_τ, U_{rr} and U_r. Then, we have

$$\begin{aligned} 0 = &\sum_{k=0}^{\infty}\frac{d}{d\tau}A_\alpha^{\langle k\rangle}(\tau)R^{\frac{\gamma-k}{\alpha}} - \frac{1}{2}\sigma(T-\tau)R^{\frac{2\alpha-1}{\alpha}}\sum_{k=0}^{\infty}\left(\left(\frac{\gamma-k}{\alpha}\right)\left(\frac{\gamma-k}{\alpha}-1\right)A_\alpha^{\langle k\rangle}(\tau)R^{\frac{\gamma-k}{\alpha}-2}\right) \\ &- \kappa(T-\tau)\left(\theta(T-\tau)R^{\frac{\alpha-1}{\alpha}} - R\right)\sum_{k=0}^{\infty}\left(\left(\frac{\gamma-k}{\alpha}\right)A_\alpha^{\langle k\rangle}(\tau)R^{\frac{\gamma-k}{\alpha}-1}\right) \end{aligned}$$

or it can be simplified as

$$\begin{aligned} 0 = &\left(\frac{d}{d\tau}A_\alpha^{\langle 0\rangle}(\tau) + \frac{\gamma}{\alpha}\kappa(T-\tau)A_\alpha^{\langle 0\rangle}(\tau)\right)R^{\frac{\gamma}{\alpha}} \\ &+ \sum_{k=1}^{\infty}\left(\frac{d}{d\tau}A_\alpha^{\langle k\rangle}(\tau) + \left(\frac{\gamma-k}{\alpha}\right)\kappa(T-\tau)A_\alpha^{\langle k\rangle}(\tau) - B_\alpha^{\langle k-1\rangle}(T-\tau)A_\alpha^{\langle k-1\rangle}(\tau)\right)R^{\frac{\gamma-k}{\alpha}}. \end{aligned}$$

Under the assumptions of the infinite series in (2.6) over $D_\alpha^{\langle\gamma\rangle}$, the above equation can be solved through the following system of ODEs,

$$\begin{cases} \dfrac{d}{d\tau}A_\alpha^{\langle 0\rangle}(\tau) + \dfrac{\gamma}{\alpha}\kappa(T-\tau)A_\alpha^{\langle 0\rangle}(\tau) = 0, \\ \dfrac{d}{d\tau}A_\alpha^{\langle k\rangle}(\tau) + \left(\dfrac{\gamma-k}{\alpha}\right)\kappa(T-\tau)A_\alpha^{\langle k\rangle}(\tau) - B_h^{\langle k-1\rangle}(T-\tau)A_\alpha^{\langle k-1\rangle}(\tau) = 0, \end{cases} \tag{2.11}$$

with their initial conditions $A_\alpha^{\langle 0\rangle}(0) = 1$ and $A_\alpha^{\langle k\rangle}(0) = 0$ for $k \in \mathbb{Z}^+$. Therefore, the coefficients in (2.6) can be directly obtained by solving the system (2.11) in the form of recursive relation, which gives the results (2.7). □

Next, examining (2.6) in Theorem 2.1 when $\gamma = n \in \mathbb{Z}^+$, the infinitely sum in (2.6) is terminated at the finite order and can be expressed as in the following theorem.

Theorem 2.2. *Suppose* R_t *follows the NLD-CEV process* (2.2) *and* $n \in \mathbb{Z}_0^+$. *Then,*

$$U_\alpha^{\langle n \rangle}(R,\tau) := E\left[R_T^{\frac{n}{\alpha}} \mid R_t = R\right] = \sum_{k=0}^{n} A_\alpha^{\langle k \rangle}(\tau) R^{\frac{n-k}{\alpha}}, \tag{2.12}$$

for all $(R,\tau) \in D_\alpha^{\langle n \rangle} \subset (0,\infty) \times [0,\infty)$ *and* $\tau = T - t \geq 0$ *where the coefficients* $A_\alpha^{\langle k \rangle}$ *in* (2.12) *are defined by* (2.7) *and* (2.8). *Moreover,* $\lim_{R \to 0^+} U_\alpha^{\langle n \rangle}(R,\tau) > 0$.

Proof. By considering (2.8), when $k = n = \gamma$, we obtain $B_\alpha^{\langle n \rangle}(\tau) = 0$ that implies the coefficients $A_\alpha^{\langle k \rangle}(\tau) = 0$ for all integers $k \geq n+1$. Thus, the infinite sum (2.6) can be reduced to the finitely sum (2.12). Moreover, $\lim_{R \to 0^+} U_\alpha^{\langle n \rangle}(R,\tau) = A_\alpha^{\langle 0 \rangle} > 0$. □

The following theorem is another one of the convergent cases of (2.6) that can be observed from Theorem 2.1.

Theorem 2.3. *Suppose* R_t *follows the NLD-CEV process* (2.2) *and*

$$\gamma = \alpha - \frac{2\alpha\kappa(\tau)\theta(\tau)}{\sigma^2(\tau)} + m, \tag{2.13}$$

for some $m \in \mathbb{Z}_0^+$ *and for all* $\tau = T - t \geq 0$. *In the other word,* γ *in* (2.13) *is a constant function. Then,*

$$U_\alpha^{\langle \gamma \rangle}(R,\tau) = \sum_{k=0}^{m} A_\alpha^{\langle k \rangle}(\tau) R^{\frac{\gamma-k}{\alpha}}, \tag{2.14}$$

for all $(R,\tau) \in D_\alpha^{\langle \gamma \rangle} \subset (0,\infty) \times [0,\infty)$, *where* $A_\alpha^{\langle k \rangle}(\tau)$ *is defined in* (2.7).

Proof. The proof is directly shown from (2.8) by inserting γ in (2.13) and getting $B_\alpha^{\langle m \rangle}(\tau) = 0$ for all $\tau \geq 0$. This makes $A_\alpha^{\langle m+1 \rangle}(\tau) = 0$ which implies that $A_\alpha^{\langle k \rangle}(\tau) = 0$ for all $k \geq m+1$. □

Next, we focus to the case that the parameters $\kappa(t)$, $\theta(t)$ and $\sigma(t)$ are constant functions. In this case, all integral functions in above theorem can be exactly integrated as presented in the following theorems.

Theorem 2.4. *Suppose* R_t *follows the NLD-CEV process* (2.2) *such that* $\kappa(t) = \kappa$, $\theta(t) = \theta$ *and* $\sigma(t) = \sigma$ *for* $R > 0$ *and* $\tau = T - t \geq 0$. *Then, the conditional* $\frac{n}{\alpha}$*-moment is given by*

$$U_\alpha^{\langle \gamma \rangle}(R,\tau) := E\left[R_T^{\frac{\gamma}{\alpha}} \mid R_t = R\right] = \sum_{k=0}^{\infty} \frac{e^{-\frac{\gamma\kappa\tau}{\alpha}}}{k!} \left(\frac{\alpha e^{\frac{\kappa\tau}{\alpha}} - \alpha}{\kappa}\right)^k \left(\prod_{j=0}^{k-1} \bar{B}_\alpha^{\langle j \rangle}\right) R^{\frac{\gamma-k}{\alpha}}, \tag{2.15}$$

for all $(R,\tau) \in D_\alpha^{\langle\gamma\rangle} \subset (0,\infty)\times[0,\infty)$ *which the infinite series in* (2.15) *uniformly converges on* $D_\alpha^{\langle\gamma\rangle}$*, where*

$$\bar{B}_\alpha^{\langle j\rangle} = \left(\frac{\gamma - j}{\alpha}\right)\left(\frac{1}{2}\left(\frac{\gamma-j}{\alpha} - 1\right)\sigma^2 + \kappa\theta\right). \tag{2.16}$$

Note that, if the product of $\bar{B}_\alpha^{\langle j\rangle}$ *in* (2.15) *runs on* j *from 0 to* -1*, the product term is defined to be 1.*

Proof. Since each coefficient $A_\alpha^{\langle k\rangle}(\tau)$ is produced from the previous coefficient starting at $A_\alpha^{\langle 0\rangle}(\tau)$, observing the coefficients order-by-order yields

$$A_\alpha^{\langle 0\rangle}(\tau) = e^{-\frac{\gamma\kappa\tau}{\alpha}},$$
$$A_\alpha^{\langle 1\rangle}(\tau) = \bar{B}_\alpha^{\langle 0\rangle}\int_0^\tau e^{-\frac{\gamma-1}{\alpha}(\tau-\eta)\kappa}A_\alpha^{\langle 0\rangle}(\eta)d\eta = \bar{B}_\alpha^{\langle 0\rangle}e^{-\frac{\gamma\kappa\tau}{\alpha}}\left(\frac{\alpha e^{\frac{\kappa\tau}{\alpha}} - \alpha}{\kappa}\right)$$

and for all $k = 2,3,4,\ldots$,

$$\begin{aligned} A_\alpha^{\langle k\rangle}(\tau) &= \bar{B}_\alpha^{\langle k-1\rangle}\int_0^\tau e^{-\frac{\gamma-k}{\alpha}(\tau-\eta)\kappa}A_\alpha^{\langle k-1\rangle}(\eta)d\eta \\ &= \frac{e^{-\frac{(\gamma-k)\kappa\tau}{\alpha}}}{(k-1)!\kappa^{k-1}}\left(\prod_{j=0}^{k-1}\bar{B}_\alpha^{\langle j\rangle}\right)\int_0^\tau e^{-\frac{k\kappa\eta}{\alpha}}\left(\alpha e^{\frac{\kappa\eta}{\alpha}} - \alpha\right)^{k-1}d\eta \\ &= \frac{e^{-\frac{(\gamma-k)\kappa\tau}{\alpha}}}{(k-1)!\kappa^{k-1}}\left(\prod_{j=0}^{k-1}\bar{B}_\alpha^{\langle j\rangle}\right)\frac{e^{-\frac{k\kappa\tau}{\alpha}}\left(\alpha e^{\frac{\kappa\tau}{\alpha}} - \alpha\right)^k}{k\kappa} \\ &= \frac{e^{-\frac{\gamma\kappa\tau}{\alpha}}}{k!}\left(\frac{\alpha e^{\frac{\kappa\tau}{\alpha}} - \alpha}{\kappa}\right)^k\left(\prod_{j=0}^{k-1}\bar{B}_\alpha^{\langle j\rangle}\right). \end{aligned}$$

Under the assumption that (2.15) converges uniformly on $D_\alpha^{\langle\gamma\rangle}$, this proof completes. □

Similarly to the previous theorem, in the case that the parameters are constants, given a positive integer n yields the closed-form formula as the following theorem.

Theorem 2.5. *Suppose* R_t *follows the NLD-CEV process* (2.2) *such that* $\kappa(t) = \kappa$, $\theta(t) = \theta$ *and* $\sigma(t) = \sigma$*. For* $n \in \mathbb{Z}_0^+$*, the conditional* $\frac{n}{\alpha}$*-moment is exactly given by*

$$U_\alpha^{\langle n\rangle}(R,\tau) := E\left[R_T^{\frac{n}{\alpha}} \mid R_t = R\right] = e^{-\frac{n\kappa\tau}{\alpha}}R^{\frac{n}{\alpha}} + \sum_{k=1}^{n}\frac{e^{-\frac{n\kappa\tau}{\alpha}}}{k!}\left(\frac{\alpha e^{\frac{\kappa\tau}{\alpha}} - \alpha}{\kappa}\right)^k\left(\prod_{j=0}^{k-1}\bar{B}_\alpha^{\langle j\rangle}\right)R^{\frac{n-k}{\alpha}}, \tag{2.17}$$

for $R > 0$ *and* $\tau = T - t \geq 0$ *where* $\bar{B}_\alpha^{\langle j\rangle}$ *is defined by* (2.16)*. Also,* $\lim_{R\to 0^+} U_\alpha^{\langle n\rangle}(R,\tau) > 0$.

Proof. The proof is rather trivial by combining Theorem 2.2 with Theorem 2.4. □

In addition, the special case of Theorem 2.3 is also provided.

Theorem 2.6. *Suppose R_t follows the NLD-CEV process* (2.2) *such that $\kappa(t) = \kappa$, $\theta(t) = \theta$, $\sigma(t) = \sigma$, and*

$$\gamma = \alpha - \frac{2\alpha\kappa\theta}{\sigma^2} + m, \tag{2.18}$$

for $m \in \mathbb{Z}_0^+$. Then,

$$U_\alpha^{\langle\gamma\rangle}(R,\tau) = e^{-\frac{\gamma\kappa\tau}{\alpha}} R^{\frac{\gamma}{\alpha}} + \sum_{k=1}^{m} \frac{e^{-\frac{\gamma\kappa\tau}{\alpha}}}{k!} \left(\frac{\alpha e^{\frac{\kappa\tau}{\alpha}} - \alpha}{\kappa}\right)^k \left(\prod_{j=0}^{k-1} \bar{B}_\alpha^{\langle j\rangle}\right) R^{\frac{\gamma-k}{\alpha}}. \tag{2.19}$$

for all $(R,\tau) \in D_\alpha^{\langle\gamma\rangle} \subset (0,\infty) \times [0,\infty)$ and $\tau = T - t \geq 0$. Note that, if the sum term in (2.19) *runs on k from 1 to 0, the sum term is defined to be 0.*

Proof. Obviously obtained from Theorem 2.3. □

Note that a simple closed-form formula of the conditional moments for the class of ECIR process in the case that $\alpha = 1$ of (2.2) agrees with that of formulas presented in [71]. Moreover, for the class of CIR process, in Theorem 2.1 of Dufresne's formula [22] also reduces to our formula (2.17) in Theorem 2.5 after simplification.

2.3.2 The conditional $\frac{\gamma}{\alpha}$-moments of the process when $\beta > 2$

This subsection is constructed in parallel to the previous subsection, and the proofs are similar to those in the previous subsection, which will be omitted depending on suitability. In the case that the process (2.3) is applied, the expression in (2.6) changes to (2.20). The proof is a straightforward analogy to the first theorem and the formula is available to use as shown in the following theorems.

Theorem 2.7. *Suppose R_t follows the NLD-CEV process* (2.3) *and the Assumptions 2.5 is assumed for all $0 < t \leq T$. The conditional $\frac{\gamma}{\alpha}$-moment for $\gamma \in \mathbb{R}$, $R > 0$ and $\tau = T - t \geq 0$ is defined by*

$$V_\alpha^{\langle\gamma\rangle}(R,\tau) := E\left[R_T^{\frac{\gamma}{\alpha}} \mid R_t = R\right] = \sum_{k=0}^{\infty} A_\alpha^{\langle k\rangle}(\tau) R^{\frac{\gamma+k}{\alpha}}, \tag{2.20}$$

for all $(R,\tau) \in D_\alpha^{\langle\gamma\rangle} \subset (0,\infty) \times [0,\infty)$ which the infinite series in (2.20) *uniformly converges on*

$D_\alpha^{\langle\gamma\rangle}$. *Then, the coefficients in* (2.20) *can be expressed by*

$$\begin{cases} A_\alpha^{\langle 0\rangle}(\tau) = e^{-\frac{\gamma}{\alpha}\int_0^\tau \kappa(T-\xi)d\xi}, \\ A_\alpha^{\langle k\rangle}(\tau) = \displaystyle\int_0^\tau e^{-\frac{\gamma+k}{\alpha}\int_\eta^\tau \kappa(T-\xi)d\xi} B_\alpha^{\langle k-1\rangle}(T-\eta) A_\alpha^{\langle k-1\rangle}(\eta)d\eta, \end{cases} \tag{2.21}$$

for $k \in \mathbb{Z}^+$, *where*

$$B_\alpha^{\langle j\rangle}(\tau) = \left(\frac{\gamma+j}{\alpha}\right)\left(\frac{1}{2}\left(\frac{\gamma+j}{\alpha}-1\right)\sigma^2(\tau) + \kappa(\tau)\theta(\tau)\right). \tag{2.22}$$

Proof. The proof is similarly to Theorem 2.1 and omitted. □

For n is a negative integer, the above theorem can be reduced to the finite sum as shown in the following theorem.

Theorem 2.8. *Suppose* R_t *follows the NLD-CEV process* (2.3) *and* $n \in \mathbb{Z}_0^-$. *Then,*

$$V_\alpha^{\langle n\rangle}(R,\tau) := E\left[R_T^{\frac{n}{\alpha}} \mid R_t = R\right] = \sum_{k=0}^{|n|} A_\alpha^{\langle k\rangle}(\tau) R^{\frac{n+k}{\alpha}}, \tag{2.23}$$

for all $(R,\tau) \in D_\alpha^{\langle n\rangle} \subset (0,\infty)\times[0,\infty)$ *and* $\tau = T-t \geq 0$ *where the coefficients* $A_\alpha^{\langle k\rangle}$ *in* (2.23) *are defined by* (2.21) *and* (2.22). *Moreover,* $\lim_{R\to 0^+} V_\alpha^{\langle n\rangle}(R,\tau) > 0$.

Proof. By considering (2.22), when $k = -n = -\gamma$, we obtain $B_\alpha^{\langle -n\rangle}(\tau) = 0$ that implies the coefficients $A_\alpha^{\langle k\rangle}(\tau) = 0$ for all integers $k \geq 1-n$. Thus, the infinitely sum (2.6) can be reduced to the finitely sum (2.12). □

The next following result looks exactly the same state as Theorem 2.3.

Theorem 2.9. *Suppose* R_t *follows the NLD-CEV process* (2.3) *and*

$$\gamma = \alpha - \frac{2\alpha\kappa(\tau)\theta(\tau)}{\sigma^2(\tau)} - m, \tag{2.24}$$

for $m \in \mathbb{Z}_0^+$ *and* $\tau = T-t \geq 0$. *In the other word,* γ *in* (2.24) *is a constant function. Then,*

$$V_\alpha^{\langle\gamma\rangle}(R,\tau) = \sum_{k=0}^{m} A_\alpha^{\langle k\rangle}(\tau) R^{\frac{\gamma+k}{\alpha}}, \tag{2.25}$$

for all $(R,\tau) \in D_\alpha^{\langle\gamma\rangle} \subset (0,\infty)\times[0,\infty)$, *where* $A_\alpha^{\langle k\rangle}(\tau)$ *is defined in* (2.21).

Proof. The proof is similarly to Theorem 2.3 and omitted. □

The specific issue that the parameters $\kappa(t) = \kappa$, $\theta(t) = \theta$ and $\sigma(t) = \sigma$ are constant functions is also given in the following theorems.

Theorem 2.10. *Suppose R_t follows the NLD-CEV process* (2.3) *such that $\kappa(t) = \kappa$, $\theta(t) = \theta$ and $\sigma(t) = \sigma$ for $R > 0$ and $\tau = T - t \geq 0$. Then, the conditional $\frac{n}{\alpha}$-moment is given by*

$$V_\alpha^{\langle\gamma\rangle}(R,\tau) := E\left[R_T^{\frac{\gamma}{\alpha}} \mid R_t = R\right]$$
$$= \sum_{k=0}^{\infty} \frac{e^{-\frac{(\gamma+k)\kappa\tau}{\alpha}}}{k!} \left(\frac{\alpha e^{\frac{\kappa\tau}{\alpha}} - \alpha}{\kappa}\right)^k \left(\prod_{j=0}^{k-1} \bar{B}_\alpha^{\langle j\rangle}\right) R^{\frac{\gamma+k}{\alpha}}, \tag{2.26}$$

for all $(R,\tau) \in D_\alpha^{\langle\gamma\rangle} \subset (0,\infty) \times [0,\infty)$ which the infinite series in (2.26) *uniformly converges on $D_\alpha^{\langle\gamma\rangle}$, where*

$$\bar{B}_\alpha^{\langle j\rangle} = \left(\frac{\gamma+j}{\alpha}\right)\left(\frac{1}{2}\left(\frac{\gamma+j}{\alpha} - 1\right)\sigma^2 + \kappa\theta\right). \tag{2.27}$$

Note that, if the product of $\bar{B}_\alpha^{\langle j\rangle}$ in (2.26) *runs on j from 0 to -1, the product term is defined to be 1.*

Proof. The proof is rather similarly to Theorem 2.4. We show only the major part, i.e.,

$$\begin{aligned}
A_\alpha^{\langle k\rangle}(\tau) &= \bar{B}_\alpha^{\langle k-1\rangle} \int_0^\tau e^{-\frac{\gamma+k}{\alpha}(\tau-\eta)\kappa} A_\alpha^{\langle k-1\rangle}(\eta)\,d\eta \\
&= \frac{e^{-\frac{(\gamma+k)\kappa\tau}{\alpha}}}{(k-1)!\kappa^{k-1}} \left(\prod_{j=0}^{k-1} \bar{B}_\alpha^{\langle j\rangle}\right) \int_0^\tau e^{-\frac{(\gamma+k)\kappa\eta}{\alpha}} \left(\alpha e^{\frac{\kappa\eta}{\alpha}} - \alpha\right)^{k-1} d\eta \\
&= \frac{e^{-\frac{(\gamma+k)\kappa\tau}{\alpha}}}{(k-1)!\kappa^{k-1}} \left(\prod_{j=0}^{k-1} \bar{B}_\alpha^{\langle j\rangle}\right) \frac{\left(\alpha e^{\frac{\kappa\tau}{\alpha}} - \alpha\right)^k}{k\kappa} \\
&= \frac{e^{-\frac{(\gamma+k)\kappa\tau}{\alpha}}}{k!} \left(\frac{\alpha e^{\frac{\kappa\tau}{\alpha}} - \alpha}{\kappa}\right)^k \left(\prod_{j=0}^{k-1} \bar{B}_\alpha^{\langle j\rangle}\right).
\end{aligned}$$

for all $k = 2, 3, 4, \ldots$. □

In the case that the parameters are constants, given a negative integer n yields the closed-form formula as the following theorem.

Theorem 2.11. *Suppose R_t follows the NLD-CEV process* (2.3) *such that $\kappa(t) = \kappa$, $\theta(t) = \theta$*

and $\sigma(t) = \sigma$*. For* $n \in \mathbb{Z}_0^-$*, the conditional* $\frac{n}{\alpha}$*-moment is exactly given by*

$$V_\alpha^{\langle n \rangle}(R,\tau) := E\left[R_T^{\frac{n}{\alpha}} \mid R_t = R\right] = e^{-\frac{n\kappa\tau}{\alpha}} R^{\frac{n}{\alpha}} + \sum_{k=1}^{|n|} \frac{e^{-\frac{(n+k)\kappa\tau}{\alpha}}}{k!} \left(\frac{\alpha e^{\frac{\kappa\tau}{\alpha}} - \alpha}{\kappa}\right)^k \left(\prod_{j=0}^{k-1} \bar{B}_\alpha^{\langle j \rangle}\right) R^{\frac{n+k}{\alpha}}, \tag{2.28}$$

for $R > 0$ *and* $\tau = T - t \geq 0$ *where* $\bar{B}_\alpha^{\langle j \rangle}$ *is defined by* (2.27)*. Also,* $\lim_{R \to 0^+} V_\alpha^{\langle n \rangle}(R,\tau) > 0$*.*

Proof. The proof is rather trivial by combining Theorem 2.8 with Theorem 2.10. □

The special case of Theorem 2.9 is also provided as follows.

Theorem 2.12. *Suppose* R_t *follows the NLD-CEV process* (2.3) *such that* $\kappa(t) = \kappa$*,* $\theta(t) = \theta$*,* $\sigma(t) = \sigma$*, and*

$$\gamma = \alpha - \frac{2\alpha\kappa\theta}{\sigma^2} - m, \tag{2.29}$$

for all $m \in \mathbb{Z}_0^+$*. Then,*

$$V_\alpha^{\langle \gamma \rangle}(R,\tau) = e^{-\frac{\gamma\kappa\tau}{\alpha}} R^{\frac{\gamma}{\alpha}} + \sum_{k=1}^{m} \frac{e^{-\frac{(\gamma+k)\kappa\tau}{\alpha}}}{k!} \left(\frac{\alpha e^{\frac{\kappa\tau}{\alpha}} - \alpha}{\kappa}\right)^k \left(\prod_{j=0}^{k-1} \bar{B}_\alpha^{\langle j \rangle}\right) R^{\frac{\gamma+k}{\alpha}}, \tag{2.30}$$

for all $(R,\tau) \in D_\alpha^{\langle \gamma \rangle} \subset (0,\infty) \times [0,\infty)$ *and* $\tau = T - t \geq 0$*. Note that, if the sum term in* (2.30) *runs on k from 1 to 0, the sum term is defined to be 0.*

Proof. Obviously obtained from Theorem 2.9. □

2.3.3 The unconditional $\frac{n}{\alpha}$-moments of the processes when $\tau \to \infty$

Under the conditions proposed in Section 2.2, this section provides two theorems which are reduced from the formula for conditional moments described in Theorems 2.5 and 2.11 to the unconditional moments as $\tau \to \infty$. This should be noted that the formulas are no longer depend on the initial value R, and they are simplified to finite product as follows.

Theorem 2.13. *Suppose* R_t *follows the NLD-CEV process* (2.2) *such that* $\kappa(t) = \kappa$*,* $\theta(t) = \theta$ *and* $\sigma(t) = \sigma$ *are satisfied the Assumptions 2.3 and 2.4. Then, for all* $n \in \mathbb{Z}^+$*,* $R > 0$ *and* $\tau = T - t$*,*

$$L_\alpha^{\langle n \rangle} := \lim_{\tau \to \infty} U_\alpha^{\langle n \rangle}(R,\tau) = \lim_{T \to \infty} E\left[R_T^{\frac{n}{\alpha}} \mid R_t = R\right] = \prod_{j=1}^{n} \frac{2\alpha\kappa\theta - \alpha\sigma^2 + j\sigma^2}{2\alpha\kappa}. \tag{2.31}$$

Proof. According to (2.17) in Theorem 2.5, we denote that the coefficient terms of $R^{\frac{n-k}{\alpha}}$ approach to 0 as $\tau \to \infty$ for $k = 0, 1, 2, \ldots, n-1$. Here, we derive the formula only for the remainder term, the case that $k = n$,

$$\begin{aligned} L_\alpha^{\langle n \rangle} &= \lim_{\tau \to \infty} \frac{e^{-\frac{n\kappa\tau}{\alpha}}}{n!} \left(\frac{\alpha e^{\frac{\kappa\tau}{\alpha}} - \alpha}{\kappa} \right)^n \left(\prod_{j=0}^{n-1} \bar{B}_\alpha^{\langle j \rangle} \right) R^{\frac{n-n}{\alpha}} \\ &= \frac{1}{n!\kappa^n} \left(\prod_{j=0}^{n-1} \bar{B}_\alpha^{\langle j \rangle} \right) \lim_{\tau \to \infty} \left(\alpha - \alpha e^{-\frac{\kappa\tau}{\alpha}} \right)^n \\ &= \frac{\alpha^n}{n!\kappa^n} \prod_{j=0}^{n-1} \bar{B}_\alpha^{\langle j \rangle}, \end{aligned}$$

where $\bar{B}_\alpha^{\langle j \rangle}$ is defined by (2.16). After substituting this expression $\bar{B}_\alpha^{\langle j \rangle}$ to the above equation, it can be reformulated to the factorization in the form of product,

$$L_\alpha^{\langle n \rangle} = \frac{\alpha^n}{n!\kappa^n} \prod_{j=0}^{n-1} \left(\frac{n-j}{\alpha} \right) \left(\frac{1}{2} \left(\frac{n-j}{\alpha} - 1 \right) \sigma^2 + \kappa\theta \right) = \prod_{j=1}^{n} \frac{2\alpha\kappa\theta - \alpha\sigma^2 + j\sigma^2}{2\alpha\kappa},$$

as required. □

Next, for the process (2.3), we obtain the following theorem.

Theorem 2.14. *Suppose R_t follows the NLD-CEV process* (2.3) *such that $\kappa(t) = \kappa$, $\theta(t) = \theta$ and $\sigma(t) = \sigma$ are satisfied the Assumptions 2.5. Then, for all $n \in \mathbb{Z}^-$, $R > 0$ and $\tau = T - t$,*

$$M_\alpha^{\langle n \rangle} := \lim_{\tau \to \infty} V_\alpha^{\langle n \rangle}(R, \tau) = \lim_{T \to \infty} E\left[R_T^{\frac{n}{\alpha}} \mid R_t = R \right] = \prod_{j=1}^{n} \frac{\alpha\kappa\theta - (j+1)\alpha\sigma^2}{\kappa}. \tag{2.32}$$

Proof. The proof is rather similar to the previous theorem and omitted here. □

2.3.4 The analysis of the convergence

This section discusses in detail the essentials of infinite sum in a way of convergence series that is necessary for Theorems 2.4 and 2.10. According to Theorem 2.4, the series (2.15) converges if and only if $\bar{B}_\alpha^{\langle j \rangle} = 0$ for some $j \in \mathbb{Z}_0^+$. For the process (2.2), there are two cases that produce $\bar{B}_\alpha^{\langle j \rangle} = 0$ for some $j \in \mathbb{Z}_0^+$, i.e., $\gamma = n \in \mathbb{Z}_0^+$ and $\gamma = \alpha - \frac{2\alpha\kappa\theta}{\sigma^2} + m$ for some $m \in \mathbb{Z}_0^+$. The convergences of the series (2.15) for each case have been shown in Theorems 2.5 and 2.6, respectively. However, the parameter γ that $\bar{B}_\alpha^{\langle j \rangle} \neq 0$ for all $j \in \mathbb{Z}_0^+$, it affects that the series (2.15) diverges as shown below. We first denote $A_\alpha^{\langle k \rangle}(\tau) := \frac{e^{-\frac{\gamma\kappa\tau}{\alpha}}}{k!} \left(\frac{\alpha e^{\frac{\kappa\tau}{\alpha}} - \alpha}{\kappa} \right)^k \left(\prod_{j=0}^{k-1} \bar{B}_\alpha^{\langle j \rangle} \right)$. Suppose that $\bar{B}_{\alpha}^{\langle j\rangle} \neq 0$ for all $j \in \mathbb{Z}_0^+$, for each R, we have

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{A_{\alpha}^{\langle k+1\rangle}(\tau)R^{\frac{\gamma-k-1}{\alpha}}}{A_{\alpha}^{\langle k\rangle}(\tau)R^{\frac{\gamma-k}{\alpha}}}\right| &= \lim_{k\to\infty}\left|\frac{\frac{e^{-\frac{\gamma\kappa\tau}{\alpha}}}{(k+1)!}\left(\frac{\alpha e^{\frac{\kappa\tau}{\alpha}}-\alpha}{\kappa}\right)^{k+1}\left(\prod_{j=0}^{k}\bar{B}_{\alpha}^{\langle j\rangle}\right)R^{\frac{\gamma-k-1}{\alpha}}}{\frac{e^{-\frac{\gamma\kappa\tau}{\alpha}}}{k!}\left(\frac{\alpha e^{\frac{\kappa\tau}{\alpha}}-\alpha}{\kappa}\right)^{k}\left(\prod_{j=0}^{k-1}\bar{B}_{\alpha}^{\langle j\rangle}\right)R^{\frac{\gamma-k}{\alpha}}}\right| \\ &= \lim_{k\to\infty}\left|\frac{\left(\alpha e^{\frac{\kappa\tau}{\alpha}}-\alpha\right)\left(\frac{\gamma-k}{\alpha}\right)\left[\frac{1}{2}\left(\frac{\gamma-k}{\alpha}-1\right)\sigma^2+\kappa\theta\right]}{(k+1)\kappa R}\right|.\end{aligned}$$

Note that the above expression is $\mathcal{O}(k)$ as $k \to \infty$, thus by ratio test (2.15) diverges.

Similarly, Theorem 2.10 in the process (2.3), there are two cases that make $\bar{B}_{\alpha}^{\langle j\rangle} = 0$ for some $j \in \mathbb{Z}_0^+$. That are $\gamma = -n \in \mathbb{Z}_0^-$ and $\gamma = \alpha - \frac{2\alpha\kappa\theta}{\sigma^2} - m$ for some $m \in \mathbb{Z}_0^+$. In each case, the convergences of the series (2.26) have been shown in Theorems 2.11 and 2.12, respectively. Similarly, for the parameter γ other than above, the series (2.26) diverges.

2.4 Experiments

In this section, the verifications of formulas in Section 2.3 are given though comparisons with Monte Carlo (MC) simulations based on the following NLD-CEV process

$$dR_t = \kappa\left(\frac{\sigma_0^2 d e^{2\sigma_1 t}}{4\kappa}R_t^{-(1-\beta)} - R_t\right)dt + \sigma_0 e^{\sigma_1 t}R_t^{\beta/2}dW_t. \tag{2.33}$$

Comparing (2.33) with (2.1) gives $\kappa(t) = \kappa$, $\theta(t) = \frac{d\sigma^2(t)}{4\kappa}$ and $\sigma(t) = \sigma_0 e^{\sigma_1 t}$, where κ and σ_0 are positive constants, σ_1 and σ_2 are nonnegative constants and d is a positive integer. Observe that the presented process (2.33) was introduced by Maghsoodi [55] for $\beta = 1$ and its transition density was also proposed.

Moreover, we have that the corresponding parameters $\kappa(t)$, $\theta(t)$ and $\sigma(t)$ in the process (2.33) can apply with Theorems 2.1 and 2.7, whose integrals can be always evaluated after substituting these parameters into $A_{\alpha}^{\langle k\rangle}$. Hence, $A_{\alpha}^{\langle k\rangle}$ also has an exact form, and we can express U and V in Theorems 2.1 and 2.7, respectively, in the following explicit forms

$$U_{\alpha}^{\langle\gamma\rangle}(R,\tau) = e^{-\frac{\gamma\kappa\tau}{\alpha}}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\prod_{j=0}^{k-1}(\gamma-j)(\alpha d+2(\gamma-j-\alpha))\right)\left(\frac{\sigma_0^2 e^{2\sigma_1(T-\tau)}\left(e^{2\sigma_1\tau+\frac{\kappa\tau}{\alpha}}-1\right)}{4\alpha(2\alpha\sigma_1+\kappa)}\right)^k R^{\frac{\gamma-k}{\alpha}}, \tag{2.34}$$

$$V_{\alpha}^{\langle\gamma\rangle}(R,\tau) = e^{-\frac{\gamma\kappa\tau}{\alpha}}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\prod_{j=0}^{k-1}(\gamma+j)(\alpha d+2(\gamma+j-\alpha))\right]\left(\frac{\sigma_0^2 e^{2\sigma_1(T-\tau)}\left(1-e^{2\sigma_1\tau-\frac{\kappa\tau}{\alpha}}\right)}{4\alpha(\kappa-2\alpha\sigma_1)}\right)^k R^{\frac{\gamma+k}{\alpha}}. \tag{2.35}$$

Here, we set that the product terms of the above formulas are 1 when an initial index $k = 0$. However, if we use parameters $\kappa(t)$, $\theta(t)$ and $\sigma(t)$ other than the above parameters, it is possible that the integral in $A_\alpha^{\langle k\rangle}$ in Theorems 2.1 and 2.7 cannot be integrable, then the numerical integrations can be utilized to approximate its value, see [8]. Note that the formulas for $U_\alpha^{\langle\gamma\rangle}(R,\tau)$ and $V_\alpha^{\langle\gamma\rangle}(R,\tau)$ presented in (2.34) and (2.35) diverge as mentioned in Subsection 2.3.4, i.e., for sufficiently large k, $\left|\frac{A_\alpha^{\langle k+1\rangle}(\tau)R^{\frac{\gamma-k-1}{\alpha}}}{A_\alpha^{\langle k\rangle}(\tau)R^{\frac{\gamma-k}{\alpha}}}\right| > 1$. However, the formula (2.34) can be reduced to the finite sum when $\gamma = n$ or $\gamma = \frac{\alpha}{2}(2-d)+n$, for $n \in \mathbb{Z}_0^+$. Similarly, the formula (2.35) is also a finite sum when $\gamma = -n$ or $\gamma = \frac{\alpha}{2}(2-d)-n$.

In this experiments, we apply Euler–Maruyama (EM) discretization to the process (2.33). The qualitatively correct numerical approximations by using EM method to the class of mean-reverting square root processes such as the ECIR process are technically provided by Higham and Mao [38]. We denote by $\hat{R}$ a time-discretized approximation to R. The EM approximation of (2.33) on the interval $[0,T]$ by discretizing N time steps, $0 = t_0 < t_1 < \ldots < t_N = T$, is defined by

$$\hat{R}_{t_{i+1}} = \hat{R}_{t_i} + \kappa(t_i)\left(\theta(t_i)\hat{R}_{t_i}^{-(1-\beta)} - \hat{R}_{t_i}\right)\Delta t + \sigma(t_i)\hat{R}_{t_i}^{\frac{\beta}{2}}\sqrt{\Delta t}Z_{i+1} \tag{2.36}$$

with the initial $\hat{R}_{t_0} = R_{t_0}$, $\Delta t = t_{i+1} - t_i$ and Z is independent N dimensional standard normal random vector. We first denote $U_\alpha^{\langle\gamma,M\rangle}$, $V_\alpha^{\langle\gamma,M\rangle}$ the approximations of $U_\alpha^{\langle\gamma\rangle}$, $V_\alpha^{\langle\gamma\rangle}$ obtained by MC simulations, respectively. To make our results more tangible, the following example illustrates the results in practice.

2.4.1 Validation of closed-form formulas for (2.2) with MC simulations

This subsection provides a major numerical example to illustrate the experimental validation of the closed-form formulas proposed in Theorem 2.2 via MC simulations. Before starting the example, we observe that the Assumptions 2.1 and 2.2 hold for the functions $\kappa(t)$, $\theta(t)$ and $\sigma(t)$ defined in (2.33).

Example 2.1. The formulas (2.34) with $\alpha = 0.5, 1, 1.5, 2$ for $\gamma = 1, 2$ and $\tau = 0.01$:
For $\gamma = 1$ yields

$$U_\alpha^{\langle 1\rangle}(R,\tau) = e^{-\frac{\kappa\tau}{\alpha}}\sum_{k=0}^{1}\frac{1}{k!}\left(\prod_{j=0}^{k-1}(1-j)(\alpha d + 2(1-j-\alpha))\right)\left(\frac{\sigma_0^2 e^{2\sigma_1(T-\tau)}\left(e^{2\sigma_1\tau+\frac{\kappa\tau}{\alpha}}-1\right)}{4\alpha(2\alpha\sigma_1+\kappa)}\right)^k R^{\frac{1-k}{\alpha}} \tag{2.37}$$

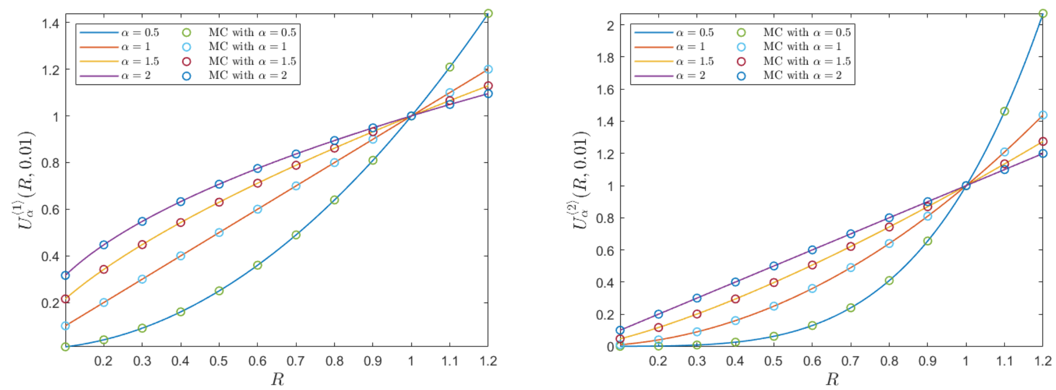
and for $\gamma = 2$ yields

$$U_\alpha^{\langle 2\rangle}(R,\tau) = e^{-\frac{2\kappa\tau}{\alpha}} \sum_{k=0}^{2} \frac{1}{k!} \left(\prod_{j=0}^{k-1} (2-j)(\alpha d + 2(2-j-\alpha)) \right) \left(\frac{\sigma_0^2 e^{2\sigma_1(T-\tau)} \left(e^{2\sigma_1\tau + \frac{\kappa\tau}{\alpha}} - 1 \right)}{4\alpha(2\alpha\sigma_1 + \kappa)} \right)^k R^{\frac{2-k}{\alpha}} \tag{2.38}$$

for all $R > 0$ and $\tau = T - t \geq 0$.

This example shows a formula based on the process (2.33) in the case of $\gamma \in \mathbb{N}$ which actually obtains to the closed-form formula. To validate the closed-form formulas presented in (2.37) and (2.38), the parameters $d = 4$, $\sigma_0 = 0.01$, $\sigma_1 = 0.02$ and $\kappa = 0.03$ in the process (2.33) are applied for the formulas and MC simulations at each initial value $R = 0.1, 0.2, \ldots, 1.2$ to generate $10,000$ sample paths of R_t, where each path consisting of $10,000$ steps over the time interval $[0, 0.01]$. The validations are performed as the comparisons between the formulas (2.37) and (2.38) with MC simulations of two different $\gamma = 1, 2$ for each $\alpha = 0.5, 1, 1.5, 2$.

As displayed in Figure 2.1(a), the results from MC simulations (colored circles) match completely with the formula (2.37) (solid lines) for each $R = 0.1, 0.2, \ldots, 1.2$. In the same way that the results from the MC simulations looked completely match the results from (2.38) as displayed in Figure 2.1(b). However, the major disadvantage of MC simulation is to consume the costly computational time for approximating the value of each initial value of R. Contrarily, our closed-form formulas can produce the exact solution at all initial values $R > 0$ and spend inexpensive time for the implementations.



(a) The conditional $\frac{1}{\alpha}$-moments **(b)** The conditional $\frac{2}{\alpha}$-moments

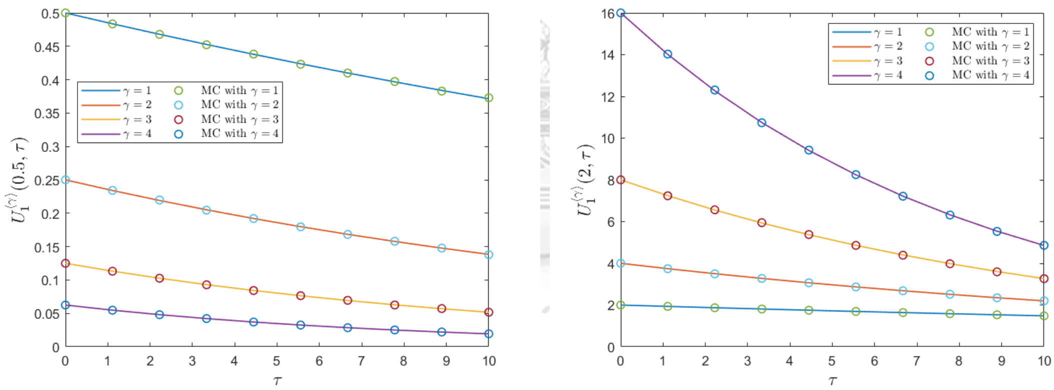
Figure 2.1: The validations of $U_\alpha^{\langle\gamma\rangle}(R, 0.01)$ for $\alpha = 0.5, 1, 1.5, 2$ and $R = 0.1, 0.2, 0.3, \ldots, 2$

Example 2.2. The ECIR process, the formula (2.34) with $\alpha = 1$ for $\gamma = 1,2,3,4$ and $R = 0.5, 2$: For $\gamma = n \in \mathbb{Z}_0^+$ yields

$$U_1^{\langle n\rangle}(R,\tau) = e^{-n\kappa\tau}\sum_{k=0}^{n}\binom{n}{k}\left(\prod_{j=0}^{k-1}(d+2(n-j-1))\right)\left(\frac{\sigma_0^2\left(e^{\tau(\kappa+2\sigma_1)}-1\right)e^{2\sigma_1(T-\tau)}}{4(\kappa+2\sigma_1)}\right)^{k} R^{n-k}, \tag{2.39}$$

for all $R > 0$ and $\tau = T - t \geq 0$.

In this example, we illustrate that our formula covers some existing special cases of the process (2.33). For the parameter $\alpha = 1$, this process is known as ECIR process. Moreover, our obtained formula with $\gamma \in \mathbb{Z}^+$ is also the closed-form formula. Hence, we validate the formula (2.39) with the same parameters used in Example 2.1 via MC simulations by varying $\tau = 0, 1, 2, \ldots, 10$ at two initial values $R = 0.5$ and $R = 2$ as depicted in Figures 2.2(a) and 2.2(b), respectively. Figure 2.2 shows that the results obtained from the formula (2.39) and MC simulations completely match.



(a) $R = 0.5$ **(b)** $R = 2$

Figure 2.2: The validations of the first-forth moments for ECIR process $U_1^{\langle\gamma\rangle}(R,\tau)$ where $\gamma = 1, 2, 3, 4$, with MC simulations for $\tau = 0, 1, 2, \ldots, 10$

Example 2.3. The formulas (2.34) with $\alpha = 0.5$ for $\gamma = \frac{\alpha}{2}(2-d) + n$:
For $d = 4$ and $\gamma = -0.5$ yields

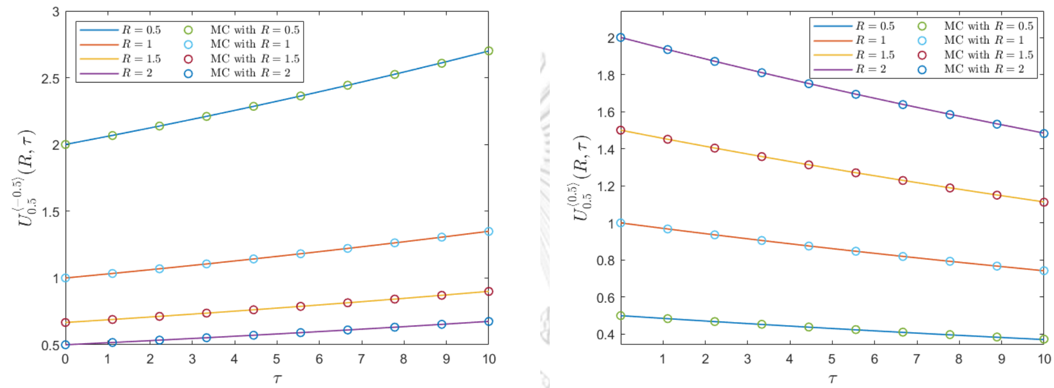
$$U_{0.5}^{\langle -0.5\rangle}(R,\tau) = e^{\kappa\tau}R^{-1}, \tag{2.40}$$

and for $d = 4$ and $\gamma = 0.5$ yields

$$U_{0.5}^{\langle 0.5\rangle}(R,\tau) = e^{\kappa\tau}R + e^{\kappa\tau}\left(\frac{\sigma_0^2 e^{2\sigma_1(T-\tau)}\left(e^{2\sigma_1\tau + \frac{\kappa\tau}{\alpha}} - 1\right)}{4\alpha(2\alpha\sigma_1+\kappa)}\right)R^{-1}, \tag{2.41}$$

for all $R > 0$ and $\tau = T - t \geq 0$.

In this example, we display the formula that still provides a closed form for some $\gamma \notin \mathbb{Z}_0^+$. For the same parameters as in Example 2.1, we obtain the closed-form formulas (2.40) when $\gamma = -0.5$ and (2.41) when $\gamma = 0.5$. In addition, we validate our formulas by comparing with the MC simulations. We can see that both formulas for $\gamma = -0.5$ and $\gamma = 0.5$ produce the results that extremely match to the MC simulations as demonstrated in Figures 2.3(a) and 2.3(b), respectively.



(a) The first inverse moments **(b)** The first moments

Figure 2.3: The validations of $U_\alpha^{\langle\gamma\rangle}(R,\tau)$ for $R = 0.5, 1, 1.5, 2$

2.4.2 Validation of closed-form formulas for (2.3) with MC simulations

This subsection provides a set of experiments using the same parameters of the previous example, except $\sigma_0 = -0.01$ and $\kappa = -0.03$ in the process (2.33), which are applied for the formulas and MC simulations with initial value $R = 0.8, 0.9, \ldots, 2$. In examples, the validations are performed as the comparisons between the formulas (2.42) and (2.43) with MC simulations of two different $\gamma = -1, -2$ for each $\alpha = 0.5, 1, 1.5, 2$.

Example 2.4. The formulas (2.35) with $\alpha = 0.5, 1, 1.5, 2$ for $\gamma = -1, -2$ and $\tau = 0.01$: For $\gamma = -1$ yields

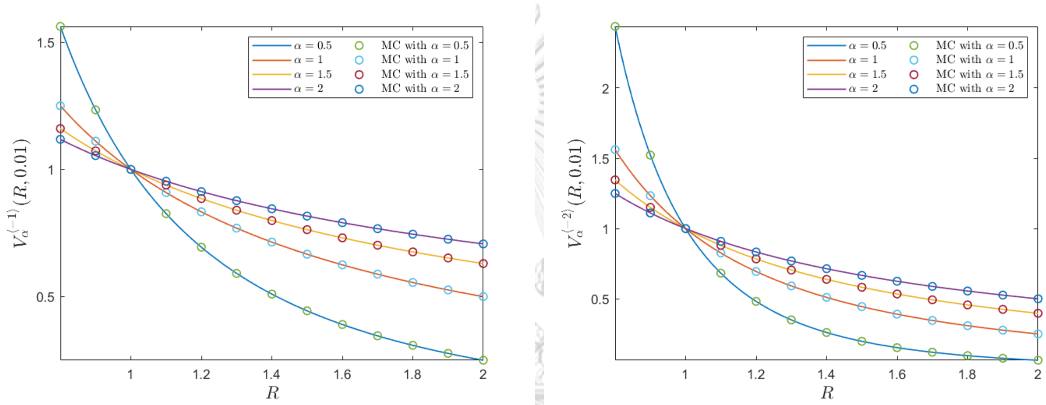
$$V_\alpha^{\langle -1\rangle}(R,\tau) = e^{\frac{\kappa\tau}{\alpha}} \sum_{k=0}^{1} \frac{1}{k!} \left(\prod_{j=0}^{k-1}(j-1)(\alpha d + 2(j-1-\alpha))\right) \left(\frac{\sigma_0^2 e^{2\sigma_1(T-\tau)}\left(1-e^{2\sigma_1\tau-\frac{\kappa\tau}{\alpha}}\right)}{4\alpha(\kappa-2\alpha\sigma_1)}\right)^k R^{\frac{k-1}{\alpha}} \tag{2.42}$$

and for $\gamma = -2$ yields

$$V_{\alpha}^{\langle -2\rangle}(R,\tau) = e^{\frac{2\kappa\tau}{\alpha}} \sum_{k=0}^{2} \frac{1}{k!} \left(\prod_{j=0}^{k-1} (j-2)(\alpha d + 2(j-2-\alpha)) \right) \left(\frac{\sigma_0^2 e^{2\sigma_1 (T-\tau)} \left(1 - e^{2\sigma_1 \tau - \frac{\kappa\tau}{\alpha}}\right)}{4\alpha(\kappa - 2\alpha\sigma_1)} \right)^k R^{\frac{k-2}{\alpha}} \tag{2.43}$$

for all $R > 0$ and $\tau = T - t \geq 0$.

The results displayed in Figure 2.4 validate our formulas, i.e., for each γ and α at each the initial value R, the MC simulations perfectly match with the results from closed-form formula.



(a) The conditional $-\frac{1}{\alpha}$-moments **(b)** The conditional $-\frac{2}{\alpha}$-moments

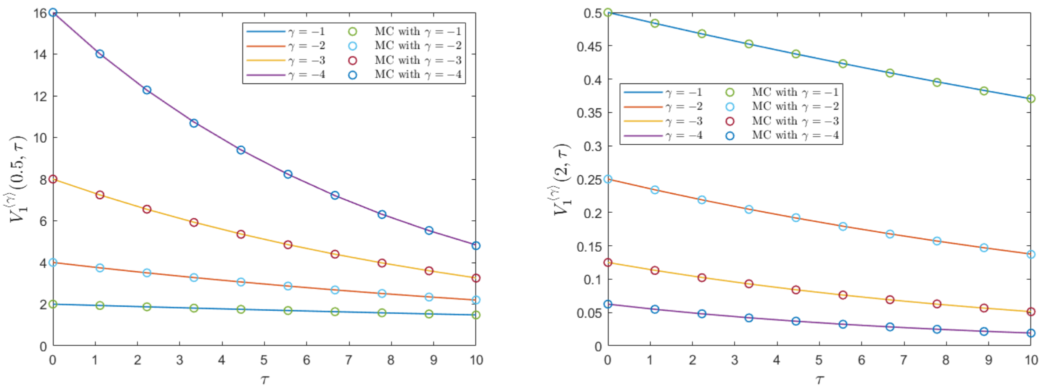
Figure 2.4: The validations of $V_{\alpha}^{\langle\gamma\rangle}(R, 0.01)$ for $\alpha = 0.5, 1, 1.5, 2$ and $R = 0.8, 0.9, 1, \ldots, 2$

Example 2.5. The EIF process (3/2-SVM), the formula (2.35) with $\alpha = 1$ for $\gamma = 1, 2, 3, 4$ and $R = 0.5, 2$: For $\gamma = n \in \mathbb{Z}_0^-$ yields

$$V_1^{\langle n\rangle}(R,\tau) = e^{n\kappa\tau} \sum_{k=0}^{|n|} \binom{|n|}{i} \left(\prod_{j=0}^{i-1} (d + 2(j-n-1)) \right) \left(\frac{\sigma_0^2 \left(1 - e^{\tau(\kappa - 2\sigma_1)}\right) e^{2\sigma_1 T - \kappa\tau}}{4(\kappa - 2\sigma_1)} \right)^i R^{n+k}, \tag{2.44}$$

for all $R > 0$ and $\tau = T - t \geq 0$.

This example shows that our formula covers some exist special cases in the process (2.33) for $\gamma \in \mathbb{Z}^-$. With parameter $\alpha = 1$, this process is called the EIF process or 3/2-SVM. Our obtained formula based on this process follows the closed form (2.44) when $\gamma \in \mathbb{Z}^-$. The validation of (2.44) is tested with the MC simulations as depicted in Figure 2.5. Moreover, we can see that the obtained results from both methods perfectly match.



(a) $R = 0.5$ **(b)** $R = 2$

Figure 2.5: The validations of the first-forth inverse moments for the EIF process $V_1^{\langle\gamma\rangle}(R,\tau)$ where $\gamma = -1,-2,-3,-4$, with MC simulations for $\tau \in \{0,1,2,\dots,10\}$

Example 2.6. The formulas (2.35) with $\alpha = 0.5$ for $\gamma = \frac{\alpha}{2}(2-d) - n$:
For $d = 4$ and $\gamma = -0.5$ yields

$$V_{0.5}^{\langle -0.5\rangle}(R,\tau) = e^{\kappa\tau}R^{-1}, \tag{2.45}$$

and for $d = 4$ and $\gamma = -1.5$ yields

$$V_{0.5}^{\langle -1.5\rangle}(R,\tau) = e^{3\kappa\tau}R^{-3} + 3e^{3\kappa\tau}\left(\frac{\sigma_0^2 e^{2\sigma_1(T-\tau)}\left(1-e^{2\sigma_1\tau-\frac{\kappa\tau}{\alpha}}\right)}{4\alpha(\kappa-2\alpha\sigma_1)}\right)R^{-1}, \tag{2.46}$$

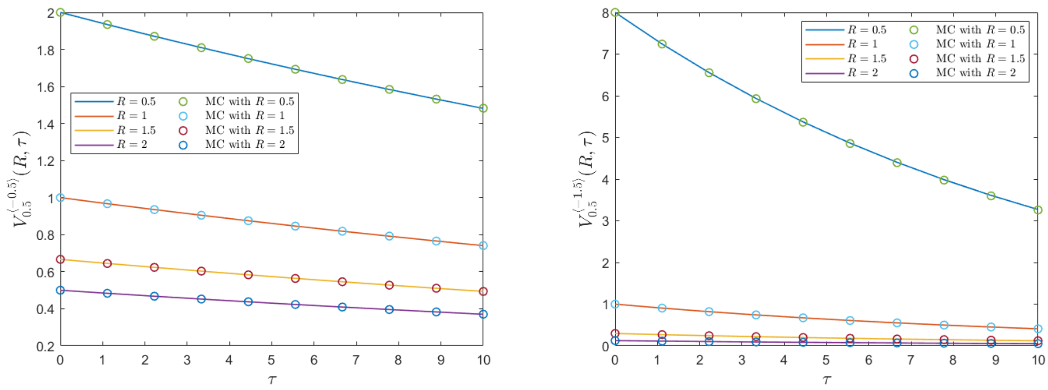
for all $R > 0$ and $\tau = T - t \geq 0$.

For Example 2.6, we illustrate the case that $\gamma \notin \mathbb{Z}_0^-$ for the process (2.33) and the formula provides a closed form. In this example, we validate our obtained formulas (2.45) and (2.45) via plotting graphs which are compared to the MC simulations as shown in Figures 2.6(a) and 2.6(b), respectively. Also, the occurrence of results from both approaches completely matches.

2.4.3 Numerical approximation of our formulas with finite sum

According to Subsection 2.3.4, each case of the processes based on β the infinite series (2.6) and (2.20) diverge except only when $B_\alpha^{\langle j\rangle} = 0$ for some $j \in \mathbb{Z}^+$. It is interesting to observe the level of accuracy of our formulas proposed in Theorems 2.1, 2.4, 2.7 and 2.10 can be obtained from their partial sum as demonstrated in the following examples.

Before studying the accuracy, for the case of process (2.2), we first denote $U_\alpha^{\langle\gamma,K\rangle}$ an approximate of $U_\alpha^{\langle\gamma\rangle}$ described by a partial sum of the infinite sum (2.6) up to order $\gamma - K$. Similarly, the case of process (2.3), $V_\alpha^{\langle\gamma,K\rangle}$ denotes an approximate of $V_\alpha^{\langle\gamma\rangle}$ as a partial sum of



(a) The first inverse moments **(b)** The third inverse moments

Figure 2.6: The validations of $V_\alpha^{\langle\gamma\rangle}(R,\tau)$ for $R = 0.5, 1, 1.5, 2$

the infinite sum (2.20) up to order $\gamma + K$.

To consider a suitable K before comparing with MC simulations, we measure the significant difference of value of $U_\alpha^{\langle\gamma,K\rangle}$ at each $K \in \mathbb{Z}^+$. In this work, that measure is defined by a sequence of absolute relative differences

$$D_{U_\alpha}^{\langle\gamma,K\rangle}(R,\tau) := \left| \frac{U_\alpha^{\langle\gamma,K\rangle}(R,\tau) - U_\alpha^{\langle\gamma,K-1\rangle}(R,\tau)}{U_\alpha^{\langle\gamma,K\rangle}(R,\tau)} \right|,$$

for all $(R,\tau) \in (0,\infty) \times [0,\infty)$. Moreover, to consider the accuracy of $U_\alpha^{\langle\gamma,K\rangle}(R,\tau)$ compared with MC for some $K \in \mathbb{Z}^+$, we define the absolute relative errors

$$E_{U_\alpha}^{\langle\gamma,K\rangle}(R,\tau) := \left| \frac{U_\alpha^{\langle\gamma,K\rangle}(R,\tau) - U_\alpha^{\langle\gamma,M\rangle}(R,\tau)}{U_\alpha^{\langle\gamma,K\rangle}(R,\tau)} \right|,$$

for all $(R,\tau) \in (0,\infty) \times [0,\infty)$. Similarly, for considering the accuracy of the process (2.3), we can perform in the same way as the process (2.2) by replacing U by V.

The sequences of absolute relative differences of $D_{U_\alpha}^{\langle\gamma,K\rangle}(R,0.01)$ are shown in Table 2.1 for $K = 1,2,3,4$ with parameters as in Example 2.1 except $R = 0.01, 1, 5$ for $\alpha = 0.5, 1, 2$ and $\gamma = 0.5, -0.5$. For the case of $D_{V_\alpha}^{\langle\gamma,K\rangle}(R,0.01)$, all parameters are the same as in Example 2.4, and the results are shown in Table 2.1.

Now, we are interested in the case of infinite sum of $U_\alpha^{\langle\gamma\rangle}(R,0.01)$ and $V_\alpha^{\langle\gamma\rangle}(R,0.01)$ by using the parameters $\alpha = 1,2$ and $\gamma = 0.5, -0.5$. From Table 2.1, we observe that the obtained absolute relative differences are improved when K increases from 1 to 4, showing that for small K, $U_\alpha^{\langle\gamma,K\rangle}$ and $V_\alpha^{\langle\gamma,K\rangle}$ are good approximations of $U_\alpha^{\langle\gamma\rangle}$ and $V_\alpha^{\langle\gamma\rangle}$, respectively. To validate this claim, the results of $U_\alpha^{\langle\gamma,K\rangle}$ and $V_\alpha^{\langle\gamma,K\rangle}$ with $K = 2$ are compared with MC simulations in the next example.

Table 2.1: The absolute relative differences $D_{U_\alpha}^{\langle\gamma,K\rangle}(R,0.01)$ and $D_{V_\alpha}^{\langle\gamma,K\rangle}(R,0.01)$

	R	K	$\alpha=1$		$\alpha=2$	
			$\gamma=0.5$	$\gamma=-0.5$	$\gamma=0.5$	$\gamma=-0.5$
$D_{U_\alpha}^{\langle\gamma,K\rangle}(R,0.01)$	0.01	1	3.7512e-05	1.2505e-05	1.5629e-06	9.3776e-07
		2	2.3453e-10	2.3454e-10	7.3282e-13	4.3970e-13
		3	2.9327e-15	1.4664e-14	2.2907e-19	2.2907e-19
		4	1.3752e-19	1.6044e-18	8.9505e-26	3.7592e-25
	1	1	3.7513e-07	1.2504e-07	1.5629e-07	9.3776e-08
		2	2.3454e-14	2.3454e-14	7.3282e-15	4.3969e-15
		3	2.9328e-21	1.4664e-20	2.2907e-22	2.2907e-22
		4	1.3752e-27	1.6044e-26	8.9505e-30	3.7592e-29
	5	1	7.5026e-08	2.5009e-08	6.9896e-08	4.1938e-08
		2	9.3816e-16	9.3816e-16	1.4656e-15	8.7939e-16
		3	2.3462e-23	1.1731e-22	2.0489e-23	2.0489e-23
		4	2.2003e-30	2.5671e-29	3.5802e-31	1.5037e-30
$D_{V_\alpha}^{\langle\gamma,K\rangle}(R,0.01)$	0.01	1	3.7513e-09	1.2504e-09	1.5629e-08	9.3776e-09
		2	3.5181e-17	2.3454e-18	5.1298e-16	7.3282e-17
		3	5.1323e-25	1.4664e-26	2.4052e-23	1.6035e-24
		4	1.0108e-32	1.6044e-34	1.4473e-30	5.6388e-32
	1	1	3.7513e-07	1.2504e-07	1.5629e-07	9.3776e-08
		2	3.5181e-13	2.3454e-14	5.1298e-14	7.3282e-15
		3	5.1323e-19	1.4664e-20	2.4052e-20	1.6035e-21
		4	1.0108e-24	1.6044e-26	1.4473e-26	5.6388e-28
	5	1	1.8757e-06	6.2522e-07	3.4948e-07	2.0969e-07
		2	8.7952e-12	5.8635e-13	2.5649e-13	3.6641e-14
		3	6.4154e-17	1.8330e-18	2.6891e-19	1.7928e-20
		4	6.3174e-22	1.0028e-23	3.6183e-25	1.4097e-26

Example 2.7. The formulas (2.34) with $\gamma=0.5,-0.5$ for $\alpha=1,2$:

For $\gamma=0.5$ yields

$$\begin{aligned}U_\alpha^{\langle 0.5,2\rangle}(R,\tau) &= A_\alpha^{\langle 0\rangle}(\tau)R^{\frac{0.5}{\alpha}} + A_\alpha^{\langle 1\rangle}(\tau)R^{-\frac{0.5}{\alpha}} + A_\alpha^{\langle 2\rangle}(\tau)R^{-\frac{1.5}{\alpha}} \\ &= e^{-\frac{\kappa\tau}{2\alpha}}R^{\frac{0.5}{\alpha}} + \frac{\sigma_0^2(\alpha(d-2)+1)\left(e^{\frac{\kappa\tau}{\alpha}+2\sigma_1\tau}-1\right)e^{2\sigma_1(T-\tau)-\frac{\kappa\tau}{2\alpha}}}{8\alpha(2\alpha\sigma_1+\kappa)}R^{-\frac{0.5}{\alpha}} \\ &\quad - \frac{\sigma_0^4(\alpha(d-2)-1)(\alpha(d-2)+1)\left(e^{\frac{\kappa\tau}{\alpha}+2\sigma_1\tau}-1\right)^2 e^{4\sigma_1(T-\tau)-\frac{\kappa\tau}{2\alpha}}}{128\alpha^2(2\alpha\sigma_1+\kappa)^2}R^{-\frac{1.5}{\alpha}},\end{aligned} \tag{2.47}$$

and for $\gamma = -0.5$ yields

$$\begin{aligned} U_\alpha^{\langle -0.5,2\rangle}(R,\tau) &= A_\alpha^{\langle 0\rangle}(\tau)R^{-\frac{0.5}{\alpha}} + A_\alpha^{\langle 1\rangle}(\tau)R^{-\frac{1.5}{\alpha}} + A_\alpha^{\langle 2\rangle}(\tau)R^{-\frac{2.5}{\alpha}} \\ &= e^{\frac{\kappa\tau}{2\alpha}}R^{-\frac{0.5}{\alpha}} - \frac{\sigma_0^2(\alpha(d-2)-1)e^{\frac{\kappa\tau}{2\alpha}}\left(e^{\frac{\kappa\tau}{\alpha}+2\sigma_1 T}-e^{2\sigma_1(T-\tau)}\right)}{8\alpha(2\alpha\sigma_1+\kappa)}R^{-\frac{1.5}{\alpha}} \\ &\quad - \frac{3\sigma_0^4(\alpha(d-2)-3)(\alpha(d-2)-1)\left(e^{\frac{\kappa\tau}{\alpha}+2\sigma_1\tau}-1\right)^2 e^{\frac{\kappa\tau}{2\alpha}+4\sigma_1(T-\tau)}}{128\alpha^2(2\alpha\sigma_1+\kappa)^2}R^{-\frac{2.5}{\alpha}}, \end{aligned} \tag{2.48}$$

for all $R > 0$ and $\tau = T - t \geq 0$.

The comparison results between the formulas $U_\alpha^{\langle 0.5,2\rangle}(R,\tau)$ and $U_\alpha^{\langle -0.5,2\rangle}(R,\tau)$ from above with the MC simulations are shown in Table 2.2. For MC simulations, we perform with 10,000, 20,000 and 40,000 sample paths using 10,000 discretized steps. Table 2.2 demonstrates the results of MC simulations that completely match (very small $E_{U_\alpha}^{\langle\gamma,2\rangle}(R, 0.01)$) with our approximate formulas (2.47) and (2.48), and more closely as the number of the sample paths increases. Suggesting that the MC simulations likely converge to our approximate formulas. This confirms that the finite-sum approximation from Example 2.7 is very accurate.

Table 2.2: The absolute relative errors $E_{U_\alpha}^{\langle\gamma,2\rangle}(R, 0.01)$ between approximations $U_\alpha^{\langle\gamma,2\rangle}(R, 0.01)$ and the MC simulations

R	No. of paths	$\alpha = 1$		$\alpha = 2$	
		$\gamma = 0.5$	$\gamma = -0.5$	$\gamma = 0.5$	$\gamma = -0.5$
0.01	10,000	4.9432e-06	6.0968e-06	1.8070e-06	2.4024e-06
	20,000	4.3787e-06	5.5227e-06	1.6572e-06	2.0408e-06
	40,000	2.8365e-06	3.5774e-06	1.7448e-06	9.9157e-07
1	10,000	4.7443e-05	6.2577e-05	2.6847e-05	3.5829e-05
	20,000	4.4541e-05	4.5552e-05	2.2873e-05	1.6153e-05
	40,000	1.9582e-05	3.1475e-05	6.9575e-06	1.3462e-05
5	10,000	1.8289e-04	2.3106e-04	7.4331e-05	9.1785e-05
	20,000	1.4254e-04	9.8655e-05	7.2443e-05	7.5774e-05
	40,000	6.2475e-05	6.5412e-05	4.5679e-05	4.4312e-05

Example 2.8. The formulas (2.35) $\gamma = 0.5, -0.5$ for $\alpha = 1, 2$:

For $\gamma = 0.5$ yields

$$\begin{aligned} V_\alpha^{\langle 0.5,2\rangle}(R,\tau) &= A_\alpha^{\langle 0\rangle}(\tau)R^{\frac{0.5}{\alpha}} + A_\alpha^{\langle 1\rangle}(\tau)R^{\frac{1.5}{\alpha}} + A_\alpha^{\langle 2\rangle}(\tau)R^{\frac{2.5}{\alpha}} \\ &= e^{-\frac{\kappa\tau}{2\alpha}}R^{\frac{0.5}{\alpha}} - \frac{\sigma_0^2(\alpha(d-2)+1)e^{-\frac{3\kappa\tau}{2\alpha}}\left(e^{2\sigma_1 T}-e^{\frac{\kappa\tau}{\alpha}+2\sigma_1(T-\tau)}\right)}{8\alpha(\kappa-2\alpha\sigma_1)}R^{\frac{1.5}{\alpha}} \\ &\quad + \frac{3\sigma_0^4(\alpha(d-2)+1)(\alpha(d-2)+3)\left(e^{\frac{\kappa\tau}{\alpha}}-e^{2\sigma_1\tau}\right)^2 e^{4\sigma_1(T-\tau)-\frac{5\kappa\tau}{2\alpha}}}{128\alpha^2(\kappa-2\alpha\sigma_1)^2}R^{\frac{2.5}{\alpha}}, \end{aligned} \tag{2.49}$$

and for $\gamma = -0.5$ yields

$$\begin{aligned} V_\alpha^{\langle -0.5,2\rangle}(R,\tau) &= A_\alpha^{\langle 0\rangle}(\tau)R^{-\frac{0.5}{\alpha}} + A_\alpha^{\langle 1\rangle}(\tau)R^{\frac{0.5}{\alpha}} + A_\alpha^{\langle 2\rangle}(\tau)R^{\frac{1.5}{\alpha}} \\ &= e^{\frac{\kappa\tau}{2\alpha}}R^{-\frac{0.5}{\alpha}} + \frac{\sigma_0^2(\alpha(d-2)-1)e^{-\frac{\kappa\tau}{2\alpha}}\left(e^{2\sigma_1 T} - e^{\frac{\kappa\tau}{\alpha}+2\sigma_1(T-\tau)}\right)}{8\alpha(\kappa-2\alpha\sigma_1)}R^{\frac{0.5}{\alpha}} \\ &\quad - \frac{\sigma_0^4(\alpha(d-2)-1)(\alpha(d-2)+1)\left(e^{\frac{\kappa\tau}{\alpha}} - e^{2\sigma_1\tau}\right)^2 e^{4\sigma_1(T-\tau)-\frac{3\kappa\tau}{2\alpha}}}{128\alpha^2(\kappa-2\alpha\sigma_1)^2}R^{\frac{1.5}{\alpha}}, \end{aligned} \tag{2.50}$$

for all $R > 0$ and $\tau = T - t \geq 0$.

For Example 2.8, we approximate the values of $V_\alpha^{\langle 0.5,2\rangle}(R,\tau)$ and $V_\alpha^{\langle -0.5,2\rangle}(R,\tau)$ via the formulas (2.49) and (2.50), respectively, for the process (2.3). These obtained approximate formulas are tested by comparing with the MC simulations that use the same parameters as in Example 2.7. This example confirms that the finite-sum approximation is very accurate similar to Example 2.7.

Table 2.3: The absolute relative errors $E_{V_\alpha}^{\langle\gamma,2\rangle}(R, 0.01)$ between approximations $V_\alpha^{\langle\gamma,2\rangle}(R, 0.01)$ and the MC simulations

R	No. of paths	$\alpha = 1$		$\alpha = 2$	
		$\gamma = 0.5$	$\gamma = -0.5$	$\gamma = 0.5$	$\gamma = -0.5$
0.01	10,000	6.3809e-06	4.6791e-06	2.5455e-06	2.8372e-06
	20,000	4.8817e-06	3.2489e-06	1.9995e-06	1.8475e-06
	40,000	3.5147e-06	1.8855e-06	8.5924e-07	9.9942e-07
1	10,000	5.2675e-05	3.6136e-05	2.0877e-05	1.6268e-05
	20,000	4.7569e-05	2.9143e-05	1.8847e-05	1.4891e-05
	40,000	3.2283e-05	1.7754e-05	9.6756e-06	9.8560e-06
5	10,000	2.1431e-04	1.2127e-04	5.4750e-05	4.1640e-05
	20,000	2.7689e-04	9.5415e-05	4.9192e-05	4.1589e-05
	40,000	1.9540e-04	7.7050e-05	4.1508e-05	3.5101e-05

2.5 Conclusion and discussion

In this work, the sufficient conditions of the existence and uniqueness for a positive pathwise strong solution are provided for the NLD-CEV process (2.1) for the cases of $\beta \in [0, 2)$ and $\beta \in (2, \infty)$. We have derived the formulas of conditional moments for the processes for the processes (2.2) and (2.3) separately based on the range of β. The derived formulas shown in Theorems 2.1 and 2.7 are presented as the infinite summation, which are reduced to finite summation for the following cases: (i) for $\gamma \in \mathbb{Z}$ as for Theorems 2.2 and 2.8 and (ii) for $\gamma \notin \mathbb{Z}$ under the conditions (2.13) and (2.24) for Theorems 2.3 and 2.9, respectively. For the case that the processes have constant parameters where the coefficients in integral forms can be exactly evaluated, the formulas can be expressed in closed forms as shown in Theorems 2.4, 2.5, and 2.6

for the process (2.2) and Theorems 2.10, 2.11, and 2.12 for the process (2.3). In addition in cases, the closed-form formulas for unconditional moments are also observed for both processes (2.2) and (2.3) as described in Theorems 2.13 and 2.14, respectively. One primary concern for the formulas (2.6) and (2.20) in Theorems 2.1 and 2.7 is the integral form of the coefficient A_α, which might not be directly evaluated for time-dependent parameters in general. In this case, a numerical integration method such as Simpson's rule, trapezoidal rule, and Newton–Cotes, is required for the approximation, e.g. one can applied an efficient method with high accuracy such as the Chebyshev integration method proposed by Boonklurb et al. [8].

The validation of accuracy and efficiency of the formulas for processes (2.2) and (2.3) is performed by comparing with MC simulations based on some experimental examples. As described in Section 2.4, the experimental results show the agreement between the proposed formulas and MC simulations: the example of process (2.2) with $\alpha = 1$ (or the ECIR process) is shown in Example 2.2; the process (2.3) with $\alpha = 1$ (or the EIF process or 3/2-SVM) is shown in Example 2.5. Note that for the case that moments γ having formula as an infinite summation, the approximation is also valid by using partial summation at suitable order as illustrated in Examples 2.7 and 2.8.

The closed-form formulas in this work would benefit market practitioners for pricing financial derivatives in which the NLD-CEV model is adopted to describe the dynamics of volatility or interest rate such as interest rate swaps, where the required conditional moments can be rapidly from the formula. In addition, the formulas can be applied for parameter estimations of the observed data such as the volatility persistence and the risk premium, for instance, the conditional mixed moments can be obtained and applied to implement the method of moments.

CHAPTER III

CLOSED-FORM FORMULAS FOR CONDITIONAL MOMENTS OF INHOMOGENEOUS PEARSON DIFFUSION PROCESSES

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Abstract

Diffusion models have been thoroughly studied for their use in seeking stochastic differential equation (SDE) solutions and investigating their properties, such as moments and conditional moments, which play significant roles in many real-world applications and are especially beneficial for estimating parameters. In fact, these moments can be directly calculated by applying the transition probability density function (PDF), which is often unknown or unavailable in closed form; the formulas for the conditional moments of the SDE may be unavailable in closed form, as well. In this work, we studied an extended case of Pearson diffusion processes, which are a class of diffusions that have squared diffusion coefficients with time-dependent parameter functions. A complete investigation was carried out for both light- and heavy-tailed Pearson diffusion processes, including Ornstein–Uhlenbeck, Cox–Ingersoll–Ross, Fisher–Snedecor, reciprocal gamma, and Student. We introduce a simple but novel approach to closed-form formulas for conditional moments of inhomogeneous Pearson diffusion processes. The approach does not require any knowledge of eigenfunctions or the transition PDF. In each class of stationary distributions reduced from Pearson diffusions, the formula is explored and presented in a concise form. The closed-form formulas obtained are also numerically validated by MC simulations.

Keywords: closed-form formula, conditional moment, Pearson diffusion, inhomogeneous diffusion, light-tailed process, heavy-tailed process

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3.1 Introduction

Pearson diffusions are diffusion models that satisfy the Pearson equation [62], and they appear in a wide variety of applications in different branches of applied science, such as physics, biology, and mathematical finance [18, 37, 39, 47]; however, investigating their properties is still challenging. In 2008, Forman and Sørensen [32] studied Pearson diffusions via stationary solutions of SDEs characterized by the mean-reverting linear drift and squared diffusion coefficients. SDEs that are studied through Pearson diffusions are now known as Pearson diffusion processes. Pearson diffusions are classified into six stationary diffusion processes according to particular characteristics, such as positive or negative, bounded or unbounded, symmetric or skewed, and light- or heavy-tailed. Some well-known examples of stationary distributions reduced from Pearson diffusion processes are Ornstein–Uhlenbeck (OU), Cox–Ingersoll–Ross (CIR), and Jacobi processes, which have been widely investigated for many applications. In contrast, heavy-tailed distributions, such as Fisher–Snedecor, reciprocal gamma, and Student processes, are rarely investigated or used in applications.

The conditional moments of Pearson diffusion processes can be derived directly by using the transition PDF. In the past several decades, transition PDFs of Pearson diffusion processes have received much attention through the Fokker–Planck equation (see [4, 46, 49–52]). If the solution to the Fokker–Planck equation is not available, conditional moments are not accessible from the transition PDF, and other efficient methods are required. Some explicit formulas for conditional polynomial moments of a class of Pearson diffusions were first applied to the GMM by Zhou [81] in 2003. In 2005, the statistical inference of Pearson diffusions was investigated by Bibby et al. [6], who also derived closed-form formulas for the first conditional moment and the correlation of diffusion processes with marginal distributions, such as the gamma distribution, variance-gamma distribution, and hyperbolic distribution. Most statistically tractable Pearson diffusion processes have been studied from a stochastic viewpoint, e.g., by Forman and Sørensen [32] in 2008, who provided statistical applications based on their explicit formulas for conditional moments and mixed moments. Under sufficient conditions, Kessler and Sørensen [43] presented formulas for conditional moments as recurrence relations involving eigenfunctions of the generators of the diffusions. However, among all classes of Pearson diffusion processes, most formulas have not yet been satisfactorily achieved.

Based on a partial differential equation (PDE) according to the Feynman–Kac formula [42], this work proposes a novel, simple closed-form formula for conditional moments and some statistical properties, such as conditional variance, mixed moment, covariance, and correlation. The formulas are derived by solving the PDE without requiring any knowledge of eigenfunctions or transition PDFs. The presented formula significantly simplifies other approaches in the literature: the direct solution approach by Forman and Sørensen [32] and the use of the transition PDF by Leonenko and Phillips [49], which is also more general in terms of time-dependent parameters. In addition, the formulas of conditional moments for all six classes of Pearson diffusions are further simplified in concise forms in terms of the original process, which could be beneficial for other statistical applications.

The rest of the paper is organized as follows. Section 3.2 provides a brief overview of Pearson diffusions and their complete classifications. The main methodology is proposed in Section 3.3 to address the relevant concepts for our main results, which are closed-form formulas for conditional moments of Pearson diffusion processes. Important properties of closed-form formulas are provided and discussed in Section 3.4. Section 3.5 is divided into six classes: OU diffusion, CIR diffusion, Jacobi diffusion, Fisher–Snedecor diffusion, reciprocal gamma diffusion, and Student diffusion. This section provides important sufficient conditions for the obtained closed-form formulas and also presents closed-form formulas for unconditional moments in special cases. The proposed formulas for Pearson diffusion processes for time-inhomogeneous cases are discussed and illustrated in Section 3.6. Furthermore, this section also presents experimental validations of the proposed formulas via MC simulations. Section 3.7 concludes the paper.

3.2 Pearson diffusion processes

A class of Pearson diffusions is defined via linear drift and quadratic squared diffusion coefficients, which satisfy a stochastic differential equation (SDE):

$$dX_t = \theta\left(\mu - X_t\right)dt + \sqrt{2\theta\left(aX_t^2 + bX_t + c\right)}\,dW_t, \tag{3.1}$$

where X_t is in the state space; $\theta > 0$ and a, b, c are real constants such that the quadratic squared diffusion term in (3.1) is well defined; and W_t is a Wiener process. The parameters in (3.1) are often described as follows: θ corresponds to the speed of adjustment to the mean of the invariant distribution μ, and a, b, c determine the state space of the diffusion and the shape of the invariant distribution.

In this work, under the probability measure P and σ-field $\mathcal{F}_t$, we first propose the integral-form formula for the conditional moment of inhomogeneous Pearson diffusion processes, where the parameters in (3.1) depend on time, in the form of

$$\mathbb{E}\left[X_T^\gamma \mid \mathcal{F}_t\right] = \mathbb{E}\left[X_T^\gamma \mid X_t = x\right], \qquad 0 \le t \le T, \tag{3.2}$$

for real order γ.

Since the process X_t in (3.1) satisfies Markov properties, the conditional density of X_t for a given X_s is known as the transition PDF, $p(X_t, t \mid X_s, s)$. Moreover, for time-dependent homogeneous processes, we have $p(X_t, t \mid X_s, s) = p(X_t, t-s \mid X_s, 0)$. In practice, the conditional moments (3.2) can be directly calculated by using the transition PDF. In 1930, Kolmogorov [45] studied the transition PDF, $p(x, t; y)$, via the Fokker–Planck equation under some known initial distributions:

$$\frac{\partial}{\partial t}p(x,t;y) = -\frac{\partial}{\partial x}\left(\varphi(x)p(x,t;y)\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\psi(x)p(x,t;y)\right), \qquad x \in \mathbb{R}, t \ge 0, \tag{3.3}$$

where the linear drift is $\varphi(x) = \theta\mu - \theta x$, and the quadratic squared diffusion $\psi(x) = 2d(x)$, where $d(x) = a\theta x^2 + b\theta x + c\theta$. Kolmogorov found that the invariant density $\mathbf{m}$ of diffusion (3.1) satisfies the Fokker–Planck equation (3.3) whenever it exists. In this case, (3.3) reduces to

$$\frac{\mathbf{m}'(x)}{\mathbf{m}(x)} = \frac{\varphi(x) - d'(x)}{d(x)} = \frac{(\mu - b) - (1+2a)x}{ax^2 + bx + c}. \tag{3.4}$$

Equation (3.4) is technically known as the Pearson equation, which satisfies Pearson diffusion processes (3.1). Since a closed-form representation of the transition PDF satisfying (3.3) and (3.4) is complicated or unavailable, the closed-form formulas for conditional moments (3.2) derived from the transition PDF are also usually complicated or unavailable (see [4, 49, 50]). Forman and Sørensen [32] defined six classes of Pearson diffusions under the conditions of stationary solutions based on the characteristic properties of the polynomial $d(x)$ in (3.4), as follows:

1. Ornstein–Uhlenbeck (OU) diffusion: $\deg(d) = 0$,
2. Squared diffusion: $\deg(d) = 1$,
3. Jacobi diffusion: $\deg(d) = 2, \quad \Delta(d) > 0, \quad a < 0$,
4. Fisher–Snedecor diffusion: $\deg(d) = 2, \quad \Delta(d) > 0, \quad a > 0$,
5. Reciprocal gamma diffusion: $\deg(d) = 2, \quad \Delta(d) = 0, \quad a > 0$,
6. Student diffusion: $\deg(d) = 2, \quad \Delta(d) < 0, \quad a > 0$,

where $\Delta(d) := b^2 - 4ac$ is the discriminant.

3.3 Conditional moments of Pearson diffusion processes

There is strong empirical evidence that extreme movements in practice tend to involve time (see [37, 39, 55]); therefore, the dynamics of diffusion processes are usually governed by time-varying parameters, called inhomogeneous Pearson diffusion:

$$dX_t = \theta(t)\,(\mu(t) - X_t)\,dt + \sqrt{2\theta(t)\,(a(t)X_t^2 + b(t)X_t + c(t))}\,dW_t, \tag{3.5}$$

where $0 \le t \le T$, and $\theta(t) > 0$, $\mu(t)$, $a(t)$, $b(t)$, and $c(t)$ are time-dependent continuous functions such that the square root in (3.5) is well defined on $[0, T]$ when X_t is in the state space. Well-known instances deduced by (3.5) are the extended Ornstein–Uhlenbeck (EOU) and extended squared diffusion (or extended Cox–Ingersoll–Ross, ECIR) processes (see Egorov et al. [23] and Hull and White [39]).

In this section, we derive an explicit formula for the conditional γ moment of (3.5) based on the solution of the PDE according to the Feynman–Kac formula [42]. According to the diffusion coefficient of (3.5), to ensure existence and uniqueness, we need the following assumption [60]:

Assumption 3.1. *The drift $\theta(t)(\mu(t) - X_t)$ and diffusion $\sqrt{2\theta(t)(a(t)X_t^2 + b(t)X_t + c(t))}$ are Borel-measurable and satisfy the local Lipschitz and linear growth conditions.*

In the following Theorems 3.1 and 3.2, we first present the integral-form formula for conditional moments of process (3.5), which is also valid for (3.1) when parameters are constant.

The idea of the theorem relies on the Feynman–Kac formula [42] by expressing the solution of the PDE as an infinite series (3.6) and solving its coefficients to obtain a closed-form formula. The motivation for the form of the conditional moment, i.e., a solution to PDE, is based on [32, 68]; since Pearson diffusion has linear drift and quadratic squared diffusion coefficients, the differential generator maps polynomials to polynomials (see more details in [32, 68]).

Theorem 3.1. *Suppose that X_t follows inhomogeneous Pearson diffusion* (3.5)*. The γ^{th} conditional moment for $\gamma \in \mathbb{R}$ is*

$$U^{\langle\gamma\rangle}(x,\tau) := \mathbb{E}\left[X_T^{\gamma} \mid X_t = x\right] = \sum_{k=0}^{\infty} P_k^{\langle\gamma\rangle}(\tau)\, x^{\gamma-k}, \tag{3.6}$$

for $\tau := T - t$, $(x,\tau) \in D^{\langle\gamma\rangle} \subset \mathbb{R} \times [0,\infty)$, $D^{\langle\gamma\rangle}$ is the domain in which the infinite series in (3.6) *converges uniformly, where the coefficients in* (3.6) *are expressed as*

$$\begin{aligned} P_0^{\langle\gamma\rangle}(\tau) &= e^{\int_0^\tau A_0^{\langle\gamma\rangle}(\xi)\; d\xi}, \\ P_1^{\langle\gamma\rangle}(\tau) &= \int_0^\tau e^{\int_\eta^\tau A_1^{\langle\gamma\rangle}(\xi)\; d\xi} B_0^{\langle\gamma\rangle}(\eta)\, P_0^{\langle\gamma\rangle}(\eta)\; d\eta, \\ P_k^{\langle\gamma\rangle}(\tau) &= \int_0^\tau e^{\int_\eta^\tau A_k^{\langle\gamma\rangle}(\xi)\; d\xi} \left(B_{k-1}^{\langle\gamma\rangle}(\eta)\, P_{k-1}^{\langle\gamma\rangle}(\eta) + C_{k-2}^{\langle\gamma\rangle}(\eta)\, P_{k-2}^{\langle\gamma\rangle}(\eta)\right) d\eta, \end{aligned} \tag{3.7}$$

for $k = 2, 3, 4, \ldots$, and

$$\begin{aligned} A_j^{\langle\gamma\rangle}(\tau) &= \theta(T-\tau)\,(\gamma - j)\,\left((\gamma - j - 1)\, a(T-\tau) - 1\right), \\ B_j^{\langle\gamma\rangle}(\tau) &= \theta(T-\tau)\,(\gamma - j)\,\left((\gamma - j - 1)\, b(T-\tau) + \mu(T-\tau)\right), \\ C_j^{\langle\gamma\rangle}(\tau) &= \theta(T-\tau)\,(\gamma - j)\,\left((\gamma - j - 1)\, c(T-\tau)\right). \end{aligned} \tag{3.8}$$

Proof. Using the Feynman–Kac formula [42], $U^{\langle\gamma\rangle}(x,\tau) := U$ in (3.6) satisfies the PDE

$$U_\tau - \theta(T-\tau)\,(\mu(T-\tau) - x)\, U_x - \theta(T-\tau)\left(a(T-\tau)x^2 + b(T-\tau)x + c(T-\tau)\right) U_{xx} = 0 \tag{3.9}$$

for all $(x,\tau) \in D^{\langle\gamma\rangle}$, subject to the initial condition

$$U^{\langle\gamma\rangle}(x,0) = \mathbb{E}\left[X_T^{\gamma} \mid X_T = x\right] = x^{\gamma}. \tag{3.10}$$

By comparing the coefficients between (3.6) and (3.10), we obtain the conditions $P_0^{\langle\gamma\rangle}(0) = 1$ and $P_k^{\langle\gamma\rangle}(0) = 0$ for $k \in \mathbb{Z}^+$. Computing (3.9) using (3.6) to find the partial derivatives U_τ, U_{xx}, and U_x, we have

$$
\begin{aligned}
0 = &\sum_{k=0}^{\infty} \frac{d}{d\tau} P_k^{\langle\gamma\rangle}(\tau)\, x^{\gamma-k} - \theta(T-\tau)\,(\mu(T-\tau)-x) \sum_{k=0}^{\infty} \Big((\gamma-k)\, P_k^{\langle\gamma\rangle}(\tau)\, x^{\gamma-k-1} \Big) \\
&- \theta(T-\tau)\,\big(a(T-\tau)\,x^2 + b(T-\tau)\,x + c(T-\tau)\big) \sum_{k=0}^{\infty} \Big((\gamma-k)\,(\gamma-k-1)\, P_k^{\langle\gamma\rangle}(\tau)\, x^{\gamma-k-2} \Big).
\end{aligned}
$$

Simplifying,

$$
\begin{aligned}
0 = &\left(\frac{d}{d\tau} P_0^{\langle\gamma\rangle}(\tau) - A_0^{\langle\gamma\rangle}(\tau) P_0^{\langle\gamma\rangle}(\tau) \right) x^{\gamma} + \left(\frac{d}{d\tau} P_1^{\langle\gamma\rangle}(\tau) - A_1^{\langle\gamma\rangle}(\tau) P_1^{\langle\gamma\rangle}(\tau) - B_0^{\langle\gamma\rangle}(\tau) P_0^{\langle\gamma\rangle}(\tau) \right) x^{\gamma-1} \\
&+ \sum_{k=0}^{\infty} \left(\frac{d}{d\tau} P_{k+2}^{\langle\gamma\rangle}(\tau) - A_{k+2}^{\langle\gamma\rangle}(\tau) P_{k+2}^{\langle\gamma\rangle}(\tau) - B_{k+1}^{\langle\gamma\rangle}(\tau) P_{k+1}^{\langle\gamma\rangle}(\tau) - C_k^{\langle\gamma\rangle}(\tau) P_k^{\langle\gamma\rangle}(\tau) \right) x^{\gamma-k-2}.
\end{aligned}
$$

Under the assumptions of the infinite series in (3.6) over $D^{\langle\gamma\rangle}$, the above equation can be solved through a system of ODEs:

$$
\begin{aligned}
0 &= \frac{d}{d\tau} P_0^{\langle\gamma\rangle}(\tau) - A_0^{\langle\gamma\rangle}(\tau) P_0^{\langle\gamma\rangle}(\tau), \\
0 &= \frac{d}{d\tau} P_1^{\langle\gamma\rangle}(\tau) - A_1(\tau) P_1^{\langle\gamma\rangle}(\tau) - B_0(\tau) P_0^{\langle\gamma\rangle}(\tau), \\
0 &= \frac{d}{d\tau} P_{k+2}^{\langle\gamma\rangle}(\tau) - A_{k+2}^{\langle\gamma\rangle}(\tau) P_{k+2}^{\langle\gamma\rangle}(\tau) - B_{k+1}^{\langle\gamma\rangle}(\tau) P_{k+1}^{\langle\gamma\rangle}(\tau) - C_k^{\langle\gamma\rangle}(\tau) P_k^{\langle\gamma\rangle}(\tau)
\end{aligned}
\tag{3.11}
$$

with initial conditions $P_0^{\langle\gamma\rangle}(0) = 1$ and $P_k^{\langle\gamma\rangle}(0) = 0$ for $k \in \mathbb{Z}^+$. Therefore, the coefficients in (3.6) are directly obtained by solving system (3.11) in the form of a recursive relation, which produces the result (3.7). □

Notice that the formula as an infinite series is obtained without solving for the eigenvalues and eigenfunctions given by Forman and Sørensen [32].

Examining (3.6) in Theorem 3.1, when $\gamma = n \in \mathbb{Z}^+$, the infinite sum in (3.6) is terminated at finite order and can be expressed as in Theorem 3.2.

Theorem 3.2. *Suppose that X_t follows inhomogeneous Pearson diffusion* (3.5). *The n^{th} conditional moment for $n \in \mathbb{Z}_0^+$ is*

$$
U^{\langle n\rangle}(x,\tau) := \mathbb{E}\left[X_T^n \mid X_t = x\right] = \sum_{k=0}^{n} P_k^{\langle n\rangle}(\tau)\, x^{n-k}, \tag{3.12}
$$

for $\tau := T - t$, $(x,\tau) \in D^{\langle n\rangle}$, where the coefficients $P_k^{\langle n\rangle}(\tau)$ in (3.12) *are defined by* (3.7) *and* (3.8).

Proof. In (3.8), when $\gamma = n$, $k = n+1$, we obtain $B_{k-1}^{\langle\gamma\rangle}(\tau) = B_n^{\langle n\rangle}(\tau) = 0$ and $C_{k-2}^{\langle\gamma\rangle}(\tau) =$

$C_{n-1}^{\langle n\rangle}(\tau) = 0$, implying $P_k^{\langle\gamma\rangle}(\tau) = P_{n+1}^{\langle n\rangle}(\tau) = 0$. Considering the coefficients at $k = n+2$, we obtain $C_{k-2}^{\langle\gamma\rangle}(\tau) = C_n^{\langle n\rangle}(\tau) = 0$. Since $P_{n+1}^{\langle n\rangle}(\tau) = 0$ and $C_n^{\langle n\rangle}(\tau) = 0$, $P_k^{\langle\gamma\rangle}(\tau) = P_{n+2}^{\langle n\rangle}(\tau) = 0$. Now, we have $P_{n+1}^{\langle n\rangle}(\tau) = 0$ and $P_{n+2}^{\langle n\rangle}(\tau) = 0$, and the coefficients $P_k(\tau)$ in (3.7) involve the previous two coefficients; therefore, $P_k^{\langle\gamma\rangle}(\tau) = 0$ for all integers $k \geq n+1$. Thus, the infinite sum (3.6) is reduced to the finite sum (3.12). □

Remark 3.1. The presented formula in Theorem 3.2 for the n^{th} conditional moment significantly simplifies the approaches in the literature by Forman and Sørensen [32] and Leonenko and Phillips [49], especially for time-dependent parameters.

If the parameters $\theta(t) = \theta$, $\mu(t) = \mu$, $a(t) = a$, $b(t) = b$, and $c(t) = c$ are constant, the integral in Theorem 3.2 can be exactly integrated, as presented in Theorem 3.3.

Theorem 3.3. *Suppose that X_t follows the Pearson diffusion process* (3.1). *The n^{th} conditional moment for $n \in \mathbb{Z}_0^+$ is*

$$U_P^{\langle n\rangle}(x,\tau) := \mathbb{E}\left[X_T^n \mid X_t = x\right] = \sum_{k=0}^{n} P_k^{\langle n\rangle}(\tau)\, x^{n-k}, \tag{3.13}$$

for $\tau := T - t$, $(x,\tau) \in D_P^{\langle n\rangle} \subset \mathbb{R}\times[0,\infty)$, where the coefficients in (3.13) *are expressed as*

$$\begin{aligned} P_0^{\langle n\rangle}(\tau) &= e^{\tau \widetilde{A}_0^{\langle n\rangle}}, \\ P_1^{\langle n\rangle}(\tau) &= \frac{\widetilde{B}_0^{\langle n\rangle} e^{\tau \widetilde{A}_0^{\langle n\rangle}}}{\widetilde{A}_0^{\langle n\rangle} - \widetilde{A}_1^{\langle n\rangle}} + \frac{\widetilde{B}_0^{\langle n\rangle} e^{\tau \widetilde{A}_1^{\langle n\rangle}}}{\widetilde{A}_1^{\langle n\rangle} - \widetilde{A}_0^{\langle n\rangle}}, \\ P_k^{\langle n\rangle}(\tau) &= \sum_{j=0}^{k} \left(\sum_{\mathcal{X}\in D_{k,j}} \frac{\prod\limits_{i\in\mathbb{Z}_{k-1}\setminus(\mathcal{X}\cup(\mathcal{X}+\{1\}))} \widetilde{B}_i^{\langle n\rangle} \cdot \prod\limits_{i\in\mathcal{X}} \widetilde{C}_i^{\langle n\rangle}}{\prod\limits_{i\in\mathbb{Z}_k\setminus(\{j\}\cup(\mathcal{X}+\{1\}))} \left(\widetilde{A}_j^{\langle n\rangle} - \widetilde{A}_i^{\langle n\rangle}\right)} \right) e^{\tau \widetilde{A}_j^{\langle n\rangle}}, \end{aligned} \tag{3.14}$$

for $k = 2, 3, \ldots, n$, where

$$D_{k,j} = \left\{\mathcal{X} \in \mathcal{P}\left(\mathbb{Z}_{k-2} \setminus \{j-1\}\right) \mid n(\mathcal{X}) \in \{0,1\} \text{ or } |u-v| \geq 2 \text{ for all } u,v \in \mathcal{X}\right\}.$$

The notation $\mathcal{P}$ refers to the power set, $\mathbb{Z}_k = \{0,1,2,\ldots,k\}$:

$$\begin{aligned} \widetilde{A}_j^{\langle n\rangle} &= \theta\,(n-j)\left((n-j-1)\,a - 1\right), \\ \widetilde{B}_j^{\langle n\rangle} &= \theta\,(n-j)\left((n-j-1)\,b + \mu\right), \\ \widetilde{C}_j^{\langle n\rangle} &= \theta\,(n-j)\left((n-j-1)\,c\right). \end{aligned} \tag{3.15}$$

According to (3.14), *when $\mathcal{X} = \emptyset$, we define the sum and product as* 0 *and* 1, *respectively.*

Proof. By applying the linear system of ODEs (3.11) for the case of the n^{th} conditional moment with constant parameters, the system of ODEs using coefficients (3.15) can be written in the form:

$$\begin{bmatrix} \frac{d}{d\tau}P_0^{\langle n\rangle}(\tau) \\ \frac{d}{d\tau}P_1^{\langle n\rangle}(\tau) \\ \frac{d}{d\tau}P_2^{\langle n\rangle}(\tau) \\ \vdots \\ \frac{d}{d\tau}P_n^{\langle n\rangle}(\tau) \end{bmatrix} = \begin{bmatrix} \widetilde{A}_0^{\langle n\rangle} & & & & & \\ \widetilde{B}_0^{\langle n\rangle} & \widetilde{A}_1^{\langle n\rangle} & & & & \\ \widetilde{C}_0^{\langle n\rangle} & \widetilde{B}_1^{\langle n\rangle} & \widetilde{A}_2^{\langle n\rangle} & & & \\ & \ddots & \ddots & \ddots & & \\ & & \widetilde{C}_{n-2}^{\langle n\rangle} & \widetilde{B}_{n-1}^{\langle n\rangle} & \widetilde{A}_n^{\langle n\rangle} \end{bmatrix} \begin{bmatrix} P_0^{\langle n\rangle}(\tau) \\ P_1^{\langle n\rangle}(\tau) \\ P_2^{\langle n\rangle}(\tau) \\ \vdots \\ P_n^{\langle n\rangle}(\tau) \end{bmatrix}, \qquad \begin{bmatrix} P_0^{\langle n\rangle}(0) \\ P_1^{\langle n\rangle}(0) \\ P_2^{\langle n\rangle}(0) \\ \vdots \\ P_n^{\langle n\rangle}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

We denote the above equation by $\frac{d}{d\tau}\mathbf{P}(\tau) = K\mathbf{P}(\tau)$ subject to the initial condition $\mathbf{P}(0) = \mathbf{P}_0$, where the solution can be given as $\mathbf{P}(\tau) = e^{\tau K}\mathbf{P}_0$. Note that the coefficient matrix K is three-band lower triangular with distinct diagonal entries, and thus, K is simple and has completely $n+1$ eigenvalues, $\widetilde{A}_j^{\langle n\rangle}$ for $j \in \mathbb{Z}_n$. The exponential matrix $e^{\tau K}$ is diagonalizable, and we can write the solution as follows:

$$\mathbf{P}(\tau) = Se^{\tau\Lambda}S^{-1}\mathbf{P}_0, \tag{3.16}$$

where Λ and S are the eigenvalue and eigenvector matrices of K, respectively;
$\Lambda = \operatorname{diag}\{\widetilde{A}_0^{\langle n\rangle}, \widetilde{A}_1^{\langle n\rangle}, \widetilde{A}_2^{\langle n\rangle}, \ldots, \widetilde{A}_n^{\langle n\rangle}\}$, and an eigenvector matrix is lower triangular $S := [S_{k,j}]$, where $k, j \in \mathbb{Z}_n$, in which the j^{th} column is an eigenvector of K corresponding to the eigenvalue $\widetilde{A}_j^{\langle n\rangle}$. We can directly compute S from K to obtain the following:

$$S_{k,j} = \begin{cases} 0 & \text{for } \; k < j, \\ \prod\limits_{i=j+1}^{n} \left(\widetilde{A}_j^{\langle n\rangle} - \widetilde{A}_i^{\langle n\rangle}\right) & \text{for } \; k = j \in \mathbb{Z}_{n-1}, \\ \prod\limits_{i=j+2}^{n} \left(\widetilde{A}_j^{\langle n\rangle} - \widetilde{A}_i^{\langle n\rangle}\right) \widetilde{B}_{k-1}^{\langle n\rangle} & \text{for } \; k = j+1 \in \mathbb{Z}_{n-1}, \\ \frac{1}{\widetilde{A}_j^{\langle n\rangle} - \widetilde{A}_k^{\langle n\rangle}} \left(\widetilde{B}_{k-1}^{\langle n\rangle} S_{k-1,j} + \widetilde{C}_{k-2}^{\langle n\rangle} S_{k-2,j}\right) & \text{for } \; k \geq j+2, \end{cases}$$

where the product from j to n for $n < j$ is defined as 1; this means that $S_{n,n} = S_{n,n-1} = 1$. Since $\mathbf{P}_0$ is a standard unit vector at the first element, the multiplication $S^{-1}\mathbf{P}_0$ produces the first column of $S^{-1} := [Q_{k,j}]$, where element $Q_{k,j}$ of S^{-1} can be calculated from the formula $S_{k,j}$:

$$Q_{k,j} = \begin{cases} 0 & \text{for } \; k < j, \\ \frac{1}{S_{k,k}} & \text{for } \; k = j, \\ \frac{-1}{S_{k,k}} \sum\limits_{i=j}^{k-1} S_{k,i}Q_{i,j} & \text{for } \; k \geq j+1. \end{cases}$$

Computing the above, we have $S^{-1}\mathbf{P}_0 = [Q_{0,0}, Q_{1,0}, Q_{2,0}, \dots, Q_{n,0}]^\top$, and (3.16) is simplified to

$$\begin{bmatrix} P_0^{\langle n\rangle}(\tau) \\ P_1^{\langle n\rangle}(\tau) \\ \vdots \\ P_n^{\langle n\rangle}(\tau) \end{bmatrix} = \begin{bmatrix} S_{0,0} & & & \\ S_{1,0} & S_{1,1} & & \\ \vdots & \vdots & \ddots & \\ S_{n,0} & S_{n,1} & \cdots & S_{n,n} \end{bmatrix} \begin{bmatrix} e^{\tau\widetilde{A}_0^{\langle n\rangle}} & & & \\ & e^{\tau\widetilde{A}_1^{\langle n\rangle}} & & \\ & & \ddots & \\ & & & e^{\tau\widetilde{A}_n^{\langle n\rangle}} \end{bmatrix} \begin{bmatrix} Q_{0,0} \\ Q_{1,0} \\ \vdots \\ Q_{n,0} \end{bmatrix}$$

$$= \begin{bmatrix} S_{0,0}Q_{0,0} & & & \\ S_{1,0}Q_{0,0} & S_{1,1}Q_{1,0} & & \\ \vdots & \vdots & \ddots & \\ S_{n,0}Q_{0,0} & S_{n,1}Q_{1,0} & \cdots & S_{n,n}Q_{n,0} \end{bmatrix} \begin{bmatrix} e^{\tau\widetilde{A}_0^{\langle n\rangle}} \\ e^{\tau\widetilde{A}_1^{\langle n\rangle}} \\ \vdots \\ e^{\tau\widetilde{A}_n^{\langle n\rangle}} \end{bmatrix}.$$

Thus, we have

$$P_k^{\langle n\rangle}(\tau) = \sum_{j=0}^{k} S_{k,j}Q_{j,0}\, e^{\tau\widetilde{A}_j^{\langle n\rangle}}. \tag{3.17}$$

For $k = 0, 1$, it is easy to obtain $P_0^{\langle n\rangle}(\tau)$ and $P_1^{\langle n\rangle}(\tau)$ by using the above formula (3.17):

$$\begin{aligned} P_0^{\langle n\rangle}(\tau) &= S_{0,0}Q_{0,0}\, e^{\tau\widetilde{A}_0^{\langle n\rangle}} = e^{\tau\widetilde{A}_0^{\langle n\rangle}}, \\ P_1^{\langle n\rangle}(\tau) &= S_{1,0}Q_{0,0}\, e^{\tau\widetilde{A}_0^{\langle n\rangle}} + S_{1,1}Q_{1,0}\, e^{\tau\widetilde{A}_1^{\langle n\rangle}} = \left(\frac{\widetilde{B}_0^{\langle n\rangle}}{\widetilde{A}_0^{\langle n\rangle} - \widetilde{A}_1^{\langle n\rangle}}\right) e^{\tau\widetilde{A}_0^{\langle n\rangle}} + \left(\frac{\widetilde{B}_0^{\langle n\rangle}}{\widetilde{A}_1^{\langle n\rangle} - \widetilde{A}_0^{\langle n\rangle}}\right) e^{\tau\widetilde{A}_1^{\langle n\rangle}}. \end{aligned}$$

For $k \geq 2$, we see that formula (3.17) depends on both $S_{k,j}$ and $Q_{j,0}$ in the form of a recurrence relation for linear difference equations. Fortunately, several schemes can be used to seek the solution to these linear difference equations (see more details in [57] and references therein). By using these schemes, we obtain the solutions of $S_{k,j}$ and $Q_{j,0}$. Finally, we simplify the multiplication $S_{k,j}Q_{j,0}$ to an the explicit form for $k \geq 2$ as follows:

$$S_{k,j}Q_{j,0} = \sum_{\mathcal{X}\in D_{k,j}} \frac{\prod_{i\in\mathbb{Z}_{k-1}\setminus(\mathcal{X}\cup(\mathcal{X}+\{1\}))} \widetilde{B}_i^{\langle n\rangle} \cdot \prod_{i\in\mathcal{X}} \widetilde{C}_i^{\langle n\rangle}}{\prod_{i\in\mathbb{Z}_k\setminus(\{j\}\cup(\mathcal{X}+\{1\}))} \left(\widetilde{A}_j^{\langle n\rangle} - \widetilde{A}_i^{\langle n\rangle}\right)}, \tag{3.18}$$

where $D_{k,j} = \{\mathcal{X} \in \mathcal{P}\left(\mathbb{Z}_{k-2} \setminus \{j-1\}\right) \mid n(\mathcal{X}) \in \{0,1\} \text{ or } |u-v| \geq 2 \text{ for all } u, v \in \mathcal{X}\}$, $\mathbb{Z}_k = \{0, 1, 2, \dots, k\}$. Consequently, after (3.18) is substituted into (3.17), we obtain $P_k^{\langle n\rangle}(\tau)$ for all $k \geq 2$, coinciding with (3.14), as in Theorem 3.3. □

In the next theorem, we present the formula of the n^{th} unconditional moment for Pearson diffusion processes (3.1) under some existing conditions on parameters. In general, unconditional moments can be calculated directly from the transition density by taking the final time T approaching infinity for fixed t to obtain the stationary density. In this work, the unconditional moments are obtained by taking the limit $T \to \infty$ of the conditional moments, as suggested in [1, 64], which can be considered to be the interchanging of limits over the integration.

Theorem 3.4. *Suppose that X_t follows the Pearson diffusion process* (3.1) *with condition* $a(n-1) \le 1$ *for* $n \in \mathbb{Z}_0^+$. *The* n^{th} *unconditional moment is given by*

$$L_P^{\langle n\rangle} := \lim_{\tau\to\infty} U_P^{\langle n\rangle}(x,\tau) = \lim_{T\to\infty} \mathbb{E}\left[X_T^n \mid X_t = x\right] = \sum_{\mathcal{X}\in D_{n,n}} \frac{\prod\limits_{i\in\mathbb{Z}_{n-1}\setminus(\mathcal{X}\cup(\mathcal{X}+\{1\}))} \widetilde{B}_i^{\langle n\rangle} \cdot \prod\limits_{i\in\mathcal{X}} \widetilde{C}_i^{\langle n\rangle}}{\prod\limits_{i\in\mathbb{Z}_n\setminus(\{n\}\cup(\mathcal{X}+\{1\}))} \left(-\widetilde{A}_i^{\langle n\rangle}\right)}. \tag{3.19}$$

Proof. From the result in Theorem 3.3, the cases of $n = 0$ and 1 are straightforward; this proof discusses only the case of $n \ge 2$. From (3.15), under the condition $a(n-1) \le 1$, $\widetilde{A}_j^{\langle n\rangle} \le 0$ for all $j \le n$ gives $\widetilde{A}_j^{\langle n\rangle} = 0$ only when $j = n$. According to Theorem 3.3, it is not difficult to see that, for fixed t, we have

$$\lim_{T\to\infty} P_k^{\langle n\rangle}(\tau) = \lim_{T\to\infty} \sum_{j=0}^{k} \left(\sum_{\mathcal{X}\in D_{k,j}} \frac{\prod\limits_{i\in\mathbb{Z}_{k-1}\setminus(\mathcal{X}\cup(\mathcal{X}+\{1\}))} \widetilde{B}_i^{\langle n\rangle} \cdot \prod\limits_{i\in\mathcal{X}} \widetilde{C}_i^{\langle n\rangle}}{\prod\limits_{i\in\mathbb{Z}_k\setminus(\{j\}\cup(\mathcal{X}+\{1\}))} \left(\widetilde{A}_j^{\langle n\rangle} - \widetilde{A}_i^{\langle n\rangle}\right)} \right) e^{\tau \widetilde{A}_j^{\langle n\rangle}} = 0, \tag{3.20}$$

for all $k < n$. For the case of $k = n$, we have

$$\begin{aligned}
\lim_{T\to\infty} P_n^{\langle n\rangle}(\tau) &= \lim_{T\to\infty} \sum_{j=0}^{n} \left(\sum_{\mathcal{X}\in D_{n,j}} \frac{\prod\limits_{i\in\mathbb{Z}_{n-1}\setminus(\mathcal{X}\cup(\mathcal{X}+\{1\}))} \widetilde{B}_i^{\langle n\rangle} \cdot \prod\limits_{i\in\mathcal{X}} \widetilde{C}_i^{\langle n\rangle}}{\prod\limits_{i\in\mathbb{Z}_n\setminus(\{j\}\cup(\mathcal{X}+\{1\}))} \left(\widetilde{A}_j^{\langle n\rangle} - \widetilde{A}_i^{\langle n\rangle}\right)} \right) e^{\tau \widetilde{A}_j^{\langle n\rangle}} \\
&= \sum_{j=0}^{n} \left(\sum_{\mathcal{X}\in D_{n,j}} \frac{\prod\limits_{i\in\mathbb{Z}_{n-1}\setminus(\mathcal{X}\cup(\mathcal{X}+\{1\}))} \widetilde{B}_i^{\langle n\rangle} \cdot \prod\limits_{i\in\mathcal{X}} \widetilde{C}_i^{\langle n\rangle}}{\prod\limits_{i\in\mathbb{Z}_n\setminus(\{j\}\cup(\mathcal{X}+\{1\}))} \left(\widetilde{A}_j^{\langle n\rangle} - \widetilde{A}_i^{\langle n\rangle}\right)} \right) \left(\lim_{T\to\infty} e^{\tau \widetilde{A}_j^{\langle n\rangle}} \right).
\end{aligned}$$

Since $\lim\limits_{T\to\infty} e^{\tau \widetilde{A}_j^{\langle n\rangle}} = 0$ for all $j < n$ and $\widetilde{A}_n^{\langle n\rangle} = 0$, $\lim\limits_{T\to\infty} e^{\tau \widetilde{A}_n^{\langle n\rangle}} = 1$, and we have

$$\lim_{T\to\infty} P_n^{\langle n\rangle}(\tau) = \sum_{\mathcal{X}\in D_{n,n}} \frac{\prod\limits_{i\in\mathbb{Z}_{n-1}\setminus(\mathcal{X}\cup(\mathcal{X}+\{1\}))} \widetilde{B}_i^{\langle n\rangle} \cdot \prod\limits_{i\in\mathcal{X}} \widetilde{C}_i^{\langle n\rangle}}{\prod\limits_{i\in\mathbb{Z}_n\setminus(\{n\}\cup(\mathcal{X}+\{1\}))} \left(-\widetilde{A}_i^{\langle n\rangle}\right)}. \tag{3.21}$$

From (3.20) and (3.21), we obtain (3.19). □

Remark 3.2. Since OU, CIR, and Jacobi processes satisfy the existing condition on parameters in Theorem 3.4, the n^{th} unconditional moments exist for all $n \in \mathbb{Z}^+$. However, for Fisher–Snedecor, reciprocal gamma, and Student processes, the n^{th} unconditional moments only exist up to order $n \le \frac{1}{a} + 1$. This existing condition agrees with the results developed by Forman and

Sørensen [68].

3.4 Mathematical Properties

This section presents the benefits of the results of Theorems 3.2 and 3.3, such as the first and second conditional moments, conditional variance, conditional mixed moments of order 2, conditional covariance of Pearson diffusion, and their combinations.

Example 3.1. The first and second conditional moments, $\mathbb{E}\left[X_T \mid X_t = x\right]$ and $\mathbb{E}\left[X_T^2 \mid X_t = x\right]$

Assume that parameters $\theta(t)$, $\mu(t)$, $a(t)$, $b(t)$, and $c(t)$ are time-dependent functions such that the inhomogeneous Pearson diffusion process (3.5) is well-defined for all $t \in [0, T]$. By employing $U^{\langle 1\rangle}(x,\tau)$ in Theorem 3.2, the first conditional moment is obtained:

$$\mathbb{E}\left[X_T \mid X_t = x\right] = P_0^{\langle 1\rangle}(\tau)\, x + P_1^{\langle 1\rangle}(\tau) = e^{-\int_0^\tau \theta(T-\xi)\, d\xi}\, x + \int_0^\tau \theta(T-\eta)\, \mu(T-\eta)\, e^{-\int_0^\eta \theta(T-\xi)\, d\xi}\, d\eta,$$

where $\tau = T - t$. The first moment does not involve parameters $a(t)$, $b(t)$, or $c(t)$, and thus, the formula of the conditional mean is the same for all classes. For constants θ, μ, a, b, and c, by applying $U_P^{\langle 1\rangle}(x,\tau)$ in Theorem 3.3, the first conditional moment becomes

$$\mathbb{E}\left[X_T \mid X_t = x\right] = e^{\tau \widetilde{A}_0^{\langle 1\rangle}} x + \left(\frac{e^{\tau \widetilde{A}_0^{\langle 1\rangle}}}{\widetilde{A}_0^{\langle 1\rangle} - \widetilde{A}_1^{\langle 1\rangle}} + \frac{e^{\tau \widetilde{A}_1^{\langle 1\rangle}}}{\widetilde{A}_1^{\langle 1\rangle} - \widetilde{A}_0^{\langle 1\rangle}} \right) \widetilde{B}_0^{\langle 1\rangle} = e^{-\tau\theta} x + \mu\left(1 - e^{-\tau\theta}\right),$$

which agrees with the results in [20, 32]. Similarly, applying $U^{\langle 2\rangle}(x,\tau)$ in Theorem 3.2 yields the second conditional moment:

$$\mathbb{E}\left[X_T^2 \mid X_t = x\right] = P_0^{\langle 2\rangle}(\tau)\, x^2 + P_1^{\langle 2\rangle}(\tau)\, x + P_2^{\langle 2\rangle}(\tau),$$

where

$$\begin{aligned}
P_0^{\langle 2\rangle}(\tau) &= e^{2\int_0^\tau \theta(T-\xi)\,(a(T-\xi)-1)\, d\xi} \\
P_1^{\langle 2\rangle}(\tau) &= 2e^{-\int_0^\tau \theta(T-\xi)\, d\xi} \int_0^\tau \theta(T-\zeta)\, \left(b(T-\zeta) + \mu(T-\zeta)\right) e^{\int_0^\zeta \theta(T-\xi)\,(2a(T-\xi)-1)\, d\xi}\, d\zeta, \\
P_2^{\langle 2\rangle}(\tau) &= \int_0^\tau 2\theta(T-\eta) \Bigg(\mu(T-\eta)\, e^{-\int_0^\eta \theta(T-\xi)\, d\xi} \int_0^\eta \theta(T-\zeta)(b(T-\zeta) \\
&\quad + \mu(T-\zeta)) e^{\int_0^\zeta \theta(T-\xi)(2a(T-\xi)-1)\, d\xi}\, d\zeta + c(T-\eta)\, e^{2\int_0^\eta \theta(T-\xi)\,(a(T-\xi)-1)\, d\xi} \Bigg)\, d\eta.
\end{aligned}$$

For the constants θ, μ, a, b, and c, by applying $U_P^{\langle 2\rangle}(x,\tau)$ in Theorem 3.3, the second conditional moment becomes

$$\mathbb{E}\left[X_T^2 \mid X_t = x\right] = e^{\tau \widetilde{A}_0^{\langle 2\rangle}} x^2 + \left(\frac{e^{\tau \widetilde{A}_0^{\langle 2\rangle}}}{\widetilde{A}_0^{\langle 2\rangle} - \widetilde{A}_1^{\langle 2\rangle}} + \frac{e^{\tau \widetilde{A}_1^{\langle 2\rangle}}}{\widetilde{A}_1^{\langle 2\rangle} - \widetilde{A}_0^{\langle 2\rangle}}\right) \widetilde{B}_0^{\langle 2\rangle} x + \left(\frac{e^{\tau \widetilde{A}_0^{\langle 2\rangle}}}{\widetilde{A}_0^{\langle 2\rangle} - \widetilde{A}_2^{\langle 2\rangle}} + \frac{e^{\tau \widetilde{A}_2^{\langle 2\rangle}}}{\widetilde{A}_2^{\langle 2\rangle} - \widetilde{A}_0^{\langle 2\rangle}}\right) \widetilde{C}_0^{\langle 2\rangle}$$
$$+ \left(\frac{e^{\tau \widetilde{A}_0^{\langle 2\rangle}}}{\left(\widetilde{A}_0^{\langle 2\rangle} - \widetilde{A}_1^{\langle 2\rangle}\right)\left(\widetilde{A}_0^{\langle 2\rangle} - \widetilde{A}_2^{\langle 2\rangle}\right)} + \frac{e^{\tau \widetilde{A}_1^{\langle 2\rangle}}}{\left(\widetilde{A}_1^{\langle 2\rangle} - \widetilde{A}_0^{\langle 2\rangle}\right)\left(\widetilde{A}_1^{\langle 2\rangle} - \widetilde{A}_2^{\langle 2\rangle}\right)} + \frac{e^{\tau \widetilde{A}_2^{\langle 2\rangle}}}{\left(\widetilde{A}_2^{\langle 2\rangle} - \widetilde{A}_0^{\langle 2\rangle}\right)\left(\widetilde{A}_2^{\langle 2\rangle} - \widetilde{A}_1^{\langle 2\rangle}\right)}\right) \widetilde{B}_0^{\langle 2\rangle} \widetilde{B}_1^{\langle 2\rangle}$$
$$= e^{2\theta(a-1)\tau} x^2 + \frac{2(b+\mu)}{2a-1}\left(e^{2\theta(a-1)\tau} - e^{-\theta\tau}\right) x + \frac{c}{a-1}\left(e^{2\theta(a-1)\tau} - 1\right)$$
$$+ \frac{\mu(b+\mu)}{(a-1)(2a-1)}\left(e^{2\theta(a-1)\tau} + 2(a-1)e^{-\theta\tau} - 2a + 1\right).$$

When $a = 1$, the second conditional moment becomes

$$\mathbb{E}\left[X_T^2 \mid X_t = x\right] = x^2 + 2(b+\mu)\left(1 - e^{-\theta\tau}\right) x + 2\theta\left(\mu(b+\mu)\left(\frac{e^{-\theta\tau} - 1}{\theta} + \tau\right) + c\tau\right)$$

and when $a = \frac{1}{2}$,

$$\mathbb{E}\left[X_T^2 \mid X_t = x\right] = e^{-\theta\tau} x^2 + 2\theta\tau(b+\mu)\, e^{-\theta\tau} x + 2e^{-\theta\tau}\left(\mu(b+\mu)\left(-\theta\tau + e^{\theta\tau} - 1\right) + c\left(e^{\theta\tau} - 1\right)\right).$$

These moments obtained by Theorem 3.3 are simpler than other results in the existing literature, such as [14, 32, 43], because they do not require solving recurrence differential equations. Moreover, this example contains useful information for investigating the other mathematical properties in the next examples.

Example 3.2. The conditional variance and n^{th} central moments

Based on inhomogeneous Pearson diffusion (3.5), the conditional variance of the diffusion can be expressed as

$$\operatorname{Var}\left[X_T \mid X_t = x\right] := \mathbb{E}\left[\left(X_T - \mathbb{E}\left[X_T \mid X_t\right]\right)^2 \mid X_t = x\right] = U^{\langle 2\rangle}(x,\tau) - \left(U^{\langle 1\rangle}(x,\tau)\right)^2,$$

where $U^{\langle 1\rangle}(x,\tau)$ and $U^{\langle 2\rangle}(x,\tau)$ are provided in Example 3.1.

For Pearson diffusion (3.1), the variance becomes

$$\operatorname{Var}\left[X_T \mid X_t = x\right] = e^{2(a-1)\theta\tau} x^2 + \frac{2(b+\mu)e^{-2\theta\tau}}{2a-1}\left(e^{2a\theta\tau} - e^{\theta\tau}\right) x - \left(e^{-\theta\tau} x + \mu\left(1 - e^{-\tau\theta}\right)\right)^2$$
$$+ \frac{\mu(b+\mu)}{2a-1}\left(\frac{e^{2(a-1)\theta\tau} - 1}{a-1} + 2\left(e^{-\theta\tau} - 1\right)\right) + \frac{c}{a-1}\left(e^{2(a-1)\theta\tau} - 1\right).$$

In general, the n^{th} moment about the mean (n^{th} central moment) is expressed as

$$\boldsymbol{\mu}_n(x,\tau) := \mathbb{E}\left[(X_T - \mathbb{E}\left[X_T \mid X_t\right])^n \mid X_t = x\right] = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \left(U^{\langle j \rangle}(x,\tau)\right) \left(U^{\langle 1 \rangle}(x,\tau)\right)^{n-j}$$

where the conditional zeroth moment $U^{\langle 0 \rangle}(x,\tau) = 1$. The first few central moments have intuitive interpretations: $\boldsymbol{\mu}_0$, known as the conditional zeroth central moment, is 1; the first central conditional moment $\boldsymbol{\mu}_1$ is 0; the conditional second central moment $\boldsymbol{\mu}_2$ is called the conditional variance; and for higher orders, the third and the fourth conditional central moments are used to define conditional standardized moments, known as the skewness and kurtosis, respectively. As a practical application, skewness and kurtosis swaps are currently used in trading, especially their moment swaps, and they have been widely studied (see Chumpong et al. [14]).

Example 3.3. The conditional mixed moment of integer order

By using the tower property for $0 \le t < T_1 \le T_2$, where $\tau_1 = T_1 - t$ and $\tau_2 = T_2 - T_1$, the conditional mixed second moment of inhomogeneous Pearson diffusion (3.5) can be expressed as

$$\mathbb{E}\left[X_{T_1} X_{T_2} \mid X_t = x\right] = \mathbb{E}\left[X_{T_1} \mathbb{E}\left[X_{T_2} \mid X_{T_1}\right] \mid X_t = x\right].$$

After applying Theorem 3.3 twice, we have

$$\begin{aligned}
\mathbb{E}\left[X_{T_1} X_{T_2} \mid X_t = x\right] &= \sum_{k=0}^{1} \sum_{j=0}^{2-k} P_k^{\langle 1 \rangle}(\tau_2)\, P_j^{\langle 2-k \rangle}(\tau_1)\, x^{2-k-j} \\
&= P_0^{\langle 1 \rangle}(\tau_2)\, P_0^{\langle 2 \rangle}(\tau_1)\, x^2 + P_0^{\langle 1 \rangle}(\tau_2)\, P_1^{\langle 2 \rangle}(\tau_1)\, x + P_0^{\langle 1 \rangle}(\tau_2)\, P_2^{\langle 2 \rangle}(\tau_1) \\
&\quad + P_1^{\langle 1 \rangle}(\tau_2)\, P_0^{\langle 1 \rangle}(\tau_1)\, x + P_1^{\langle 1 \rangle}(\tau_2)\, P_1^{\langle 1 \rangle}(\tau_1), \qquad (3.22)
\end{aligned}$$

where parameter functions dependent on time are given in Example 3.1. Moreover, the generality of the formula for conditional mixed moments, $\mathbb{E}\left[X_{T_1}^{n_1} X_{T_2}^{n_2} \cdots X_{T_k}^{n_k} \mid X_t = x\right]$, where $n_1, n_2, \ldots n_k \in \mathbb{Z}^+$ and $0 \le t < T_1 \le T_2 \le \cdots \le T_k$, can be analytically derived by using Theorem 3.3, as well. The advantage of the mixed moment (3.22) is its simple closed form, which can be used to estimate functions of the powers of the observed processes presented in Sørensen [70], Leonenko and Šuvak [51, 52], and Avram et al. [4]. In addition, in order to study integrated Pearson diffusion processes, the mixed moments need to be calculated; however, the closed-form formula is not available (see Forman and Sørensen [32]), and thus, (3.22) can be applied directly. This result is also useful for the estimation of parameters; for instance, Gouriéroux and Valéry [34] used conditional mixed moments to implement the method of moments and thereby captured various features of observed data, such as the risk premium and possible volatility persistence.

Example 3.4. The conditional covariance and correlation

The conditional covariance of inhomogeneous Pearson diffusion (3.5) is defined by

$$\begin{aligned}\operatorname{Cov}\left[X_{T_1}, X_{T_2} \mid X_t = x\right] &:= \mathbb{E}\left[\left(X_{T_1} - \mathbb{E}\left[X_{T_1} \mid X_t\right]\right)\left(X_{T_2} - \mathbb{E}\left[X_{T_2} \mid X_t\right]\right) \mid X_t = x\right] \\ &= \mathbb{E}\left[X_{T_1} X_{T_2} \mid X_t = x\right] - \mathbb{E}\left[X_{T_1} \mid X_t = x\right]\mathbb{E}\left[X_{T_2} \mid X_t = x\right]\end{aligned}$$

for $0 \leq t < T_1 \leq T_2$, where $\tau = T_2 - t$, $\tau_1 = T_1 - t$, and $\tau_2 = T_2 - T_1$. Applying the results from Examples 3.1 and 3.3 yields

$$\operatorname{Cov}\left[X_{T_1}, X_{T_2} \mid X_t = x\right] = \sum_{k=0}^{1}\sum_{j=0}^{2-k} P_k^{\langle 1\rangle}(\tau_2)\, P_j^{\langle 2-k\rangle}(\tau_1)\, x^{2-k-j} - U^{\langle 1\rangle}(x,\tau_1)\, U^{\langle 1\rangle}(x,\tau_2). \quad (3.23)$$

The conditional correlation of the diffusion is defined by

$$\operatorname{Corr}\left[X_{T_1}, X_{T_2} \mid X_t = x\right] := \frac{\operatorname{Cov}\left[X_{T_1}, X_{T_2} \mid X_t = x\right]}{\sqrt{\operatorname{Var}[X_{T_1} \mid X_t = x]}\sqrt{\operatorname{Var}[X_{T_2} \mid X_t = x]}}.$$

Applying the results in Examples 3.2 and (3.23) yields

$$\operatorname{Corr}\left[X_{T_1}, X_{T_2} \mid X_t = x\right] = \frac{\sum_{k=0}^{1}\sum_{j=0}^{2-k} P_k^{\langle 1\rangle}(\tau_2)\, P_j^{\langle 2-k\rangle}(\tau_1)\, x^{2-k-j} - U^{\langle 1\rangle}(x,\tau_1)\, U^{\langle 1\rangle}(x,\tau_2)}{\sqrt{U^{\langle 2\rangle}(x,\tau_1) - \left(U^{\langle 1\rangle}(x,\tau_1)\right)^2}\sqrt{U^{\langle 2\rangle}(x,\tau) - \left(U^{\langle 1\rangle}(x,\tau)\right)^2}}. \quad (3.24)$$

We can generalize (3.23) and (3.24) by using Examples 3.1, 3.2, and 3.3 as the closed forms of $\operatorname{Cov}\left[X_{T_1}^{n_1}, X_{T_2}^{n_2} \mid X_t = x\right]$ and $\operatorname{Corr}\left[X_{T_1}^{n_1}, X_{T_2}^{n_2} \mid X_t = x\right]$, where n_1 and n_2 are positive integers. Note that Examples 3.1–3.4 provide explicit forms of the estimators of unknown parameters, which can be obtained by the GMM (see [4, 31, 51, 52, 70]).

Example 3.5. Approximation using Taylor expansion

Suppose that f is a real analytic function on an open interval and infinitely differentiable at x_0 in its domain such that the Taylor series at x_0 converges pointwise to $f(x)$ for x in a neighborhood of x_0. For example, by Taylor expansion, $f(x) = \sqrt{x}$ can be expressed as

$$\sqrt{x} = \sqrt{x_0} + \frac{(x-x_0)}{2\sqrt{x_0}} - \frac{(x-x_0)^2}{8\sqrt{x_0}^3} + \frac{(x-x_0)^3}{16\sqrt{x_0}^5} - \frac{5(x-x_0)^4}{128\sqrt{x_0}^7} + \mathcal{O}\left(\left(\frac{x-x_0}{x_0}\right)^5\right).$$

Let X_t follow inhomogeneous Pearson diffusion (3.5) for $0 \leq t \leq T$. Based on the first three terms of the expansion, substituting $x = X_T$ and $x_0 = \mathbb{E}\left[X_T \mid X_t = x\right]$ and taking the conditional expectation yields

$$\mathbb{E}\left[\sqrt{X_T} \mid X_t = x\right] \approx \sqrt{U^{\langle 1\rangle}(x,\tau)} + \frac{\boldsymbol{\mu}_1(x,\tau)}{2\sqrt{U^{\langle 1\rangle}(x,\tau)}} - \frac{\boldsymbol{\mu}_2(x,\tau)}{8\sqrt{U^{\langle 1\rangle}(x,\tau)}^3},$$

where $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ are given in Example 3.2. Since $\boldsymbol{\mu}_1 = 0$, and $\boldsymbol{\mu}_2$ is the conditional variance, we have

$$\mathbb{E}\left[\sqrt{X_T} \mid X_t = x\right] \approx \sqrt{U^{\langle 1\rangle}(x,\tau)} - \frac{\mathrm{Var}(x,\tau)}{8\sqrt{U^{\langle 1\rangle}(x,\tau)}^3}.$$

This approach can also be applied to other functions, such as exponential, logarithmic, or trigonometric functions. There is no empirical analysis for the convergence condition of utilizing the Taylor expansion in a stochastic environment in practice. Moreover, there is no empirical evidence that higher-order Taylor expansions will provide better accuracy to approximate functions of stochastic variables.

3.5 Classification of Pearson diffusion processes

In this section, concise forms of the formulas in Section 3.3 are further investigated for all six classes of Pearson diffusion processes according to Theorems 3.3 and 3.4. Since the formulas proposed in this section are the consequences of Theorems 3.3 and 3.4, validation based on MC is not necessary and is omitted.

3.5.1 Ornstein–Uhlenbeck diffusion

As mentioned in the first section, the OU diffusion process has the polynomial degree $d = 0$, corresponding to $a = b = 0$ in (3.1), which has the form

$$dX_t = \theta\,(\mu - X_t)\,dt + \sqrt{2\theta c}\,dW_t, \tag{3.25}$$

where $\mu \in \mathbb{R}$ and $\theta > 0$. The classical process of Vasicek [76] can describe the short rate through the OU process (3.25). With the mean reversion property of the drift term, X_t is pulled toward the μ level, and the OU process becomes a basic model in pricing applications, for which μ can be interpreted as the long-run (mean) interest rate, and θ serves as the speed of reversion. The conditional moments of the OU diffusion process can be useful for investors who want to price financial derivatives based on assets described by the Schwartz model [69]. In 2017, Weraprasertsakun and Rujivan [78] proposed only the first and second conditional moments by solving recursive ODEs. Recently, a simple formula for conditional moments was presented by Chumpong et al. [14]. Even though their formulas are considered very general for integer moments, solving the system of recursive equations is required, which might be complicated when computing higher moments. In contrast to formulas in the literature, our formula in Theorem 3.3

is applied without solving any recursive equations, thus efficiently resolving the issue. Based on Examples 3.1 and 3.2, we obtain the following for OU diffusion:

$$\mathbb{E}\left[X_T \mid X_t = x\right] = e^{-\theta\tau}x + \mu\left(1 - e^{-\tau\theta}\right),$$
$$\mathbb{E}\left[X_T^2 \mid X_t = x\right] = e^{-2\theta\tau}x^2 + 2\mu e^{-2\theta\tau}\left(e^{\theta\tau} - 1\right)x + e^{-2\theta\tau}\left(c\left(e^{2\theta\tau} - 1\right) + \mu^2\left(e^{\theta\tau} - 1\right)^2\right),$$
$$\operatorname{Var}\left[X_T \mid X_t = x\right] = c\left(1 - e^{-2\theta\tau}\right).$$

Note that conditional moments can be obtained directly from the transition PDF, which is known to have the form of Hermite polynomials as orthogonal eigenfunctions with respect to the standard Gaussian density (see [12, 49]). For the OU process (3.25), the transition PDF can be rewritten in terms of a Gaussian distribution as

$$p(y, T \mid x_t, t) = \frac{1}{\sqrt{2c\pi\left(1 - e^{-2\theta\tau}\right)}} \exp\left(-\frac{\left(y - \mu - (x_t - \mu)e^{-\theta\tau}\right)^2}{2c\left(1 - e^{-2\theta\tau}\right)}\right), \tag{3.26}$$

where $\tau = T - t$. Alternatives for obtaining conditional and unconditional moments of the OU process derived from the transition PDF are described in the following theorems.

Theorem 3.5. *Suppose that X_t follows the OU process (3.25). The n^{th} conditional moment for $n \in \mathbb{Z}_0^+$ is*

$$\hat{U}_O^{\langle n\rangle}(x, \tau) := \mathbb{E}\left[X_T^n \mid X_t = x\right] = \sum_{k=0}^{n}\left(\frac{1 + (-1)^{n-k}}{2}\right)\binom{n}{k}\frac{\Gamma\left(\frac{n-k+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}A^k B^{\frac{n-k}{2}}, \tag{3.27}$$

for $\tau = T - t \geq 0$, where $A = \mu + (x - \mu)e^{-\theta\tau}$ and $B = 2c\left(1 - e^{-2\theta\tau}\right)$.

Proof. Suppose that $u = \frac{y-A}{\sqrt{B}}$; then, we have

$$\begin{aligned}
\mathbb{E}\left[X_T^n \mid X_t = x\right] &= \int_{-\infty}^{\infty} y^n \, p(y, T \mid x, t)\, dy \\
&= \int_{-\infty}^{\infty} y^n \frac{1}{\sqrt{\pi B}} e^{-\frac{(y-A)^2}{B}}\, dy \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(A + \sqrt{B}u\right)^n e^{-u^2}\, du \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \left(\sqrt{B}u\right)^{n-k} A^k e^{-u^2}\, du \\
&= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{n} \binom{n}{k} A^k B^{\frac{n-k}{2}} \int_{-\infty}^{\infty} u^{n-k} e^{-u^2}\, du.
\end{aligned}$$

With respect to the integrand, $u^{n-k}e^{-u^2}$ is an even function when $n - k$ is even, and $u^{n-k}e^{-u^2}$

is an odd function when $n-k$ is odd. Thus,

$$\begin{aligned}\mathbb{E}\left[X_T^n \mid X_t = x\right] &= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{n} \binom{n}{k} A^k B^{\frac{n-k}{2}} \left(1 + (-1)^{n-k}\right) \int_0^\infty u^{n-k} e^{-u^2}\, du \\ &= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{n} \binom{n}{k} A^k B^{\frac{n-k}{2}} \left(\frac{1 + (-1)^{n-k}}{2}\right) \int_0^\infty \left(u^2\right)^{\frac{n-k+1}{2}-1} e^{-u^2}\, d\left(u^2\right).\end{aligned}$$

Applying the Euler integral of the second kind [40] to the integral term, we obtain (3.27) as the result. □

Theorem 3.6. *Suppose that X_t follows the OU process* (3.25)*. The n^{th} unconditional moment at equilibrium, for all $n \in \mathbb{Z}_0^+$ and $\tau = T - t \geq 0$, is given by*

$$\begin{aligned}\hat{L}_O^{\langle n\rangle} := \lim_{\tau\to\infty} \hat{U}_O^{\langle n\rangle}(x,\tau) &= \lim_{T\to\infty} \mathbb{E}\left[X_T^n \mid X_t = x\right] \\ &= \sum_{k=0}^{n} \left(\frac{1 + (-1)^{n-k}}{2}\right) \binom{n}{k} \frac{\Gamma\left(\frac{n-k+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \mu^k (2c)^{\frac{n-k}{2}}. \end{aligned} \tag{3.28}$$

Proof. The proof follows from Theorem 3.5, since $A = \mu$ and $B = 2c$ as $\tau \to \infty$. □

3.5.2 Squared diffusion

In 1951, Feller [25] studied a class of diffusion processes, including the square-root diffusion, later called the Cox–Ingersoll–Ross (CIR) process:

$$dX_t = \theta\left(\mu - X_t\right) dt + \sqrt{2\theta b X_t}\, dW_t. \tag{3.29}$$

The CIR process was introduced by Cox, Ingersoll, and Ross [18] as a process to describe the short rate. As with the OU process (3.25), the drift terms mean that the short rates X_t are pulled back toward μ at the speed of adjustment θ. In contrast to the OU process, in the CIR process, X_t approaches zero as well as its diffusion term. Because of this feature of the process, X_t avoids non-negative values, which is the reason that the process quickly became famous for studying the behavior of interest rates, such as in Hull and White [39], Filipović [26, 27], Filipović and Mayerhofer [30], and Alfonsi [2]. For the conditional moments of the CIR process, Dufresne [22] proposed a closed-form formula for $\mathbb{E}^p\left[X_T^\gamma \mid \mathcal{F}_t\right] = \mathbb{E}^p\left[X_T^\gamma \mid X_t = x\right]$ for real $\gamma > -\frac{\mu}{b}$. An analytical approach presented by Rujivan [64] extended Dufresne's work to the ECIR process for real $\gamma \in \mathbb{R}$. Recently, an exact formula for conditional expectations of the product of polynomial and exponential functions of the affine transform $\mathbb{E}^P\left[X_T^n e^{\alpha X_T} \mid X_t = x\right]$ was proposed by Sutthimat et al. [72] for the ECIR process for any real γ and α, which generalizes the Rujivan result [64]. However, this section provides an alternative formula for conditional moments, which is simpler than Rujivan's result [64].

In order to ensure the existence of a pathwise unique strong solution for the process X_t in (3.29) and to avoid zero almost surely with respect to probability measure P for all $t \in [0, T]$, the two following assumptions proposed by Maghsoodi [55] are required (see details in Theorems 2.1 and 2.4 in [55]). In practice, for the ECIR process (3.5), where $a(t) = c(t) = 0$, we require the functions $\theta(t), \mu(t)$, and $b(t)$ to be strictly positive and smooth on $[0, T]$, with $\frac{1}{b(t)}$ being locally bounded and $\mu(t) \geq b(t)$ on $[0, T]$.

The following theorems are consequences of Theorems 3.3 and 3.4 for the CIR process.

Theorem 3.7. *Suppose that X_t follows the CIR process* (3.29). *The n^{th} conditional moment for $n \in \mathbb{Z}_0^+$ is defined by*

$$U_C^{\langle n \rangle}(x, \tau) := \mathbb{E}\left[x_T^n \mid X_t = x\right] = e^{-n\theta\tau} x^n + \sum_{k=1}^{n} \frac{e^{-n\theta\tau}\left(e^{\theta\tau} - 1\right)^k}{k!\,\theta^k} \left(\prod_{i=0}^{k-1} \widetilde{B}_i^{\langle n \rangle}\right) x^{n-k}, \tag{3.30}$$

for $x > 0$ and $\tau = T - t \geq 0$.

Proof. We show that, from Theorem 3.3, the coefficients $P_k^{\langle n \rangle}$ in (3.13) are those defined in (30) for the CIR process (29). The cases of k = 0, 1 are straightforward according to the coefficients in (3.14) and (3.15) when $a = c = 0$. Here, we consider the case of $k = 2, 3, \ldots, n$. Since $a = c = 0$ in (3.15), $\widetilde{A}_j^{\langle n \rangle} = -\theta\,(n - j)$ and $\widetilde{C}_j^{\langle n \rangle} = 0$ for all $j \leq n$; then,

$$P_k^{\langle n \rangle}(\tau) = \sum_{j=0}^{k} \left(\sum_{\mathcal{X} \in D_{k,j}} \frac{\displaystyle\prod_{i \in \mathbb{Z}_{k-1} \setminus (\mathcal{X} \cup (\mathcal{X} + \{1\}))} \widetilde{B}_i^{\langle n \rangle} \cdot \prod_{i \in \mathcal{X}} \widetilde{C}_i^{\langle n \rangle}}{\displaystyle\prod_{i \in \mathbb{Z}_k \setminus (\{j\} \cup (\mathcal{X} + \{1\}))} \left(\widetilde{A}_j^{\langle n \rangle} - \widetilde{A}_i^{\langle n \rangle}\right)} \right) e^{\tau \widetilde{A}_j^{\langle n \rangle}} = 0, \tag{3.31}$$

for $\mathcal{X} \neq \emptyset$. For the case of $\mathcal{X} = \emptyset$, the term $\prod_{i \in \mathcal{X}} \widetilde{C}_i^{\langle n \rangle} = 1$ and (3.31) can be reduced to

$$\begin{aligned}
P_k^{\langle n \rangle}(\tau) &= \sum_{j=0}^{k} \frac{e^{\tau \widetilde{A}_j^{\langle n \rangle}}}{\prod\limits_{\substack{i=0 \\ i \neq j}}^{k} \left(\widetilde{A}_j^{\langle n \rangle} - \widetilde{A}_i^{\langle n \rangle}\right)} \prod_{i=0}^{k-1} \widetilde{B}_i^{\langle n \rangle} \\
&= \sum_{j=0}^{k} \frac{e^{-\theta\tau(n-j)}}{\prod\limits_{\substack{i=0 \\ i \neq j}}^{k} \theta(j-i)} \prod_{i=0}^{k-1} \widetilde{B}_i^{\langle n \rangle} \\
&= \frac{e^{-n\theta\tau}}{\theta^k} \sum_{j=0}^{k} \frac{e^{j\theta\tau}}{j!\,(k-j)!(-1)^{j-k}} \prod_{i=0}^{k-1} \widetilde{B}_i^{\langle n \rangle} \\
&= \frac{e^{-n\theta\tau}}{k!\,\theta^k} \sum_{j=0}^{k} \frac{k!(-1)^{k-j} e^{j\theta\tau}}{j!\,(k-j)!} \prod_{i=0}^{k-1} \widetilde{B}_i^{\langle n \rangle} \\
&= \frac{e^{-n\theta\tau}\left(e^{\theta\tau} - 1\right)^k}{k!\,\theta^k} \prod_{i=0}^{k-1} \widetilde{B}_i^{\langle n \rangle},
\end{aligned} \tag{3.32}$$

which yields (3.30). □

Based on Theorem 3.7 and Example 3.2, we obtain the following for CIR diffusion:

$$\begin{aligned}
&\mathbb{E}\left[X_T \mid X_t = x\right] = e^{-\theta\tau}x + \mu\left(1 - e^{-\tau\theta}\right),\\
&\mathbb{E}\left[X_T^2 \mid X_t = x\right] = e^{-2\theta\tau}x^2 + 2(b+\mu)e^{-2\theta\tau}\left(e^{\theta\tau}-1\right)x + \mu(b+\mu)e^{-2\theta\tau}\left(e^{\theta\tau}-1\right)^2,\\
&\mathrm{Var}\left[X_T \mid X_t = x\right] = 2be^{-2\theta\tau}\left(e^{\theta\tau}-1\right)x + \mu b e^{-2\theta\tau}\left(e^{\theta\tau}-1\right)^2.
\end{aligned}$$

Theorem 3.8. *Suppose that X_t follows the CIR process* (3.29)*. The n^{th} unconditional moment at equilibrium, for all $n \in \mathbb{Z}_0^+$, $x > 0$, and $\tau = T - t \geq 0$, is given by*

$$L_C^{\langle n\rangle} := \lim_{\tau\to\infty} U_C^{\langle n\rangle}(x,\tau) = \lim_{T\to\infty} \mathbb{E}\left[X_T^n \mid X_t = x\right] = \prod_{j=0}^{n-1}(\mu + jb). \tag{3.33}$$

Proof. According to (3.30) in Theorem 3.7, the coefficient terms of x^{n-k} approach 0 as $\tau \to \infty$ for $k = 0, 1, 2, \ldots, n-1$, except in the case of $k = n$. Thus,

$$\begin{aligned}
L_C^{\langle n\rangle} &= \lim_{\tau\to\infty} \frac{e^{-n\theta\tau}\left(e^{\theta\tau}-1\right)^n}{n!\,\theta^n}\left(\prod_{i=0}^{n-1}\widetilde{B}_i^{\langle n\rangle}\right)\\
&= \frac{1}{n!\,\theta^n}\prod_{i=0}^{n-1}\widetilde{B}_i^{\langle n\rangle}\lim_{\tau\to\infty}\left(1 - e^{-\theta\tau}\right)^n\\
&= \frac{1}{n!\,\theta^n}\prod_{i=0}^{n-1}\widetilde{B}_i^{\langle n\rangle}.
\end{aligned}$$

By substituting $\widetilde{B}_i^{\langle n\rangle}$ in (3.15) into the equation above, the product can be simplified to obtain the form of (3.33). □

Similar to the OU process, the conditional moments of the CIR process can be derived directly from the transition PDF, which is known to have the form of a Laguerre polynomial and the gamma density function (see [12, 49]). The significant difference between OU and CIR is that the transition PDF for the CIR process is not in a concise form. The eigenvalue expansion for the transition PDF of the CIR process can be written in the form

$$p(y, T \mid x_t, t) = c_\tau\, e^{-(u+v)}\left(\frac{v}{u}\right)^{q/2} I_q\left(2\sqrt{uv}\right), \tag{3.34}$$

where

$$\tau = T - t, \quad c_\tau = \frac{1}{b\left(1 - e^{-\theta\tau}\right)}, \quad u = c_\tau\, x_t\, e^{-\theta\tau}, \quad v = c_\tau\, y, \quad q = \frac{\mu}{b} - 1, \tag{3.35}$$

and $I_q(\cdot)$ is the Bessel function of the first kind:

$$I_q(y) = \sum_{k=0}^{\infty} \left(\frac{y}{2}\right)^{2k+q} \frac{1}{\Gamma(k+1)\,\Gamma(k+q+1)}.$$

Based on the transition PDF, an alternative for conditional moments of the CIR process is given in the following theorem.

Theorem 3.9. *Suppose that X_t follows the CIR process* (3.29). *The n^{th} conditional moment for $n \in \mathbb{Z}_0^+$ is*

$$\hat{U}_C^{\langle n \rangle}(x,\tau) := \mathbb{E}\left[X_T^n \mid X_t = x\right] = c_\tau^{-n}\, e^{-u} \sum_{k=0}^{\infty} \frac{u^k\, \Gamma(n+k+q+1)}{\Gamma(k+1)\,\Gamma(k+q+1)}, \tag{3.36}$$

for $x > 0$, $\tau = T - t$, and $u = c_\tau\, e^{-\theta\tau} x$, where c_τ is given in (3.35).

Proof. By using the transition PDF, we have

$$\begin{aligned}
\mathbb{E}\left[X_T^n \mid X_t = x\right] &= \int_0^\infty y^n p(y,T \mid x,t)\, dy \\
&= \int_0^\infty c_\tau^{-n} v^n c_\tau\, e^{-(u+v)} \left(\frac{v}{u}\right)^{q/2} I_q\left(2\sqrt{uv}\right) \frac{1}{c_\tau}\, dv \\
&= \int_0^\infty c_\tau^{-n} v^n\, e^{-(u+v)} \left(\frac{v}{u}\right)^{q/2} \sum_{k=0}^{\infty} (uv)^{k+q/2} \frac{1}{\Gamma(k+1)\,\Gamma(k+q+1)}\, dv \\
&= c_\tau^{-n}\, e^{-u} \sum_{k=0}^{\infty} \frac{u^k}{\Gamma(k+1)\,\Gamma(k+q+1)} \int_0^\infty e^{-v} v^{n+k+q}\, dv.
\end{aligned}$$

Applying the Euler integral of the second kind to the integral term, we obtain the result (3.36).

□

Next, by considering the marginal density of x given in (3.34) as $\tau \to \infty$, we have

$$c_\tau \to \frac{1}{b}, \quad u \to 0, \quad v \to \frac{x}{b}.$$

Thus, the steady-state density function is the gamma distribution with the shape parameter $\frac{\mu}{b}$ and scale parameter b,

$$\pi(y) := \lim_{\tau\to\infty} p\left(y,T \mid x_t,t\right) = \frac{1}{b}\left(\frac{v}{u}\right)^{\frac{q}{2}} I_q\left(2\sqrt{uv}\right) = \frac{e^{-\frac{y}{b}}}{b\,\Gamma\left(\frac{\mu}{b}\right)}\left(\frac{y}{b}\right)^{\frac{\mu}{b}-1}, \tag{3.37}$$

and the unconditional moment for x is in the following theorem.

Theorem 3.10. *Suppose that X_t follows the CIR process* (3.29). *The n^{th} unconditional moment* *at equilibrium, for all $n \in \mathbb{Z}_0^+$, $x > 0$, and $\tau = T - t \geq 0$, is given by*

$$\hat{L}_C^{\langle n\rangle} := \lim_{T\to\infty} \mathbb{E}\left[X_T^n \mid X_t = x\right] = \frac{b^n\,\Gamma\left(n+\frac{\mu}{b}\right)}{\Gamma\left(\frac{\mu}{b}\right)}. \tag{3.38}$$

Proof. The proof follows from (3.37). □

Remark 3.3. Note that formula (3.36) obtained from the transition PDF involves the infinite sum, in contrast to formula (3.30) in Theorem 3.7. We further investigated formula (3.36) and reduced it to a finite sum by employing a confluent hypergeometric function ${}_1F_1$. We show that formula (3.36) can be transformed into formula (3.33) as follows.

$$\begin{aligned}\hat{U}_C^{\langle n\rangle}(x,\tau) &= c_\tau^{-n}\,e^{-u}\sum_{k=0}^{\infty}\frac{u^k\,\Gamma(n+k+q+1)}{\Gamma(k+1)\,\Gamma(k+q+1)}\\ &= \frac{c_\tau^{-n}\,\Gamma\left(n+\frac{\mu}{b}\right)e^{-c_\tau e^{-\theta\tau}x}}{\Gamma\left(\frac{\mu}{b}\right)}\,{}_1F_1\left(n+\frac{\mu}{b};\frac{\mu}{b};c_\tau e^{-\theta\tau}x\right).\end{aligned}$$

By referring to the identity ${}_1F_1(r;s;z) = e^z\,{}_1F_1(s-r;s;-z)$ and the relation between Laguerre polynomials and the confluent hypergeometric function (see more details Lebedev [48]), we have

$$\begin{aligned}\hat{U}_C^{\langle n\rangle}(x,\tau) &= \frac{c_\tau^{-n}\,\Gamma\left(n+\frac{\mu}{b}\right)e^{-c_\tau e^{-\theta\tau}}x}{\Gamma\left(\frac{\mu}{b}\right)}e^{c_\tau e^{-\theta\tau}x}\,{}_1F_1\left(-n;\frac{\mu}{b};-c_\tau e^{-\theta\tau}x\right)\\ &= \frac{c_\tau^{-n}\,\Gamma\left(n+\frac{\mu}{b}\right)}{\Gamma\left(\frac{\mu}{b}\right)}\sum_{k=0}^{n}\frac{(-n)_k\left(-c_\tau\,e^{-\theta\tau}x\right)^k}{\left(\frac{\mu}{b}\right)_k k!}\\ &= b^n\left(1-e^{-\theta\tau}\right)^n\frac{\Gamma\left(n+\frac{\mu}{b}\right)}{\Gamma\left(\frac{\mu}{b}\right)}\sum_{k=0}^{n}\frac{(-n)_k}{\left(\frac{\mu}{b}\right)_k}\frac{(-1)^k}{b^k\left(e^{\theta\tau}-1\right)^k k!}x^k\\ &= b^n e^{-n\theta\tau}\left(e^{\theta\tau}-1\right)^n\left(\frac{\mu}{b}\right)_n\sum_{k=0}^{n}\frac{(-n)_{n-k}}{\left(\frac{\mu}{b}\right)_{n-k}}\frac{(-1)^{n-k}\left(e^{\theta\tau}-1\right)^{-(n-k)}}{b^{n-k}(n-k)!}x^{n-k}\\ &= \sum_{k=0}^{n}e^{-n\theta\tau}\left(e^{\theta\tau}-1\right)^k b^k(-1)^{n-k}\left(\frac{\left(\frac{\mu}{b}\right)_n}{\left(\frac{\mu}{b}\right)_{n-k}}\frac{(-n)_{n-k}}{(n-k)!}\right)x^{n-k}.\\ &= \sum_{k=0}^{n}\frac{e^{-n\theta\tau}\left(e^{\theta\tau}-1\right)^k}{k!}\left(\prod_{i=0}^{k-1}(n-i)\left((n-i-1)\,b+\mu\right)\right)x^{n-k}\\ &= U_C^{\langle n\rangle}(x,\tau)\end{aligned}$$

where $(\lambda)_k = \lambda(\lambda+1)\cdots(\lambda+k-1)$ denotes the rising factorial, also known as the Pochhammer polynomial. This shows that our proposed formula (3.30) is identical to the formula from the transition PDF.

Remark 3.4. Similarly, one can show that formula (3.38) in Theorem 3.10 can be transformed into the closed-form formula (3.33) in Theorem 3.8.

3.5.3 Jacobi diffusion

The Jacobi diffusion process is considered one class of the Pearson diffusion process, which is sometimes called the generalized Jacobi diffusion process. It is associated with Jacobi polynomials through the Jacobi diffusion generator's eigenfunctions. The Jacobi diffusion process is a class of solvable diffusion processes whose solution follows the SDE:

$$dX_t = \theta\left(\mu - X_t\right)dt + \sqrt{2b\theta\, X_t\left(1 - X_t\right)}\,dW_t, \tag{3.39}$$

obtained from process (3.1) when $a = -b$ and $c = 0$ with $a < 0$. The values produced by this process are always in $[0, 1]$ with mean-reverting μ, and the boundary $\{0, 1\}$ is inaccessible if and only if $b \leq \mu \leq 1 - b$ (see Veraart and Veraart [77]). In the context of finance, Larsen and Sørensen [47] used the Jacobi diffusion process to price the central bank's exchange rates when the pay-off is expressed in log-prices. Moreover, Larsen and Sørensen [47] also used the diffusion process to model the dynamics of a correlation coefficient by setting $X_t = \frac{\rho_t + 1}{2}$. By Itô's lemma,

$$d\rho_t = \theta\left((2\mu - 1) - \rho_t\right)dt + \sqrt{2b\theta\,\left(1 + \rho_t\right)\left(1 - \rho_t\right)}\,dW_t.$$

Note that the values produced by the process above are always in $[-1, 1]$ with mean-reverting $2\mu - 1$. In particular, if we set $X_t = \frac{\rho_t - \rho_{\min}}{\rho_{\max} - \rho_{\min}}$, the obtained process ρ_t will have values in $[\rho_{\min}, \rho_{\max}]$.

The next two theorems for the Jacobi process (3.39) are consequences deduced from Theorem 3.3 for process (3.1) when $c = 0$.

Theorem 3.11. *Suppose that X_t follows the Jacobi process* (3.39). *The n^{th} conditional moment for $n \in \mathbb{Z}_0^+$ is*

$$U_J^{\langle n \rangle}(x, \tau) := \mathbb{E}\left[X_T^n \mid X_t = x\right] = e^{\tau \tilde{A}_0^{\langle n \rangle}} x^n + \sum_{k=1}^{n}\left(\sum_{j=0}^{k} \frac{e^{\tau \tilde{A}_j^{\langle n \rangle}}}{\prod_{\substack{l=0 \\ l \neq j}}^{k}\left(\tilde{A}_j^{\langle n \rangle} - \tilde{A}_l^{\langle n \rangle}\right)} \prod_{j=0}^{k-1} \tilde{B}_j^{\langle n \rangle}\right) x^{n-k}, \tag{3.40}$$

for $0 < x < 1$, $\tau = T - t \geq 0$, where $\tilde{A}_j^{\langle n \rangle}$ and $\tilde{B}_j^{\langle n \rangle}$ are given in (3.15) *with $a = -b$.*

Proof. We show that, from Theorem 3.3, the coefficients $P_k^{\langle n \rangle}$ in (3.13) are those defined in (3.40). The cases of $k = 0, 1$ are straightforward according to the coefficient functions in (3.14) and (3.15), so we explore the cases of $k = 2, 3, \ldots, n$. Since $c = 0$ in (3.15), $\widetilde{C}_j^{\langle n \rangle} = 0$ for all

$j \le n$,

$$P_k^{\langle n \rangle}(\tau) = \sum_{j=0}^{k} \left(\sum_{\mathcal{X} \in D_{k,j}} \frac{\prod\limits_{i \in \mathbb{Z}_{k-1} \setminus (\mathcal{X} \cup (\mathcal{X}+\{1\}))} \widetilde{B}_i^{\langle n \rangle} \cdot \prod\limits_{i \in \mathcal{X}} \widetilde{C}_i^{\langle n \rangle}}{\prod\limits_{i \in \mathbb{Z}_k \setminus (\{j\} \cup (\mathcal{X}+\{1\}))} \left(\widetilde{A}_j^{\langle n \rangle} - \widetilde{A}_i^{\langle n \rangle} \right)} \right) e^{\tau \widetilde{A}_j^{\langle n \rangle}} = 0, \tag{3.41}$$

for $\mathcal{X} \neq \emptyset$. For the case of $\mathcal{X} = \emptyset$, the term $\prod\limits_{i \in \mathcal{X}} \widetilde{C}_i^{\langle n \rangle} = 1$; thus, (3.41) can be reduced to

$$P_k^{\langle n \rangle}(\tau) = \sum_{j=0}^{k} \frac{e^{\tau \widetilde{A}_j^{\langle n \rangle}}}{\prod\limits_{\substack{l=0 \\ l \neq j}}^{k} \left(\widetilde{A}_j^{\langle n \rangle} - \widetilde{A}_l^{\langle n \rangle} \right)} \prod_{i=0}^{k-1} \widetilde{B}_i^{\langle n \rangle}, \tag{3.42}$$

which yields (3.40) in this case. □

Based on Theorem 3.11, we obtain the following for Jacobi diffusion:

$$\begin{aligned}
\mathbb{E}\left[X_T \mid X_t = x\right] &= e^{-\theta\tau} x + \mu \left(1 - e^{-\tau\theta}\right), \\
\mathbb{E}\left[X_T^2 \mid X_t = x\right] &= e^{-2(b+1)\theta\tau} x^2 + \frac{2(b+\mu) e^{-2(b+1)\theta\tau}}{2b+1} \left(e^{(2b+1)\theta\tau} - 1 \right) x \\
&\quad + \frac{\mu(b+\mu)}{(b+1)(2b+1)} \left(b \left(2 - 2e^{-\theta\tau}\right) + e^{-2(b+1)\theta\tau} - 2e^{-\theta\tau} + 1 \right).
\end{aligned}$$

The variance is obtained according to Example 3.2, which does not have a simplified form as in OU and CIR diffusions.

Theorem 3.12. *Suppose that X_t follows the Jacobi process* (3.39). *The n^{th} unconditional moment at equilibrium for $n \in \mathbb{Z}_0^+$, $0 < x < 1$, and $\tau = T - t \ge 0$ is given by*

$$L_J^{\langle n \rangle} := \lim_{\tau \to \infty} U_J^{\langle n \rangle}(x, \tau) = \lim_{T \to \infty} \mathbb{E}\left[X_T^n \mid X_t = x\right] = \prod_{j=0}^{n-1} \frac{\mu + jb}{1 - ja}, \tag{3.43}$$

where $a = -b$.

Proof. According to (3.40) in Theorem 3.11, since $\widetilde{A}_j^{\langle n \rangle} < 0$ for all $j < n$, the coefficient terms of x^{n-k} given in (3.42) approach 0 as $\tau \to \infty$ for $k = 0, 1, 2, \ldots, n-1$, except in the case of $k = j = n$, $\widetilde{A}_n^{\langle n \rangle} = 0$, and thus,

$$L_J^{\langle n \rangle} = \lim_{\tau \to \infty} \frac{e^{\tau \widetilde{A}_n^{\langle n \rangle}}}{\prod\limits_{\substack{l=0 \\ l \neq n}}^{n} \left(\widetilde{A}_n^{\langle n \rangle} - \widetilde{A}_l^{\langle n \rangle} \right)} \prod_{i=0}^{n-1} \widetilde{B}_i^{\langle n \rangle} = (-1)^n \prod_{j=0}^{n-1} \frac{\widetilde{B}_j^{\langle n \rangle}}{\widetilde{A}_j^{\langle n \rangle}} = \prod_{j=0}^{n-1} \frac{\mu + jb}{1 - ja}$$

as required. □

Note that one can obtain conditional moments directly using the transition PDF, which is the expansion in the form of Jacobi polynomials and the beta function (see [12, 49]). In this case, we discuss only the classical case given in (3.39), where eigenvalue expansion for the transition PDF can be rewritten as

$$p(y,T \mid x_t,t) = beta(y)\sum_{j=0}^{\infty}\frac{e^{-\lambda_j(T-t)}}{w_j}P_j^{(\alpha,\beta)}\left(2x_t-1\right)P_j^{(\alpha,\beta)}\left(2y-1\right). \tag{3.44}$$

The invariant distribution is $beta(y) = \dfrac{y^{\beta}\,(1-y)^{\alpha}}{\mathcal{B}\,(\alpha+1,\beta+1)}$, where $\mathcal{B}(\alpha,\beta) = \dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ is the beta function, and

$$\begin{aligned}
&\alpha = \frac{1}{b}-\frac{\mu}{b}-1 > -1, \qquad \beta = \frac{\mu}{b}-1 > -1, \qquad \lambda_j = jb\theta\left(j-1+\frac{1}{b}\right), \qquad w_0 = 1\\
&w_j = \frac{\Gamma(\alpha+2)\,\Gamma(\beta+2)\,\Gamma(\alpha+\beta+4-j)}{\Gamma(\alpha+2-j)\,\Gamma(\beta+2-j)\,\Gamma(\alpha+\beta+3)\,(2j+\alpha+\beta+1)\,j!},\\
&P_j^{(\alpha,\beta)}(z) = \frac{\Gamma(\alpha+j+1)}{\Gamma(\alpha+\beta+j+1)\,j!}\sum_{k=0}^{j}\binom{j}{k}\frac{\Gamma(\alpha+\beta+j+k+1)}{\Gamma(\alpha+k+1)}\left(\frac{z-1}{2}\right)^{k}.
\end{aligned} \tag{3.45}$$

The conditional moments based on the transition PDF require the following lemma.

Lemma 3.1. *Let m, n, and j be in $\mathbb{Z}_0^+$. For $m < j$, we have*

$$\sum_{k=0}^{j}(-1)^k\binom{j}{k}k^m = 0. \tag{3.46}$$

Otherwise, it is greater than zero. Moreover, for $n+1 \le j$,

$$\int_0^1 y^n\, beta(y)\, P_j^{(\alpha,\beta)}\left(2y-1\right)\, dy = 0, \tag{3.47}$$

where invariant distribution $beta(y)$ is defined as in (3.44).

Proof. The key idea of the proof can be understood in terms of counting rather than using induction by counting all onto functions from a domain with m members and a range with j members. It is obvious that (3.46) is 0 for $m < j$, since there is no onto function. For the same reason, for $m \ge j$, (3.46) is greater than zero. By considering the left-hand side of (3.47) for $n+1 \le j$,

$$\begin{aligned}
&\int_0^1 y^n\, beta(y)\, P_j^{(\alpha,\beta)}\left(2y-1\right)\, dy\\
&\quad= \frac{\Gamma(\alpha+\beta+2)\,\Gamma(\alpha+j+1)}{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(\alpha+\beta+j+1)\,j!}\sum_{k=0}^{j}(-1)^k\binom{j}{k}\frac{\Gamma(\alpha+\beta+j+k+1)}{\Gamma(\alpha+k+1)}\mathcal{B}\left(n+\beta+1,\alpha+k+1\right)\\
&\quad= \frac{\Gamma\left(\alpha+\beta+2\right)\Gamma\left(\alpha+j+1\right)\Gamma\left(n+\beta+1\right)}{\Gamma\left(\alpha+1\right)\Gamma\left(\beta+1\right)\Gamma\left(\alpha+\beta+j+1\right)j!}\sum_{k=0}^{j}(-1)^k\binom{j}{k}\frac{\Gamma\left(\alpha+\beta+j+k+1\right)}{\Gamma\left(\alpha+\beta+n+k+2\right)}.
\end{aligned}$$

Since $\frac{\Gamma(\alpha+\beta+j+k+1)}{\Gamma(\alpha+\beta+n+k+2)}$ is a polynomial function in k up to degree $j-n-1$, which is less than j, (3.46) implies (3.47). $\square$

Theorem 3.13. *Suppose that X_t follows the Jacobi process* (3.39). *The n^{th} conditional moment for $n \in \mathbb{Z}_0^+$ is*

$$\hat{U}_J^{\langle n\rangle}(x,\tau) := \mathbb{E}\left[X_T^n \mid X_t = x\right]$$
$$= \sum_{j=0}^{n} \left(\frac{e^{-\lambda_j \tau} P_j^{(\alpha,\beta)}(2x-1)\, \Gamma\left(\frac{1}{b}\right) \Gamma\left(\frac{1}{b} - \frac{\mu}{b} + j\right) \Gamma\left(n + \frac{\mu}{b}\right)}{w_j \;\Gamma\left(\frac{1}{b} - \frac{\mu}{b}\right) \Gamma\left(\frac{\mu}{b}\right) \Gamma\left(j+1\right) \Gamma\left(\frac{1}{b} + j - 1\right)} \sum_{k=0}^{j} (-1)^k \binom{j}{k} \frac{\Gamma\left(\frac{1}{b} + j + k - 1\right)}{\Gamma\left(n + \frac{1}{b} + k\right)} \right), \tag{3.48}$$

for all $0 < x < 1$ and $\tau = T - t$, where $P_j^{(\alpha,\beta)}(2x-1)$, λ_j, w_j, α, and β are given in (3.45).

Proof. By using the transition PDF,

$$\begin{aligned}\mathbb{E}\left[X_T^n \mid X_t = x\right] &= \int_0^1 y^n p(y, T \mid x, t)\, dy \\ &= \sum_{j=0}^{\infty} \frac{e^{-\lambda_j \tau}}{w_j} P_j^{(\alpha,\beta)}(2x-1) \int_0^1 y^n \, beta(y)\, P_j^{(\alpha,\beta)}(2y-1)\, dy.\end{aligned}$$

From Lemma 3.1,

$$\begin{aligned}&\mathbb{E}\left[X_T^n \mid X_t = x\right] \\ &= \sum_{j=0}^{n} \frac{e^{-\lambda_j \tau}}{w_j} P_j^{(\alpha,\beta)}(2x-1) \int_0^1 y^n \, beta(y)\, P_j^{(\alpha,\beta)}(2y-1)\, dy \\ &= \sum_{j=0}^{n} \left(\frac{e^{-\lambda_j \tau} P_j^{(\alpha,\beta)}(2x-1)\, \Gamma\left(\frac{1}{b}\right) \Gamma\left(\frac{1}{b} - \frac{\mu}{b} + j\right)}{w_j\, \Gamma\left(\frac{1}{b} - \frac{\mu}{b}\right) \Gamma\left(\frac{\mu}{b}\right) \Gamma\left(\frac{1}{b} + j - 1\right) j!} \sum_{k=0}^{j} (-1)^k \binom{j}{k} \frac{\Gamma\left(\frac{1}{b} + j + k - 1\right)}{\Gamma\left(\frac{1}{b} - \frac{\mu}{b} + k\right)} \mathcal{B}\left(n + \frac{\mu}{b}, \frac{1}{b} - \frac{\mu}{b} + k\right) \right) \\ &= \sum_{j=0}^{n} \left(\frac{e^{-\lambda_j \tau} P_j^{(\alpha,\beta)}(2x-1)\, \Gamma\left(\frac{1}{b}\right) \Gamma\left(\frac{1}{b} - \frac{\mu}{b} + j\right) \Gamma\left(n + \frac{\mu}{b}\right)}{w_j\; \Gamma\left(\frac{1}{b} - \frac{\mu}{b}\right) \Gamma\left(\frac{\mu}{b}\right) \Gamma\left(\frac{1}{b} + j - 1\right) j!} \sum_{k=0}^{j} (-1)^k \binom{j}{k} \frac{\Gamma\left(\frac{1}{b} + j + k - 1\right)}{\Gamma\left(n + \frac{1}{b} + k\right)} \right),\end{aligned}$$

as required. $\square$

By considering the marginal density of x given in (3.44), if $\tau \to \infty$, we have the unconditional moments for x in the following theorem.

Theorem 3.14. *Suppose that X_t follows the Jacobi process* (3.39). *The n^{th} unconditional moment at equilibrium, for all $n \in \mathbb{Z}_0^+$, $0 < x < 1$, and $\tau = T - t \geq 0$, is given by*

$$\hat{L}_J^{\langle n\rangle} := \lim_{T\to\infty} \mathbb{E}\left[X_T^n \mid X_t = x\right] = \frac{\Gamma\left(\frac{1}{b}\right) \Gamma\left(n + \frac{\mu}{b}\right)}{\Gamma\left(\frac{\mu}{b}\right) \Gamma\left(n + \frac{1}{b}\right)}.$$

Proof. Note that $a = -b < 0$, and thus, $\lambda_j > 0$ for all $j = 1, 2, \ldots, n$, but $\lambda_0 = 0$. According to (3.48) in Theorem 3.13, its partial sum from $j = 1$ to $j = n$ approaches 0 as $\tau \to \infty$, except in the case of $j = 0$. Thus,

$$\hat{L}_J^{\langle n \rangle} = \frac{P_0^{(\alpha,\beta)}(2x-1)\,\Gamma\left(\frac{1}{b}\right)\Gamma\left(\frac{1}{b}-\frac{\mu}{b}\right)\Gamma\left(n+\frac{\mu}{b}\right)\Gamma\left(\frac{1}{b}-1\right)}{w_0\ \Gamma\left(\frac{1}{b}-\frac{\mu}{b}\right)\Gamma\left(\frac{\mu}{b}\right)\Gamma\left(\frac{1}{b}-1\right)\Gamma\left(n+\frac{1}{b}\right)}.$$

Since $w_0 = 1$ and $P_0^{(\alpha,\beta)}(2x-1) = 1$, this completes the proof. □

Remark 3.5. It is not difficult to confirm that the proposed formula (3.40) is identical to the formula obtained from the transition PDF. For example, one can check this by using built-in functions in Wolfram Mathematica 9 software. Since formula (3.48) involves Jacobi polynomials, it is not easy to apply this result (3.48) in certain cases, such as the conditional mixed moment, covariance, and correlation mentioned in Section 3.4.

Remark 3.6. In the literature, a generalized case of the Jacobi process is transformed using an affine transformation. Indeed, applying $\rho_t = \rho_{\min} + (\rho_{\max} - \rho_{\min}) X_t$ in (3.39), Itô's lemma gives

$$\begin{aligned} d\rho_t &= (\rho_{\max} - \rho_{\min})\, dX_t \\ &= \theta\left((\rho_{\max} - \rho_{\min})\,\mu + \rho_{\min} - \rho_t\right) dt + \sqrt{2b\theta\,(\rho_{\max} - \rho_t)(\rho_t - \rho_{\min})}\, dW_t \end{aligned}$$

(see also [19]). By applying the binomial theorem, the conditional moments of ρ_t are obtained as

$$\begin{aligned} \mathbb{E}\left[\rho_T^n \mid \rho_t = \rho\right] &= \mathbb{E}\left[(\rho_{\min} + (\rho_{\max} - \rho_{\min}) X_T)^n \mid \rho_t = \rho\right] \\ &= \rho_{\min}^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\rho_{\max} - \rho_{\min}}{\rho_{\min}}\right)^k \mathbb{E}\left[X_T^k \mid X_t = x\right], \end{aligned}$$

where $x = \frac{\rho - \rho_{\min}}{\rho_{\max} - \rho_{\min}}$. In the general case, we can apply the above formula to evolution problems of a Jacobi-type process using the results of Theorems 3.11, 3.12, 3.13, and 3.14.

3.5.4 Fisher–Snedecor diffusion

The last three diffusion processes, Fisher–Snedecor, reciprocal gamma, and Student, belong to the class with heavy-tailed invariant distributions described by the Pearson family distributions (see Pearson [62]). These diffusions are also called heavy-tailed Kolmogorov–Pearson diffusions. Their properties in statistical analysis are the subject of great interest at present, especially the study of their transition PDFs, which are useful in facilitating the estimation of parameters (see more details in [4, 20, 46]). Since closed-form formulas for the transition PDF are unavailable, the statistical analysis and properties are difficult to obtain. However, in this paper, the study of these properties is possible without using their transition PDFs.

The Fisher–Snedecor process X_t is defined as the solution of the nonlinear SDE given by

$$dX_t = \theta\left(\mu - X_t\right)dt + \sqrt{2\theta X_t\left(\frac{X_t}{\beta/2-1} + \frac{\mu}{\alpha/2}\right)}\,dW_t \tag{3.49}$$

for all $t \geq 0$, where $\alpha > 0, \beta > 2$ (see also [4, 46]). To guarantee that process (3.49) satisfies Fisher–Snedecor diffusion, we check the following conditions: $\deg(d) = 2$, $\Delta(d) > 0$, and $a > 0$. According to the study by Leonenko et al. [49], in the particular case of $\mu = \frac{\beta}{\beta-2}$, process (3.49) is reformulated into

$$dX_t = \theta\left(\frac{\beta}{\beta-2} - X_t\right)dt + \sqrt{\frac{4\theta}{\alpha\left(\beta-2\right)}X_t\left(\alpha X_t + \beta\right)}\,dW_t. \tag{3.50}$$

The Fisher–Snedecor process is one of the Pearson diffusion process cases in which its transition PDE is associated with the eigenfunctions produced by Rodrigues's formula, Fisher–Snedecor polynomials, and the solutions of the Sturm–Liouville equation (see more details in equations (36)–(40) of Leonenko et al.'s work [49]). Since the transition PDE has a complicated solution form, the conditional moments and their consequences are challenging to obtain in a concise form.

Process (3.1) becomes (3.50) when $a = \frac{2}{\beta-2}$, $b = \frac{2\beta}{\alpha(\beta-2)}$, and $c = 0$. The next two theorems are consequences deduced from Theorem 3.3. Observe that the only difference between Jacobi (3.39) and Fisher–Snedecor processes (3.50) is the condition on a of the polynomial $d(x)$. This suggests that their closed-form formulas for conditional moments appear to be the same.

Theorem 3.15. *Suppose that X_t follows the Fisher–Snedecor process* (3.50). *The n^{th} conditional moment for $n \in \mathbb{Z}_0^+$ is*

$$U_F^{\langle n\rangle}(x,\tau) := \mathbb{E}\left[X_T^n \mid X_t = x\right] = e^{\tau\tilde{A}_0^{\langle n\rangle}}x^n + \sum_{k=1}^{n}\left(\sum_{j=0}^{k}\frac{e^{\tau\tilde{A}_j^{\langle n\rangle}}}{\prod_{\substack{l=0\\l\neq j}}^{k}\left(\tilde{A}_j^{\langle n\rangle} - \tilde{A}_l^{\langle n\rangle}\right)}\prod_{j=0}^{k-1}\tilde{B}_j^{\langle n\rangle}\right)x^{n-k}, \tag{3.51}$$

for $\tau = T - t \geq 0$, where $\tilde{A}_j^{\langle n\rangle}$ and $\tilde{B}_j^{\langle n\rangle}$ are given in (3.15) *with $a = \frac{2}{\beta-2}$ and $b = \frac{2\beta}{\alpha(\beta-2)}$.*

Proof. The proof is the same as that of Theorem 3.11. □

Based on Theorem 3.15, we obtain the following for Fisher–Snedecor diffusion:

$$\begin{aligned}\mathbb{E}\left[X_T \mid X_t = x\right] &= e^{-\theta\tau}x + \mu\left(1 - e^{-\tau\theta}\right),\\ \mathbb{E}\left[X_T^2 \mid X_t = x\right] &= e^{-\frac{2(\beta-4)\theta\tau}{\beta-2}}x^2 + \frac{2(\alpha+2)(\beta-2)\mu e^{\frac{(10-3\beta)\theta\tau}{\beta-2}}}{\alpha(\beta-6)}\left(e^{\frac{2(\beta-4)\theta\tau}{\beta-2}} - e^{\theta\tau}\right)x\\ &\quad + \frac{(\alpha+2)(\beta-2)\mu^2}{\alpha(\beta-6)(\beta-4)}\left(-2(\beta-4)e^{-\theta\tau} + (\beta-2)e^{-\frac{2(\beta-4)\theta\tau}{\beta-2}} + \beta - 6\right).\end{aligned}$$

The variance is obtained according to Example 3.2, which does not have a simplified form as in OU and CIR diffusions.

According to Theorem 3.4 and Remark 3.2 for OU, CIR, and Jacobi processes with a condition on parameter a, the n^{th} unconditional moment at equilibrium exists for all $n \in \mathbb{Z}^+$, while this not true for the Fisher–Snedecor, reciprocal gamma, and Student processes.

In fact, process (3.50) has unconditional moments of order n satisfying the condition $\frac{2(n-1)}{\beta-2} = a(n-1) \leq 1$, or $n \leq \frac{\beta}{2} + 1$. For example, the first and second unconditional moments exist but not for higher orders when $\beta = 3$. Their closed-form formulas are given in the following theorem.

Theorem 3.16. *Suppose that X_t follows the Fisher–Snedecor process* (3.50) *with $n \leq \frac{\beta}{2} + 1$. The n^{th} unconditional moment at equilibrium for all $n \in \mathbb{Z}^+$ and $\tau = T - t \geq 0$ is given by*

$$L_F^{\langle n\rangle} := \lim_{\tau\to\infty} U_F^{\langle n\rangle}(x,\tau) = \lim_{T\to\infty} \mathbb{E}\left[X_T^n \mid X_t = x\right] = \prod_{j=0}^{n-1} \frac{\mu + jb}{1 - ja}, \tag{3.52}$$

where $a = \frac{2}{\beta-2}$ and $b = \frac{2\beta}{\alpha(\beta-2)}$.

Proof. The proof is similar to that of Theorem 3.12. □

3.5.5 Reciprocal gamma diffusion

The reciprocal gamma diffusion process X_t is defined as the solution of the nonlinear SDE given by

$$dX_t = \theta\left(\frac{\alpha}{\beta-1} - X_t\right)dt + \sqrt{\frac{2\theta}{\beta-1}X_t^2}\,dW_t, \tag{3.53}$$

where $\alpha > 0$ and $\beta > 1$ (see [51]). The reciprocal gamma diffusion process (3.53) has the following conditions: $\deg(d) = 2$, $\Delta(d) = 0$, and $a > 0$. This diffusion process was first studied by Linetsky [54] as an ergodic diffusion process that is widely studied, especially for its application to the maximum likelihood method. One process deduced using the reciprocal gamma is known as the Dothan model [21], which became well known after it was used for the valuation formula of the default-free bond and its properties.

Similar to Fisher–Snedecor diffusion, the reciprocal gamma process has a transition PDE associated with eigenfunctions produced by Rodrigues's formula, Bessel polynomials, and the solutions of the Sturm–Liouville equation, and it has quite a complicated form for the calculation of conditional moments.

Process (3.1) becomes (3.53) when $a = \frac{1}{\beta-1}$ and $b = c = 0$, and the two theorems are consequences deduced from Theorem 3.3. Note that the only difference between the Fisher–Snedecor (3.50) and reciprocal gamma processes (3.53) is the condition on the discriminant $\Delta(d)$ of the polynomial $d(x)$. Thus, the obtained closed-form formula for conditional moments for process (3.53) is also similar to that of (3.51).

Theorem 3.17. *Suppose that X_t follows the reciprocal gamma process* (3.53)*. The n^{th} conditional moment for $n \in \mathbb{Z}_0^+$ is*

$$U_r^{\langle n\rangle}(x,\tau) := \mathbb{E}\left[X_T^n \mid X_t = x\right] = e^{\tau \tilde{A}_0^{\langle n\rangle}} x^n + \sum_{k=1}^{n} \left(\sum_{j=0}^{k} \frac{e^{\tau \tilde{A}_j^{\langle n\rangle}}}{\prod_{\substack{l=0 \\ l\neq j}}^{k} \left(\tilde{A}_j^{\langle n\rangle} - \tilde{A}_l^{\langle n\rangle}\right)} \frac{\theta^k \mu^k n!}{(n-k)!} \right) x^{n-k}, \tag{3.54}$$

for $\tau = T - t \geq 0$, where $\tilde{A}_j^{\langle n\rangle}$ and $\tilde{B}_j^{\langle n\rangle}$ are given in (3.15)*, with $a = \frac{1}{\beta-1}$.*

Proof. The proof is similar to that of Theorem 3.11 with the condition $b = 0$. □

Based on Theorem 3.17, we obtain the following for reciprocal gamma diffusion:

$$\begin{aligned}
\mathbb{E}\left[X_T \mid X_t = x\right] &= e^{-\theta\tau} x + \frac{\alpha}{\beta-1}\left(1 - e^{-\tau\theta}\right), \\
\mathbb{E}\left[X_T^2 \mid X_t = x\right] &= e^{2\left(\frac{1}{\beta-1}-1\right)\theta\tau} x^2 + \frac{2\alpha}{\beta-3}\left(e^{-\theta\tau} - e^{-\frac{2(\beta-2)\theta\tau}{\beta-1}}\right) x \\
&\quad + \frac{\alpha^2}{\beta-3}\left(\frac{e^{-\frac{2(\beta-2)\theta\tau}{\beta-1}} - 1}{\beta-2} + \frac{2\left(1-e^{-\theta\tau}\right)}{\beta-1}\right).
\end{aligned}$$

The variance is obtained according to Example 3.2, which does not have a simplified form as in OU and CIR diffusions.

Referring to Theorem 3.4, as stated in Remark 3.2, the n^{th} unconditional moment at equilibrium only exists for order n satisfying the condition $\frac{n-1}{\beta-1} = a(n-1) \leq 1$ or $n \leq \beta$, given in the following theorem.

Theorem 3.18. *Suppose that X_t follows the reciprocal gamma process* (3.53) *for $n \in \mathbb{Z}^+$ with $n \leq \beta$. The n^{th} unconditional moment at equilibrium for $\tau = T - t \geq 0$ is given by*

$$L_r^{\langle n \rangle} := \lim_{\tau \to \infty} U_r^{\langle n \rangle}(x, \tau) = \lim_{T \to \infty} \mathbb{E}\left[X_T^n \mid X_t = x\right] = \prod_{j=0}^{n-1} \frac{\alpha}{\beta - j - 1}. \tag{3.55}$$

Proof. The proof is the same as that of Theorem 3.12, with $b = 0$, $a = \frac{1}{\beta - 1}$, and $\mu = \frac{\alpha}{\beta - 1}$. □

3.5.6 Symmetric Student diffusion

This section focuses on the symmetric Student process. This is a symmetric diffusion process that has a heavy-tailed distribution according to the classification of Pearson diffusions. As a practical application, the process is widespread in areas of financial mathematics (see [52]). This diffusion was first studied by Wong [79], and applications for parameter estimation were recently studied by Leonenko et al. [49, 52]. One well-known particular case of this diffusion is the hypergeometric diffusion presented as a model of spectral expansions for Asian options by Linetsky. Based on Leonenko et al. [49, 54], this symmetric Student process is given by

$$dX_t = \theta\left(\mu - X_t\right) dt + \sqrt{\frac{2\theta\delta^2}{v - 1}\left(1 + \left(\frac{\mu - X_t}{\delta}\right)^2\right)}\, dW_t, \tag{3.56}$$

where $\delta > 0$, $v > 1$, and $\mu \in \mathbb{R}$. The transition PDF for the symmetric Student process is associated with eigenfunctions in terms of Romanovski polynomials and the Sturm–Liouville equation solutions (see details in equations (48)–(52) of Leonenko et al.'s work [49]), and its closed form is unavailable; thus, statistical analysis is quite complicated. However, a general version of the Student process is also proposed in [49] as the Skew-Student diffusion process, defined in the following form:

$$dX_t = \theta\left(\mu - X_t\right) dt + \sqrt{2\alpha\theta\left(\delta^2\left(\nu - X_t\right)^2\right)}\, dW_t, \tag{3.57}$$

where μ, $\nu \in \mathbb{R}$, $\alpha > 0$, and $\delta > 0$. This process becomes a symmetric case when $\mu = \nu$ and $v+1 = \frac{1}{\alpha}+2$. Note also that Theorem 3.3 can be applied to this process to obtain the closed-form formula.

In the cases of (3.56) and (3.57), the formulas in Theorems 3.3 and 3.4 can be applied to obtain the conditional and unconditional moments and other mathematical properties similar to those described in Section 3.4.

Based on Example 3.1, we obtain the following for Student diffusion (3.57):

$$\begin{aligned}
\mathbb{E}\left[X_T \mid X_t = x\right] &= e^{-\theta\tau}x + \mu\left(1 - e^{-\tau\theta}\right), \\
\mathbb{E}\left[X_T^2 \mid X_t = x\right] &= e^{2\theta\tau\left(\alpha\delta^2-1\right)}x^2 + \frac{2e^{-2\theta\tau}}{2\alpha\delta^2 - 1}\left(2\nu\alpha\delta^2 - \mu\right)\left(e^{\theta\tau} - e^{2\alpha\delta^2\theta\tau}\right)x \\
&\quad + \frac{\nu^2\alpha\delta^2}{\alpha\delta^2 - 1}\left(e^{2\theta\tau\left(\alpha\delta^2-1\right)} - 1\right) - \frac{\mu\left(2\nu\alpha\delta^2 - \mu\right)}{2\alpha\delta^2 - 1}\left(\frac{e^{2\theta\tau\left(\alpha\delta^2-1\right)} - 1}{\alpha\delta^2 - 1} + 2e^{-\theta\tau} - 2\right).
\end{aligned}$$

The variance is obtained according to Example 3.2, which does not have a simplified form as in OU and CIR diffusions.

3.6 Inhomogeneous Pearson diffusion processes

Based on extended time-dependent parameters, the closed-form formula (3.12) in Theorem 3.2 has a greater advantage over formulas presented in the previous section and in the literature [39]. In this section, we only discuss the details of the EOU and ECIR processes, as presented by Egorov et al. [23]. To validate the formulas, the values are compared with the results of Monte Carlo simulations with 10,000 sample paths, where each path has 10,000 steps; the model parameter values are taken from Table 1 in [23].

3.6.1 The extended Ornstein–Uhlenbeck process

The dynamics of the EOU process presented by Egorov et al. are governed by time-varying parameters, given as

$$dX_t = -\theta X_t \, dt + \sigma_0 e^{\sigma_1 t} \, dW_t, \tag{3.58}$$

where θ, σ_0 are positive, and σ_1 is a real number with a Gaussian transition PDF:

$$p(y, T \mid x_t, t) = \mathcal{N}\left(e^{-\theta\tau}x_t, \frac{\sigma_0^2}{2\left(\theta + \sigma_1\right)}\left(e^{2\sigma_1 T} - e^{2\left(\sigma_1 t - \theta\tau\right)}\right)\right), \tag{3.59}$$

where $\tau = T - t$. The n^{th} conditional moment for $n \in \mathbb{Z}_0^+$ is obtained directly by Theorem 3.2 as

$$\mathbb{E}\left[X_T^n \mid X_t = x\right] = \sum_{k=0}^{n} P_k^{\langle n \rangle}(\tau)\, x^{n-k}$$

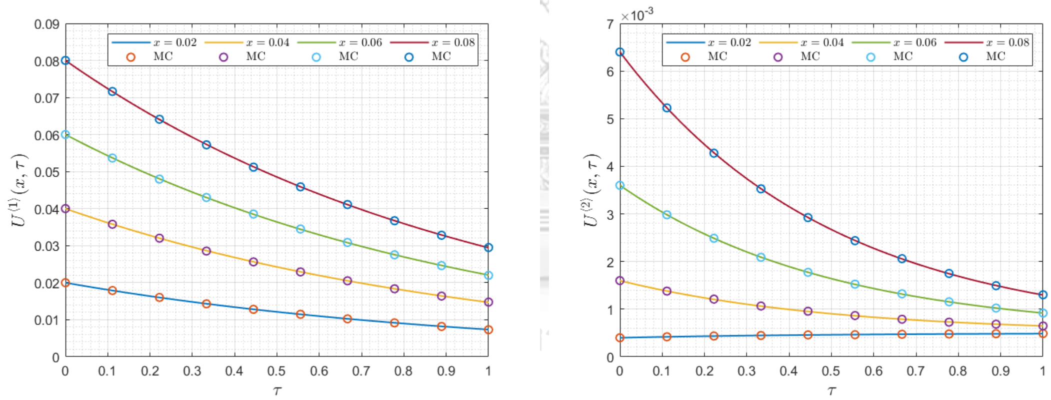
for $\tau = T - t \geq 0$, where

$$P_k^{\langle n \rangle}(\tau) = \begin{cases} e^{-n\theta\tau}, & \text{for } k = 0, \\ e^{-n\theta\tau + k(T-\tau)\sigma_1}\left(\frac{\sigma_0^2 e^{2(\sigma_1+\theta)\tau} - \sigma_0^2}{\sigma_1 + \theta}\right)^{\frac{k}{2}} \frac{k!\binom{n}{k}}{2^k\left(\frac{k}{2}\right)!}, & \text{for positive even integer } k, \\ 0, & \text{for positive odd integer } k. \end{cases} \tag{3.60}$$

Furthermore, the fundamental properties presented in Section 3.4, such as the conditional variance, mixed moments, covariance, and correlation, are quickly produced by using the coefficients in (3.60). Based on Examples 3.1 and 3.2, we obtain the following for EOU diffusion (3.58):

$$\mathbb{E}\left[X_T \mid X_t = x\right] = e^{-\theta\tau}x,$$
$$\mathbb{E}\left[X_T^2 \mid X_t = x\right] = e^{-2\theta\tau}x^2 + e^{-2\theta\tau+2(T-\tau)\sigma_1}\left(\frac{\sigma_0^2 e^{2(\sigma_1+\theta)\tau} - \sigma_0^2}{2\sigma_1 + 2\theta}\right),$$
$$\mathrm{Var}\left[X_T \mid X_t = x\right] = e^{-2\theta\tau+2(T-\tau)\sigma_1}\left(\frac{\sigma_0^2 e^{2(\sigma_1+\theta)\tau} - \sigma_0^2}{2\sigma_1 + 2\theta}\right).$$

To validate formula (3.12) in Theorem 3.2 with coefficients (3.60), we denote the formula for conditional moments of the EOU process (3.58) by $U^{\langle n\rangle}(x,\tau)$. Comparisons between formula (3.12) and MC simulations were performed on process (3.58) presented in [23] with parameters $\theta = 1$, $\sigma_0 = 0.001$, and $\sigma_1 = -0.001$. Figure 3.1 displays the first and second conditional moments from our formula (solid lines) compared with MC simulations (colored circles), showing their agreement for each initial $x = 0.02, 0.04, 0.06, 0.08$.



(a) The first conditional moments **(b)** The second conditional moments

Figure 3.1: Comparisons of $U^{\langle\gamma\rangle}(x,\tau)$ and MC simulations for $x = 0.02, 0.04, 0.06, 0.08$ at $\tau = 0, 0.1, 0.2, \ldots, 1$

Note that a more general version of the EOU process (3.58), known as the Hull–White process, is defined by

$$dX_t = \theta\left(\beta + \zeta t - X_t\right)dt + \sigma_0 e^{\sigma_1 t}\,dW_t,$$

where θ and σ_0 are positive numbers, and β, ζ, and σ_1 are real numbers with the Gaussian transition PDF given in [23]. In this case, the proposed formula (3.12) in Theorem 3.2 can be applied to obtain conditional moments with slightly more complicated forms of coefficients than those shown in (3.60), and they are omitted here.

3.6.2 The extended Cox–Ingersoll–Ross process

The dynamics of the ECIR process, sometimes called the ECIR(d) process, presented by Egorov et al. are governed by time-varying parameters as

$$dX_t = \theta\left(\frac{\sigma_0^2 d}{4\theta}e^{2\sigma_1 t} - X_t\right)dt + \sigma_0 e^{\sigma_1 t}\sqrt{X_t}\,dW_t, \tag{3.61}$$

where θ, σ_0, d are positive, and σ_1 is a real number with the transition PDF:

$$p(y, T \mid x_t, t) = \frac{1}{2}Ge^{-\frac{\lambda+Gy}{2}}\left(\frac{Gy}{\lambda}\right)^{\frac{d-2}{4}} I_{\frac{d}{2}-1}\left(\lambda Gy\right), \tag{3.62}$$

where $\lambda = x_t v$, $G = e^{\theta\tau}v$, $v = \frac{8\sigma_1}{\sigma_0^2}e^{-\theta\tau}\left(e^{2\sigma_1 T} - e^{2\sigma_1 t}\right)$, $\tau = T - t$, and $I_q(\cdot)$ is the modified Bessel function of the first kind order q, which was first proposed by Maghsoodi [55]. Moreover, he also provided sufficient conditions for the existence of a pathwise unique strong solution with a non-attainable zero almost everywhere with respect to the probability measure (see details in Theorems 2.1 and 2.4 of [55]). For process (3.61), the condition is rapidly deduced for $d \geq 2$. The n^{th} conditional moment for $n \in \mathbb{Z}_0^+$ obtained directly by Theorem 3.2 is

$$\mathbb{E}\left[X_T^n \mid X_t = x\right] = \sum_{k=0}^{n} P_k^{\langle n\rangle}(\tau)\, x^{n-k}$$

for all $x > 0$ and $\tau = T - t \geq 0$, where

$$P_k^{\langle n\rangle}(\tau) = \begin{cases} e^{-n\theta\tau}, & \text{for } k = 0, \\ \frac{e^{-n\theta\tau}}{k!}\left(\prod_{j=0}^{k-1}(n-j)(d+2(n-j-1))\right)\left(\frac{\sigma_0^2 e^{2\sigma_1(T-\tau)}\left(e^{2\sigma_1\tau+\theta\tau}-1\right)}{4(2\sigma_1+\theta)}\right)^k, & \text{for } k \geq 1. \end{cases} \tag{3.63}$$

Note that the fundamental properties presented in Section 3.4 are quickly obtained by the coefficients in (3.63). Unlike the previous case, where the transition PDF of (3.58) is given in closed form as (3.59), formula (3.62) is not a closed-form transition PDF for (3.61). In general, the conditional moments for (3.61) can be derived by using (3.62); however, the formula obtained by using the transition PDF is difficult to apply in some cases, e.g., when we need to compute the mathematical properties mentioned in Section 3.4. Moreover, the transition PDF (3.62) behaves like a Dirac delta function when the increment $\tau = T - t$ is very small, for instance, $\tau = 0.01$. Because the transition PDF may produce a large spike at the initial value X_0, using its transition PDF may produce inaccurate results when applying the usual integration methods numerically (see [23]). This suggests that the formula based on the coefficients (3.63) produced by Theorem 3.2 are more applicable than those obtained by using the transition PDF. Based on

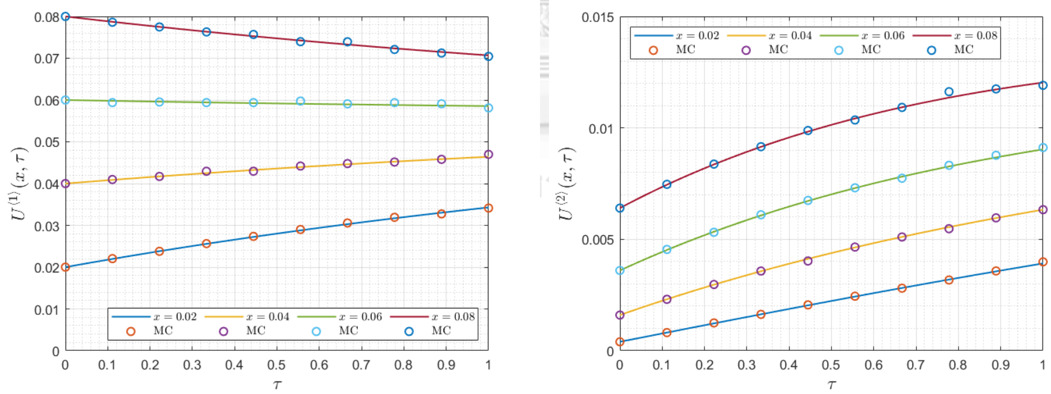
Examples 3.1 and 3.2, we obtain the following for ECIR diffusion (3.61):

$$\mathbb{E}\left[X_T \mid X_t = x\right] = e^{-\theta\tau}x + e^{-\theta\tau}d\left(\frac{\sigma_0^2 e^{2\sigma_1(T-\tau)}\left(e^{2\sigma_1\tau+\theta\tau}-1\right)}{8\sigma_1+4\theta}\right),$$

$$\mathbb{E}\left[X_T^2 \mid X_t = x\right] = e^{-2\theta\tau}x^2 + e^{-2\theta\tau}\left(2d+4\right)\left(\frac{\sigma_0^2 e^{2\sigma_1(T-\tau)}\left(e^{2\sigma_1\tau+\theta\tau}-1\right)}{8\sigma_1+4\theta}\right)x$$
$$+ e^{-2\theta\tau}d(d+2)\left(\frac{\sigma_0^2 e^{2\sigma_1(T-\tau)}\left(e^{2\sigma_1\tau+\theta\tau}-1\right)}{8\sigma_1+4\theta}\right)^2,$$

$$\mathrm{Var}\left[X_T \mid X_t = x\right] = 4e^{-2\theta\tau}\left(\frac{\sigma_0^2 e^{2\sigma_1(T-\tau)}\left(e^{2\sigma_1\tau+\theta\tau}-1\right)}{8\sigma_1+4\theta}\right)x$$
$$+ 2de^{-2\theta\tau}\left(\frac{\sigma_0^2 e^{2\sigma_1(T-\tau)}\left(e^{2\sigma_1\tau+\theta\tau}-1\right)}{8\sigma_1+4\theta}\right)^2.$$

To validate formula (3.12) in Theorem 3.2 with coefficients (3.63), we denote the formula for conditional moments of the ECIR process (3.61) by $U^{\langle n\rangle}(x,\tau)$. Comparisons between formula (3.12) and MC simulations are performed on process (3.61) given in [23] with parameters $d = 5$, $\theta = 0.5$, $\sigma_0 = 0.15$, and $\sigma_1 = 0.001$. Figure 3.2 displays the first and second conditional moments from our formula (solid lines) compared with MC simulations (colored circles), showing their agreement for each initial $x = 0.02, 0.04, 0.06, 0.08$.



(a) The first conditional moments **(b)** The second conditional moments

Figure 3.2: The comparisons of $U^{\langle\gamma\rangle}(x,\tau)$ and MC simulations for $\tau = 0.01, 1$ at $x = 0.1, 0.2, 0.3, \ldots, 1.6$

Remark 3.7. One primary concern for our proposed formula in Theorem 3.2 is that the coefficients $P_k^{\langle n\rangle}(\tau)$ in (3.7) may not be directly integrable to obtain the exact integration, and some numerical integration methods are required to manipulate the integral terms, such as a trapezoidal rule, Simpson's rule, and Newton–Cotes. One efficient method that we use to handle the integral terms is the Chebyshev integration method introduced by Boonklurb et al. [8], which has been demonstrated to produce higher accuracy than the other mentioned integration methods when using the same discretizing nodes.

3.7 Conclusion

This work proposes formulas for conditional moments in integral forms for time inhomogeneous Pearson diffusion processes (3.5) introduced by Forman and Sørensen [32]. First, the derived formula of process (3.5) in Theorem 3.1 is in the form of an infinite sum, and then it is reduced to a finite sum for the case of $\gamma \in \mathbb{Z}_0^+$ in Theorem 3.2. Furthermore, we present more concise formulas when the integral terms in the coefficients are analytically evaluated for the case of constant parameters, as provided in Theorem 3.3. Additionally, the closed-form formula for unconditional moments is also observed for processes in the case of constant parameters in Theorem 3.4. The closed-form statistical inference is feasible for all time-inhomogeneous cases of Pearson diffusion processes, as presented in Section 3.4.

Closed-form formulas are available for conditional and unconditional moments in all classes of Pearson diffusion processes, including the OU, CIR, Jacobi, Fisher–Snedecor, reciprocal gamma, and Student diffusion processes. The conditional moments of the class of light-tailed Pearson diffusion processes, namely, OU, CIR, and Jacobi processes, are provided directly using their transition PDFs to compare them with the formulas solved by the Feynman–Kac formula. The proposed formulas for each diffusion process were validated by comparing them with MC simulations via several experimental examples, as presented in Section 3.5. We show the advantage of the proposed formula for computing the conditional moments of time-inhomogeneous processes, such as EOU and ECIR processes, as given in (3.60) and (3.63).

CHAPTER IV

ANALYTICAL FORMULA FOR CONDITIONAL EXPECTATIONS OF PATH-DEPENDENT PRODUCT OF POLYNOMIAL AND EXPONENTIAL FUNCTIONS OF EXTENDED COX–INGERSOLL–ROSS PROCESS

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Abstract

This paper proposes an analytical formula for the conditional expectations of path dependent product of polynomial and exponential function in the form of

$$\left(\sum_{j=0}^{n}\lambda_j^{(l)}r_{t_l}^j\right)e^{\sum_{k=1}^{m}\alpha_k^{(l)}r_{t_k}}$$

for $n, m \in \mathbb{N}$, $l = 1, 2, ..., m$, $0 \leq t_1 < t_2 < \cdots < t_m = T < \infty$ and $\lambda_j^{(l)}, \alpha_k^{(l)} \in \mathbb{R}$, where $\{r_t\}_{t\in[0,T]}$ corresponds to the extended Cox–Ingersoll–Ross (ECIR) process. The validation of the analytical formula is illustrated for several examples by comparing the results from the formula with those from Monte Carlo (MC) simulations. The efficiency of the formula is also presented via the computational run-times as compared with MC simulation. Moreover, the application of the analytical formula of this work is demonstrated for pricing arrears interest rate swaps under the ECIR process.

Keywords: analytical formula, conditional expectation, ECIR process, path-dependent

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4.1 Introduction

An extended Cox–Ingersoll–Ross (ECIR) process [39] is one of the most widely used processes in financial mathematics, which was firstly considered in 1990 by Hull and White [39] to generalize models constructed by Vasicek (1977), see [76]. The ECIR process is usually applied to price financial derivatives, such as zero-coupon bond, ex-coupon, interest rate swaps (IRSs), and options, which often involves evaluation of conditional expectations, see e.g. [5, 35, 56]. Moreover, the process is a continuous Markov process that posses some useful properties including mean reversion and analytical formulas for its expectation and variance, in which there are a number of methods readily available for the calibration of the ECIR process parameters, see more details in Yang [80]. Thus, mathematical properties of the ECIR process are challenging topics for observing and applying in financial applications.

Let $(\Omega, \mathcal{F}_t, \{\mathcal{F}_t\}_{0\leq t\leq T}, P)$ be a filtered probability space generated by an adapted stochastic process $\{r_t\}_{0\leq t\leq T}$ satisfying the ECIR process which is governed by the stochastic differential equation

$$dr_t = \kappa(t)(\theta(t) - r_t)dt + \sigma(t)\sqrt{r_t}dW_t, \quad 0 \leq t \leq T, \tag{4.1}$$

where t and T are the initial and terminal times, respectively, $\theta(t)$ is the mean-reverting level, $\kappa(t)$ is the speed of adjustment, $\sigma(t)$ is the state space of the diffusion, and $\{W_t\}_{t\in[0,T]}$ is a

Brownian motion. The drift factor $\kappa(t)(\theta(t) - r_t)$ is identical to that of the extended Vasicek process [39], respectively. Comparing (4.1) with Hull and White's work [39], $\kappa(t)$ and $\theta(t)$ in (4.1) represent $a(t)$ and $\theta(t)/a(t) + b$ in the extended Vasicek process. The only difference between the ECIR and the extended Vasicek process is the standard deviation factor $\sigma(t)\sqrt{r_t}$ which prevents r_t from being negative [44]. In contrast to the extended Vasicek process, the ECIR process has r_t approaches zero as well as the diffusion term. This characteristic of the ECIR process makes r_t to have non-negative value, which is a main reason the process becomes famous for the study of the behavior of interest rates. The case where the parameters are constant the process (4.1) becomes the well-known CIR process [18]. Both processes are frequently applied to describe the dynamics of observed data such as the interest rate, see e.g. [17, 39]. However, many empirical evidences strongly suggest that parameters of the process should dependent on time, see e.g. [37, 39, 55].

Many researchers focus on modeling the term-structure movements of interest rates such as Kijima [44], Hull and White [39], Maghsoodi [55], Filipović [26, 27], Filipović and Mayerhofer [55], Alfonsi [2] and Thamrongrat and Rujivan [75]. In 2001, Dufresne [22] proposed a closed-form formula for $E^p[r_T^\gamma \mid F_t] = E^p[r_T^\gamma \mid r_t = r]$ for $\gamma > -\frac{2\kappa\theta}{\sigma^2}$, in which r_t is assumed to follow the CIR process. An analytical approach for $E^p[r_T^\gamma \mid r_t = r]$ presented by Rujivan [64] is extended from Dufresne's work [22] to ECIR process for any $\gamma \in \mathbb{R}$. Recently, for ECIR process, an exact formula for conditional expectations of product of polynomial and exponential function of affine transform $E^P\left[r_T^n e^{\alpha r_T} \mid r_t = r\right]$ was introduced by Sutthimat et al. [72] for nonnegative integer n and any real number α, which also cover the results in those of Rujivan [64] in the case of $\alpha = 0$. The formulas provided in [72] are useful, for instance, Rujivan [65], Rujivan and Rakwongwan [66], Rujivan and Zhu [67] applied the first and second conditional moments of the CIR process to obtain an explicit formula for a pricing discretely-sampled variance swap of the Heston model.

In this work, we provide an extension of the recent results by Sutthimat et al. [72] and propose an analytical formula for a conditional expectation of a path-dependent product of polynomial and exponential functions described in the form

$$E^p\left[\left(\sum_{j=0}^{n} \lambda_j^{(l)} r_{t_l}^j\right) e^{\sum\limits_{k=1}^{m} \alpha_k^{(l)} r_{t_k}} \mid r_{t_1} = r\right], \tag{4.2}$$

where $n, m \in \mathbb{N}$, $l = 1, 2, \ldots, m$, $0 \leq t_1 < t_2 < \cdots < t_m = T < \infty$, $\lambda_j^{(l)}, \alpha_k^{(l)} \in \mathbb{R}$ and $\{r_t\}_{0 \leq t \leq T}$ is assumed to follow the ECIR process. In this work, we focus primarily on the case that the exponential term depends on the values $r_{t_1}, r_{t_2}, \ldots, r_{t_m}$ at a fixed set of times $t_1, t_2, \ldots, t_m$, as described in (4.2). With this idea, it is usually possible to apply the result to value some financial products, e.g., the discussion of the valuation of IRS is mentioned in Section 4.5, which, in principle, depends on the complete path $\{r_t\}_{0\leq t\leq T}$ over an interval $[0,T]$.

The remainder of the paper is structured as follows. The background knowledge that is essential to understanding the problem and getting the aims of this paper is proposed in Section 4.2. In Section 4.3, we provide the main results of the conditional expectation (4.2) in an analytical form. Furthermore, the obtained results are also explored in some special cases. Section 4.4 discusses in details the advantages of the analytical formulas compared with the Monte Carlo (MC) simulations for the ECIR process. The formulas for pricing the arrears swap is provided in Section 4.5. The aims of the paper are recapitulated and concluded in Section 4.6.

4.2 Background knowledge

In order to ensure that there exists a path-wise unique strong solution for the process r_t in (4.1) and to avoid zero almost surely with respect to probability measure $\mathcal{P}$ for all $t\in[0,T]$, the two following assumptions proposed by Rogers and Williams [63] and Ekström et al. [24], respectively, are extremely needed.

Assumption 4.1. *The functions $\theta(t),\kappa(t)$ and $\sigma(t)$ in the ECIR process* (4.1) *are strictly positive, smooth and continuous time-dependent functions on $[0,T]$. Moreover, $\frac{\kappa(t)}{\sigma^2(t)}$ is locally bounded on $[0,T]$.*

Assumption 4.2. *The process r_t defined by* (4.1) *holds the inequality, $2\kappa(t)\theta(t)>\sigma(t)^2$.*

Suppose that r_t follows ECIR process (4.1) and both Assumptions 4.1 and 4.2 hold. By Feynman-Kac theorem under the suitable constructions, an explicit formula of $E^P\left[r_T^n e^{\alpha r_T}|r_t=r\right]$ is proposed by Sutthimat et al. [72], where the formula is denoted by $U_E^{(n,\alpha)}(r,\tau)$. Note that the subscripts E and C are used to denote the formula corresponding to the ECIR and CIR processes, respectively. The proposed formula for ECIR process is given in the following theorem.

Theorem 4.1. *Suppose that r_t follows the ECIR process* (4.1) *with $\alpha\in\mathbb{R}$. Let n be a nonnegative integer and $0\leq t\leq T$. Then, for $r>0$ with $\tau=T-t\geq 0$,*

$$U_E^{(n,\alpha)}(r,\tau):=E^P\left[r_T^n\, e^{\alpha r_T}\mid r_t=r\right]=\ e^{rB(\tau,\alpha)}\sum_{j=0}^{n}A_j(\tau,\alpha)\,r^j, \tag{4.3}$$

where

$$A_n(x,y) = \exp\left[\int_0^x \left(n\sigma^2(T-u)B(u,y) + \kappa(T-u)\theta(T-u)B(u,y) - n\kappa(T-u)\right) du\right],$$

$$A_j\left(x,y\right) = \exp\left[\int_0^x Q_j(T-u,y)\,du\right]\int_0^x \exp\left[-\int_0^s Q_j(T-u,y)\,du\right] P_{j+1}(T-s)A_{j+1}(s,y)\,ds,$$

$$P_{j+1}(x) = (j+1)\left(\frac{1}{2}j\sigma^2(x) + \kappa(x)\theta(x)\right),$$

$$Q_j(x,y) = j\sigma^2(x)B(T-x,y) + \kappa(x)\theta(x)B(T-x,y) - j\kappa(x),$$

for $j = 0, 1, 2, \ldots, n-1$*. The function* B *is given by*

$$B\left(x,y\right) = \frac{2y\exp\left[-\int_0^x \kappa\left(T-u\right)du\right]}{2 - y\int_0^x \sigma^2\left(T-s\right)\exp\left[-\int_0^s \kappa\left(T-u\right)du\right]ds}.$$

Note that the explicit formula which is described by (4.3) converges if

$$\alpha \in \left(-\infty, \frac{2}{\delta\left(\tau\right)}\right), \text{ where } \delta\left(\tau\right) = \int_0^\tau \sigma^2\left(T-s\right)\exp\left[-\int_0^s \kappa\left(T-u\right)du\right]ds. \tag{4.4}$$

Moreover, for any initial $r > 0$ and nonnegative integer n, the monotone convergence theorem gives that $U_E^{(n,\alpha)}\left(r,\tau\right) \to \infty$ as $\alpha \to \frac{2}{\delta(\tau)}$. Calculating the expectation (4.3), when $\kappa\left(t\right) = \kappa$, $\theta\left(t\right) = \theta$ and $\sigma\left(t\right) = \sigma$ are constants, the explicit formula is reduced into the CIR process, denoted by $U_C^{(n,\alpha)}\left(r,\tau\right)$, as stated in the following theorem.

Theorem 4.2. *Suppose that* r_t *follows the CIR process with* $\alpha \in R$*. Let* n *be a non-negative integer and* $0 \le t \le T$*. Then, for* $r > 0$ *with* $\tau = T - t \ge 0$*,*

$$\begin{aligned}
U_C^{(n,\alpha)}\left(r,\tau\right) &:= E^P\left[r_T^n e^{\alpha r_T} \mid r_t = r\right] \\
&= \exp\left[\frac{2\alpha\kappa}{\alpha\sigma^2 + e^{\kappa\tau}\left(2\kappa - \alpha\sigma^2\right)}r + n\kappa\tau + \frac{2\theta\kappa^2\tau}{\sigma^2}\right] \\
&\quad \times \left(\frac{2\kappa}{\alpha\sigma^2 + e^{\kappa\tau}\left(2\kappa - \alpha\sigma^2\right)}\right)^{\frac{2}{\sigma^2}\left(n\sigma^2 + \kappa\theta\right)} r^n \\
&\quad + \sum_{j=0}^{n-1}\exp\left[\frac{2\alpha\kappa}{\alpha\sigma^2 + e^{\kappa\tau}\left(2\kappa - \alpha\sigma^2\right)}r + j\kappa\tau + \frac{2\theta\kappa^2\tau}{\sigma^2}\right] \\
&\quad \times \prod_{m=1}^{n-j}\overline{P}_{n-m+1}\frac{2^{n-j}}{(n-j)!}\left(\frac{2\kappa}{\alpha\sigma^2 + e^{\kappa\tau}\left(2\kappa - \alpha\sigma^2\right)}\right)^{\frac{2}{\sigma^2}\left(j\sigma^2 + \kappa\theta\right)} \\
&\quad \times \left(\frac{e^{\kappa\tau} - 1}{\alpha\sigma^2 + e^{\kappa\tau}\left(2\kappa - \alpha\sigma^2\right)}\right) r^j,
\end{aligned}$$

where $\overline{P}_{n-m+1} = (n-m+1)(\frac{1}{2}(n-m)\sigma^2 + \kappa\theta)$ *when* $m \in \{1, 2, \ldots, n-j\}$*, for* $j \in \{1, 2, \ldots, n-1\}$*.*

As for Theorem 4.1, the convergent set of α in (4.4) can be applied to Theorem 4.2 directly.

Furthermore, for CIR process, Theorem 2.1 in Dufresne [22] can be rewritten into the form of $U_C^{(n,0)}(r,\tau)$.

4.3 Main results

This paper uniquely determines the law of the process r_t by discretizing the continuous time $[0,T]$ into $0 \le t_1 < t_2 < \cdots < t_m = T$, $m > 1$, and define functions for all $i = 1, 2, \ldots, m-1$ as the following

- $\tau(i) = t_{m-i+1} - t_{m-i}$,
- $\phi^{(l)}(i) = \alpha_{m-i}^{(l)} + B\left(\tau(i), \phi^{(l)}(i-1)\right)$, where $\phi^{(l)}(0) = \alpha_m^{(l)}$ for $l = 1, 2, \ldots, m$,
- $A_0\left(\tau(0), \phi^{(l)}(-1)\right) = 1$,

where the functions A_0 and B are defined in Theorem 4.1.

The analytical formulas presented in the following is extended from the approaches proposed in [72]. By applying the tower property, also known as nested conditional expectation (see [28] for more details), and adopting the result in Theorem 4.1, we immediately obtain The following result.

Theorem 4.3. *Suppose that r_t follows the ECIR process* (4.1)*, $h, l, m, n \in \mathbb{N}$ with $h < l \le m$ and $\alpha_k^{(l)} = 0$ for $k = 1, 2, \ldots, l-1$. Then the conditional expectation* (4.2) *can be reduced into the form*

$$V^*(r, t_l, t_h) := E^p\left[\left(\sum_{j=0}^{n} \lambda_j^{(l)} r_{t_l}^j\right) e^{\sum\limits_{k=l}^{m} \alpha_k^{(l)} r_{t_k}} \mid r_{t_h} = r\right],$$

where $0 \le t_1 < t_2 < \cdots < t_m = T$ and $\lambda_j^{(l)}, \alpha_k^{(l)} \in \mathbb{R}$, and can be expressed as

$$V^*(r, t_l, t_h) = \left[\prod_{i=0}^{m-l} A_0\left(\tau(i), \phi^{(l)}(i-1)\right)\right]\left[\sum_{j=0}^{n} \lambda_j^{(l)} U_E^{\left(j, \phi^{(l)}(m-l)\right)}(r, t_l - t_h)\right]. \quad (4.5)$$

Proof. The proof is separated into two cases. For the case of $m = l$, by using Theorem 4.1,

$V^*\left(r,t_l,t_h\right)$ can be rewritten as

$$
\begin{aligned}
E^p\left[\left(\sum_{j=0}^{n}\lambda_j^{(l)}r_{t_l}^j\right)e^{\alpha_l^{(l)}r_{t_l}}\mid r_{t_h}=r\right] &= \sum_{j=0}^{n}\lambda_j^{(l)}\;E^p\left[r_{t_l}^j e^{\alpha_l^{(l)}r_{t_l}}\mid r_{t_h}=r\right]\\
&= \sum_{j=0}^{n}\lambda_j^{(l)}U_E^{\left(j,\alpha_l^{(l)}\right)}\left(r,t_l-t_h\right)\\
&= \sum_{j=0}^{n}\lambda_j^{(l)}U_E^{\left(j,\phi^{(l)}(0)\right)}\left(r,t_l-t_h\right). \qquad (4.6)
\end{aligned}
$$

This satisfies Equation (4.5). The remaining is to prove the latter case, $m>l$. The tower property is firstly applied with Theorem 4.1 to yield

$$
\begin{aligned}
&V^*\left(r,t_l,t_h\right) = E^p\left[\left(\sum_{j=0}^{n}\lambda_j^{(l)}r_{t_l}^j\right)e^{\sum\limits_{k=l}^{m}\alpha_k^{(l)}r_{t_k}}\mid r_{t_h}=r\right]\\
&= E^p\left[\left(\sum_{j=0}^{n}\lambda_j^{(l)}r_{t_l}^j\right)e^{\sum\limits_{k=l}^{m-1}\alpha_k^{(l)}r_{t_k}}E^p\left[e^{\alpha_m^{(l)}r_{t_m}}\mid r_{t_{m-1}}\right]\mid r_{t_h}=r\right]\\
&= E^p\left[\left(\sum_{j=0}^{n}\lambda_j^{(l)}r_{t_l}^j\right)e^{\sum\limits_{k=l}^{m-1}\alpha_k^{(l)}r_{t_k}}U_E^{\left(0,\alpha_m^{(l)}\right)}\left(r_{t_{m-1}},t_m-t_{m-1}\right)\mid r_{t_h}=r\right]\\
&= E^p\left[\left(\sum_{j=0}^{n}\lambda_j^{(l)}r_{t_l}^j\right)e^{\sum\limits_{k=l}^{m-1}\alpha_k^{(l)}r_{t_k}}e^{r_{t_{m-1}}B\left(t_m-t_{m-1},\alpha_m^{(l)}\right)}A_0\left(t_m-t_{m-1},\alpha_m^{(l)}\right)\mid r_{t_h}=r\right]\\
&= A_0\left(t_m-t_{m-1},\alpha_m^{(l)}\right)E^p\left[\left(\sum_{j=0}^{n}\lambda_j^{(l)}r_{t_l}^j\right)e^{\sum\limits_{k=l}^{m-1}\alpha_k^{(l)}r_{t_k}}e^{r_{t_{m-1}}B\left(t_m-t_{m-1},\alpha_m^{(l)}\right)}\mid r_{t_h}=r\right]\\
&= A_0\left(\tau\left(1\right),\phi^{(l)}\left(0\right)\right)E^p\left[\left(\sum_{j=0}^{n}\lambda_j^{(l)}r_{t_l}^j\right)e^{\sum\limits_{k=l}^{m-1}\alpha_k^{(l)}r_{t_k}}e^{r_{t_{m-1}}B\left(\tau(1),\phi^{(l)}(0)\right)}\mid r_{t_h}=r\right]\\
&= A_0\left(\tau\left(1\right),\phi^{(l)}\left(0\right)\right)E^p\left[\left(\sum_{j=0}^{n}\lambda_j^{(l)}r_{t_l}^j\right)e^{\sum\limits_{k=l}^{m-2}\alpha_k^{(l)}r_{t_k}}e^{r_{t_{m-1}}\left(\alpha_{m-1}^{(l)}+B\left(\tau(1),\phi^{(l)}(0)\right)\right)}\mid r_{t_h}=r\right]\\
&= \left[\prod_{i=0}^{1}A_0\left(\tau\left(i\right),\phi^{(l)}\left(i-1\right)\right)\right]E^p\left[\left(\sum_{j=0}^{n}\lambda_j^{(l)}r_{t_l}^j\right)e^{\sum\limits_{k=l}^{m-2}\alpha_k^{(l)}r_{t_k}}\;e^{r_{t_{m-1}}\phi^{(l)}(1)}\mid r_{t_h}=r\right].
\end{aligned}
$$

Note that, by applying the tower property in the first time to the fifth line in the above equality, the summation in the exponent term from $k=l$ to $k=m-1$ is reduced to the summation from $k=l$ to $k=m-2$. It is not difficult to see that, by applying the tower property $(m-l)^{th}$ time, the summation will run from $k=l$ to $k=m-(m-l)=l$, which has only one term. Thus, we obtain the followings.

$$
\begin{aligned}
V^{*}\left(r,t_l,t_h\right) &= \left[\prod_{i=0}^{m-l} A_0\left(\tau\left(i\right),\phi^{(l)}\left(i-1\right)\right)\right] E^{p}\left[\left(\sum_{j=0}^{n}\lambda_j^{(l)} r_{t_l}^{j}\right) e^{r_{t_l}\phi^{(l)}(m-l)} \mid r_{t_h}=r\right] \\
&= \left[\prod_{i=0}^{m-l} A_0\left(\tau\left(i\right),\phi^{(l)}\left(i-1\right)\right)\right] \left[\sum_{j=0}^{n}\lambda_j^{(l)} E^{p}\left[r_{t_l}^{j} e^{r_{t_l}\phi^{(l)}(m-l)} \mid r_{t_h}=r\right]\right] \\
&= \left[\prod_{i=0}^{m-l} A_0\left(\tau\left(i\right),\phi^{(l)}\left(i-1\right)\right)\right] \left[\sum_{j=0}^{n}\lambda_j^{(l)} U_E^{\left(j,\phi^{(l)}(m-l)\right)}\left(r,t_l-t_h\right)\right].
\end{aligned}
$$

This completes the proof. □

In the case that $\alpha_k^{(l)} \in \mathbb{R}$ for all $k=1,2,\ldots,m$ with $l \le m$, an extension of Theorem 4.3 can be derived as shown in the following theorem.

Theorem 4.4. *Suppose that r_t follows the ECIR process* (4.1)*, $l,m,n \in \mathbb{N}$ and $l \le m$. Then, the conditional expectation*

$$
V\left(r,t_l\right) := E^{p}\left[\left(\sum_{j=0}^{n}\lambda_j^{(l)} r_{t_l}^{j}\right) e^{\sum_{k=1}^{m}\alpha_k^{(l)} r_{t_k}} \mid r_{t_1}=r\right], \tag{4.7}
$$

for $l=1,2,3,\ldots,m$, where $0 \le t_1 < t_2 < \cdots < t_m = T$ and $\lambda_j^{(l)},\alpha_k^{(l)} \in \mathbb{R}$, can be expressed as

$$
\begin{aligned}
V\left(r,t_l\right) = e^{r\alpha_1^{(l)}} &\left[\prod_{i=0}^{m-l} A_0\left(\tau\left(i\right),\phi^{(l)}\left(i-1\right)\right)\right] \\
&\times\left[\sum_{j_1=0}^{n}\lambda_{j_1}^{(l)}\left(\sum_{j_2=0}^{j_1}\sum_{j_3=0}^{j_2}\cdots\sum_{j_{l-1}=0}^{j_{l-2}}\left(\prod_{s=1}^{l-2}A_{j_{s+1}}(\tau(m-l+s),\phi^{(l)}(m-l+s-1))\right)\right.\right. \\
&\left.\left.\times U_E^{(j_{l-1},\phi^{(l)}(m-2))}(r,\tau(m-1))\right)\right].
\end{aligned} \tag{4.8}
$$

Proof. In what starts, for the case $k=l,l+1,\ldots,m$, the proof mainly work with the tower property and the result obtained in Theorem 4.3.

$$
\begin{aligned}
V\left(r,t_l\right) &= E^p\left[\left(\sum_{j=0}^{n}\lambda_j^{(l)} r_{t_l}^{j}\right) e^{\sum\limits_{k=1}^{m}\alpha_k^{(l)} r_{t_k}} \mid r_{t_1}=r\right] \\
&= E^p\left[e^{\sum\limits_{k=1}^{l-1}\alpha_k^{(l)} r_{t_k}}\left(\sum_{j=0}^{n}\lambda_j^{(l)} r_{t_l}^{j}\right) e^{\sum\limits_{k=l}^{m}\alpha_k^{(l)} r_{t_k}} \mid r_{t_1}=r\right] \\
&= E^p\left[e^{\sum\limits_{k=1}^{l-1}\alpha_k^{(l)} r_{t_k}} E^p\left[\left(\sum_{j=0}^{n}\lambda_j^{(l)} r_{t_l}^{j}\right) e^{\sum\limits_{k=l}^{m}\alpha_k^{(l)} r_{t_k}} \mid r_{t_{l-1}}\right] \mid r_{t_1}=r\right]. \qquad (4.9)
\end{aligned}
$$

So, Theorem 4.3 is applied here, the inner conditional expectation term of (4.9) is equal to $V^*\left(r,t_l,t_{l-1}\right)$.

$$
\begin{aligned}
V\left(r,t_l\right) &= E^p\left[e^{\sum\limits_{k=1}^{l-1}\alpha_k^{(l)} r_{t_k}} V^*\left(r,t_l,t_{l-1}\right) \mid r_{t_1}=r\right] \\
&= \left[\prod_{i=0}^{m-l} A_0\left(\tau\left(i\right),\phi^{(l)}\left(i-1\right)\right)\right] \\
&\quad\times\left[\sum_{j_1=0}^{n}\lambda_{j_1}^{(l)} E^p\left[e^{\sum\limits_{k=1}^{l-1}\alpha_k^{(l)} r_{t_k}} U_E^{\left(j_1,\phi^{(l)}(m-l)\right)}\left(r_{t_{l-1}},\tau\left(m-l+1\right)\right) \mid r_{t_1}=r\right]\right]. \qquad (4.10)
\end{aligned}
$$

To address the right-hand side of Equation (4.10), only the conditional expectation term is considered and the tower property is firstly applied.

$$
\begin{aligned}
&E^p\left[e^{\sum\limits_{k=1}^{l-1}\alpha_k^{(l)} r_{t_k}} U_E^{\left(j_1,\phi^{(l)}(m-l)\right)}\left(r_{t_{l-1}},\tau\left(m-l+1\right)\right) \mid r_{t_1}=r\right] \\
&= E^p\left[e^{\sum\limits_{k=1}^{l-1}\alpha_k^{(l)} r_{t_k}} \sum_{j_2=0}^{j_1} A_{j_2}\left(\tau\left(m-l+1\right),\phi^{(l)}\left(m-l\right)\right) r_{t_{l-1}}^{j_2} e^{r_{t_{l-1}} B\left(\tau(m-l+1),\phi^{(l)}(m-l)\right)} \mid r_{t_1}=r\right] \\
&= \sum_{j_2=0}^{j_1} A_{j_2}\left(\tau\left(m-l+1\right),\phi^{(l)}\left(m-l\right)\right) E^p\left[e^{\sum\limits_{k=1}^{l-1}\alpha_k^{(l)} r_{t_k}} r_{t_{l-1}}^{j_2} e^{r_{t_{l-1}} B\left(\tau(m-l+1),\phi^{(l)}(m-l)\right)} \mid r_{t_1}=r\right] \\
&= \sum_{j_2=0}^{j_1} A_{j_2}\left(\tau\left(m-l+1\right),\phi^{(l)}\left(m-l\right)\right) E^p\left[e^{\sum\limits_{k=1}^{l-2}\alpha_k^{(l)} r_{t_k}} r_{t_{l-1}}^{j_2} e^{r_{t_{l-1}}\left(\alpha_{l-1}^{(l)}+B\left(\tau(m-l+1),\phi^{(l)}(m-l)\right)\right)} \mid r_{t_1}=r\right] \\
&= \sum_{j_2=0}^{j_1} A_{j_2}\left(\tau\left(m-l+1\right),\phi^{(l)}\left(m-l\right)\right) E^p\left[e^{\sum\limits_{k=1}^{l-2}\alpha_k^{(l)} r_{t_k}} r_{t_{l-1}}^{j_2} e^{r_{t_{l-1}}\phi^{(l)}(m-l+1)} \mid r_{t_1}=r\right] \\
&= \sum_{j_2=0}^{j_1} A_{j_2}\left(\tau\left(m-l+1\right),\phi^{(l)}\left(m-l\right)\right) E^p\left[e^{\sum\limits_{k=1}^{l-2}\alpha_k^{(l)} r_{t_k}} E^p\left[r_{t_{l-1}}^{j_2} e^{r_{t_{l-1}}\phi^{(l)}(m-l+1)} \mid r_{t_{l-2}}\right] \mid r_{t_1}=r\right] \\
&= \sum_{j_2=0}^{j_1} A_{j_2}\left(\tau\left(m-l+1\right),\phi^{(l)}\left(m-l\right)\right) E^p\left[e^{\sum\limits_{k=1}^{l-2}\alpha_k^{(l)} r_{t_k}} U_E^{\left(j_2,\phi^{(l)}(m-l+1)\right)}\left(r_{t_{l-2}},\tau\left(m-l+2\right)\right) \mid r_{t_1}=r\right].
\end{aligned}
$$

Continue the process through the remaining statements until applying the tower property at $(l-2)^{th}$ time to get

$$
\begin{aligned}
&E^p\left[e^{\sum\limits_{k=1}^{l-1}\alpha_k^{(l)}r_{t_k}}\,U_E^{\left(j_1,\phi^{(l)}(m-l)\right)}\left(r_{t_{l-1}},\tau\left(m-l+1\right)\right)\mid r_{t_1}=r\right]\\
&=\sum_{j_2=0}^{j_1}\sum_{j_3=0}^{j_2}\cdots\sum_{j_{l-1}=0}^{j_{l-2}}\left(\prod_{s=1}^{l-2}A_{j_{s+1}}\left(\tau\left(m-l+s\right),\phi^{(l)}\left(m-l+s-1\right)\right)\right)\\
&\quad\times E^p\left[e^{\alpha_1^{(l)}r_{t_1}}\,U_E^{\left(j_{l-1},\phi^{(l)}(m-2)\right)}\left(r_{t_1},\tau\left(m-1\right)\right)\mid r_{t_1}=r\right]\\
&=e^{r\alpha_1^{(l)}}\left(\sum_{j_2=0}^{j_1}\sum_{j_3=0}^{j_2}\cdots\sum_{j_{l-1}=0}^{j_{l-2}}\left(\prod_{s=1}^{l-2}A_{j_{s+1}}\left(\tau\left(m-l+s\right),\phi^{(l)}\left(m-l+s-1\right)\right)\right)\right.\\
&\quad\left.\times\,U_E^{\left(j_{l-1},\phi^{(l)}(m-2)\right)}\left(r,\tau\left(m-1\right)\right)\right).
\end{aligned}
\tag{4.11}
$$

Since (4.11) is the term on the right side of (4.10), by inserting (4.11) into (4.10) yields (4.8), and hence the theorem is proved. □

Remark 4.1. Given $l\in\mathbb{N}$. The result in Theorem 4.4 can show mathematically that whenever $\alpha_k^{(l)}=0$ for $k=1,2,\ldots,l-1$, it reduces (4.8) into (4.5) of Theorem 4.3.

4.4 Numerical results and discussions

In this section, the verifications of results in Section 4.3 are given though comparisons with MC simulations based on the following ECIR process

$$
dr_t=\kappa\left(\frac{\sigma_0^2d\,e^{2\sigma_1\left(t+\sigma_2\sin\left(2\pi\sqrt{t}\right)\right)}}{4\kappa}-r_t\right)dt+\sigma_0e^{\sigma_1\left(t+\sigma_2\sin\left(2\pi\sqrt{t}\right)\right)}\sqrt{r_t}\,dW_t. \tag{4.12}
$$

Comparing (4.12) with (4.1) gives $\kappa(t)=\kappa$, $\theta(t)=\sigma_0^2d\,e^{2\sigma_1\left(t+\sigma_2\sin\left(2\pi\sqrt{t}\right)\right)}/4\kappa$ and $\sigma(t)=\sigma_0e^{\sigma_1\left(t+\sigma_2\sin\left(2\pi\sqrt{t}\right)\right)}$, where κ and σ_0 are positive constants, σ_1 and σ_2 are nonnegative constants and d is a positive integer. In fact, the process is specifically called ECIR(d) process, where d is the dimension.

The transition density of ECIR(d) process X_t, represented by $p_X\left(x,s+h\mid x_s,s\right)$ for $x>0$, is introduced by Maghsoodi [55] and mentioned again by Egorov et al. [23]. Indeed, for example, a special case of Theorem 4.4 can be computed as follows.

$$
\begin{aligned}
V\left(k,t_2\right)&=E^p\left[r_{t_2}e^{-r_{t_2}-r_{t_3}}\mid r_{t_1}=k\right]\\
&=\int_0^\infty\int_0^\infty xe^{-x-y}\,p_r\left(y,t_3\mid r_{t_2}=x,r_{t_1}=k\right)p_r\left(x,t_2\mid r_{t_1}=k\right)dy\,dx.
\end{aligned}
$$

Remark that the transition density p_X behaves like a Dirac delta function when the increment h is small, for instant $h = 0.01$. Because p_X may give a tall spike at the initial value X_0, using p_X may produce inaccurate results if applying the usual integration methods numerically (e.g. the trapezoidal rule), see [23, 64]. Generally, the MC is widely used in practice and computationally faster than calculating the transition density, especially, in the cases where the path needs to be sampled at any specific points in order to approximate a conditional expectation of the path-dependence.

The qualitatively correct approximations by using Euler-Maruyama (EM) discretization to the mean-reverting square root process, such as the ECIR process, are deterministically provided by Higham and Mao [38]. In this paper, a module of EM method in MATLAB is applied numerically to obtain the values defined in (4.2). To make the results more tangible, the following three examples illustrate that how to use the results in practice. Implementation of EM method is straightforward based on (4.12), unless parameters $\kappa(t), \theta(t)$ and $\sigma(t)$ are difficult to evaluate.

Example 4.1. Consider the conditional expectation $V(r, t_m)$ with parameters $m = 3$, $n = 1$, $l = 2$, $\alpha_1^{(2)} = 0$, $\alpha_2^{(2)} = -1$, $\alpha_3^{(2)} = -1$, $\lambda_0^{(2)} = 0$ and $\lambda_1^{(2)} = 1$. From Theorem 4.4, (4.7) can be given explicitly as

$$V(r, t_2) := E^p \left[r_{t_2} e^{-r_{t_2} - r_{t_3}} \mid r_{t_1} = r \right]. \tag{4.13}$$

In this example, we verify the accuracy and efficiency of the result in Theorem 4.4 by comparing with MC simulations via EM method without any variance reduction techniques. Moreover, the ECIR process (4.12) of this example is immediately deduced to the CIR process by setting the parameters $\sigma_0 \in \mathbb{R}^+$ and $\sigma_1 = \sigma_2 = 0$. Thus, all integral terms in Theorem 4.4 can be completely evaluated. Therefore, the obtained result (4.13) is the exact formula,

$$V(r, t_2) = A_0 (t_3 - t_2, -1) U_C^{(1, \phi^{(2)}(1))} (r, t_2 - t_1) = \omega_0 (\omega_1 r + \omega_2) \tag{4.14}$$

for all $r > 0$ and $0 < t_2 < t_3$, where $\zeta = 2\kappa + \sigma_0^2$ and

$$\begin{aligned}
\delta_1 &= \frac{-2\kappa}{\zeta e^{\kappa(t_3-t_2)} - \sigma_0^2} \\
\delta_2 &= \frac{2\kappa - \sigma_0^2 + \zeta e^{\kappa(t_3-t_2)}}{2\kappa\sigma_0^2 - \sigma_0^4 - \zeta^2 e^{\kappa(t_3-t_1)} + \zeta\sigma_0^2 e^{\kappa(t_3-t_2)} + \sigma_0^4 e^{\kappa(t_2-t_1)}} \\
\omega_0 &= \left(\frac{2\kappa e^{\kappa(t_3-t_2)}}{\zeta e^{\kappa(t_3-t_2)} - \sigma_0^2}\right)^{\frac{d}{2}} \\
\omega_1 &= e^{2\kappa\delta_2 r + \kappa(t_2-t_1)\left(1+\frac{d}{2}\right)} \left(\frac{2\kappa\delta_2}{\delta_1 - 1}\right)^{2+\frac{d}{2}} \\
\omega_2 &= \left(\frac{\sigma_0^2 d\delta_2\left(e^{\kappa(t_2-t_1)} - 1\right)}{2\left(\delta_1 - 1\right)}\right) e^{2\kappa\delta_2 r + \kappa(t_2-t_1)\frac{d}{2}} \left(\frac{2\kappa\delta_2}{\delta_1 - 1}\right)^{\frac{d}{2}}.
\end{aligned}$$

To validate the exact formula (4.14) the following parameters $d = 2$, $\kappa = 0.3$ and $\sigma_0 = 0.01$ are used for MC simulations at each initial value $r \in \{0.1, 0.2, \ldots, 2\}$ to generate sample paths of r_t, where each path consisting of $10,000$ steps over $[t_1, t_3]$. The validations are shown as the comparisons between the formula (4.14) and MC simulations for two different interval times $[0, 0.01]$, $\{t_1 = 0, t_2 = 0.005, t_3 = 0.01\}$ and $[0, 5]$, $\{t_1 = 0, t_2 = 2.5, t_3 = 5\}$.

In this section, the numerical computations for obtaining comparison results are implemented by using MATLAB R2021a running on a laptop computer configured with the following details: Intel(R) Core(TM) i7-5700HQ, CPU @2.70GHz, 16.0GB RAM, Windows 8, 64-bit Operating System.

The comparison results between the formula (4.14) and the MC simulations with $10,000$ sample paths are shown in Figure 4.1. Figure 4.1 demonstrates that the results of MC simulations completely match with the analytical formula (4.14), which validates the accuracy of the analytical formula (4.14) obtained from Theorem 4.4. Moreover, Table 4.1 shows the mean absolute percentage error (MAPE) between the formula (4.14) and MC simulations and the average runtime (AVRT) of the MC simulations for different numbers of sample paths from $5,000$ to $25,000$, to validate the accuracy and efficiency of the formula. These illustrative AVRTs are the average of consuming times to compute the MC simulations at each initial value $r \in \{0.1, 0.2, \ldots, 2\}$. In each of the terminal times t_3, Table 4.1 is concluded the increasing of the sample path numbers that the MAPEs do not only decrease, but the AVRTs do also increase. We can see that the accuracy from the MAPEs in Table 4.1 are quite to vanish. Moreover, we can see the efficiency of our obtained formula (4.14) from Theorem 4.4 that it provides the value of $V(r, T)$ exactly for arbitrary value r, and also employs a small computational time around 0.3154 seconds.

Example 4.2. Consider the conditional expectation $V(r, t_m)$ with the same parameters in

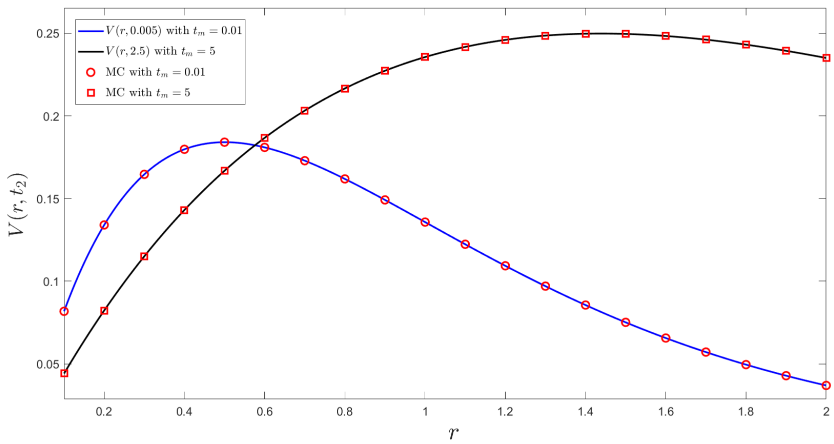

Figure 4.1: The comparisons between the formula (4.14) and MC simulations with 10,000 sample paths for $t_3 = 0.01$ and $t_3 = 5$.

Table 4.1: The MAPEs and AVRTs between formula (4.14) and the MC simulations with various numbers of sample paths for $t_3 = 0.01$ and $t_3 = 5$.

No. of paths	Terminal time $t_3 = 0.01$		Terminal time $t_3 = 5$	
	MAPEs (%)	AVRTs (s)	MAPEs (%)	AVRTs (s)
5,000	1.8000E-03	24.3244	1.8207E-02	25.0730
10,000	9.6656E-04	46.7345	1.2415E-02	50.6503
15,000	8.4001E-04	70.2077	1.0702E-02	73.6469
20,000	7.0942E-04	96.3160	9.4505E-03	98.0842
25,000	5.8531E-04	123.1318	8.8071E-03	126.1908

Example 4.1. From Theorem 4.4, (4.7) can be given explicitly as

$$V(r, t_2) := E^p \left[r_{t_2} e^{-r_{t_2} - r_{t_3}} \mid r_{t_1} = r \right]. \tag{4.15}$$

This example provides two sets of experiments using the same parameters of the previous example, except $\sigma_1 = 0.02$ and $\sigma_2 = 0.03$. The main difference of this example and the previous example is that the process (4.12) is still ECIR process and some integral terms in Theorem 4.4 cannot be directly evaluated. Thus, (4.15) cannot be expressed as an exact formula. In order to evaluate (4.15) by using Theorem 4.4, a numerical integration is required such as trapezoidal rule, Simpson's rule, Newton-Cotes, etc. Anywise, one efficient method that we choose to handle the integral terms is the Chebyshev integration method (CIM) introduced by Boonklurb et al. [8], which has been illustrated to produce a much higher accuracy than the other mentioned integration methods when using the same discretizing nodes.

Table 4.2 shows the comparisons between the approximate results from Theorem 4.4 by the CIM with 15 discretizing nodes and the MC simulations for several numbers of sample paths from $5,000$–$25,000$, in terms of the average difference (AVDF) and the AVRT. Additionally, this AVDF is measured by $\frac{1}{n}\sum_{i=1}^{n}|V(r_i)-M(r_i)|$, where $V(r)$ and $M(r)$ are the results from Theorem 4.4 via CIM and the MC simulations, respectively, and n is the number of initial values r. The AVDF value is also used for measuring the validity of the formula in addition to the MAPE. The closer the AVDF to zero, the better the approximation based on the proposed formula, when compared with MC simulation.

The results produced in Table 4.2 for $r \in \{0.1, 0.2, \ldots, 2\}$ suggest that the proposed formula in Theorem 4.4 is very accurate when compared with MC simulation as shown by the very small values of AVDFs. Based on the AVRTs of MC, the computation for the expectation usually quite expensive for MC simulation, especially when using a lot of of sample paths, as compared to the proposed formula in Theorem 4.4 which only take around 1 second to produce the result. In practice, MC simulation is one basic technique for obtaining accurate information of observed data when time is allowed. The advantage of the proposed formula in this work is clearly the time efficiency for obtaining required valued in financial application as illustrated in the next section.

Table 4.2: The comparisons between approximate results from Theorem 4.4 and the MC simulations.

No. of paths	Terminal time $t_3 = 0.01$		Terminal time $t_3 = 5$	
	AVDFs	AVRTs (s)	AVDFs	AVRTs (s)
$5,000$	1.2787E-06	26.7712	1.8194E-04	29.1465
$10,000$	1.0734E-06	54.7781	1.8150E-04	57.8478
$15,000$	1.0640E-06	77.3101	1.8101E-04	77.6856
$20,000$	1.0253E-06	95.4816	1.8084E-04	99.0211
$25,000$	8.9991E-07	122.9427	1.8016E-04	128.6961

Example 4.3. Consider the special case of conditional expectation subject to the parameters followed in Example 4.1 on terminal time $t_3 = 5$ which is defined by

$$B(t,T) := E\left[e^{-\int_t^T r(u)\,du}\right]. \tag{4.16}$$

In this example, Theorem 4.4 is applied to approximate (4.16) using the idea of the left Riemann sum approximation. Technically speaking, the forward rate $f(t,T)$ represents the instantaneous continuously compounded rate contracted at time t to maturity T, and $r(t) := f(t,t)$ is the short rate at time t. The time-t price of a bond paying 1 maturing at $T > t$ is given in (4.16), where $r(t)$ follows ECIR process. Let $\hat{f}(t_i, t_j)$ denote the discretized forward rate for maturity t_j as of time t_i, $i \le j$, and $\hat{B}(t_i, t_j)$ denote the corresponding bond price,

$$\hat{B}(t_i, t_j) := e^{-\sum\limits_{l=i}^{j-1} \hat{f}(t_i, t_l)(t_{l+1} - t_l)},$$

where the initial values of the discretized bonds $\hat{B}(0, t_j)$ to coincide with the exact values $B(0, t_j)$ for all maturities t_j, see [33] for more details. By uniquely considering at each path of $\hat{f}$, the discretized short rate $\hat{r}$ are extracted as

$$\hat{r}(t_1) = \hat{f}(t_1, t_1), \quad \hat{r}(t_2) = \hat{f}(t_2, t_2), \quad \ldots, \quad \hat{r}(t_m) = \hat{f}(t_m, t_m),$$

where $0 = t_1 < t_2 < \cdots < t_m = T$. In the case, a discount factor can be calculated as

$$\hat{D}(t_m) := e^{-\sum\limits_{i=1}^{m-1} \hat{r}(t_i)(t_{i+1} - t_i)},$$

for the maturity $t_m = T$. Suppose this repeats over n independent paths and let $\hat{D}^{(i)}(T)$ be discount factors calculated at i^{th} path. By a consequence of the strong law of large numbers in term on n and the martingale property, it is almost surely that

$$\frac{1}{n} \sum_{i=1}^{n} \hat{D}^{(i)}(T) \to E\left[\hat{D}(T)\right] = \hat{B}(0, T) = B(0, T) = E\left[e^{-\int_0^T r(u)\, du}\right]. \tag{4.17}$$

This example is setup with parameters $\sigma_1 = 0.02$ and $\sigma_2 = 0$, which is still ECIR process. By discretizing with $m = 2^n$ steps for $n \in \{1, 2, 3, 4, 5\}$ and $\Delta t = t_{i+1} - t_i = \frac{t_m}{m-1}$ for all $i \in \{1, 2, \ldots, m-1\}$, (4.17) can be rewritten as

$$B(0, t_m; r) := E\left[e^{-\int_0^{t_m} r(u)\, du} \mid r_{t_1} = r\right] \approx E\left[e^{-\Delta t \sum\limits_{i=1}^{m-1} \hat{r}_{t_i}} \mid r_{t_1} = r\right]. \tag{4.18}$$

Applying a consequence in Theorem 4.4 for $r \in [0.1, 2]$, which the integral terms therein can be completely evaluated. Thus, we have the exact value of right-hand-side term in (4.18) that is used to estimate the value of expectation in its left-hand-side term. The values of $B(0, t_m; r)$ for various discretizing m are shown in Figure 4.2. However, this example can be together performed by the MC simulations, but it uses more computational time to obtain the values especially, for the large value of discretized partitions m. Furthermore, we can see that the right-hand-side term of (4.18) occurs from approximating the integral term by the left Riemann sum with step size Δt or discretizing node m. It is well known that the smaller Δt or larger m, the closer the value of the approximation to the real value of integration. In Figure 4.2, the graphical results at different values $m = 2^n$ for $n = 1, 2, 3, 4, 5$, show the behavior of results

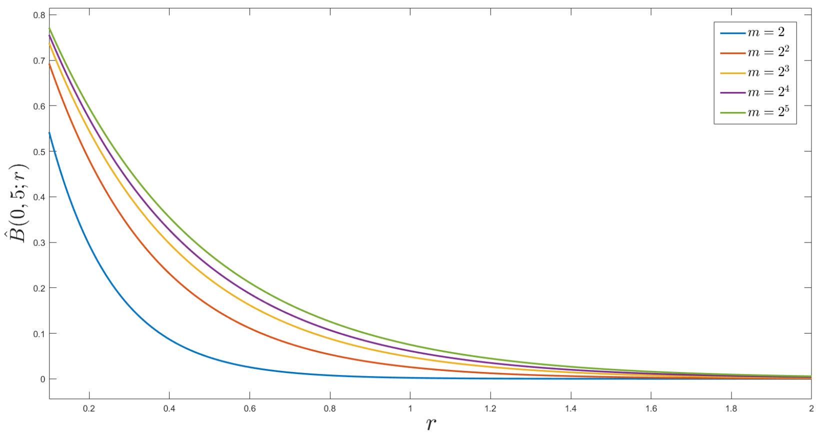

Figure 4.2: Approximate values of bond prices (4.18) obtained from Theorem 4.4 for various m.

ordering from left to right in each of the curve lines. We observe that a distant between each curve is decreasing, so large value m should give the decent approximation. Therefore, we can conclude that the green curve ($m = 2^5$) provides the best approximation of (4.18) than others in Figure 4.2.

4.5 Interest rate swap pricing

A swap is a derivative contract for two parties involving the exchange a series of cash flows. In this section, we consider a fixed-to-floating interest rate swap (IRS) where a buyer agrees to pay a floating interest rate on a predetermined principle, called a notional principle P, in order to receive a fixed one from a seller over a specified period of time $[t, T]$; see more details in [56]. The IRSs are the most traded swaps at present and have many potential uses in practice, such as in hedging, portfolio management, and speculation. For instance, Company A borrowing 5 million dollars from Bank B with an interest rate of 3% plus the London Interbank Offered Rate (LIBOR) can sell a fixed-to-floating IRS to an IRS buyer to hedge against the exposure to fluctuations in interest rates, see Figure 4.3. By selling the fixed-to-floating IRS, Company A can pay fixed rate to the buyer to receive the floating rate. It then use the money received from the buyer to pay Bank B. Instead of paying the floating rate whose value is uncertain, Company A can now fix the interest rate it needs to pay. This may be much easier for Company A to come up with a plan to allocate the capital to repay the debt.

As the IRSs are over-the-counter products, they can be customized differently to fulfil buyers' needs. We consider an IRS contract that the fixed and floating interest rate payments

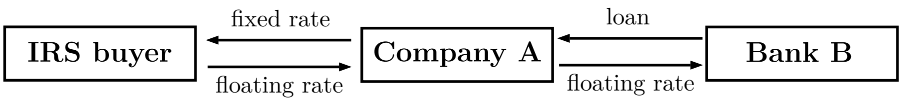

Figure 4.3: Mechanic of a fix-to-floating IRS.

are exchanged at every specified period of time, such as every three, four, or six months, over a pre-determined time, such as two, three, or ten years. A classic example of such swap is an arrears swap. A floating payment for an arrears swap is based on an interest rate at a payment time for a discrete time observation sampled at $t = T_0, T_1, \ldots, T_N = T$ with the increment time of Δt. Thus, the payment date and the reset time coincide. This section provides an analytical formula for pricing the arrears swap under the ECIR process (4.1) from the perspective of a buyer, a company who pays a floating interest rate to receive a fixed rate.

Let P denote a notional principle, r_{fix} denote a fixed rate, and r_t denote a floating rate at time t which follows the ECIR process (4.1). Suppose that an arrears swap has a initial time $t = T_0$, maturity T and M payment dates at $T_{i_1}, T_{i_2}, \ldots, T_{i_M} = T_N = T$ in a fixed increment time of Δt^*. Figure 4.4 illustrates the payment mechanism of the arrears swap. At each payment time, buyer pays an interest on the notional principle P specified by a floating interest rate at the time of the payment and receives a fixed interest at the same time. In other words, the payment time coincides with the reset time.

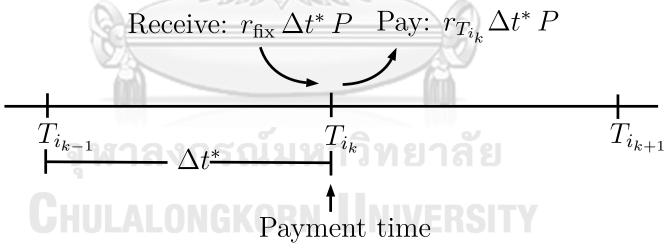

Figure 4.4: Interest payment of an arrears swap at time T_{i_k}.

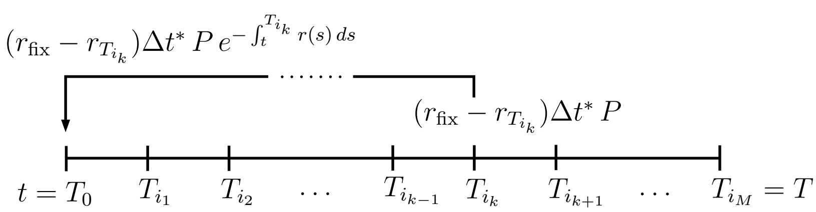

Figure 4.5: Present value of interest payment of an arrears swap at time T_i.

The payoff of this swap from a buyer's point of view at the k^{th} payment time is $(r_{\text{fix}} - r_{T_{i_k}})\Delta t^* P$, which is just the difference between the interest on a notional principle P determined by the fixed interest rate r_{fix} and by the floating interest rate at time T_{i_k}, $r_{T_{i_k}}$; see Figure 4.4. Thus, the present value of such payoff is

$$\left(r_{\text{fix}} - r_{T_{i_k}}\right) \Delta t^* \, P \, e^{-\int_t^{T_{i_k}} r(s)\,ds},$$

see Figure 4.5. By the fundamental theorem of asset pricing, the no-arbitrage price for the arrears swap with the short rate discount, r_t, can be expressed as

$$\begin{aligned}\text{Valuation of swap} &:= \mathbb{E}^Q\left[\sum_{k=1}^{M}\left(r_{\text{fix}} - r_{T_{i_k}}\right)\Delta t^* \, P \, e^{-\int_t^{T_{i_k}} r(s)\,ds} \mid r_t = r\right] \\ &\approx \Delta t^* \, P \, \mathbb{E}^Q\left[\sum_{k=1}^{M}\left(r_{\text{fix}} - r_{T_{i_k}}\right) e^{-\Delta t \sum\limits_{j=1}^{i_k-1} r_{T_j}} \mid r_t = r\right],\end{aligned}$$

where $\{T_{i_1}, T_{i_2}, \ldots, T_{i_M}\} \subset \{T_1, T_2, \ldots, T_N\}$ is the set of payment times.

Theorem 4.4 can be directly applied to the above expression to evaluate the price of the swap. Moreover, the fixed rate r_{fix} resulting in the fair valuation of the swap is

$$r_{\text{fix}} = \frac{\mathbb{E}^Q\left[\sum\limits_{k=1}^{M} r_{T_{i_k}} \, e^{-\Delta t^* \sum\limits_{j=1}^{i_k-1} r_{T_j}} \mid r_t = r\right]}{\mathbb{E}^Q\left[\sum\limits_{k=1}^{M} e^{-\Delta t^* \sum\limits_{j=1}^{i_k-1} r_{T_j}} \mid r_t = r\right]}. \tag{4.19}$$

Remark 4.2. This idea can be extended to price a premium for the buyer when the fixed interest rate r_{fix} does not correspond to the result in (4.19). Moreover, hedging by using higher-order moment swaps such as the skewness and kurtosis swaps are nowadays traded. The idea of the ECIR process presented here can be extended to other processes such as the Schwartz's model, for instance, Chumpong et al. [15] provided an analytical formula for pricing discretely-sampled skewness and kurtosis swaps for commodities. However, the discounted rate in their work is a constant and not governed by any stochastic process.

4.6 Conclusion

In this paper, the analytical formulas for conditional expectations of path-dependent product of polynomial and exponential functions based on the ECIR process (4.1) have been proposed via directly extending the recent research in [72]. The major development of the extension is utilizing the tower property to transform the analytical formula for the conditional expectation of exponential function from a single step to multiple steps. In practically, for CIR process, our analytical formula in Theorem 4.4 can be expressed to the exact formula by using the result in Theorem 4.2.

To validate these formulas with MC simulation, we implement through MATLAB software in order to illustrate the accuracy via the MAPE and the AVDF, and also show efficiency via the AVRT. As a results, the obtained formulas extremely agree with the MC simulations as depicted in several examples.

Finally, an application of our proposed formula in finance is illustrated by deriving an analytical pricing formula for interest rate swap, namely, arrears swap under the ECIR process (4.1). This suggests that the proposed formula in this work could be useful for the investor in the market who wants a sufficient formula for hedging, portfolio management, and speculation.

CHAPTER V

CONCLUSION AND FUTURE WORKS

5.1 Conclusion

Diffusion model has been thoroughly studied for its use in seeking a solution of an SDE and investigating its properties, such as moments and conditional moments, which play significant roles in many real-world applications and are especially beneficial for estimating parameters, pricing financial derivatives, etc. In fact, these moments can be directly calculated by applying the transition PDF. However, it is often unknown or unavailable in closed form. In addition, the formulas for the conditional moments of the SDE may be unavailable in closed form, as well.

In this research, we study two generalized CIR processes: (i) a class of diffusions that have nonlinear diffusion coefficients (NLD-CEV) and (ii) the Pearson diffusion processes. A complete investigation was carried out for both light- and heavy-tailed processes, including 3/2-SVM, Ornstein–Uhlenbeck, Cox–Ingersoll–Ross, Fisher–Snedecor, reciprocal gamma and Student processes. Then, we introduce a simple but novel approach to find closed-form formulas for conditional moments of the two generalized CIR processes. The main idea to obtain these formulas is based on the Feynman–Kac representation. Particularly, this approach does not require any knowledge of eigenfunctions or the transition PDF. In each class of stationary distributions reduced from the two processes, the formulas are explored and presented in a very concise form. Also, the closed-form formulas obtained are numerically validated by comparison with MC simulations.

The obtained results from all three research articles can be summarized in the followings. In Chapter 2, we provide the sufficient conditions of the existence and uniqueness for a positive pathwise strong solution of the NLD-CEV process (2.1) for $\beta \in [0, 2)$ and $\beta \in (2, \infty)$. Based on this process, we have derived the formulas of conditional moments for each range of β. Moreover, in the case of constant parameters, the derived formulas can be expressed in closed forms. As a consequence, the closed-form formulas obtained are also produced the formulas of unconditional moments.

In Chapter 3, we propose the formulas of conditional and unconditional moments for time-inhomogeneous Pearson diffusion processes (3.5), including the OU, CIR, Jacobi, Fisher–Snedecor, reciprocal gamma and Student diffusion processes. Also, the proposed formulas can be reduced to concise closed forms in the case of constant parameters which are validated by comparison with MC simulations. Furthermore, the advantage of these formulas is able to compute the conditional moments of EOU and ECIR processes.

Finally, in Chapter 4, the analytical formulas for conditional expectations based on the ECIR process (4.1) have been presented. As a consequence, the obtained formulas extremely agree with the MC simulations as demonstrated via several examples. In addition, an application in finance of the presented formula is shown in deriving an analytical formula for pricing IRS, namely, arrears swap. Lastly, this suggests that the proposed formula could be useful for the investor who wants a sufficient formula for hedging, portfolio management and speculation.

5.2 Future works

In this section, we provide some possible future works related to this dissertation. Similar to this work, the idea can be extended to obtain closed-form formulas for the conditional moments of multi-dimensional stochastic diffusion processes, e.g., the extended Heston-CEV hybrid model, the Schwartz's two-factor model and etc. Moreover, based on the processes discussed in this work, one can also further investigate some essential mathematical and statistical properties, namely, the conditional variance, covariance, central moment, mixed moment and correlation. Finally, another aspect that could be possible for future research is to apply results in this work for financial applications, such as calculating the fair strike price for a variance swap, pricing interest rate swaps like arrears and vanilla, and so on.

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