การกำกับกลเอสของกราฟสามส่วนบริบูรณ์บางประเภท

นางสาวศรารัตน์ นุใหม่

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2564

S-MAGIC LABELINGS OF SOME COMPLETE TRIPARTITE GRAPHS

Ms. Sararat Numai

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Master Degree Program in Mathematics
Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Academic Year 2021

Thesis Title	S-MAGIC LABELINGS OF SOME COMPLETE TRIPARTITE GRAPHS
By	Ms. Sararat Numai
Field of Study	Mathematics
Thesis Advisor	Assistant Professor Kirati Sriamorn, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master Degree

. . . . Dean of the Faculty of Science
(Professor Polkit Sangvanich, Ph.D.)

THESIS COMMITTEE

. . . . Chairman
(Associate Professor Chariya Uiyyasathian, Ph.D.)

. . . . Thesis Advisor
(Assistant Professor Kirati Sriamorn, Ph.D.)

. . . . Examiner
(Assistant Professor Teeradej Kittipassorn, Ph.D.)

. . . . External Examiner
(Assistant Professor Tanawat Wichianpaisarn, Ph.D.)

ศรารัตน์ นุใหม่ : การกำกับกลเอสของกราฟสามส่วนบริบูรณ์บางประเภท (S-MAGIC LABELINGS OF SOME COMPLETE TRIPARTITE GRAPHS)
อ.ที่ปรึกษาวิทยานิพนธ์หลัก : ผศ.ดร. กีรติ ศรีอมร 38 หน้า

ในวิทยานิพนธ์นี้ เรากล่าวถึงนิยามของกราฟซิกมา การกำกับกลซิกมาและดัชนีระยะทาง ในการศึกษานี้เราจะเรียกกราฟ $G = (V, E)$ ว่าเป็นกราฟกลเอสก็ต่อเมื่อมีเซตของจำนวนเต็มบวก T มีฟังก์ชันหนึ่งต่อหนึ่งทั่วถึง $f : V \to T$ และมีจำนวนเต็มบวก k ที่ทำให้ $\sum_{u \in N(v)} f(u) = k$ สำหรับทุกจุด $v \in V(G)$ เมื่อ $N(v)$ คือย่านใกล้เคียงของ v โดยเราจะเรียก T ว่าเซตกำกับกลเอสของกราฟ G และเรียก k ว่าค่าคงที่กล นอกจากนี้กำหนดให้ $i(G) = \min_{T \in \mathcal{S}} \alpha(T)$ โดยที่ $\alpha(T) = \max(T)$ และ $\mathcal{S} = \{T \subset \mathbb{N} : T$ เป็นเซตกำกับกลเอสของ $G\}$ เราศึกษาฟังก์ชัน $i(G)$ สำหรับ G ที่สอดคล้องกับเงื่อนไขต่อไปนี้

1. $G = K_{m_1, m_2, \ldots, m_r}$ เป็นกราฟ r ส่วนบริบูรณ์ที่ทุกส่วนมีจำนวนจุดเท่ากัน
2. $G = K_{1, m_2, m_3}$ เป็นกราฟสามส่วนบริบูรณ์และ $2 \leq m_2 \leq m_3$
3. $G = K_{2, m_2, m_3}$ เป็นกราฟสามส่วนบริบูรณ์และ $2 \leq m_2 \leq m_3$.

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต ....
สาขาวิชา คณิตศาสตร์ ลายมือชื่อ อ.ที่ปรึกษาหลัก ....
ปีการศึกษา 2564

6270103423 : MAJOR MATHEMATICS
KEYWORDS : S-magic graph, S-magic labeling, S-magic constant, Distance magic index

SARARAT NUMAI : S-MAGIC LABELINGS OF SOME COMPLETE TRIPARTITE GRAPHS
ADVISOR : ASSIST. PROF. KIRATI SRIAMORN, Ph.D., 38 pp.

In this thesis, we recall the definitions of Σ-graph, Σ-labeling Σ-constant and distance magic index of graph. A graph $G = (V, E)$ is said to be an S-magic graph if there exist a set T of positive integers with $|T| = |V|$, a bijection $\phi : V \to T$, and a positive integer k such that $\sum_{u \in N(v)} \phi(u) = k$ for all $v \in V$. We call k an S-magic constant, ϕ an S-magic labeling, and T an S-magic labeling set. Define $i(G) = \min_{T \in \mathcal{S}} \alpha(T)$ where $\mathcal{S} = \{T \subset \mathbb{N} : T$ is an S-magic labeling set of $G\}$ and $\alpha(T) = \max(T)$.

In this study, we determine $i(G)$ for G that satisfies the following conditions:

1. $G = K_{m_1,m_2,\ldots,m_r}$ is a complete r-partite graph and $m_1 = m_2 = \ldots = m_r \geq 2$
2. $G = K_{1,m_2,m_3}$ is a complete tripartite graph and $2 \leq m_2 \leq m_3$
3. $G = K_{2,m_2,m_3}$ is a complete tripartite graph and $2 \leq m_2 \leq m_3$.

Department : Mathematics and Computer Science Student's Signature ....
Field of Study : Mathematics Advisor's Signature ....
Academic Year : 2021

ACKNOWLEDGEMENTS

I would like to express my sincere thanks to my thesis advisors: Assistant Professor Kirati Sriamorn, Ph.D. for his invaluable help, suggestions and encouragement throughout doing this thesis. The time I spent studying it, while not very long, is extremely valuable and full of the happiness. I am most grateful for his teaching and advice, not only the Mathematics methodologies but also many other methodologies in life.

Furthermore, I would like to thank my thesis committee for their suggestions and valuable comments to improve my work, and to my teachers and lecturers at Chulalongkorn university who have taught me various knowledge in mathematics. Their guidances and questions were very helpful and insightful, that is partial for making the plenary work.

I take this opportunity to express gratitude to the Development and Promotion of Science and Technology Project for the scholarship.

In particular, I thank to my parents for the eternal encouragement and attention throughout my life. For that reason I am forever grateful.

Last but not least, I would like to thank all my friends for helpful suggestions and valuable support. It is full of fun, happiness and beautiful memory throughout the period of the master's course.

CONTENTS

LIST OF FIGURES

CHAPTER I
INTRODUCTION

By a graph $G = (V, E)$, we mean a finite undirected graph containing no loops or multiple edges. Furthermore, we assume that G has no isolated vertices.

In 1994, Vilfred [2] introduced the concept of Σ-labeling: A Σ-labeling of a graph $G = (V, E)$ of order n is a bijection $f : V \to \{1, 2, ..., n\}$ such that $\sum\limits_{u \in N(v)} f(u) = k$ for all $v \in V$, where $N(v)$ is the neighborhood of v. The constant k is called the magic constant of the labeling f. A graph which admits a Σ-labeling is called a Σ-graph. The Σ-labeling is also known as the 1-vertex-magic vertex labeling [3] and the distance magic labeling [4].

In 2015, Godinho and Singh [1] introduced the concept of S-magic graph. A graph $G = (V, E)$ is said to be an **S-magic graph** if there exist a set T of positive integers with $|T| = |V|$, a bijection $\phi : V \to T$, and a positive integer k such that $\sum\limits_{u \in N(v)} \phi(u) = k$ for all $v \in V$. We call k an **S-magic constant**, ϕ an **S-magic labeling**, and T an **S-magic labeling set**. It follows that a Σ-graph is an S-magic graph. Moreover, if G is an S-magic graph, then each S-magic labeling set T has a unique corresponding S-magic constant, i.e., for any two S-magic labelings $\phi_1 : V \to T$ and $\phi_2 : V \to T$, we have $\sum\limits_{u \in N(v)} \phi_1(u) = \sum\limits_{u \in N(v)} \phi_2(u)$ for all $v \in V$. We denote the set of all S-magic constants that can be obtained through different S-magic labelings of G by $M(G)$. Moreover, they observed that the complete r-partite graph $G = K_{m_1, m_2, ..., m_r}$, where $m_1 \leq m_2 \leq \cdots \leq m_r$ is an S-magic graph if and only if $m_2 \geq 2$.

In 2018, Godinho and Singh [4] studied the function $i(G) = \min\limits_{T \in \mathcal{S}} \alpha(T)$, where $\mathcal{S} = \{T \subset \mathbb{N} : T \text{ is an } S\text{-magic labeling set of } G\}$ and $\alpha(T) = \max(T)$. The distance magic index of G is defined by $i(G) - n$ and is denoted by $\theta(G)$.

In this thesis, we determine $i(G)$ for G which satisfies the conditions:

1. $G = K_{m_1, m_2, ..., m_r}$ is a complete r-partite graph and $m_1 = m_2 = ... = m_r \geq 2$
2. $G = K_{1, m_2, m_3}$ is a complete tripartite graph and $2 \leq m_2 \leq m_3$
3. $G = K_{2, m_2, m_3}$ is a complete tripartite graph and $2 \leq m_2 \leq m_3$.

CHAPTER II
PRELIMINARIES

In this chapter, we review some definitions, theorems, lemmas, corollaries, and examples used in this work. For more details, see in [1], [4] and [5].

2.1 S-magic graph

Definition 2.1. [1] A **Σ-labeling** of a graph $G = (V, E)$ of order n is a bijection $f : V \to \{1, 2, ..., n\}$ such that $\sum_{u \in N(v)} f(u) = k$ for all $v \in V$, where $N(v)$ is the neighborhood of v and where $k \in \mathbb{N}$. The constant k is called the **magic constant** of the labeling f. A graph G is called a **Σ-graph**.

Definition 2.2. [1] Let $G = (V, E)$ be an undirected graph with neither loops nor multiple edges. A graph $G = (V, E)$ is said to be an **S-magic graph** if there exist a set T of positive integers with $|T| = |V|$, a bijection $\phi : V \to T$, and a positive integer k such that $\sum_{u \in N(v)} \phi(u) = k$ for all $v \in V$. We call k an **S-magic constant**, ϕ an **S-magic labeling**, and T an **S-magic labeling set**.

Definition 2.3. [1] If a graph G is S-magic then magic spectrum of G is defined to be the set of all magic constants that can be obtained through different S-magic labeling of G and is denoted by $M(G)$.

Example 2.4. [1] A path P_3 has 3 vertices x, y and z. Let $deg(x) = 1, deg(y) = 2$ and $deg(z) = 1$. We will show that an S-magic labeling set T of P_3 must be in the form $T = \{a, a + b, b\}$ where a, b are distinct positive integers. It is obvious that if we define $f : V \to T$ by $f(x) = a, f(y) = a + b$ and $f(z) = b$, then f is an S-magic labeling. Therefore $T = \{a, a + b, b\}$ is an S-magic labeling set of P_3. Now we assume that $T = \{a, b, c\}$ is an S-magic labeling set of P_3, and let $f : V \to T$ by $f(x) = a, f(y) = c$ and $f(z) = b$. Then c must be equal to $a + b$. It follow that the S-magic constant of P_3 is $a + b$. Since a and b are distinct positive integers, $a + b \geq 1 + 2 = 3$. Hence, the path P_3 is an S-magic graph where $M(P_3) = \{3, 4, 5, 6, \ldots\}$.

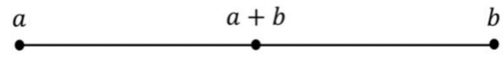

Figure 2.1: A labeling of P_3 where S-magic constant is $a+b$.

Example 2.5. [1] For a cycle C_4, if we label a pair of the opposite vertices with the same summation, we get that C_4 is an S-magic graph. It is not hard to see that $T = \{1, 2, i, i+1\}$ is an S-magic labeling set of C_4 where $i = 3, 4, 5, \ldots$ with $5, 6, 7, \ldots$ as magic constants. Since C_4 has 4 vertices, there is one vertex such that the labeling assigned to its neighborhoods are at least 4 and another number. Thus the magic constant of C_4 greater than 4. Hence C_4 is an S-magic graph where $M(C_4) = \{5, 6, \ldots\}$.

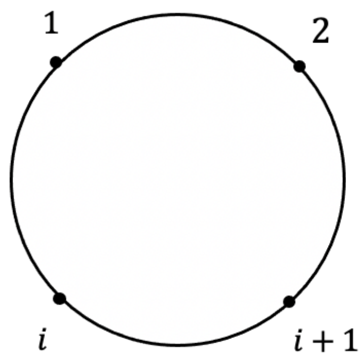

Figure 2.2: An S-magic labeling $T = \{1, 2, i, i+1\}$ of C_4 where S-magic constant is $i+2$.

Definition 2.6. [1] A vertex of degree 1 is a leaf, and a vertex that adjacient to a leaf is called a support vertex.

Remark 2.7. [1] If G contains two distinct support vertices u and v, then G is not an S-magic graph.

Proof. Suppose G is an S-magic graph, and G has two distinct support vertices u and v. There are a leaf a adjacent to u and a leaf b adjacent to v, it implies the numbers that label to u and v are equal. This is a contradiction. □

Theorem 2.8. *[1] A tree T is an S-magic graph if and only if $T = K_{1,r}$ where $r \geq 2$.*

Theorem 2.9. *[1] If there exist two vertices u and v in G such that* $|\,(N(u) \setminus N(v)) \cup (N(v) \setminus N(u))\,| = 2$, *then G is not an S-magic graph.*

Corollary 2.10. *[1] The complete graph K_n is not S-magic for $n \geq 2$.*

Lemma 2.11. *[1] The complete r-partite graph $G = K_{m_1,m_2,...,m_r}$ is S-magic if and only if the sum of the labels of all vertices in any two partite sets are equal.*

Theorem 2.12. *[1] The complete r-partite graph $G = K_{m_1,m_2,...,m_r}$, $m_1 \leq m_2 \leq \cdots \leq m_r$ is S-magic if and only if $m_2 \geq 2$.*

Lemma 2.13. *[1] If G is S-magic, then the smallest S-magic constant corresponds to the S-magic labeling set T for which $\sum_{i \in T} i$ is minimum.*

2.2 Distance magic index

Definition 2.14. [4] Let $i(G) = \min_{T \in \mathcal{S}} \alpha(T)$, where $\mathcal{S} = \{T \subset \mathbb{N} : T$ is an S-magic labeling set of $G\}$ and $\alpha(T) = \max(T)$. The distance magic index of G, denoted by $\theta(G)$ is defined by $i(G) - n$.

Theorem 2.15. *[4] A tree T is S-magic if and only if $T = K_{1,r}$, where $r \geq 2$. Furthermore, $\theta(K_{1,r})$ is $\frac{r(r-1)}{2} - 1$.*

Lemma 2.16. *If G is an S-magic graph of order n with distance magic index θ, then*

$$\frac{\delta(2(n+\theta) - \delta + 1) - \Delta(\Delta+1)}{2} \geq 0.$$

Proof. Since the distance magic of G is θ, there is a set $T \subset \{1, 2, ..., n+\theta\}$ with $|T| = n$ and an S-magic labeling $f : V \to T$ with a magic constant k. Let $v_1, v_2 \in V(G), deg(v_1) = \delta$ and $deg(v_2) = \Delta$. Thus

$$\sum_{u \in N(v_1)} f(u) \geq 1 + 2 + \cdots + \Delta = \frac{\Delta(\Delta+1)}{2}$$

and

$$\sum_{u \in N(v_2)} f(u) \leq (n+\theta) + (n+\theta-1) + \cdots + (n+\theta-\delta+1) = \frac{\delta(2(n+\theta) - \delta + 1)}{2}$$

. Since $\sum_{u \in N(v_1)} f(u) = \sum_{u \in N(v_2)} f(u) = k$, we get

$$\frac{\delta(2(n+\theta) - \delta + 1)}{2} \geq \frac{\Delta(\Delta+1)}{2}.$$

Therefore
$$\frac{\delta(2(n+\theta)-\delta+1)-\Delta(\Delta+1)}{2}\geq 0.$$
Let
$$g(x)=\frac{\delta(2(n+x)-\delta+1)-\Delta(\Delta+1)}{2}.$$
then $g(x)$ is a strictly increasing function of x. If there exist a non-negative integer a satisfying
$$\frac{\delta(2(n+\theta)-\delta+1)-\Delta(\Delta+1)}{2}<0,$$
it implies $\theta(G)>a$. Also that if a is a smallest integer such that $g(a)\geq 0$, then $\theta(G)\geq a$. So,
$$g(0)=\frac{\delta(2n-\delta+1)-\Delta(\Delta+1)}{2}. \tag{2.1}$$
□

Lemma 2.17. *Let G be a graph of order n such that $g(0)<0$. Then $\theta(G)\geq \left\lceil\frac{|g(0)|}{\delta}\right\rceil$.*

Proof. Let $|g(0)|=q\delta+r, 0\leq r<\delta$. Since $g(0)<0$, we have
$$g(0)=\frac{\delta(2n-\delta+1)-\Delta(\Delta+1)}{2}=-q\delta-r.$$
Then
$$\frac{\delta(2n-\delta+1)-\Delta(\Delta+1)}{2}+q\delta=-r$$
$$\frac{\delta(2n-\delta+1)-\Delta(\Delta+1)+2q\delta}{2}=-r.$$
$$\frac{\delta(2(n+q)-\delta+1)-\Delta(\Delta+1)}{2}=-r.$$
It implies that if $r=0$, q is a smallest value of x that $g(x)\geq 0$. Then $\theta(G)\geq q$. If $r>0$, then $\theta(G)>q$ and
$$\frac{\delta(2(n+q)-\delta+1)-\Delta(\Delta+1)+2r}{2}=0.$$
Since $r<\delta$,
$$\frac{\delta(2(n+q)-\delta+1)-\Delta(\Delta+1)+2r}{2}<\frac{\delta(2(n+q)-\delta+1)-\Delta(\Delta+1)+2\delta}{2}.$$
Hence
$$\frac{\delta(2(n+(q+1))-\delta+1)-\Delta(\Delta+1)}{2}>0.$$

Therefore, $q+1$ is the smallest value of x that $g(x) \geq 0$. Thus $\theta(G) \geq q+1$.
Observation that if $G = K_{m_1,m_2}$ is a complete bipartite graph where $2 \leq m_1 \leq m_2$. We apply $\delta = m_1, \Delta = m_2$ and $n = m_1 + m_2$. By (2.1), we get

$$\begin{aligned} g(0) &= \frac{m_1(2n - m_1 + 1) - m_2(m_2+1)}{2} \\ &= \frac{m_1(2(m_1+m_2) - m_1 + 1) - m_2(m_2+1)}{2} \\ &= \frac{m_1^2 + 2m_1m_2 + m_1 - (m_2^2 + m_2)}{2} \\ &= \frac{n(n+1)}{2} - m_2(m_2+1). \end{aligned} \tag{2.2}$$

□

Theorem 2.18. *[4] Let G be a complete bipartite graph K_{m_1,m_2} where $2 \leq m_1 \leq m_2$ and $n = m_1 + m_2$. Let $g(0) = \frac{n(n+1)}{2} - m_2(m_2+1)$. Then*

$$\theta(G) = \begin{cases} 0, & n(n+1) \geq 2m_2(m_2+1) \text{ and } n \equiv 0 \text{ or } 3 \pmod 4 \\ 1, & n(n+1) \geq 2m_2(m_2+1) \text{ and } n \equiv 1 \text{ or } 2 \pmod 4 \\ \left\lceil \dfrac{|g(0)|}{m_1} \right\rceil, & n(n+1) < 2m_2(m_2+1). \end{cases}$$

Proof. **Case** $n(n+1) \geq 2m_2(m_2+1)$ **and** $n \equiv 0$ **or** $3 \pmod 4$. It is completed by Theorem 1.6 in [4].

Case $n(n+1) \geq 2m_2(m_2+1)$ **and** $n \equiv 1$ **or** $2 \pmod 4$. Since a sum of elements in a set $\{1, 2, \ldots, m_1+m_2\}$ is equal to $\frac{(m_1+m2)(m_1+m_2+1)}{2}$ and $m_1 + m_2 \equiv 1$ or $2 \pmod 4$, this sum is not divided by 2. Then $\theta(G) > 0$. Let $S(L_1)$ and $S(L_2)$ be the sums of the labelings assigned to V_1 and V_2, repectively. We label $L_1 = \{m_2+1, m_2+2, \ldots, m_2+m_1\}$ to V_1 and $L_2 = \{1, 2, \ldots, m_2\}$ to V_2. Then $S(L_1) = m_1m_2 + \frac{m_1(m_1+1)}{2}$ and $S(L_2) = \frac{m_2(m_2+1)}{2}$. Thus

$$S(L_1) - S(L_2) = \frac{n(n+1)}{2} - m_2(m_2+1).$$

Since $n \equiv 1$ or $2 \pmod 4$, it follows that $\frac{n(n+1)}{2} \equiv 1 \pmod 2$. Furthermore, $m_2(m_2+1) \equiv 0 \pmod 2$, and then

$$\frac{n(n+1)}{2} - m_2(m_2+1) \equiv 1 \pmod 2.$$

Let $S(L_1) - S(L_2) = 2p - 1$ where $p = (m_1-1)q + r > 0$ and $r \geq 0$. So,

$$S(L_1) - p + 1 = S(L_2) + p. \tag{2.3}$$

Now, we proceed to attain equality in the sum of the labelings for the two partite set. We divide into 2 cases.

For $r = 0$: we label the vertices in V_1 and V_2 with the labeling sets $L_1^{'} = \{m_2+1-q, m_2+2-q, \ldots, m_2+m_1-1-q, m_2+m_1+1\}$ and $L_2^{'} = \{1, 2, \ldots, m_2 - q, m_2 - q + 1 + (m_1 - 1), m_2 - q + 2 + (m_1 - 1), \ldots, m_2 + (m_1 - 1)\}$, respectively. Thus $S(L_1^{'}) = S(L_2^{'})$ by using the relation in (2.3). See the labeling in Figure 2.3 .

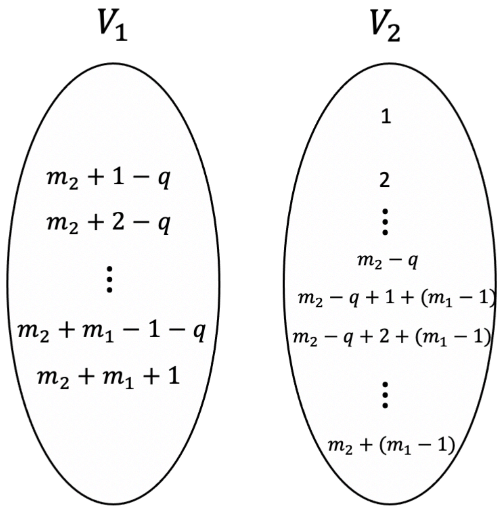

Figure 2.3: A labeling of K_{m_1,m_2} where m_1 and m_2 satisfy $n(n+1) \geq 2m_2(m_2 + 1)$ and $n \equiv 1$ or $2 \pmod 4$ for $r = 0$.

To see that all elements in $L_1^{'}$ except $m_2 + m_1 + 1$ are the numbers between $m_2 - q$ and $m_2 - q + 1 + (m_1 - 1)$ in $L_2^{'}$. Moreover, it obvious that $m_2 + m_1 + 1$ greater than all elements in $L_2^{'}$. Hence all elements in $L_1^{'}$ and $L_2^{'}$ are distinct.

For $r > 0$: we label the vertices in V_1 and V_2 with the labeling sets $L_1^{'} = \{m_2 - q, m_2 - q + 1, \ldots, m_2 - q + (r-1), m_2 - q + (r+1), m_2 - q + (r+2), \ldots, m_2 - q + (m_1 - 1).m_2 + m_1 + 1\}$ and $L_2^{'} = \{1, 2, \ldots, m_2 - q, m_2 - q + 1 + (m_1 - 1), m_2 - q + 2 + (m_1 - 1), \ldots, m_2 + (m_1 - 1)\}$. Thus $S(L_1^{'}) = S(L_2^{'})$ by using the relation in (2.3). See the labeling in Figure 2.4

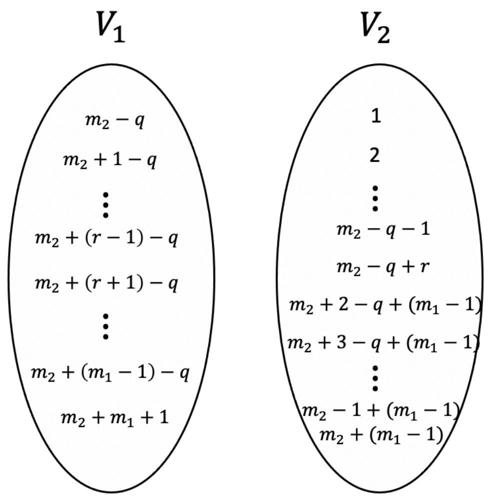

Figure 2.4: A labeling of K_{m_1,m_2} where m_1 and m_2 satisfy $n(n+1) \geq 2m_2(m_2+1)$ and $n \equiv 1$ or $2 \pmod 4$ for $r > 0$.

To see that all elements in $L_1^{'}$ except $m_2 + m_1 + 1$ are the numbers between $m_2 - q$ and $m_2 + 2 - q + (m_1 - 1)$ in $L_2^{'}$. Moreover, it obvious that $m_2 + m_1 + 1$ greater than all elements in $L_2^{'}$. Hence all elements in $L_1^{'}$ and $L_2^{'}$ are distinct. Therefore, the set $\{1, 2, \ldots, m_1 + m_2 - 1, m_1 + m_2 + 1\}$ is an S-magic labeling set of G, this implies $\theta(G) = 1$.

Case $n(n+1) < 2m_2(m_2+1)$. We have

$$\begin{aligned}(m_1 + m_2)(m_1 + m_2 + 1) &< 2(m_2 + 1)\\ m_1^2 + m_2^2 + 2m_1m_2 + m_1 + m_2 &< 2m_2^2 + 2m_2\\ 2m_1m_2 + m_1(m_1+1) &< m_2^2 + m_2\\ m_1m_2 + \frac{m_1(m_1+1)}{2} &< \frac{m_2^2 + m_2}{2}. \end{aligned} \tag{2.4}$$

By Lemma 2.17 and (2.2), $\theta(G) \geq \left\lceil \frac{|g(0)|}{m_1} \right\rceil$. We claim that $\theta(G) = \left\lceil \frac{|g(0)|}{m_1} \right\rceil$. Let $S(L_1)$ and $S(L_2)$ be the sums of the labelings assigned to V_1 and V_2, repectively. We label the sets $L_1 = \{m_2+1, m_2+2, \ldots, m_2+m_1\}$ to V_1 and $L_2 = \{1, 2, \ldots, m_2\}$ to V_2. Then $S(L_1) = m_1m_2 + \frac{m_1(m_1+1)}{2}$ and $S(L_2) = \frac{m_2(m_2+1)}{2}$. By (2.4), we get $S(L_1) < S(L_2)$. Let $K = S(L_2) - S(L_1) = m_1q + r$ where $r \geq 0$ and $q < m_1$. So

$$S(L_2) - (S(L_1) + m_1q + r) = 0. \tag{2.5}$$

For $r = 0$: we label the vertices in V_1 and V_2 with the labeling sets $L_1^{'} = \{m_2+1+q, m_2+2+q, \ldots, m_2+m_1-1+q, m_2+m_1+q\}$ and $L_2^{'} = L_2 = \{1, 2, \ldots, m_2\}$, respectively, see in Figure 2.5.

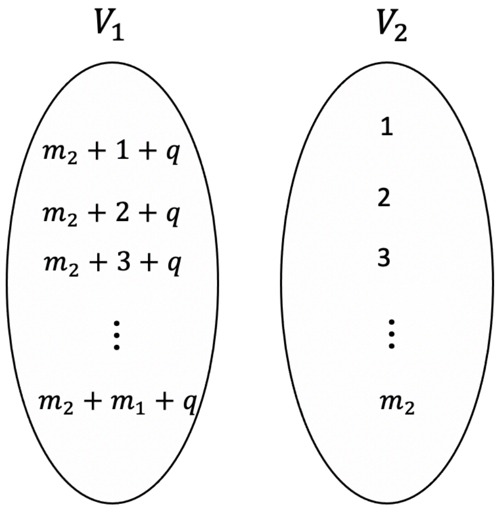

Figure 2.5: A labeling of K_{m_1,m_2} where m_1 and m_2 satisfy $n(n+1) < 2m_2(m_2+1)$ for $r = 0$.

Thus $S(L_1^{'}) = S(L_2^{'})$ by using the relation in (2.5). Therefore, $\theta(G) = q = \left\lceil \frac{|g(0)|}{m_1} \right\rceil$.
For $r > 0$: we label the vertices in V_1 and V_2 with the labeling sets $L_1^{'} = \{m_2+1+q, m_2+2+q, \ldots, m_2+m_1-r+q, m_2+m_1-r+2+q, \ldots, m_2+m_1+q, m_2+m_1+q+1\}$ and $L_2^{'} = L_2 = \{1, 2, \ldots, m_2\}$, respectively, see in Figure 2.6.

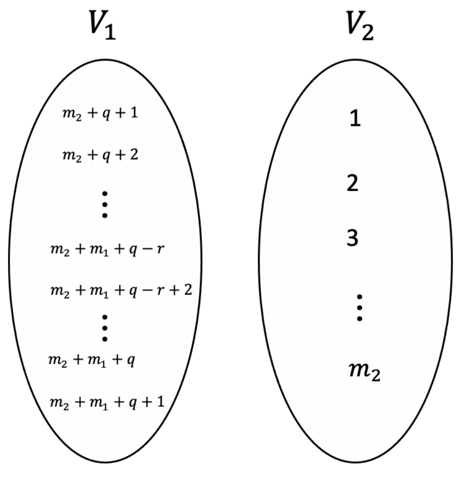

Figure 2.6: A labeling of K_{m_1,m_2} where m_1 and m_2 satisfy $n(n+1) < 2m_2(m_2+1)$ for $r > 0$.

Thus $S(L_1^{'}) = S(L_2^{'})$ by using the relation in (2.5). Therefore, we get $\theta(G) = q + 1 = \left\lceil \frac{|g(0)|}{m_1} \right\rceil$. □

Example 2.19. Let $G = K_{m_1,m_2}$ where $m_1 = 3$ and $m_2 = 5$. Then m_1, m_2 satisfies the condition $n(n+1) \geq 2m_2(m_2+1)$ and $n \equiv 0$ or $3 \pmod 4$. Then $G = K_{3,5}$ is an S-magic graph with an S-magic labeling set $T = \{1, 2, 3, 4, 5, 6, 7, 8\}$. See the labeling in Figure 2.7. Then $\theta(G) = 0$.

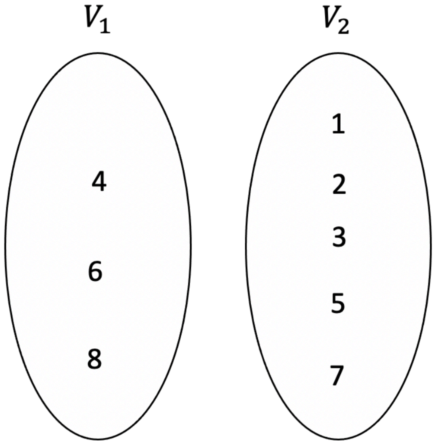

Figure 2.7: A labeling of $K_{3,5}$ and $\theta(K_{3,5}) = 0$.

Example 2.20. Let $G = K_{m_1,m_2}$ where $m_1 = 3$ and $m_2 = 6$. Then m_1, m_2 satisfies the condition $n(n+1) \geq 2m_2(m_2+1)$ and $n \equiv 1$ or $2 \pmod 4$. By Theorem 2.18, $\theta(G) = 1$. Then $G = K_{3,6}$ is an S-magic graph with an S-magic labeling set $T = \{1, 2, 3, 4, 5, 6, 7, 8, 10\}$. See the labeling in Figure 2.8.

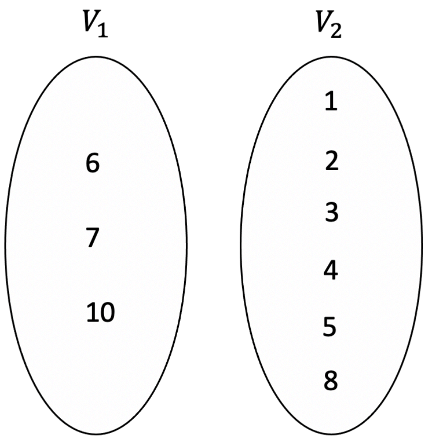

Figure 2.8: A labeling of $K_{3,6}$ and $\theta(K_{3,6}) = 1$.

Example 2.21. Let $G = K_{m_1,m_2}$ where $m_1 = 3$ and $m_2 = 10$. Then m_1, m_2 satisfies the condition $n(n+1) < 2m_2(m_2+1)$. By Theorem 2.18, $\theta(G) = 7$. Then $G =$

$K_{3,6}$ is an S-magic graph with an S-magic labeling set $T = \{1, 2, \ldots, 10, 17, 18, 20\}$ that can see in Figure 2.9.

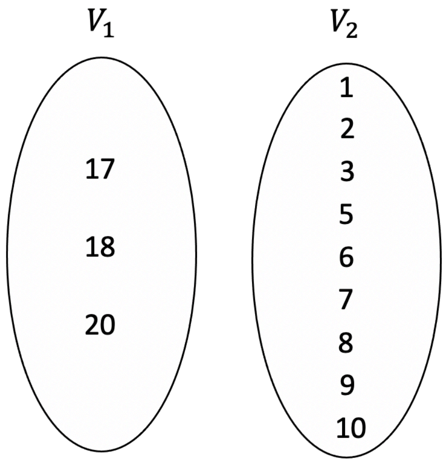

Figure 2.9: A labeling of $K_{3,10}$ and $\theta(K_{3,10}) = 7$.

In the next chapter, we determine $i(G)$ for $G = K_{m_1,m_2,m_3}$ is a complete tripartite graph and satisfies the condition $m_1 = m_2 = m_3 \geq 2$ and determine $i(G)$ for $G = K_{m_1,m_2,m_3}$ satisfies the following conditions:

1. $G = K_{m_1,m_2,\ldots,m_r}$ is a complete r-partite graph and $m_1 = m_2 = \ldots = m_r \geq 2$
2. $G = K_{1,m_2,m_3}$ is a complete tripartite graph and $2 \leq m_2 \leq m_3$
3. $G = K_{2,m_2,m_3}$ is a complete tripartite graph and $2 \leq m_2 \leq m_3$.

CHAPTER III
MAIN RESULTS

Theorem 3.1. *Let $m_1, m_2, \ldots, m_r$ be positive integers where $2 \leq m_1 = m_2 = \cdots = m_r$, and let $G = K_{m_1,m_2,\ldots,m_r}$ be a complete r-partite graph. If m is even, then G is an S-magic graph and $\theta(G) = 0$.*

Proof. Let $m_1, m_2, ..., m_r$ be positive integers where $2 \leq m_1 = m_2 = \cdots = m_r = m$, and let $G = K_{m_1,m_2,...,m_r}$ be a complete r-partite graph. Let $V_1, V_2, \ldots, V_r$ be the partite sets of G. For $i \in S_1$, $i \in V_k$ where $k = 1, 2, \ldots, r$ if and only if $i \equiv k$ or $(1-k) \pmod{2r}$. Figure 3.1 shows the labeling $f_1 : V(G) \to S_1$ with a labeling set $S_1 = \{1, 2, \ldots, rm\}$.

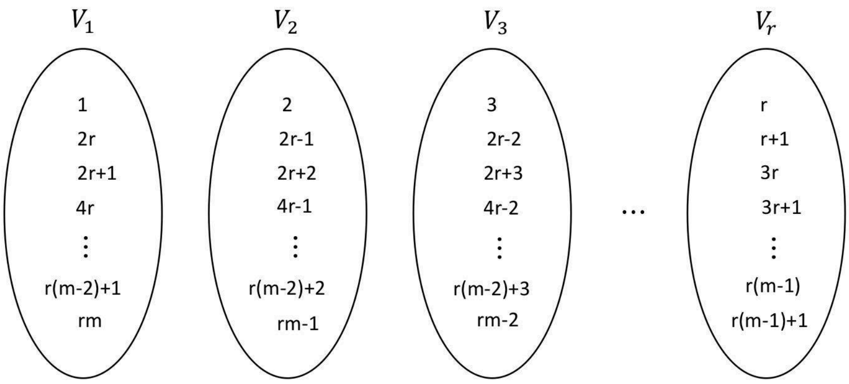

Figure 3.1: A Labeling of G with a label set $S_1 = \{1, 2, \ldots, rm\}$

Consider the sum of the labelings assigned to each partite V_k. Then the sum is equal to

$$\begin{aligned}
\sum_{n=0}^{\frac{m-2}{2}}(2rn+k) + \sum_{n=1}^{\frac{m}{2}}(2rn+1-k) &= k + (2r+k) + (4r+k) + \cdots + (r(m-2)+k) \\
&\quad + (2r+1-k) + (4r+1-k) + \cdots + (rm+1-k) \\
&= (k+rm+1-k) + (2r+k+r(m-2)+1-k) \\
&\quad + \cdots + (r(m-2)+k+2r+1-k) \\
&= \frac{m}{2}(rm+1).
\end{aligned}$$

This show that the sum of the labelings assigned to each partite is equal to $\frac{m(rm+1)}{2}$. Then $i(G) = rm$. Hence $\theta(G) = 0$. □

Lemma 3.2. *Let* $S = \{rm-3r+1, rm-3r+2, ..., rm\}$ *where* m *and* r *are odd. Then*
$A = \{rm-3r+1, rm-3r+3, ..., rm-2r\}$,
$B = \{rm-(\frac{3r-1}{2}), rm-(\frac{3r-1}{2})-1, ..., rm-2r+1\}$,
$C = \{rm, rm-1, ..., rm-(\frac{r-1}{2})\}$,
$D = \{rm-3r+2, rm-3r+4, ..., rm-2r-1\}$,
$E = \{rm-r, rm-r-1, ..., rm-\frac{3}{2}(r-1)\}$,
$F = \{rm-(\frac{r-1}{2})-1, rm-(\frac{r-1}{2})-2, ..., rm-r+1\}$ *partition* S.

Proof. Let $S = \{rm-3r+1, rm-3r+2, ..., rm\}$ where m and r are odd. We divide all elements in S into 6 sets: $A = \{rm-3r+1, rm-3r+3, ..., rm-2r\}$,
$B = \{rm-\frac{3r-1}{2}, rm-\frac{3r-1}{2}-1, ..., rm-2r+1\}$,
$C = \{rm, rm-1, ..., rm-\frac{r-1}{2}\}$,
$D = \{rm-3r+2, rm-3r+4, ..., rm-2r-1\}$,
$E = \{rm-r, rm-r-1, ..., rm-\frac{3}{2}(r-1)\}$,
$F = \{rm-(\frac{r-1}{2})-1, rm-(\frac{r-1}{2})-2, ..., rm-r+1\}$.
We will show that A, B, C, D, E and F are 6 partitions of S. Note that A and D contain an increasing sequence. The others contain a decreasing sequence. Then $\max A < \min B$, $\min C > \max F$ and $\max F > \max E$. Moreover, $C \cap F \cap E \cap D = \varnothing$ and $A \cap B = \varnothing$. We only need to show that $A \cap D = \varnothing$. Since A contains only odd positive integers and D contains only even positive integers, then $A \cap D = \varnothing$. In the last, we will show $|A|+|B|+|C|+|D|+|E|+|F| = |S| = 3r$. Consider

$$|A| = \frac{rm-2r-(rm-3r+1)+2}{2} = \frac{r+1}{2}$$
$$|B| = rm-\frac{3r-1}{2}-(rm-2r+1)+1 = \frac{r+1}{2}$$
$$|C| = rm-(rm-\frac{r-1}{2})+1 = \frac{r+1}{2}$$
$$|D| = \frac{rm-2r-1-(rm-3r+2)+2}{2} = \frac{r-1}{2}$$
$$|E| = rm-r-(rm-\frac{3}{2}(r-1))+1 = \frac{r-1}{2}$$
$$|F| = rm-(\frac{r-1}{2})-1-(rm-r+1)+1 = \frac{r-1}{2}.$$

Therefore $|A|+|B|+|C|+|D|+|E|+|F| = 3(\frac{r+1}{2})+3(\frac{r-1}{2}) = 3r$. Hence, A, B, C, D, E and F are the partitions of S.

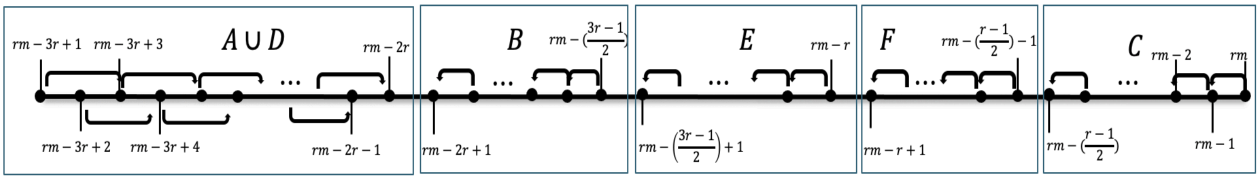

Figure 3.2: The partition of $S = \{rm - 3r + 1, rm - 3r + 2, ..., rm\}$.

$\square$

Lemma 3.3. *Let* $S = \{rm - 3r + 1, rm - 3r + 2, rm - \frac{r}{2}, rm - \frac{r}{2} + 2, rm - \frac{r}{2} + 3, ..., rm + 1\}$ *where* m *is odd, and* r *is even. Then*
$A = \{rm - 3r + 1, rm - 3r + 3, ..., rm - 2r - 1\}$
$B = \{rm - \frac{3r}{2}, rm - \frac{3r}{2} - 1, ..., rm - 2r + 1\}$
$C = \{rm + 1, rm, ..., rm - \frac{r}{2} + 2\}$
$D = \{rm - 3r + 2, rm - 3r + 4, ..., rm - 2r\}$
$E = \{rm - r, rm - r - 1, ..., rm - \frac{3r}{2} + 1\}$
$F = \{rm - (\frac{r}{2}), rm - (\frac{r}{2}) - 1, ..., rm - r + 1\}$ *partition* S.

Proof. Let $S = \{rm - 3r + 1, rm - 3r + 2, rm - \frac{r}{2}, rm - \frac{r}{2} + 2, rm - \frac{r}{2} + 3, ..., rm + 1\}$ where m is odd and r is even. Figure 3.3 shows how we put elements in S into 3 sets; A, B and C.

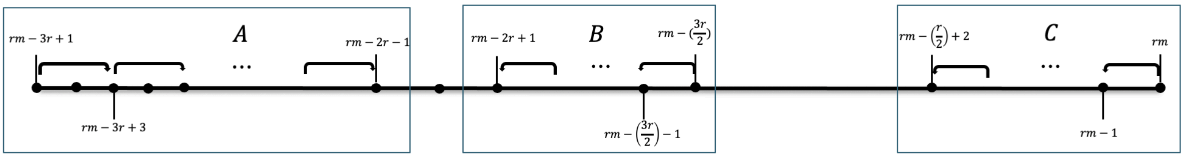

Figure 3.3: Subsets A, B and C of S.

We will divide $S \setminus (A \cup B \cup C)$ into 3 sets. Figure 3.4 shows how we put elements in $S \setminus (A \cup B \cup C)$ into 3 sets; D, E and F.

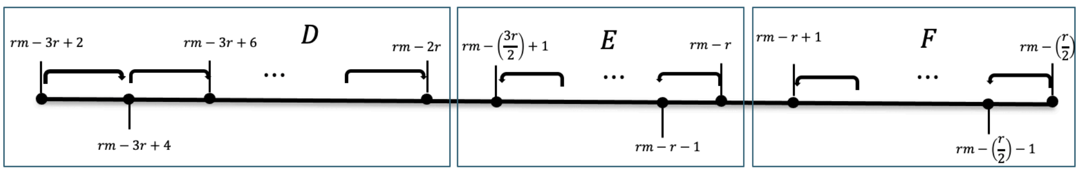

Figure 3.4: Subsets D, E and F of S.

Now, we divide all elements in S into 6 sets: $A = \{rm-3r+1, rm-3r+3, ..., rm-2r-1\}$,
$B = \{rm-\frac{3r}{2}, rm-\frac{3r}{2}-1, ..., rm-2r+1\}$,
$C = \{rm+1, rm, ..., rm-\frac{r}{2}+2\}$,
$D = \{rm-3r+2, rm-3r+4, ..., rm-2r\}$,
$E = \{rm-r, rm-r-1, ..., rm-\frac{3r}{2}+1\}$,
$F = \{rm-(\frac{r}{2}), rm-(\frac{r}{2})-1, ..., rm-r+1\}$.
We will shows that A, B, C, D, E and F are 6 partitions of S. Note that A and D contain an increasing sequence. The others contain a decreasing sequence. Then $\max A < \min B$, $\min C > \max F$ and $\max F > \max E$. Furthermore, all partition not contain $rm-\frac{r}{2}+1$. Thus, $C \cap F \cap E \cap D = \varnothing$ and $A \cap B = \varnothing$. We only need to show that $A \cap D = \varnothing$. Since A is a sequence of odd integers and D is a sequence of even integers, then $A \cap D = \varnothing$. In the last, we will show that $|A|+|B|+|C|+|D|+|E|+|F| = |S| = 3r$. Consider

$$\begin{aligned}
|A| &= \frac{rm-2r-1-(rm-3r+1)+2}{2} = \frac{r}{2} \\
|B| &= rm-\frac{3r}{2}-(rm-2r+1)+1 = \frac{r}{2} \\
|C| &= rm+1-(rm-\frac{r}{2}+2)+1 = \frac{r}{2} \\
|D| &= \frac{rm-2r-(rm-3r+2)+2}{2} = \frac{r}{2} \\
|E| &= rm-r-(rm-\frac{3r}{2}+1)+1 = \frac{r}{2} \\
|F| &= rm-(\frac{r}{2})-(rm-r+1)+1 = \frac{r}{2}.
\end{aligned}$$

Therefore $|A|+|B|+|C|+|D|+|E|+|F| = 6(\frac{r}{2}) = 3r$. Hence, A, B, C, D, E and F are the partitions of S.

□

Theorem 3.4. *Let $m_1, m_2, \ldots, m_r$ be positive integers where $2 \le m_1 = m_2 = \cdots = m_r = m$, and let $G = K_{m_1,m_2,\ldots,m_r}$ be a complete r-partite graph. If m is odd, then G is an S-magic graph and* $\theta(G) = \begin{cases} 0, & \text{if } r \text{ is odd} \\ 1, & \text{if } r \text{ is even.} \end{cases}$

Proof. Let $V_1, V_2, \ldots, V_r$ be partite sets of G. In the beginning, we use the label set $\{1, 2, \ldots, rm-3r\}$ to label $m-3$ rows of G, as shown in Figure 3.5.

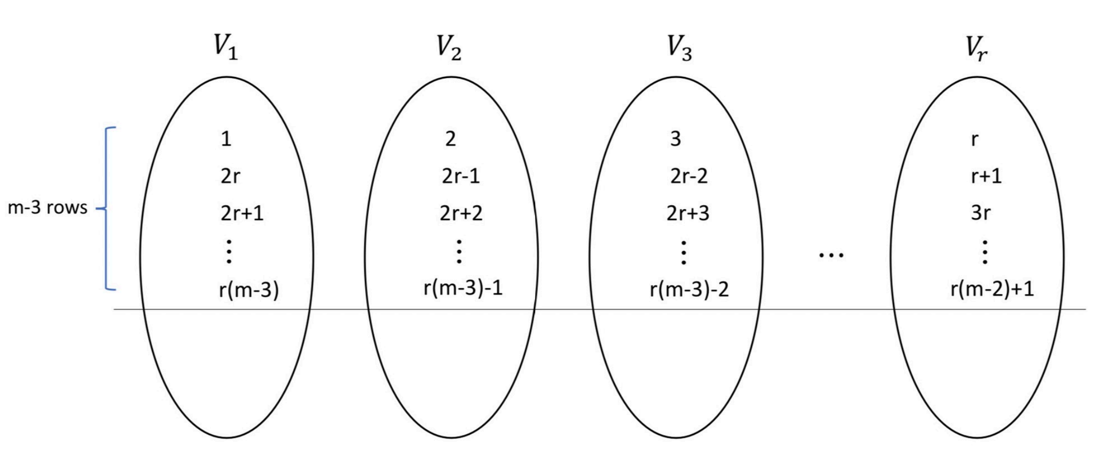

Figure 3.5: A labeling $m-3$ rows of G with label set $\{1,2,\ldots,rm-3r\}$

Case I: r is odd.

Firstly, we demonstrate how to divide $A=\{rm-3r+1, rm-3r+2,\ldots,rm-1,rm\}$ into r sets with three elements and the same sum, which is $3rm+\frac{3-9r}{2}$.

Let $a_n=(rm-3r+1)+2(n-1)$, $b_n=rm-(\frac{3r-1}{2})-(n-1)$, $c_n=rm-(n-1)$ and $P_n=\{a_n,b_n,c_n\}$. Observation that b_n, c_n are decreasing and a_n is increasing. We consider carefully about the largest value of n satisfies $a_n<b_n$. Consider if $a_n<b_n$, then

$$\begin{aligned}(rm-3r+1)+2(n-1) &< rm-(\frac{3r-1}{2})-(n-1)\\ 3(n-1) &< 3r-(\frac{3r-1}{2})-1\\ &< \frac{3r-1}{2}\\ n-1 &< \frac{3r-1}{6}\\ n &< \frac{3r+5}{6}\\ n &\leq \frac{r+1}{2}.\end{aligned}$$

As a result, we get $\frac{r+1}{2}$ sets from A which are $P_1,P_2,\ldots,P_{\frac{r+1}{2}}$. By Lemma 3.2, it easy to see that $a_n\in A, b_n\in B$ and $c_n\in C$ where $n=1,2,\ldots,\frac{r+1}{2}$. Thus we get all elements in $\bigcup\limits_{n=1}^{\frac{r+1}{2}} P_n$ are distinct. Next, consider a set $A\setminus\Big(P_1\cup\cdots\cup P_{\frac{r+1}{2}}\Big)$; $\{rm-3r+2, rm-3r+4, rm-3r+6, ..., rm-2r-1, rm-(\frac{3r-1}{2})+1, rm-(\frac{3r-1}{2})+2, ..., rm-(\frac{r-1}{2})-1\}$.

Let $d_n = rm - r - (n-1)$. Then d_n is decreasing. Choose

$$Q_n = \{a_n + 1, d_n, c_{\frac{r+1}{2}+n}\} \text{ for } n = 1, 2, \ldots, \frac{r-1}{2}.$$

By Lemma 3.2, it easy to see that $a_n + 1 \in D, d_n \in E$ and $c_{\frac{r+1}{2}+n} \in F$ where $n = 1, 2, \ldots, \frac{r-1}{2}$. Thus we get that all elements in $\bigcup_{n=1}^{\frac{r-1}{2}} Q_n$ are distinct. Finally, we get r sets with three elements and the same sum, which is $3rm + \frac{3-9r}{2}$ to labels in each V_i of G. Hence $\theta(G) = 0$, and we complete the proof.

Case II: r is even.
Let $m = 2p+1, r = 2q$ where p, q are positive integers.
Let $B = \{1, 2, ..., rm\}$. Consider

$$\sum_{b \in B} b = \frac{rm(rm+1)}{2} = 8p^2q^2 + 8pq^2 + 2q^2 + 2pq + q.$$

By lemma 2.11, B can be an S-magic labeling set of G under the condition the summation of all elements in B is divided by r. We have

$$\frac{\sum_{b \in B} b}{r} = \frac{rm(rm+1)}{2r} = 4p^2q + 4pq + q + p + \frac{1}{2}$$

is not an integer. It implies that B is not an S-magic labeling set of G, i.e. $\theta(G) > 0$. Moreover, we get $\frac{rm(rm+1)}{2} + \frac{r}{2} \equiv 0 \pmod{r}$.
We claim that $\{1, 2, \ldots, rm - \frac{r}{2}, rm - \frac{r}{2} + 2, \ldots, rm, rm+1\}$ is an S-magic labeling set of G. In the begining, we use the label set $\{1, 2, \ldots, rm - 3r\}$ to labels $n-3$ rows of G, as shown in Figure 3.5. Next, we demonstrate how to divide $C = \{rm - 3r + 1, rm - 3r + 2, \ldots, rm - \frac{r}{2}, rm - \frac{r}{2} + 2, \cdots, rm, rm+1\}$ into r sets with three elements and the same sum, which is $3rm + 2 - \frac{9r}{2}$. Let $x_n = (rm - 3r + 1) + 2(n-1)$, $y_n = rm - (\frac{3r}{2}) - (n-1)$, $z_n = rm + 1 - (n-1)$, and $P_n = \{x_n, y_n, z_n\}$. Observation that y_n, z_n are decreasing and x_n is increasing.
We be careful about the largest value of n that satisfies $x_n < y_n$. Consider if $x_n < y_n$, then

$$\begin{aligned}
(rm - 3r + 1) + 2(n-1) &< rm - (\frac{3r}{2}) - (n-1) \\
3(n-1) &< \frac{3r}{2} - 1 \\
&< \frac{3r-2}{2} \\
n - 1 &< \frac{3r-2}{6}
\end{aligned}$$

$$n < \frac{3r+4}{6}$$
$$n \leq \frac{r}{2}.$$

As a result, we get $\frac{r}{2}$ sets from C which are $P_1, P_2, \ldots, P_{\frac{r}{2}}$. By Lemma 3.3, it easy to see that $x_n \in A, y_n \in B$ and $z_n \in C$ where $n = 1, 2, \ldots, \frac{r}{2}$. Thus we get that all elements in $\bigcup_{n=1}^{\frac{r}{2}} P_n$ are distinct. Consider a set $C \setminus \left(P_1 \cup \cdots \cup P_{\frac{r}{2}}\right)$; $\{rm-3r+2, rm-3r+4, \ldots, rm-2r, rm-\frac{3r}{2}+1, rm-\frac{3r}{2}+2, \cdots, rm-\frac{r}{2}\}$. Let $w_n = rm - r - (n-1)$. Then w_n is decreasing. Choose

$$Q_n = \{x_n + 1, w_n, z_{\frac{r}{2}+(n+1)}\} \text{ for } n = 1, 2, \ldots, \frac{r}{2}.$$

By Lemma 3.3, it easy to see that $x_n + 1 \in D, w_n \in E$ and $z_{\frac{r}{2}+n+1} \in F$ where $n = 1, 2, \ldots, \frac{r}{2}$. Thus we get that all elements in $\bigcup_{n=1}^{\frac{r}{2}} Q_n$ are distinct. Hence $\{1, 2, \ldots, rm-\frac{r}{2}, rm-\frac{r}{2}+2, \ldots, rm, rm+1\}$ is an S-magic labeling set of G, and then $i(G) = rm+1$. It implies $\theta(G) = 1$. This completes the proof. □

Definition 3.5. **A minimal S-magic labeling set** T of G is an S-magic labeling set of G such that $\sum_{i \in T} i$ is minimum.

Lemma 3.6. *Let m_1 and m_2 be two positive integers where $m_1 \leq m_2$. Suppose $G = K_{m_1,m_2}$ is an S-magic graph with a labeling set $T = \{t_1, t_2, \ldots, t_{m_1+m_2}\}$ and $n = m_1 + m_2$. Then we have the following results.*

(I) If m_1, m_2 and n satisfy $n(n+1) \geq 2m_2(1+m_2)$ and $n \equiv 0$ or $3 \pmod 4$, then

$$\sum_{i=1}^{m_1+m_2} t_i \geq 1 + 2 + 3 + \cdots + (m_1 + m_2).$$

(II) If m_1, m_2 and n satisfy $n(n+1) \geq 2m_2(1+m_2)$ and $n \equiv 1$ or $2 \pmod 4$, then

$$\sum_{i=1}^{m_1+m_2} t_i \geq (1 + 2 + 3 + \cdots + m_1 + m_2) + 1.$$

(III) If m_1, m_2 and n satisfy $n(n+1) < 2m_2(1+m_2)$, then

$$\sum_{i=1}^{m_1+m_2} t_i \geq 2(1 + 2 + 3 + \cdots + m_2).$$

Proof. Let $G = K_{m_1,m_2}$ be an S-magic graph. Let V_1 and V_2 be partite sets of G. Let $T = \{t_1, t_2, \ldots, t_{m_1+m_2}\}$ and a labeling $f : V(G) \to T$ which $\sum_{x_i \in V_1} f(x_i) = \sum_{y_j \in V_2} f(y_j)$.

For case (I): By the proof of Theorem 2.18 and $\theta(G) = 0$ implies $\{1, 2, \ldots, m_1 + m_2\}$ is an S-magic labeling set of G. Thus

$$\sum_{t_i \in T} t_i \geq 1 + 2 + 3 + \cdots + (m_1 + m_2).$$

For case (II): By the proof of Theorem 2.18 and $\theta(G) = 1$ implies $\{1, 2, \ldots, m_1 + m_2 - 1, m_1 + m_2 + 1\}$ is a minimal labeling set of G. Thus

$$\sum_{t_i \in T} t_i \geq 1 + 2 + 3 + \cdots + (m_1 + m_2) + 1.$$

For case (III): In this case, the minimal labeling set for V_2 is $\{1, 2, \ldots, m_2\}$. Then

$$\sum_{y_j \in V_2} f(y_j) \geq 1 + 2 + 3 + \cdots + m_2.$$

By Lemma 2.11, the sum of the labelings assigned to each partite is equal implies

$$\begin{aligned}\sum_{t_i \in T} t_i &= \sum_{x_i \in V_1} f(x_i) + \sum_{y_j \in V_2} f(y_j) \\ &\geq (1 + 2 + 3 + \cdots + m_2) + (1 + 2 + 3 + \cdots + m_2) \\ &= 2(1 + 2 + 3 + \cdots + m_2).\end{aligned}$$

This completes the proof. □

Lemma 3.7. *Let m_2 and m_3 be two positive integers. Let $G = K_{2,m_2,m_3}$ be an S-magic graph, and T be a minimal labeling set of G. Then $i(G) \geq \lceil \frac{S(L)+1}{2} \rceil$ where $S(L)$ is the sum of the labelings assigned to each partite of G by a labeling set T.*

Proof. Let m_2 and m_3 be two positive integers, and let $G = K_{2,m_2,m_3}$ be an S-magic graph. Let V_1, V_2 and V_3 be partite sets of G, and $S(L_i)$ be the sum of the labelings assigned to each V_i for $i = 1, 2, 3$. Let T' be any S-magic labeling set of G, and let $f : V(G) \to T'$ be an S-magic labeling with $|V(G)| = |T'|$. Let $V_1(G) = \{x_1, x_2\}$ and $f(x_1) = a, f(x_2) = b$ with $a < b$. Then $S(L_1) = a + b$. Since G is an S-magic graph, by Lemma 2.11, $S(L_1) = S(L_2) = S(L_3) = a + b$.

Since $a < b$ and $a + b < 2b$, $b > \frac{S(L_1)}{2} \geq \frac{S(L)}{2}$. Then $\max(T') \geq b > \frac{S(L)}{2}$. Hence $i(G) > \frac{S(L)}{2}$, it follows that $i(G) \geq \lceil \frac{S(L)+1}{2} \rceil$. □

Notation: We divide the relation between m_2 and m_3 into 3 cases:
Case I: $(m_2 + m_3)(m_2 + m_3 + 1) \geq 2m_3(m_3 + 1)$ and $m_2 + m_3 \equiv 0$ or $3 \pmod 4$
Case II: $(m_2 + m_3)(m_2 + m_3 + 1) \geq 2m_3(m_3 + 1)$ and $m_2 + m_3 \equiv 1$ or $2 \pmod 4$
Case III: $(m_2 + m_3)(m_2 + m_3 + 1) < 2m_3(m_3 + 1)$.

Theorem 3.8. *For two positive integers m_2 and m_3 where $2 \leq m_2 \leq m_3$, let $G = K_{1,m_2,m_3}$ be an S-magic graph.*
If G satisfies case I, then $i(G) = \frac{(m_2+m_3)(m_2+m_3+1)}{4}$.
If G satisfies case II, then $i(G) = \frac{(m_2+m_3)(m_2+m_3+1)+2}{4}$.
If G satisfies case III, then $i(G) = \frac{m_3(m_3+1)}{2}$.

Proof. Let V_1, V_2 and V_3 be the partite sets of G. Since $|V_1(G)| = 1$, V_1 contains the maximum number in a labeling set of G. Since $G = K_{1,m_1,m_2}$ is an S-magic graph, the sum of the labelings assigned to V_1, V_2 and V_3 are equal.

For case I: By the proof of Theorem 2.18 [4], $\{1, 2, \ldots, m_2 + m_3\}$ is a labeling set for V_2, V_3, and the sum of the labelings of each partite is $\frac{(m_2+m_3)(m_2+m_3+1)}{4}$. Then label V_1 with a labeling set $\{\frac{(m_2+m_3)(m_2+m_3+1)}{4}\}$. This labeling is S-magic. If $i(G) < \frac{(m_2+m_3)(m_2+m_3+1)}{4}$, then the sum of the labelings assigned to each partite less than $\frac{(m_2+m_3)(m_2+m_3+1)}{4}$, but it is impossible. Hence, $i(G) = \frac{(m_2+m_3)(m_2+m_3+1)}{4}$.

For case II: By the proof of Theorem 2.18 [4], $\{1, 2, \ldots, m_2+m_3-1, m_2+m_3+1\}$ is a labeling set for V_2 and V_3, and the sum of each partite is $\frac{(m_2+m_3)(m_2+m_3+1)+2}{4}$. Then label V_1 with a label set $\{\frac{(m_2+m_3)(m_2+m_3+1)+2}{4}\}$. This labeling is S-magic. If $i(G) < \frac{(m_2+m_3)(m_2+m_3+1)+2}{4}$, then the sum of the labelings assigned to each partite less than $\frac{(m_2+m_3)(m_2+m_3+1)+2}{4}$, but it is impossible. Hence, $i(G) = \frac{(m_2+m_3)(m_2+m_3+1)+2}{4}$.

For case III: By the proof of Theorem 2.18 [4], we label the vertices in V_3 by the elements in $\{1, 2, ..., m_3\}$, and there exists a labeling set for V_2. Since G is an S-magic graph, the sum of the labelings assigned to V_1 is equal to the sum of the labelings assigned to V_3. Then we label V_1 with a labeling set $\{\frac{m_3(m_3+1)}{2}\}$. By Lemma 3.6, $i(G) = \frac{m_3(m_3+1)}{2}$. This completes the proof. □

Theorem 3.9. *Let m_2 and m_3 be two positive integers with $2 \leq m_2 \leq m_3$.*
If m_2 and m_3 satisfy case I or case II and $m_2 + m_3 > 8$, then $G = K_{2,m_2,m_3}$ is an S-magic graph and

$$i(G) = \begin{cases} \left\lceil \frac{(m_2+m_3)(m_2+m_3+1)+4}{8} \right\rceil, & \textit{for case } I \\ \left\lceil \frac{(m_2+m_3)(m_2+m_3+1)+6}{8} \right\rceil, & \textit{for case } II. \end{cases}$$

Proof. If $m_2 = 2$ and $m_2 + m_3 > 8$, then $m_3 > 6$. It implies that m_2 and m_3 satisfy case III. We omit this case. Let $G = K_{2,m_2,m_3}$ with $3 \leq m_2 \leq m_3$ and $m_2+m_3 > 8$. Let $S(L_i)$ be the sum of the labelings assigned to V_i where $i = 1, 2, 3$.

For case I:

By the proof of Theorem 2.18, $\{1, 2, \ldots, m_2+m_3\}$ is a labeling set for V_2 and V_3 with $S(L_2) = S(L_3)$. It implies $S(L_2) = S(L_3) = \frac{(m_2+m_3)(m_2+m_3+1)}{4}$, i.e. $\frac{(m_2+m_3)(m_2+m_3+1)}{4}$ is an integer. We divide into 2 cases;

Case 1: $\frac{(m_2+m_3)(m_2+m_3+1)}{4}$ is even.

We claim that $T_1 = \{1, 2, \ldots, m_2+m_3, \frac{(m_2+m_3)(m_2+m_3+1)}{8} - 1, \frac{(m_2+m_3)(m_2+m_3+1)}{8} + 1\}$ is an S-magic labeling set of G. Since $m_2+m_3 > 8$, $\frac{m_2+m_3}{8} > 1$. Then $\frac{(m_2+m_3)(m_2+m_3+1)}{8}$ $> m_2 + m_3 + 1$. It implies $\frac{(m_2+m_3)(m_2+m_3+1)}{8} - 1 > m_2 + m_3$. It implies all elements in T_1 are distinct. Furthermore, Figure 3.6 shows the labeling of G with the label set $T_1 = \{1, 2, \ldots, m_2 + m_3, \frac{(m2+m_3)(m_2+m_3+1)}{8} - 1, \frac{(m_2+m_3)(m_2+m_3+1)}{8} + 1\}$, and the sum of the labelings assigned to each partite is equal to $\frac{(m_2+m_3)(m_2+m_3+1)}{4}$.

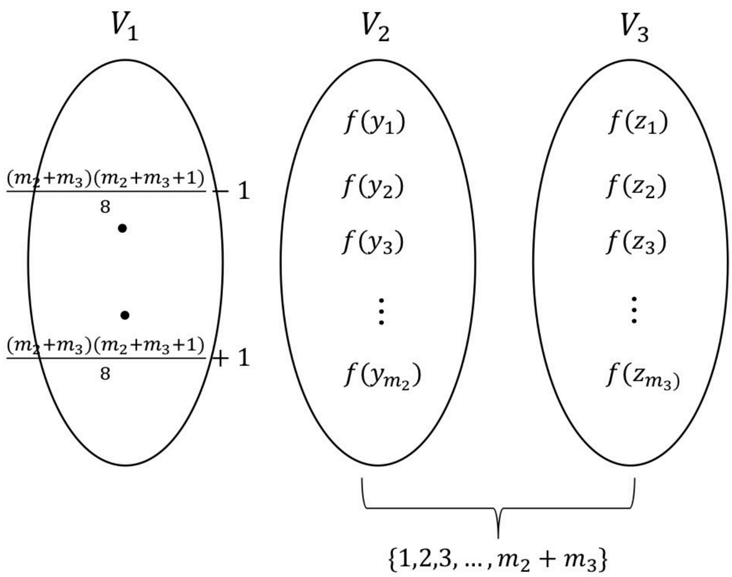

Figure 3.6: A labeling of K_{2,m_2,m_3} where m_2 and m_3 satisfy case I and $\frac{(m_2+m_3)(m_2+m_3+1)}{4}$ is even.

Therefore $G = K_{2,m_2,m_3}$ is an S-magic graph. Since the sum of each partite is $\frac{(m_2+m_3)(m_2+m_3+1)}{4}$, and this is a minimum sum, then T_1 is a minimal S-magic labeling set for this case. We have $S(L) = \frac{(m_2+m_3)(m_2+m_3+1)}{4}$. By Lemma 3.7, $i(G) \geq \frac{(m_2+m_3)(m_2+m_3+1)}{8} + 1$, and Figure 3.6 shows the labeling with $i(G) = \frac{(m_2+m_3)(m_2+m_3+1)}{8} + 1$. Moreover, if $i(G) < \frac{(m_2+m_3)(m_2+m_3+1)}{8} + 1$ it implies the sum of the labelings assigned to V_2 and V_3 less than $\frac{(m_2+m_3)(m_2+m_3+1)}{4}$, but it is impossible. In conclusion, if $\frac{(m_2+m_3)(m_2+m_3+1)}{4}$ is even, then $i(G) = \left\lceil \frac{(m_2+m_3)(m_2+m_3+1)}{8} + 1 \right\rceil$.

Case 2: $\frac{(m_2+m_3)(m_2+m_3+1)}{4}$ is odd.
We claim that $T_2 = \{1, 2, \ldots, m_2+m_3, \frac{(m_2+m_3)(m_2+m_3+1)+4}{8} - 1, \frac{(m_2+m_3)(m_2+m_3+1)+4}{8}\}$ is an S-magic labeling set of G. By the proof of case 1, $\frac{(m_2+m_3)(m_2+m_3+1)}{8} > m_2 + m_3 + 1$ implies $\frac{(m_2+m_3)(m_2+m_3+1)+4}{8} > m_2 + m_3 + 1$. Furthermore, Figure 3.7 shows the labeling of G with the label set $T_2 = \{1, 2, \ldots, m_2 + m_3, \frac{(m_2+m_3)(m_2+m_3+1)+4}{8} - 1, \frac{(m_2+m_3)(m_2+m_3+1)+4}{8}\}$, and the sum of the labelings assigned to each partite is equal to $\frac{(m_2+m_3)(m_2+m_3+1)}{4}$.

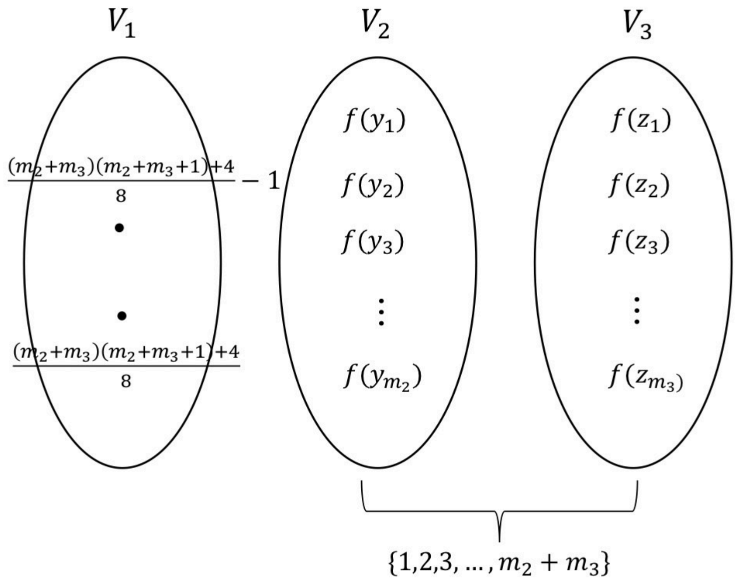

Figure 3.7: A labeling of K_{2,m_2,m_3} where m_2 and m_3 satisfy case I and $\frac{(m_2+m_3)(m_2+m_3+1)}{4}$ is odd.

Therefore G is S-magic. Since the sum of each partite is $\frac{(m_2+m_3)(m_2+m_3+1)}{4}$, and this is a minimum sum, then T_2 is a minimal S-magic labeling set for this case. We have $S(L) = \frac{(m_2+m_3)(m_2+m_3+1)}{4}$. By Lemma 3.7, $i(G) \geq \frac{(m_2+m_3)(m_2+m_3+1)+4}{8}$, and Figure 3.9 shows the labeling with $i(G) = \frac{(m_2+m_3)(m_2+m_3+1)+4}{8}$. Moreover, if $i(G) < \frac{(m_2+m_3)(m_2+m_3+1)+4}{8}$ it implies the sum of the labelings assigned to V_2 and V_3 less than $\frac{(m_2+m_3)(m_2+m_3+1)}{4}$, but it is impossible. In conclusion, if $\frac{(m_2+m_3)(m_2+m_3+1)}{4}$ is odd, then $i(G) = \left\lceil \frac{(m_2+m_3)(m_2+m_3+1)+4}{8} \right\rceil$. Hence $i(G) = \left\lceil \frac{(m_2+m_3)(m_2+m_3+1)+4}{8} \right\rceil$ for case I.

<u>For case II:</u>

By the proof of Theorem 2.18, $\{1, 2, \ldots, m_2 + m_3 - 1, m_2 + m_3 + 1\}$ is a labeling set for V_2 and V_3 with $S(L_2) = (SL_3)$. It implies $S(L_2) = S(L_3) = \frac{(m_2+m_3)(m_2+m_3+1)+2}{4}$, i.e. $\frac{(m_2+m_3)(m_2+m_3+1)+2}{4}$ is an integer. We divide into 2 cases;

Case 1: $\frac{(m_2+m_3)(m_2+m_3+1)+2}{4}$ is even.
We claim that $T_3 = \{1, 2, \ldots, m_2 + m_3 - 1, m_2 + m_3 + 1, \frac{(m_2+m_3)(m_2+m_3+1)+2}{8} - 1, \frac{(m_2+m_3)(m_2+m_3+1)+2}{8} + 1\}$ is an S-magic labeling set of G. Consider

$$\frac{(m_2 + m_3)(m_2 + m_3 + 1) + 2}{8} - 1 > m_2 + m_3 + 1 - \frac{6}{8} \geq m_2 + m_3 + 1.$$

If $\frac{(m_2+m_3)(m_2+m_3+1)+2}{8} - 1 = m_2 + m_3 + 1$, then

$$((m_2 + m_3) - 8)((m_2 + m_3) + 1) = 6.$$

It implies $m_2 + m_3$ is not an integer. Thus $\frac{(m_2+m_3)(m_2+m_3+1)+2}{8} - 1 > m_2 + m_3 + 1$. Furthermore, Figure 3.8 shows the labeling of G with $T_3 = \{1, 2, \ldots, m_2 + m_3 - 1, m_2 + m_3 + 1, \frac{(m_2+m_3)(m_2+m_3+1)+2}{8} - 1, \frac{(m_2+m_3)(m_2+m_3+1)+2}{8} + 1\}$, and the sum of the labelings assigned to each partite is $\frac{(m_2+m_3)(m_2+m_3+1)+2}{4}$.

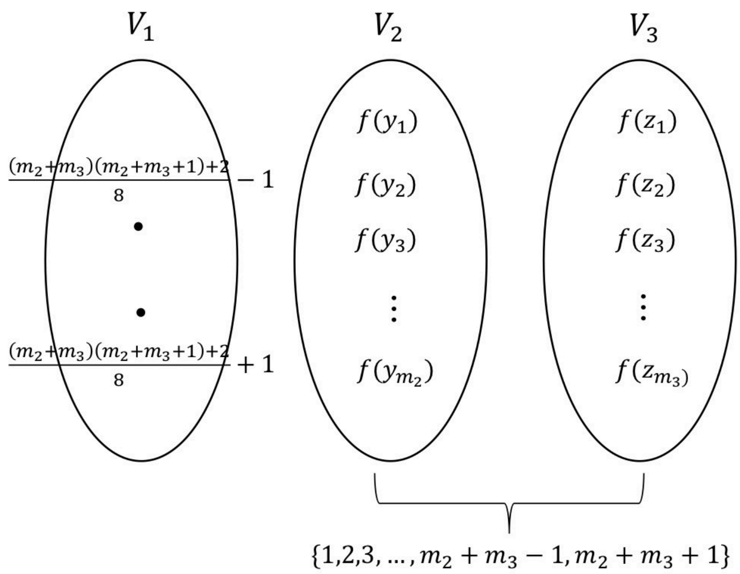

Figure 3.8: A labeling of K_{2,m_2,m_3} where m_2 and m_3 satisfy case II and $\frac{(m_2+m_3)(m_2+m_3+1)+2}{4}$ is even.

Therefore G is S-magic. Since the sum of each partite is $\frac{(m_2+m_3)(m_2+m_3+2)}{4}$, and this is a minimum sum, then T_3 is a minimal S-magic labeling set for this case. We have $S(L) = \frac{(m_2+m_3)(m_2+m_3+1)+2}{4}$. By Lemma 3.7, $i(G) \geq \frac{(m_2+m_3)(m_2+m_3+1)+2}{8} + 1$. Moreover, if $i(G) < \frac{(m_2+m_3)(m_2+m_3+1)+2}{8} + 1$ it implies the sum of the labelings assigned to V_2 and V_3 less than $\frac{(m_2+m_3)(m_2+m_3+1)+2}{4}$, but it is impossible. In conclusion, if $\frac{(m_2+m_3)(m_2+m_3+1)+2}{4}$ is even, then $i(G) = \left\lceil \frac{(m_2+m_3)(m_2+m_3+1)+2}{8} + 1 \right\rceil$.

Case 2: $\frac{(m_2+m_3)(m_2+m_3+1)+2}{4}$ is odd.
We will prove that $T_4 = \{1, 2, \ldots, m_2+m_3-1, m_2+m_3+1, \frac{(m_2+m_3)(m_2+m_3+1)-2}{8}, \frac{(m_2+m_3)(m_2+m_3+1)+6}{8}\}$ is an S-magic labeling set of G.
From the above, we found that

$$\frac{(m_2+m_3)(m_2+m_3+1)+2}{8} > m_2+m_3+1.$$

Furthermore, Figure 3.9 shows the labeling of G with $T_4 = \{1, 2, \ldots, m_2+m_3-1, m_2+m_3+1, \frac{(m_2+m_3)(m_2+m_3+1)-2}{8}, \frac{(m_2+m_3)(m_2+m_3+1)+6}{8}\}$, and the sum of the labelings assigned to each partite is $\frac{(m_2+m_3)(m_2+m_3+1)+2}{4}$.

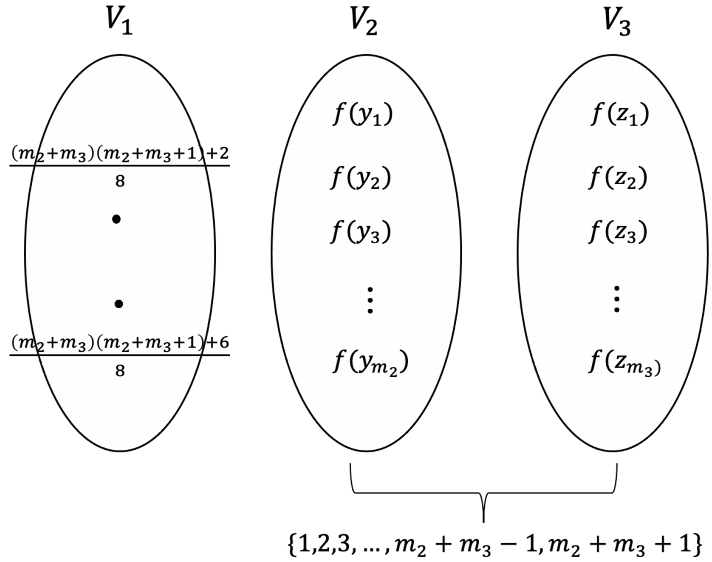

Figure 3.9: A labeling of K_{2,m_2,m_3} where m_2 and m_3 satisfy case II and $\frac{(m_2+m_3)(m_2+m_3+1)+2}{4}$ is odd.

Therefore G is S-magic. Since the sum of each partite is $\frac{(m_2+m_3)(m_2+m_3+1)+2}{4}$, and this is a minimum sum, then T_4 is a minimal S-magic labeling set for this case. We have $S(L) = \frac{(m_2+m_3)(m_2+m_3+1)+2}{4}$. By Lemma 3.7, $i(G) \geq \frac{(m_2+m_3)(m_2+m_3+1)+6}{8}$. Moreover, if $i(G) < \frac{(m_2+m_3)(m_2+m_3+1)+6}{8}$ it implies the sum of the labelings assigned to V_2 and V_3 less than $\frac{(m_2+m_3)(m_2+m_3+1)+2}{4}$, but it is impossible. In conclusion, if $\frac{(m_2+m_3)(m_2+m_3+1)+2}{4}$ is odd, then $i(G) = \lceil \frac{(m_2+m_3)(m_2+m_3+1)+6}{8} \rceil$. Hence $i(G) = \lceil \frac{(m_2+m_3)(m_2+m_3+1)+6}{8} \rceil$ for case II. The proof is completed. □

Theorem 3.10. *Let m_2 and m_3 be two positive integers with $2 \leq m_2 \leq m_3$. Suppose $G = K_{2,m_2,m_3}$ is an S-magic graph where m_2 and m_3 satisfy case I or case II, and $m_2 + m_3 \leq 8$.*

$(I.)$ $\quad m_2 + m_3 = 4$

$\{1, 2, 3, 4, 5, 6\}$ *is an S-magic labeling set of $K_{2,2,2}$ and $i(G) = 6$.*

$(II.)$ $\quad m_2 + m_3 = 5$

$\{1, 2, 3, 4, 5, 7, 8\}$ *is an S-magic labeling set of $K_{2,2,3}$ and $i(G) = 8$.*

$(III.)$ $\quad m_2 + m_3 = 6$

$\{1, 2, 3, 4, 5, 6, 7, 8\}$ *is an S-magic labeling set of $K_{2,2,4}, K_{2,3,3}$, and $i(G) = 8$.*

$(IV.)$ $\quad m_2 + m_3 = 7$

$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ *is an S-magic labeling set of $K_{2,2,5}, K_{2,3,4}$, and $i(G) = 9$.*

$(V.)$ $\quad m_2 + m_3 = 8$

$\{1, 2, 3, 4, 5, 6, 7, 8, 10, 11\}$ *is an S-magic labeling set of $K_{2,3,5}, K_{2,4,4}$, and $i(G) =$ 11.*

Proof. For $(I), (III), (IV)$, it is clear by the proof of Theorem 2.18 , see in Figure 3.10, Figure 3.11, Figure 3.12, Figure 3.13 and Figure 3.14, as shown below.

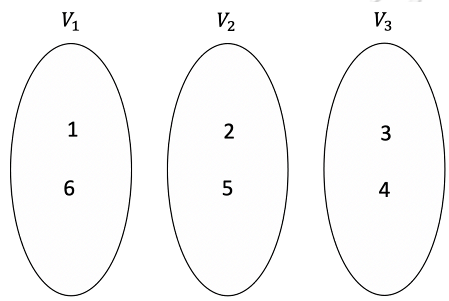

Figure 3.10: A Labeling of $K_{2,2,2}$.

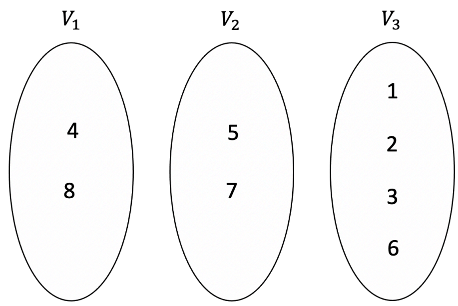

Figure 3.11: A Labeling of $K_{2,2,4}$.

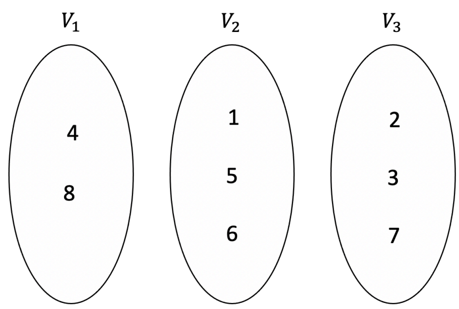

Figure 3.12: A Labeling of $K_{2,3,3}$.

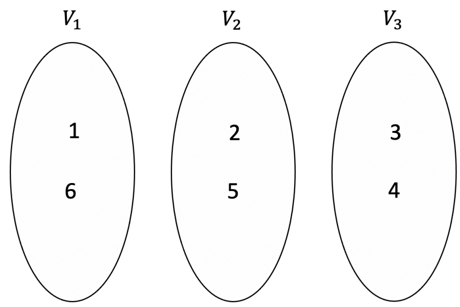

Figure 3.13: A Labeling of $K_{2,2,5}$.

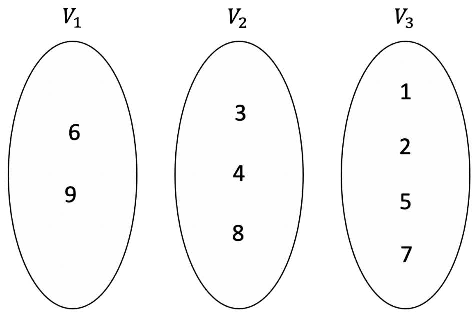

Figure 3.14: A Labeling of $K_{2,3,4}$.

For (II): Note that $1+2+3+4+5+6+7 = 28$. Since $28 \equiv 1 \pmod 3$, $\{1,2,3,4,5,6,7\}$ is not an S-labeling set of G. Then $i(G) \geq 8$. Figure 3.15 shows the labeling of $f : V(K_{2,2,3}) \to \{1,2,3,4,5,7,8\}$. Hence $i(G) = 8$.

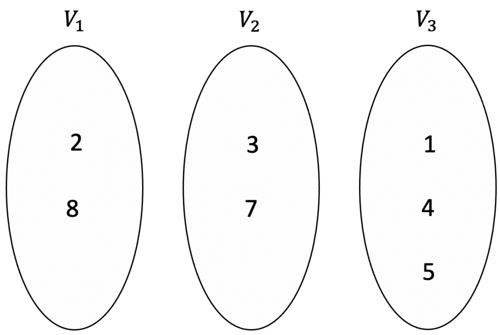

Figure 3.15: A Labeling of $K_{2,2,3}$.

For (V): Note that $1+2+3+4+5+6+7+8+9+10 = 55$. Since $55 \equiv 1 \pmod 3$, $\{1,2,3,4,5,6,7,8,9,10\}$ is not an S-labeling set of G. Then $i(G) \geq 11$. Figure 3.16 and Figure 3.17 show the labelings of $K_{2,3,5}$ and $K_{2,4,4}$ with a labeling set $\{1,2,3,4,5,6,7,8,10,11\}$, respectively. Hence $i(G) = 11$.

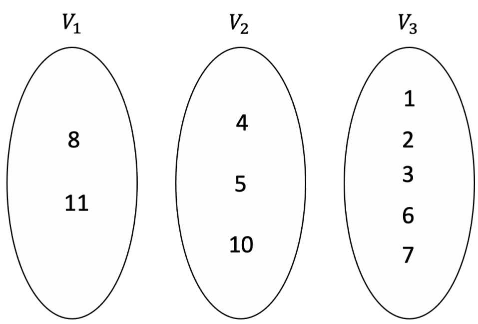

Figure 3.16: A Labeling of $K_{2,3,5}$.

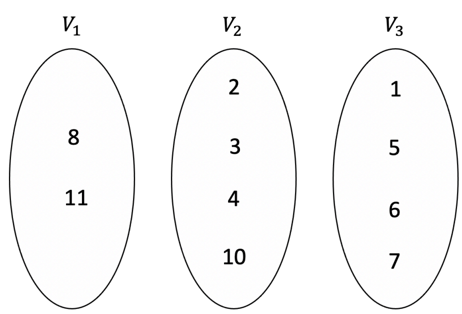

Figure 3.17: A Labeling of $K_{2,4,4}$.

□

By using an elemantary calculation, we obtain the following lemma that will be useful in the proof of Theorem 3.12.

Lemma 3.11. *Let m_2 and m_3 be positive integers. If $m_3 > -\frac{1}{2}+\frac{\sqrt{8m_3^3+9m_2^2-52m_2+4}}{2(m_2-2)}$, then $\frac{m_3(m_3+1)}{4}-1 > \frac{m_2^2+m_3^2+m_2+m_3}{2m_2}$.*

Proof. Suppose $m_3 > -\frac{1}{2}+\frac{\sqrt{8m_3^3+9m_2^2-52m_2+4}}{2(m_2-2)}$. Then

$$m_3 > \frac{-(m_2+2)+\sqrt{(m_2-2)^2-4(m_2-2)(-(2m_2^2+6m_2))}}{2(m_2-2)}.$$

Hence,

$$(m_2-2)m_3^2+(m_2-2)m_3-(2m_2^2+6m_2) > 0.$$

Therefore,

$$\begin{aligned} m_2m_3^2+m_2m_3-4m_2 &> 2m_2^2+2m_3^2+2m_2+2m_3 \\ \frac{(m_3^2+m_3)m_2}{4m_2}-\frac{4m_2}{4m_2} &> \frac{2m_2^2+2m_3^2+2m_2+2m_3}{4m_2} \\ \frac{m_3(m_3+1)}{4}-1 &> \frac{m_2^2+m_3^2+m_2+m_3}{2m_2}. \end{aligned}$$

□

Theorem 3.12. *Let m_2 and m_3 be two positive integers and $3 \leq m_2 \leq m_3$. If m_2 and m_3 satisfy case III, then $G = K_{2,m_2,m_3}$ is an S-magic graph and $i(G) = \left\lceil \frac{m_3(m_3+1)+2}{4} \right\rceil$.*

Proof. Suppose $S(L_i)$ is the sum of the labelings assigned to V_i for $i = 1, 2, 3$. Because m_2 and m_3 satisfy case III, by Lemma 3.6, and Lemma 3.7, we get that $S(L_1) = S(L_2) = S(L_3) \geq \frac{m_3(m_3+1)}{2}$ and $i(G) \geq \lceil \frac{m_3(m_3+1)+2}{4} \rceil$. Now, we demonstrate a labeling of G with $S(L_1) = S(L_2) = S(L_3) = \frac{m_3(m_3+1)}{2}$ and $i(G) = \lceil \frac{m_3(m_3+1)+2}{4} \rceil$. First, label V_2 and V_3 with labeling sets $L_2 = \{m_3+1, m_3+2, \dots, m_3+m_2\}$ and $L_3 = \{1, 2, \dots, m_3\}$, respectively as in Figure 3.18.

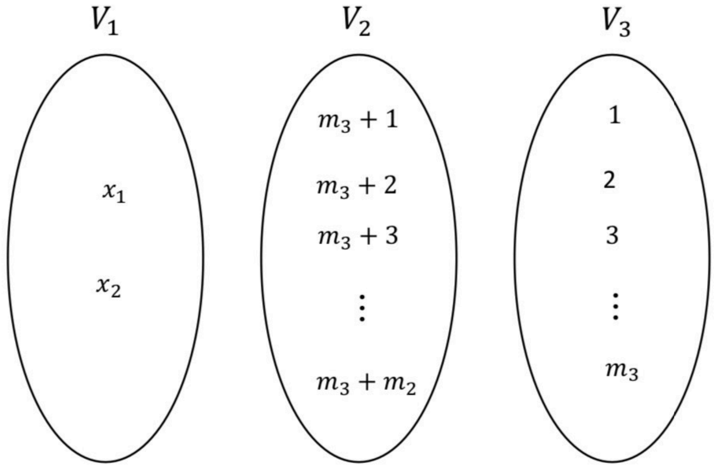

Figure 3.18: Label V_2 and V_3 with label sets $L_2 = \{m_3 + 1, \ldots, m_3 + m_2\}$ and $L_3 = \{1, 2, \ldots, m_3\}$, respectively.

By the proof of Case $n(n+1) < 2m_2(m_2+1)$ of theorem 2.18, $K = S(L_3) - S(L_2) = m_2 q + r$, for $q, r \geq 0$ and $r < m_2$.

For $r = 0$: $K = m_2 q$, we now replace the label set L_2 by $L_2' = \{m_3 + 1 + q, m_3 + 2 + q, \ldots, m_3 + m_2 + q\}$ and leave L_3 unchanged as in Figure 3.19.

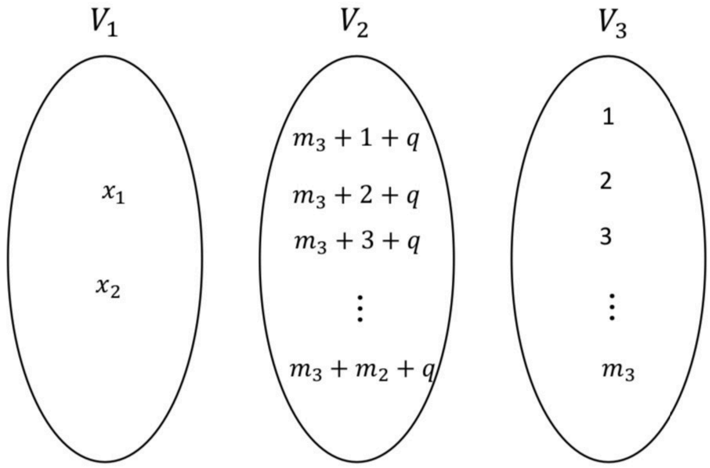

Figure 3.19: Replace the label set L_2 by L_2' for $r = 0$

For $r > 0$: $K = m_2 q + r$, we now replace the label set L_2 by $L_2' = \{m_3 + q + 1, m_3 + q + 2, \ldots, m_3 + m_2 + q - r, m_3 + m_2 + q - r + 2, \ldots, m_3 + m_2 + q, m_3 + m_2 + q + 1\}$ and leave L_3 unchanged as in Figure 3.20.

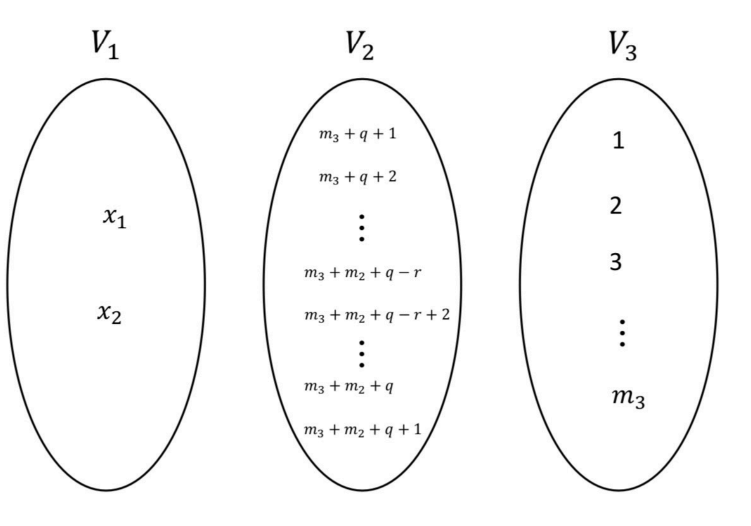

Figure 3.20: Replace the label set L_2 by L_2' for $r > 0$.

By the proof of theorem 2.18, $S(L_2) = S(L_3) = \frac{m_3(m_3+1)}{2}$. Next, we will show that $i(G) = \left\lceil \frac{m_3(m_3+1)+2}{4} \right\rceil$ by labeling V_1 so that $S(L_1) = \frac{m_3(m_3+1)}{2}$. Consider the following situations.

Case 1: $\frac{m_3(m_3+1)}{2}$ is even. Label V_1 with label set $L_1 = \{\frac{m_3(m_3+1)}{4} - 1, \frac{m_3(m_3+1)}{4} + 1\}$, see Figure 3.21 and Figure 3.22 for $r = 0$ and $r > 0$, respectively.

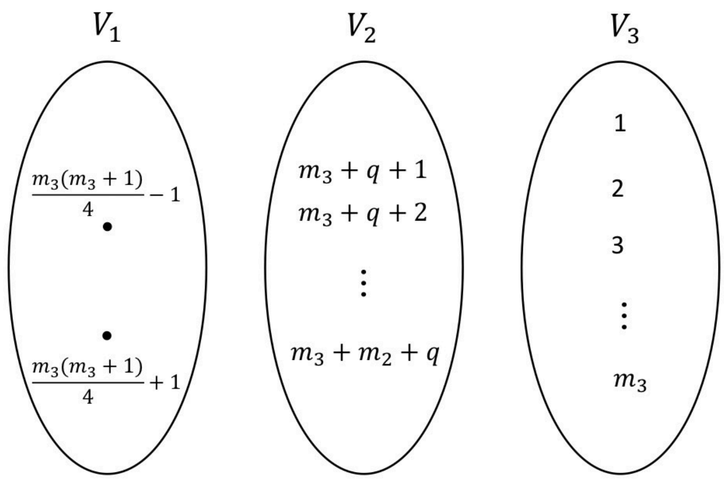

Figure 3.21: Label V_1 with a label set $L_1 = \{\frac{m_3(m_3+1)}{4} - 1, \frac{m_3(m_3+1)}{4} + 1\}$ for $r = 0$.

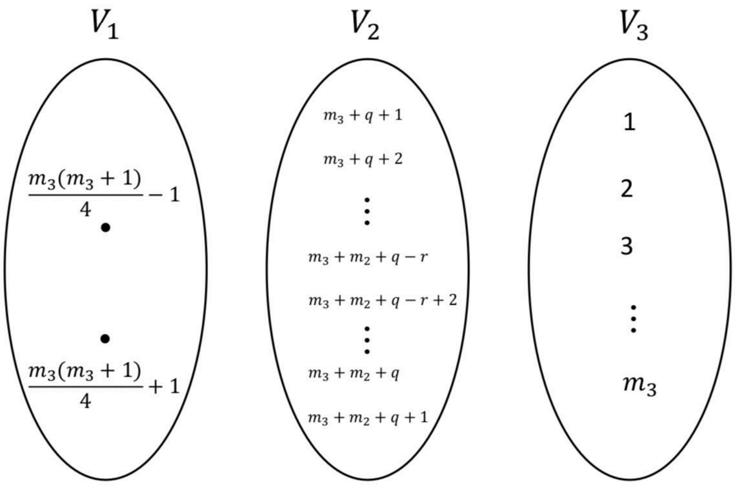

Figure 3.22: Label V_1 with a label set $L_1 = \{\frac{m_3(m_3+1)}{4} - 1, \frac{m_3(m_3+1)}{4} + 1\}$ for $r > 0$.

Denote the labelings in Figure 3.21 and Figure 3.22 by f_1 and f_2, respectively. We will show that f_1 and f_2 are S-magic labelings of G for $r = 0$ and $r > 0$, respectively by showing

$$\frac{m_3(m_3+1)}{4} - 1 > m_2 + m_3 + q + 1. \tag{3.1}$$

Note that

$$\begin{aligned} q &= \frac{S(L_3) - S(L_3) - r}{m_2} \\ q &= \frac{\frac{m_3(m_3+1)}{2} - (m_2m_3 + \frac{m_2(m_2+1)}{2}) - r}{m_2} \\ q &= \frac{m_3^2 + m_3}{2m_2} - \left(\frac{2m_2m_3 + m_2^2 + m_2}{2m_2}\right) - \frac{r}{m_2} \\ m_2 + m_3 + q + 1 &= m_2 + m_3 + \frac{m_3^2 + m_3}{2m_2} - \left(\frac{2m_2m_3 + m_2^2 + m_2}{2m_2}\right) - \frac{r}{m_2} + 1 \\ &= \frac{2m_2m_3 + 2m_2^2 + m_3^2 + m_3 - 2m_2m_3 - m_2^2 + m_2 - 2r}{2m_2} \\ &\leq \frac{m_2^2 + m_3^2 + m_2 + m_3}{2m_2}. \end{aligned}$$

Then we will show that

$$\frac{m_3(m_3+1)}{4} - 1 > \frac{m_2^2 + m_3^2 + m_2 + m_3}{2m_2}. \tag{3.2}$$

Since m_2 and m_3 satisfy case III, $(m_2+m_3)(m_2+m_3+1) < 2m_3(m_3+1)$. So

$$\begin{aligned} m_3^2 - m_2^2 - 2m_2m_3 + m_3 - m_2^2 - m_2 &> 0 \\ m_3^2 - (2m_2-1) - (m_2^2+m_2) &> 0. \end{aligned}$$

Thus

$$m_3 > -\frac{1}{2} + \frac{2m_2 + \sqrt{8m_2^2+1}}{2}. \tag{3.3}$$

Then, if we can show that $\frac{\sqrt{8m_2^3+9m_2^2-52m_2+4}}{2(m_2-2)} < \frac{2m_2+\sqrt{8m_2^2+1}}{2}$, by Lemma 3.11, we complete this case. Consider

$$\begin{aligned} 12m_2^2 + 24\sqrt{2}m_2 + 1 &> 8m_2^2 + 25m_2 - 2 \\ 12m_2^2 + 24\sqrt{2}m_2 + 1 &> \frac{8m_2^2 + 25m_2 - 2}{m_2 - 2} \\ (2m_2 + \sqrt{8m_2^2+1})^2 &> \left(\sqrt{\frac{8m_2^2 + 25m_2 - 2}{m_2 - 2}}\right)^2 \\ 2k + \sqrt{8m_2^2+1} &> \sqrt{\frac{(m_2-2)(8m_2^2 + 25m_2 - 2)}{(m_2-2)^2}} \\ &= \sqrt{\frac{8m_2^3 + 9m_2^2 - 52m_2 + 4}{(m_2-2)^2}} \\ &= \frac{\sqrt{8m_2^3 + 9m_2^2 - 52m_2 + 4}}{(m_2-2)} \\ \frac{2m_2 + \sqrt{8m_2^2+1}}{2} &> \frac{\sqrt{8m_2^3 + 9m_2^2 - 52m_2 + 4}}{2(m_2-2)}. \end{aligned} \tag{3.4}$$

By Lemma 3.11 and (3.4), (3.1) holds. Then f_1 and f_2 are S-magic. Hence G is an S-magic graph, and $i(G) = \frac{m_3(m_3+1)}{4} + 1 = \left\lceil \frac{m_3(m_3+1)+2}{4} \right\rceil$ when $\frac{m_3(m_3+1)}{2}$ is even.

Case 2: $\frac{m_3(m_3+1)}{2}$ is odd. Label V_1 with label set $L_1 = \{\frac{m_3(m_3+1)+2}{4} - 1, \frac{m_3(m_3+1)+2}{4}\}$, see in Figure 3.23 and Figure 3.24 for $r = 0$ and $r > 0$, respectively.

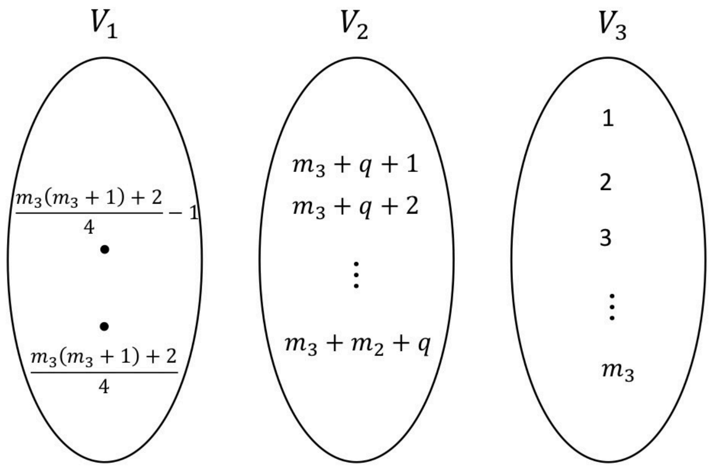

Figure 3.23: Label V_1 with label set $L_1 = \{\frac{m_3(m_3+1)+2}{4} - 1, \frac{m_3(m_3+1)+2}{4}\}$ for $r = 0$.

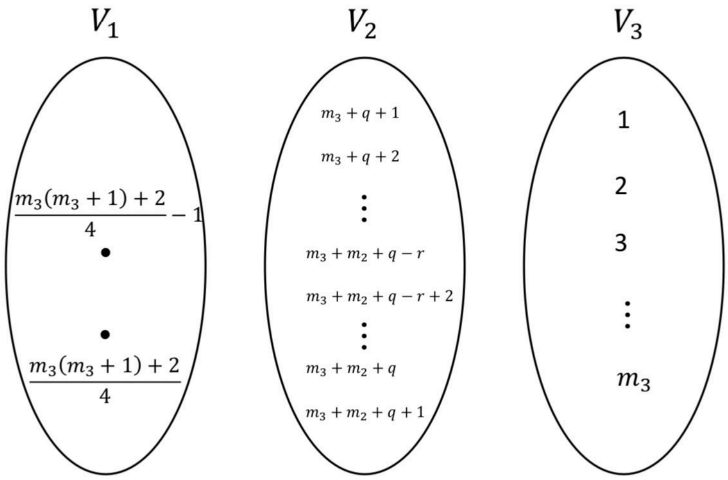

Figure 3.24: Label V_1 with label set $L_1 = \{\frac{m_3(m_3+1)+2}{4} - 1, \frac{m_3(m_3+1)+2}{4}\}$ For $r > 0$.

Denote the labelings in Figure 3.23 and Figure 3.24 by f_3 and f_4, respectively. We will show that f_3 and f_4 are S-magic labelings of G for $r = 0$, and $r > 0$, respectively by showing $\frac{m_3(m_3+1)+2}{4} - 1 > \frac{m_2^2+m_3^2+m_2+m_3}{2m_2}$. It is completed in case 1. Hence f_3 and f_4 are S-magic. Therefore, G is an S-magic graph, and $i(G) = \frac{m_3(m_3+1)+2}{4} = \left\lceil \frac{m_3(m_3+1)+2}{4} \right\rceil$ when $\frac{m_3(m_3+1)}{2}$ is odd. In conclusion, $i(G) = \left\lceil \frac{m_3(m_3+1)+2}{4} \right\rceil$. □

Theorem 3.13. *Let m_2 and m_3 be two positive integers and $3 \le m_3$. If m_3 satisfies case III, then $G = K_{2,2,m_3}$ is an S-magic graph and $i(G) = \left\lceil \frac{m_3(m_3+1)+2}{4} \right\rceil + 1$.*

Proof. Let m_3 be a positive integer with $2 \leq m_3$. Suppose $S(L_i)$ is the sum of the labelings assigned to $V_i, i = 1, 2, 3$. Now, we divide into 2 cases;

Case 1: $\frac{m_3(m_3+1)}{2}$ is even.
We claim that a labeling $f : V \to \{1, 2, \ldots, m_3, \frac{m_3(m_3+1)}{2} - 2, \frac{m_3(m_3+1)}{2} - 1, \frac{m_3(m_3+1)}{2} + 1, \frac{m_3(m_3+1)}{2} + 2\}$ is an S-magic labeling of G with $S(L_1) = S(L_2) = S(L_3) = \frac{m_3(m_3+1)}{2}$. Since m_2 and m_3 satisfy case III, $m_3 \geq 5$. If $m_3 \geq 6$, $m_3^2 + m_3 - 8 \geq m_3^2 - 2 > 4m_3$, and if $m_3 = 5$, it is obvious that $\frac{5(6)}{4} - 2 > 5$. So, $\frac{m_3(m_3+1)}{4} - 2 > m_3$. Figure 3.25 shows the labeling of G with $T = \{1, 2, \ldots, m_3, \frac{m_3(m_3+1)}{2} - 2, \frac{m_3(m_3+1)}{2} - 1, \frac{m_3(m_3+1)}{2} + 1, \frac{m_3(m_3+1)}{2} + 2\}$, and the sum the the labelings assigned to each partite is equal to $\frac{m_3(m_3+1)}{2}$.

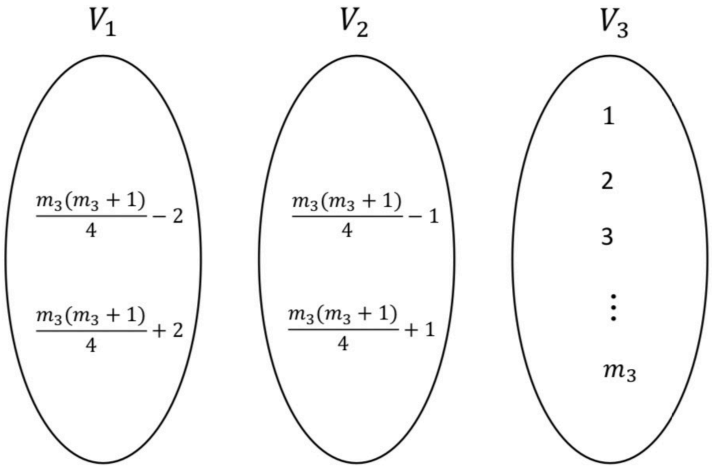

Figure 3.25: A labeling of $K_{2,2,m_3}$ with m_3 satisfies case III and $\frac{m_3(m_3+1)}{2}$ is even with $i(G) = \frac{m_3(m_3+1)}{2} + 2$.

Then G is an S-magic graph. By Lemma 3.6, T has a minimum sum of elements. Then T is a minimal S-magic labeling set. By lemma 3.7, $i(G) \geq \frac{m_3(m_3+1)}{4} + 1$. Suppose $i(G) = \frac{m_3(m_3+1)}{4} + 1$, there is a labeling set T_1 with $\max(T_1) = \frac{m_3(m_3+1)}{4} + 1$. Then 4 maximum elements that can be in T_1 are $\frac{m_3(m_3+1)}{4} - 2, \frac{m_3(m_3+1)}{4} - 1, \frac{m_3(m_3+1)}{4}$ and $\frac{m_3(m_3+1)}{4} + 1$. Since the sum of the labelings assigned to each partite are equal, the only possible labeling sets for V_1 and V_2 are $L_1' = \{\frac{m_3(m_3+1)}{4} + 1, \frac{m_3(m_3+1)}{4} - 2\}$ and $L_2' = \{\frac{m_3(m_3+1)}{4}, \frac{m_3(m_3+1)}{4} - 1\}$, respectively. Then $S(L_1') = S(L_2') \leq \frac{m_3(m_3+1)}{2} - 1$. By Lemma 3.6, $S(L_3) \geq \frac{m_3(m_3+1)}{2}$. This is a contradiction. Hence $i(G) \geq \frac{m_3(m_3+1)}{4} + 2$, and then $i(G) = \left\lceil \frac{m_3(m_3+1)}{4} + 2 \right\rceil$ for this case.

Case 2: $\frac{m_3(m_3+1)}{2}$ is odd.

We claim that a labeling $f : V \to \{1, 2, \ldots, m_3, \frac{m_3(m_3+1)+2}{2} - 2, \frac{m_3(m_3+1)+2}{2} - 1, \frac{m_3(m_3+1)+2}{2}, \frac{m_3(m_3+1)+2}{2} + 1\}$ is an S-magic labeling of G with $S(L_1) = S(L_2) = S(L_3) = \frac{m_3(m_3+1)}{2}$. Since m_2 and m_3 satisfy case III, $m_3 \geq 5$. Then $m_3^2 + m_3 - 6 \geq m_3^2 - 1 > 4m_3$. So, $\frac{m_3(m_3+1)+2}{4} - 2 > m_3$. Figure 3.26 shows the labeling of G with $T = \{1, 2, \ldots, m_3, \frac{m_3(m_3+1)+2}{2} - 2, \frac{m_3(m_3+1)+2}{2} - 1, \frac{m_3(m_3+1)+2}{2}, \frac{m_3(m_3+1)+2}{2} + 1\}$, and the sum the the labelings assigned to each partite is equal to $\frac{m_3(m_3+1)}{2}$.

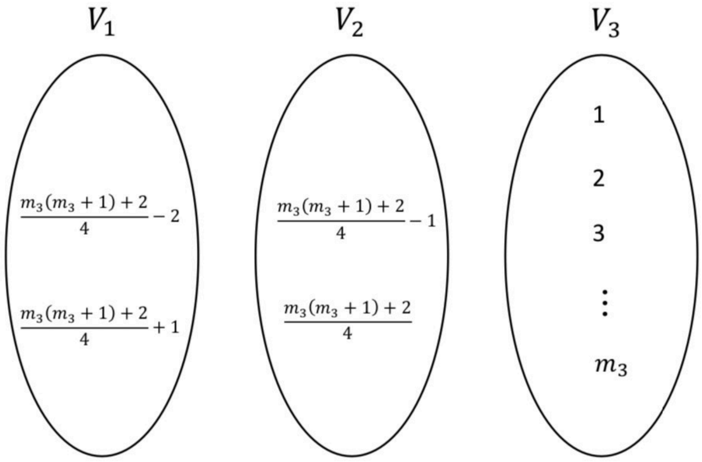

Figure 3.26: A labeling of $K_{2,2,m_3}$ with m_3 satisfies case III and $\frac{m_3(m_3+1)}{2}$ is odd with $i(G) = \frac{m_3(m_3+1)+2}{2} + 1$.

Then G is an S-magic graph. By Lemma 3.6, T has a minimum sum of elements. Then T is a minimal S-magic labeling set. By Lemma 3.7, $i(G) \geq \frac{m_3(m_3+1)+2}{4}$. Suppose $i(G) = \frac{m_3(m_3+1)+2}{4}$. There is a labeling set T_2 with $\max(T_2) = \frac{m_3(m_3+1)+2}{4}$. Then 4 maximum elements that can be in T_2 are $\frac{m_3(m_3+1)+2}{4} - 3, \frac{m_3(m_3+1)+2}{4} - 2, \frac{m_3(m_3+1)+2}{4} - 1$ and $\frac{m_3(m_3+1)+2}{4}$. Since the sum of the labelings assigned to each partite are equal, the only possible labeling sets for V_1 and V_2 are $L_1'' = \{\frac{m_3(m_3+1)+2}{4} - 3, \frac{m_3(m_3+1)+2}{4} - 1\}$ and $L_2'' = \{\frac{m_3(m_3+1)+2}{4} - 2, \frac{m_3(m_3+1)+2}{4}\}$, respectively. Then $S(L_1'') = S(L_2'') \leq \frac{m_3(m_3+1)}{2} - 2$. By Lemma 3.6, $S(L_3) \geq \frac{m_3(m_3+1)}{2}$. This is a contradiction. Hence $i(G) \geq \frac{m_3(m_3+1)+2}{4} + 1$, and then $i(G) = \frac{m_3(m_3+1)+2}{4} + 1 = \left\lceil \frac{m_3(m_3+1)+2}{4} \right\rceil + 1$ for this case. $\square$

CHAPTER IV
CONCLUSION AND SCOPE

In this thesis, we recall the concept of S-magic graph and distance magic indices of graphs. We obtain $i(G)$ for the complete r-partite graph $K_{m_1,m_2,\ldots,m_r}$ with all m_i are equal where $i = 1, 2, \ldots, r$ as follows:

Theorem 3.1. Let $m_1, m_2, \ldots, m_r$ be positive integers where $2 \leq m_1 = m_2 = \cdots = m_r$, and let $G = K_{m_1,m_2,\ldots,m_r}$ be a complete r-partite graph. If m is even, then G is an S-magic graph and $\theta(G) = 0$.

Theorem 3.4. Let $m_1, m_2, \ldots, m_r$ be positive integers where $2 \leq m_1 = m_2 = \cdots = m_r = m$, and let $G = K_{m_1,m_2,\ldots,m_r}$ be a complete r-partite graph. If m is odd, then G is an S-magic graph and $\theta(G) = \begin{cases} 0, & \text{if r is odd} \\ 1, & \text{if r is even.} \end{cases}$

Moreover, we obtain $i(G)$ for the complete tripartite graph K_{m_1,m_2,m_3} that satisfies $m_1 = 1, 2$ and $2 \leq m_2 \leq m_3$ as follows:

Theorem 3.8. For two positive integers m_2 and m_3 where $2 \leq m_2 \leq m_3$, let $G = K_{1,m_2,m_3}$ be an S-magic graph.
If G satisfies case I, then $i(G) = \frac{(m_2+m_3)(m_2+m_3+1)}{4}$.
If G satisfies case II, then $i(G) = \frac{(m_2+m_3)(m_2+m_3+1)+2}{4}$.
If G satisfies case III, then $i(G) = \frac{m_3(m_3+1)}{2}$.

Theorem 3.9. Let m_2 and m_3 be two positive integers with $2 \leq m_2 \leq m_3$. If m_2 and m_3 satisfy case I or case II and $m_2 + m_3 > 8$, then $G = K_{2,m_2,m_3}$ is an S-magic graph and

$$i(G) = \begin{cases} \lceil \frac{(m_2+m_3)(m_2+m_3+1)+4}{8} \rceil, & \text{for case I} \\ \lceil \frac{(m_2+m_3)(m_2+m_3+1)+6}{8} \rceil, & \text{for case II.} \end{cases}$$

Theorem 3.10. Let m_2 and m_3 be two positive integers with $2 \leq m_2 \leq m_3$. Suppose $G = K_{2,m_2,m_3}$ is an S-magic graph where m_2 and m_3 satisfy case I or case II, and $m_2 + m_3 \leq 8$.

$(I.)$ $m_2 + m_3 = 4$

$\{1, 2, 3, 4, 5, 6\}$ is an S-magic labeling set $K_{2,2,2}$ and $i(G) = 6$.

$(II.)$ $m_2 + m_3 = 5$

$\{1, 2, 3, 4, 5, 7, 8\}$ is an S-magic labeling set of $K_{2,2,3}$ and $i(G) = 8$.

$(III.)$ $m_2 + m_3 = 6$

$\{1,2,3,4,5,6,7,8\}$ is an S-magic labeling set of $K_{2,2,4}, K_{2,3,3}$ and $i(G) = 8$.

$(IV.)$ $m_2 + m_3 = 7$

$\{1,2,3,4,5,6,7,8,9\}$ is an S-magic labeling set of $K_{2,2,5}, K_{2,3,4}$ and $i(G) = 9$.

$(V.)$ $m_2 + m_3 = 8$

$\{1,2,3,4,5,6,7,8,10,11\}$ is an S-magic labeling set of $K_{2,3,5}, K_{2,4,4}$ and $i(G) = 11$.

Theorem 3.12. Let m_2 and m_3 be two positive integers and $3 \leq m_2 \leq m_3$. If m_2 and m_3 satisfy case III, then $G = K_{2,m_2,m_3}$ is an S-magic graph and $i(G) = \left\lceil \frac{m_3(m_3+1)+2}{4} \right\rceil$.

Theorem 3.13. Let m_2 and m_3 be two positive integers and $3 \leq m_3$. If m_3 satisfies case III, then $G = K_{2,2,m_3}$ is an S-magic graph and $i(G) = \left\lceil \frac{m_3(m_3+1)+2}{4} \right\rceil + 1$.

The following problems naturally arise.

Problem 4.1 For complete tripartite graph K_{m_1,m_2,m_3} with $m_1 \geq 3$, determine $i(G)$.

Problem 4.2 For a complete tripartite $G = K_{2,m_2,m_2}$ with $2 \leq m_2 \leq m_3$, dertermine $M(G)$.

REFERENCES

[1] A. Godinho and T. Singh, *On S-magic graphs*, Electronic Notes in Discrete Mathematics, **48**, (2015) 267–273.

[2] M. Miller, C. Rodger and R. Simanjuntak, *Distance magic labelings of graphs*, Australas. J. Combin., **28**, (2003), 305–315.

[3] K.A. Sugeng, D. Froncek, M. Miller, J. Ryan and J. Walker, *On distance magic labeling of graphs*, J. Combin. Math. Combin. Comput., **71**, (2009), 39–48. Electron. Notes Discrete Mathematics , **48**, (2015), 267–273.

[4] A. Godinho, T. Singh and S. Arumugam, *The distance magic index of a graph*,Discussiones Mathematicae Graph Theorey , **38**, (2018), 135–142.

[5] S. Arumugam, N. Kamatchi and G.R. Vijayakumar, *On the uniqueness of D-vertex magic constant*, Discussions Mathematicae Graph Theorey, **34**, (2014), 279–286.

[6] G. Chartrand and L. Lesniak, *Graph and Digraphs*, (4th ed.), Chapman and hall,CRC, 2005.

[7] V. Vilfred, Σ-*labelled graph and circulant Graphs*, Ph.D. Thesis, University of Kerala, Trivandrum, India, 1994.

VITA

Name	:	Ms. Sararat Numai
Date of Birth	:	18 April 1997
Place of Birth	:	Nan, Thailand
Education	:	B.Sc. (Mathematics), (First Class Honors), Chiang Mai University, 2019
Scholarship	:	Development and Promotion of Science and Technology Talents Project (DPST)