ผลเฉลยแบบยอมได้ชนิดพิเศษและผลเฉลยแบบควบคุมชนิดพิเศษของระบบสมการเชิงเส้นแบบช่วง

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SPECIAL TYPES OF TOLERANCE AND CONTROL SOLUTIONS TO SYSTEM OF INTERVAL LINEAR EQUATIONS

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A Thesis Submitted in Partial Fulfillment of the Requirements

for the Degree of Master of Science Program in Applied Mathematics and Computational Science

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วิทยานิพนธ์ฉบับนี้ศึกษาระบบสมการเชิงเส้นแบบช่วงในรูปแบบ $\mathbb{A}x = \mathbb{b}$ ซึ่งประกอบด้วยเมทริกซ์สัมประสิทธิ์และเวกเตอร์ด้านขวามือในรูปแบบของช่วง โดยที่ผลเฉลยแต่ละประเภทของระบบสมการเชิงเส้นแบบช่วงนี้ถูกสร้างขึ้นเพื่อให้เหมาะสมกับสถานการณ์จริงที่แตกต่างกัน เช่น ผลเฉลยแบบยอมได้และผลเฉลยแบบควบคุม เป็นต้น เราสนใจการตีความของระบบสมการที่ให้ความสำคัญกับขอบของช่วงสัมประสิทธิ์และนำมารวมกับการตีความของผลเฉลยแบบยอมได้และผลเฉลยแบบควบคุม เพื่อนำเสนอผลเฉลยประเภทใหม่ของระบบสมการเชิงเส้นแบบช่วงในรูปแบบ $\mathbb{A}x = \mathbb{b}$ ได้แก่ ผลเฉลยแบบยอมได้ชนิดพิเศษทางซ้าย, ผลเฉลยแบบยอมได้ชนิดพิเศษทางขวา, ผลเฉลยแบบควบคุมชนิดพิเศษทางซ้ายและผลเฉลยแบบควบคุมชนิดพิเศษทางขวา ยิ่งไปกว่านั้น เรายังศึกษาคุณลักษณะเฉพาะของผลเฉลยประเภทใหม่ดังกล่าวและนำมาประยุกต์ใช้ในการแก้ปัญหาการจัดตารางสอนที่มีเงื่อนไขข้อจำกัดภาระงานแบบช่วง

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WARINTORN PONGSUMRANKUL : SPECIAL TYPES OF TOLERANCE AND CONTROL SOLUTIONS TO SYSTEM OF INTERVAL LINEAR EQUATIONS. ADVISOR : ASSOC. PROF. PHANTIPA THIPWIWATPOTJANA, Ph.D., **??** pp.

In this thesis, we study a system of interval linear equations of the form $\mathbb{A}x = \mathbb{b}$ which includes an interval coefficient matrix and an interval right hand side vector. There are many types of solutions of this system which serve the different semantics in real-life situations such as tolerance and control solutions. We pay attention to the semantics that some boundaries of intervals are important. Moreover, we combine these semantics together with the semantics of tolerance and control solutions in order to present the new types of solutions to $\mathbb{A}x = \mathbb{b}$, called special tolerance left, special tolerance right, special control left and special control right solutions. In addition, we study the characteristics of the new solution and apply them to solve the course assignment problem with interval workload constraints.

Department :	Mathematics and Computer Science	Student's Signature	....
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CONTENTS

CHAPTER I

INTRODUCTION

A system of interval linear equations $\mathbb{A}x = \mathbb{b}$ comprises an interval matrix $\mathbb{A}$ and an interval column vector $\mathbb{b}$ as coefficients and right hand side terms, respectively. Since many real-life problems can be written in the form of this system, there are many studies about $\mathbb{A}x = \mathbb{b}$. For instance, in 2015, Lodwick and Dubois [?] studied the extension of the interval linear equations system with its fuzzy coefficients. In 2018, Garajova et al. [?] proposed the general transformations and their effects on the interval system of the interval linear programming in order to solve it.

By the different interpretations of the equal sign of the system $\mathbb{A}x = \mathbb{b}$, there are many types of its solutions such as weak, tolerance and control solutions, which are showed in [?] and [?]-[?]. In 1964, Oettli and Prager [?] provided the characteristic to approximate the exact solution of an interval linear equations system. In 1994, Shary [?] studied the tolerance problem for an interval linear system in order to expand the solvability theory for this problem in term of its required characteristics. Moreover, Shary [?] also provided the adequate conditions for a control solution in 1997. There are solution types combining the concept of the multiple solution types to suit the real-life problems in various situations. For example, Tian et al.[?] presented a solution type which is the combination of the semantics of tolerance and control solutions, called a tolerance-control solution. Leela-apiradee et al.[?] combined the ideas of tolerance and control solutions with the idea of localized solution to propose two new solution types of an interval linear equations system, called tolerance-localized and control-localized solutions.

In this thesis, we consider the situation where some boundaries of intervals are important. Consider an equation $[\underline{a}, \overline{a}]^T x = [\underline{a^T x}, \overline{a^T x}] = [\underline{b}, \overline{b}]$ when the semantic is that $\underline{b}$ is very important. Then $a^T x$ has to be at least $\underline{b}$ for all $a \in [\underline{a}, \overline{a}]$. This semantic leads to the special left concept. On the other hand, we create the special right concept from the semantic of the above equation when $\overline{b}$ is very important, thus $a^T x$ needs to be at most

$\overline{b}$ for all $a \in [\underline{a}, \overline{a}]$. Hence, we combine the special left and special right concepts together with the semantics of tolerance and control solutions in order to present the four new types of solutions to $\mathbb{A}x = \mathbb{b}$, called a special tolerance left, a special tolerance right, a special control left and a special control right solutions. The characteristics of these new types of solutions are also studied.

In [?] and [?], Thipwiwatpotjana et al. explained how to use the idea of the interval linear equations system in a course assignment problem. Moreover, Leela-apiradee et al. [?] used the concept of a tolerance-localized solution to deal with a course assignment problem to get the solution under tolerance-localized semantics. We carry on using a course assignment problem as an application of a special tolerance left solution when the interval workload constraint behaves like special tolerance semantics.

The preliminary knowledge of an interval linear equations system along with the definitions and characteristics of some solution types of the system are described in Chapter 2. In Chapters 3 and 4, we present the special types of tolerance and control solutions and their properties. For Chapter 5, we illustrate a course assignment problem as an application of a special tolerance left solution. In the last chapter, we propose the conclusion of this thesis.

CHAPTER II

BACKGROUND KNOWLEDGE

In this chapter, we present the preliminary knowledge of the interval linear equations system together with the definitions and characteristics of weak, tolerance and control solution types of the system.

2.1 The interval linear equations system

Let m and n be positive integers. The set of all (interval) matrices of size $m \times n$ over $\mathbb{R}$ and the set of all column (interval) vectors of size m over $\mathbb{R}$ are defined by $\mathbb{R}^{m\times n}(\mathbb{IR}^{m\times n})$ and $\mathbb{R}^m(\mathbb{IR}^m)$, respectively. An interval matrix $\mathbb{A} \in \mathbb{IR}^{m\times n}$ is defined by

$$\mathbb{A} = [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n} : \underline{A} \leq A \leq \overline{A}\}$$

where $\underline{A}$ and $\overline{A}$ are the left and right boundaries matrices of $\mathbb{A}$, respectively, and can be written as follows:

$$\underline{A} = \begin{bmatrix} \underline{a}_{11} & \underline{a}_{12} & \cdots & \underline{a}_{1n} \\ \underline{a}_{21} & \underline{a}_{22} & \cdots & \underline{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{a}_{m1} & \underline{a}_{m2} & \cdots & \underline{a}_{mn} \end{bmatrix} \text{ and } \overline{A} = \begin{bmatrix} \overline{a}_{11} & \overline{a}_{12} & \cdots & \overline{a}_{1n} \\ \overline{a}_{21} & \overline{a}_{22} & \cdots & \overline{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a}_{m1} & \overline{a}_{m2} & \cdots & \overline{a}_{mn} \end{bmatrix}.$$

An interval column vector $\mathbb{b} \in \mathbb{IR}^m$ is defined by

$$\mathbb{b} = [\underline{b}, \overline{b}] = \{b \in \mathbb{R}^m : \underline{b} \leq b \leq \overline{b}\}$$

where $\underline{b}$ and $\overline{b}$ are the left and right boundaries vectors of the interval vector $\mathbb{b}$, respectively, and can be written as follows:

$$\underline{b} = (\underline{b}_1, \underline{b}_2, \cdots, \underline{b}_m)^T \text{ and } \overline{b} = (\overline{b}_1, \overline{b}_2, \cdots, \overline{b}_m)^T.$$

The interval linear equations system of the form

$$\mathbb{A}x = \mathbb{b}$$

can be written as follows:

$$\begin{aligned}
[\underline{a}_{11}, \overline{a}_{11}]x_1 + [\underline{a}_{12}, \overline{a}_{12}]x_2 + \cdots + [\underline{a}_{1n}, \overline{a}_{1n}]x_n &= [\underline{b}_1, \overline{b}_1] \\
[\underline{a}_{21}, \overline{a}_{21}]x_1 + [\underline{a}_{22}, \overline{a}_{22}]x_2 + \cdots + [\underline{a}_{2n}, \overline{a}_{2n}]x_n &= [\underline{b}_2, \overline{b}_2] \\
&\vdots \\
[\underline{a}_{m1}, \overline{a}_{m1}]x_1 + [\underline{a}_{m2}, \overline{a}_{m2}]x_2 + \cdots + [\underline{a}_{mn}, \overline{a}_{mn}]x_n &= [\underline{b}_m, \overline{b}_m].
\end{aligned}$$

In addition, for each row index $i = 1, 2, \ldots, m$

$$[\underline{a}_{i1}, \overline{a}_{i1}]x_1 + [\underline{a}_{i2}, \overline{a}_{i2}]x_2 + \cdots + [\underline{a}_{in}, \overline{a}_{in}]x_n = [\underline{b}_i, \overline{b}_i],$$

We define it as $(\mathbb{A}x)_i = (\mathbb{b})_i$, for short. Next, the center matrix and vector are defined by

$$A_c = \frac{1}{2}(\overline{A} + \underline{A}) \text{ and } b_c = \frac{1}{2}(\overline{b} + \underline{b}),$$

respectively, and the radius matrix and vector are defined by

$$\Delta = \frac{1}{2}(\overline{A} - \underline{A}) \text{ and } \delta = \frac{1}{2}(\overline{b} - \underline{b}),$$

respectively. Hence $\mathbb{A}$ and $\mathbb{b}$ can be written in the form:

$$\mathbb{A} = [A_c - \Delta, A_c + \Delta] \text{ and } \mathbb{b} = [b_c - \delta, b_c + \delta],$$

respectively.

Since the inequality $\underline{A}x \leq \overline{A}x$ is not always true, the set $\mathbb{A}x$ cannot be written by $[\underline{A}x, \overline{A}x]$. It can be represented by an explicit interval as shown in the theorem below.

Theorem 2.1. *(See [?]) Let* $\mathbb{A} = [\overline{A}, \underline{A}] \in \mathbb{IR}^{m\times n}$ *and* $x \in \mathbb{R}^n$*. Then*

$$\mathbb{A}x = [A_c x - \Delta|x|, A_c x + \Delta|x|]$$

where $|x|$ *is a vector which has the component* $|x_j|$ *for each* $j \in \{1, 2, \ldots, n\}$.

From Theorem **??**, we define $\underline{Ax} = A_c x - \Delta|x|$ and $\overline{Ax} = A_c x + \Delta|x|$. Moreover, $(\underline{Ax})_i = (A_c x)_i - (\Delta|x|)_i$ and $(\overline{Ax})_i = (A_c x)_i + (\Delta|x|)_i$ for all $i \in \{1, 2, \ldots, m\}$.

2.2 Some definitions and properties of solutions to system of interval linear equations

For the interval linear equations system of the form $\mathbb{A}x = \mathbb{b}$, there are many types of its solutions which serve the different semantics in real-life situations. In this section, we present the definitions and the characterizations of the solution types of the interval linear equations system, called weak, tolerance and control solutions.

Definition 2.1. (See [?]) A vector $x \in \mathbb{R}^n$ is called a weak solution of $\mathbb{A}x = \mathbb{b}$ if it satisfies $Ax = b$ for some $A \in \mathbb{A}, b \in \mathbb{b}$.

Theorem 2.2. *(See [?]) A vector* $x \in \mathbb{R}^n$ *is called a weak solution of* $\mathbb{A}x = \mathbb{b}$ *if and only if* x *satisfies*

$$|A_c x - b_c| \leq \Delta|x| + \delta.$$

In 2013, Li Tian et al. presented the special types of the weak solutions such as the tolerance solutions and control solutions which are shown in the definitions below.

Definition 2.2. (See [?]) A vector $x \in \mathbb{R}^n$ is called a tolerance solution of $\mathbb{A}x = \mathbb{b}$ if for each $A \in \mathbb{A}$ there exists $b \in \mathbb{b}$ such that $Ax = b$.

Lemma 2.1. *(See [?]) The set of all tolerance solutions of* $\mathbb{A}x = \mathbb{b}$*, defined by* $\sum_{\forall\exists}(\mathbb{A}, \mathbb{b})$*, can be written as follows:*

$$\sum_{\forall\exists}(\mathbb{A}, \mathbb{b}) = \{x \in \mathbb{R}^n : \mathbb{A}x \subseteq \mathbb{b}\}.$$

For $x \in \mathbb{R}$, the set of all tolerance solutions of $\mathbb{A}x = \mathbb{b}$ shows in Figure **??**.

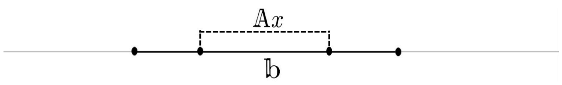

Figure 2.1: $\mathbb{A}x \subseteq \mathbb{b}$, where $x \in \mathbb{R}$.

Theorem 2.3. *(See [?]) A vector $x \in \mathbb{R}^n$ is a tolerance solution of $\mathbb{A}x = \mathbb{b}$ if and only if x satisfies*

$$|A_c x - b_c| \leq -\Delta|x| + \delta.$$

Remark 2.1. A vector $x \in \mathbb{R}^n$ satisfies $|A_c x - b_c| \leq -\Delta|x| + \delta$ if and only if $\underline{Ax} \geq \underline{b}$ and $\overline{Ax} \leq \overline{b}$.

Proof. Suppose that $x \in \mathbb{R}^n$ which satisfies

$$|A_c x - b_c| \leq -\Delta|x| + \delta.$$

Then

$$\Delta|x| - \delta \leq A_c x - b_c \leq -\Delta|x| + \delta.$$

Hence,

$$\underline{Ax} = A_c x - \Delta|x| \geq b_c - \delta = \underline{b} \text{ and } \overline{Ax} = A_c x + \Delta|x| \leq b_c + \delta = \overline{b}.$$

Reverse this proof to complete the proof of this Remark. □

Definition 2.3. (See [?]) A vector $x \in \mathbb{R}^n$ is called a control solution of $\mathbb{A}x = \mathbb{b}$ if for each $b \in \mathbb{b}$ there exists $A \in \mathbb{A}$ such that $Ax = b$.

Lemma 2.2. *(See [?]) The set of all control solutions of $\mathbb{A}x = \mathbb{b}$, defined by $\sum_{\exists\forall}(\mathbb{A}, \mathbb{b})$ can be written as follows:*

$$\sum_{\exists\forall}(\mathbb{A}, \mathbb{b}) = \{x \in \mathbb{R}^n : \mathbb{A}x \supseteq \mathbb{b}\}.$$

For $x \in \mathbb{R}$, the set of all control solutions of $\mathbb{A}x = \mathbb{b}$ is shown in Figure **??**.

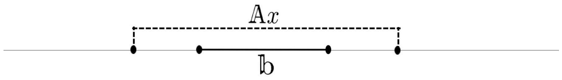

Figure 2.2: $\mathbb{A}x \supseteq \mathbb{b}$, where $x \in \mathbb{R}$.

Theorem 2.4. *(See [?]) A vector $x \in \mathbb{R}^n$ is a control solution of $\mathbb{A}x = \mathbb{b}$ if and only if x satisfies*

$$|A_c x - b_c| \leq \Delta|x| - \delta.$$

CHAPTER III

SPECIAL TOLERANCE LEFT AND SPECIAL CONTROL LEFT SOLUTIONS

In this chapter, we present the special left concept and combine it with the concept of tolerance and control solutions, respectively.

3.1 Special left concept

Let $\mathbb{A} = [\underline{a}, \overline{a}]$ be an interval matrix which consists of the column vector $[\underline{a}_{ij}, \overline{a}_{ij}]$ for each column index $j = 1, 2, \ldots, n$ and row index $i = 1, 2, \ldots, m$. Consider an interval system of linear equations

$$\mathbb{A}x = [\underline{a}, \overline{a}]^T x = [\underline{a^T x}, \overline{a^T x}] = [\underline{b}, \overline{b}] = \mathbb{b}.$$

For the semantics of this equation that $\underline{b}$ is very important and a solution x to the system must serve the requirement that no matter a realization "a" in the interval $[\underline{a}, \overline{a}]$ is, $a^T x$ needs to be at least $\underline{b}$, then we have

$$\underline{a^T x} = \underline{b}.$$

Next, we present the combination of the special left concept and the concept of tolerance solution in the next subsection.

3.1.1 Special tolerance left solution

Definition 3.1. A vector $x \in \mathbb{R}^n$ is called a *special tolerance left solution* of $\mathbb{A}x = \mathbb{b}$ if x is a tolerance solution to $(\mathbb{A}x)_i = \mathbb{b}_i$ and $(\underline{Ax})_i = \underline{b}_i$, for all $i \in \{1, 2, \ldots, m\}$.

Theorem 3.1. *A vector $x \in \mathbb{R}^n$ is a special tolerance left solution of $\mathbb{A}x = \mathbb{b}$ if and only if it satisfies*

$$\overline{Ax} \leq \overline{b} \text{ and } \underline{Ax} = \underline{b}. \tag{3.1}$$

Proof. Suppose that x is a special tolerance left solution of $\mathbb{A}x = \mathbb{b}$. Let $i \in \{1, 2, \ldots, m\}$. Then

$$(\mathbb{A}x)_i \subseteq \mathbb{b}_i \text{ and } (\underline{Ax})_i = \underline{b}_i. \tag{3.2}$$

The statement (**??**) implies

$$[(\underline{Ax})_i, (\overline{Ax})]_i = (\mathbb{A}x)_i \subseteq \mathbb{b}_i = [\underline{b}_i, \overline{b}_i].$$

Thus,

$$\underline{b}_i = (\underline{Ax})_i \leq (\overline{Ax})_i \leq \overline{b}_i.$$

Therefore, (**??**) holds for all $i \in \{1, 2, \ldots, m\}$.

Conversely, let x satisfy the condition (**??**). Then

$$\overline{Ax} \leq \overline{b} \tag{3.3}$$

and

$$\underline{Ax} = \underline{b}. \tag{3.4}$$

Let $i \in \{1, 2, \ldots, m\}$. From (**??**) and (**??**), we have

$$\underline{b}_i = (\underline{Ax})_i \leq (\overline{Ax})_i \leq \overline{b}_i.$$

Then

$$[(\underline{Ax})_i, (\overline{Ax})_i] = (\mathbb{A}x)_i \subseteq \mathbb{b}_i = [\underline{b}_i, \overline{b}_i].$$

Therefore, x satisfying (**??**) is a special tolerance left solution of $\mathbb{A}x = \mathbb{b}$. The proof is completed. $\square$

By Theorem **??**, a special tolerance left solution $x \in \mathbb{R}^n$ of $\mathbb{A}x = \mathbb{b}$ satisfies

$$\overline{Ax} \leq \overline{b}, \tag{3.5}$$

$$\underline{Ax} = \underline{b}, \tag{3.6}$$

and from Theorem **??**, Inequality (**??**) and Equation (**??**) can be transformed as follows:

$$A_c x + \Delta|x| \leq \overline{b}, \tag{3.7}$$

$$A_c x - \Delta|x| = \underline{b}, \tag{3.8}$$

respectively. By adding slack variable $s = (s_1, s_2, \ldots, s_m) \geq (0, 0, \ldots, 0)$ to (**??**), the system becomes Equation (**??**)

$$A_c x + \Delta|x| + s = \overline{b}. \tag{3.9}$$

The Equation system (**??**) and (**??**) involve $|x|$. So it is difficult to present the existence of the solutions. If we transform Equations (**??**) and (**??**) as follows:

$$A_c(x^+ - x^-) + \Delta(x^+ + x^-) + s = \overline{b} \tag{3.10}$$

$$A_c(x^+ - x^-) - \Delta(x^+ + x^-) = \underline{b}, \tag{3.11}$$

where

$$x_i^+ = \begin{cases} x_i & \text{if } x_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$x_i^- = \begin{cases} -x_i & \text{if } x_i < 0 \\ 0 & \text{otherwise,} \end{cases}$$

it should not be hard to see from (**??**) and (**??**) that $|x| = x^+ + x^-$ if and only if x^+ and x^- are nonnegative and $x^+ \cdot x^- = 0$. We write (**??**) and (**??**) with the restrictions on variables x^+, x^- and s in terms of $\overline{A}, \underline{A}, \overline{b}$ and $\underline{b}$ as follows:

$$(A_c + \Delta)x^+ - (A_c - \Delta)x^- + s = \overline{A}x^+ - \underline{A}x^- + s = \overline{b}, \tag{3.12}$$

$$(A_c - \Delta)x^+ - (A_c + \Delta)x^- = \underline{A}x^+ - \overline{A}x^- = \underline{b} \tag{3.13}$$

where $x^+, x^-, s \geq 0$ and $x^+ \cdot x^- = 0$. Now consider the system

$$\begin{bmatrix} \overline{A} & -\underline{A} & I_m \\ \underline{A} & -\overline{A} & 0_m \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix} = \begin{bmatrix} \overline{b} \\ \underline{b} \end{bmatrix}, \tag{3.14}$$

where I_m is the identity matrix of size $m \times m$, 0_m is the zero matrix of size $m \times m$. Therefore, the linear equations system (**??**) has at least one solution which satisfy $x^+, x^- \geq 0, x^+ \cdot x^- = 0$ and $s \geq 0$ if and only if there exist at least one special tolerance left solution of $\mathbb{A}x = \mathbb{b}$. The linear equations system (**??**) has at least one solution (does not need to be nonnegative) when $Rank(\mathbf{A}) = 2m$ where $\mathbf{A} = \begin{bmatrix} \overline{A} & -\underline{A} & I_m \\ \underline{A} & -\overline{A} & 0_m \end{bmatrix}$, based on the well-known theorem in linear algebra below. However, we can consider only the rank of the matrix $\begin{bmatrix} \overline{A} & \underline{A} \end{bmatrix}$ to check the existence of the solution of (**??**), which shows in Lemma **??**.

Theorem 3.2. *(See [**?**]) Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, where $m \leq n$. A system $Ax = b$ has at least one solution x for every b if and only if the columns of A span $\mathbb{R}^m$, i.e., $Rank(A) = m$. In this case, there exists right-inverse $C \in \mathbb{R}^{n \times m}$ such that $AC = I_m$, the identity matrix of order m.*

Lemma 3.1. *Let $\overline{A}$ and $\underline{A}$ be the matrices of size $m \times n$ where $m \leq n$. $Rank(\begin{bmatrix} \overline{A} & \underline{A} \end{bmatrix}) = m$ if and only if $Rank(\mathbf{A}) = 2m$ where $\mathbf{A} = \begin{bmatrix} \overline{A} & -\underline{A} & I_m \\ \underline{A} & -\overline{A} & 0_m \end{bmatrix}$, when I_m is the $m \times m$ identity matrix and 0_m is the $m \times m$ zero matrix.*

Proof. Assume that $Rank(\begin{bmatrix} \overline{A} & \underline{A} \end{bmatrix}) = m$. Then, all of m rows of the matrix $\begin{bmatrix} \underline{A} & -\overline{A} & 0_m \end{bmatrix}$ are linearly independent. Also, since $\begin{bmatrix} \overline{A} & -\underline{A} & I_m \end{bmatrix}$ has the last m columns as the identity matrix of size m, it implies that all m rows of this matrix are linearly independent and they are also linearly independent to all m rows of the matrix $\begin{bmatrix} \underline{A} & -\overline{A} & 0_m \end{bmatrix}$. Therefore, $\begin{bmatrix} \overline{A} & -\underline{A} & I_m \\ \underline{A} & -\overline{A} & 0_m \end{bmatrix}$ has $2m$ linearly independent rows. Hence, $Rank(\mathbf{A}) = 2m$ where $\mathbf{A} = \begin{bmatrix} \overline{A} & -\underline{A} & I_m \\ \underline{A} & -\overline{A} & 0_m \end{bmatrix}$.

Conversely, suppose that $Rank(\mathbf{A}) = 2m$. Then, all $2m$ rows of $\mathbf{A}$ are linearly independent. Moreover, all m rows of $\begin{bmatrix} \underline{A} & -\overline{A} & 0_m \end{bmatrix}$ are linearly independent. Therefore, $Rank(\begin{bmatrix} \underline{A} & -\overline{A} \end{bmatrix}) = m$. It implies that $Rank(\begin{bmatrix} \overline{A} & \underline{A} \end{bmatrix}) = m$. □

Remark 3.1. Note that if $Rank(\underline{A}) = m$ or $Rank(\overline{A}) = m$, then $Rank(\begin{bmatrix} \overline{A} & \underline{A} \end{bmatrix}) = m$. Therefore, if $Rank(\underline{A}) = m$ (or $Rank(\overline{A}) = m$) is true, it suffices to say that there exists at least one solution of the linear equations system (**??**).

Next, we present the combination of the special left concept and the concept of control solution in the next subsection.

3.1.2 Special control left solution

Definition 3.2. A vector $x \in \mathbb{R}^n$ is called a *special control left solution* of $\mathbb{A}x = \mathbb{b}$ if x is a control solution to $(\mathbb{A}x)_i = \mathbb{b}_i$ and $(\underline{Ax})_i = \underline{b}_i$, for all $i \in \{1, 2, \ldots, m\}$.

Theorem 3.3. *A vector $x \in \mathbb{R}^n$ is a special control left solution of $\mathbb{A}x = \mathbb{b}$ if and only if it satisfies*

$$\overline{Ax} \geq \overline{b} \text{ and } \underline{Ax} = \underline{b}. \tag{3.15}$$

Proof. Suppose that x is a special control left solution of $\mathbb{A}x = \mathbb{b}$. Let $i \in \{1, 2, \ldots, m\}$. Then

$$(\mathbb{A}x)_i \supseteq \mathbb{b}_i \text{ and } (\underline{Ax})_i = \underline{b}_i. \tag{3.16}$$

The statement (**??**) implies

$$[\underline{Ax}_i, \overline{Ax}_i] = (\mathbb{A}x)_i \supseteq \mathbb{b}_i = [\underline{b}_i, \overline{b}_i].$$

Thus,

$$(\underline{Ax})_i = \underline{b}_i \leq \overline{b}_i \leq (\overline{Ax})_i.$$

Therefore, (**??**) holds for all $i \in \{1, 2, \ldots, m\}$.

Conversely, let x satisfy the condition (**??**). Then

$$\overline{Ax} \geq \overline{b} \tag{3.17}$$

and

$$\underline{Ax} = \underline{b}. \tag{3.18}$$

Let $i \in \{1, 2, \ldots, m\}$. From (**??**) and (**??**), we have

$$(\underline{Ax})_i = \underline{b}_i \leq \overline{b}_i \leq (\overline{Ax})_i.$$

Then

$$[(\underline{Ax})_i, (\overline{Ax})_i] = (\mathbb{A}x)_i \supseteq (\mathbb{b})_i = [\underline{b}_i, \overline{b}_i].$$

Therefore, x satisfying (**??**) is a special control left solution of $\mathbb{A}x = \mathbb{b}$. The proof is completed. □

We are also able to transform (**??**) to

$$\overline{A}x^+ - \underline{A}x^- \geq \overline{b} \tag{3.19}$$

and

$$\underline{A}x^+ - \overline{A}x^- = \underline{b}, \tag{3.20}$$

in a similar fashion as the approach to the systems (**??**) and (**??**). Then we can write

Inequality (**??**) and Equation (**??**) into the linear equations system as follows:

$$\begin{bmatrix} \overline{A} & -\underline{A} & -I_m \\ \underline{A} & -\overline{A} & 0_m \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix} = \begin{bmatrix} \overline{b} \\ \underline{b} \end{bmatrix}, \tag{3.21}$$

where I_m is the $m \times m$ identity matrix, 0_m is the $m \times m$ zero matrix and s is the vector of the surplus variables. Therefore, the linear equations system (**??**) has at least one solution which satisfy $x^+, x^- \geq 0, x^+ \cdot x^- = 0$ and $s \geq 0$ if and only if there exist at least one special control left solution of $\mathbb{A}x = \mathbb{b}$.

In the same manner as in Lemma **??**, we can consider only the presence of the full row rank of the matrix $\begin{bmatrix} \overline{A} & \underline{A} \end{bmatrix}$, or the condition that $Rank(\underline{A}) = m$ or $Rank(\overline{A}) = m$ to make sure that the linear equations system (**??**) has at least one solution.

CHAPTER IV

SPECIAL TOLERANCE RIGHT AND SPECIAL CONTROL RIGHT SOLUTIONS

In this chapter, we present the special right concept and combine it with the concept of tolerance and control solutions, respectively. We design the writing pattern of this chapter in the same fashion as in Chapter 3.

4.1 Special right concept

Consider an interval system of linear equations

$$\mathbb{A}x = [\underline{a}, \overline{a}]^T x = [\underline{a^T x}, \overline{a^T x}] = [\underline{b}, \overline{b}] = \mathbb{b}.$$

For the semantics of this equation that (i) $\overline{b}$ is very important and (ii) a solution x to the system must serve the requirement that no matter a realization "a" in the interval $[\underline{a}, \overline{a}]$ is, $a^T x$ needs to be at most $\overline{b}$, then we have

$$\overline{a^T x} = \overline{b}.$$

Next, we present the combination of the special right concept and the concept of tolerance solution in the next subsection.

4.1.1 Special tolerance right solution

Definition 4.1. A vector $x \in \mathbb{R}^n$ is called a *special tolerance right solution* of $\mathbb{A}x = \mathbb{b}$ if x is a tolerance solution to $(\mathbb{A}x)_i = \mathbb{b}_i$ and $(\overline{Ax})_i = \overline{b}_i$, for all $i \in \{1, 2, \cdots, m\}$.

Theorem 4.1. *A vector $x \in \mathbb{R}^n$ is a special tolerance right solution of $\mathbb{A}x = \mathbb{b}$ if and* *only if it satisfies*

$$\overline{Ax} = \overline{b} \text{ and } \underline{Ax} \geq -\underline{b}. \tag{4.1}$$

Proof. Suppose that x is a special tolerance right solution of $\mathbb{A}x = \mathbb{b}$. Let $i \in \{1, 2, \cdots, m\}$. Then

$$(\mathbb{A}x)_i \subseteq \mathbb{b}_i \text{ and } (\overline{Ax})_i = \overline{b}_i. \tag{4.2}$$

The statement (**??**) implies

$$[(\underline{Ax})_i, (\overline{Ax})_i] = (\mathbb{A}x)_i \subseteq (\mathbb{b})_i = [\underline{b}_i, \overline{b}_i].$$

Thus,

$$\underline{b}_i \leq (\underline{Ax})_i \leq (\overline{Ax})_i = \overline{b}_i.$$

Therefore, (**??**) holds for all $i \in \{1, 2, \ldots, m\}$.

Conversely, let x satisfy the condition (**??**). Then

$$\underline{Ax} \geq \underline{b} \tag{4.3}$$

and

$$\overline{Ax} = \overline{b}. \tag{4.4}$$

Let $i \in \{1, 2, \ldots, m\}$. From (**??**) and (**??**), we have

$$\underline{b}_i \leq (\underline{Ax})_i \leq (\overline{Ax})_i = \overline{b}_i.$$

Then

$$[(\underline{Ax})_i, (\overline{Ax})_i] = (\mathbb{A}x)_i \subseteq (\mathbb{b})_i = [\underline{b}_i, \overline{b}_i].$$

Therefore, x satisfying (**??**) is a special tolerance right solution of $\mathbb{A}x = \mathbb{b}$. The proof is completed. □

We are also able to transform (??) to

$$\overline{A}x^+ - \underline{A}x^- = \overline{b} \tag{4.5}$$

and

$$\underline{A}x^+ - \overline{A}x^- \geq \underline{b}, \tag{4.6}$$

in the similar fashion as the approach to the systems (??) and (??). Then we can form Inequality (??) and Equation (??) into the linear equations system as follows:

$$\begin{bmatrix} \overline{A} & -\underline{A} & 0_m \\ \underline{A} & -\overline{A} & -I_m \end{bmatrix} \begin{bmatrix} x^+ \\ x^- \\ s \end{bmatrix} = \begin{bmatrix} \overline{b} \\ \underline{b} \end{bmatrix}, \tag{4.7}$$

where I_m is the $m \times m$ identity matrix, 0_m is the $m \times m$ zero matrix and s is the vector of the surplus variables. Therefore, the linear equations system (??) has at least one solution which satisfy $x^+, x^- \geq 0, x^+ \cdot x^- = 0$ and $s \geq 0$ if and only if there exist at least one special tolerance right solution of $\mathbb{A}x = \mathbb{b}$.

Using the same idea from Lemma ??, the restriction that $Rank(\begin{bmatrix} \overline{A} & \underline{A}j \end{bmatrix}) = m$ can be used to confirm that the linear equations system (??) has at least one solution. Moreover, if the system (??) satisfies the restriction that $Rank(\underline{A}) = m$ or $Rank(\overline{A}) = m$, which makes it also satisfies the condition that $Rank(\begin{bmatrix} \overline{A} & \underline{A} \end{bmatrix}) = m$, then it also confirms the above result in (??) as well.

Next, we present the combination of the special right concept and the concept of control solution in the next subsection.

4.1.2 Special control right solution

Definition 4.2. A vector $x \in \mathbb{R}^n$ is called a *special control right solution* of $\mathbb{A}x = \mathbb{b}$ if x is a control solution to $(\mathbb{A}x)_i = \mathbb{b}_i$ and $(\overline{Ax})_i = \overline{b}_i$, for all $i \in \{1, 2, \ldots, m\}$.

Theorem 4.2. *A vector $x \in \mathbb{R}^n$ is a special control right solution of $\mathbb{A}x = \mathbb{b}$ if and only* *if it satisfies*

$$\underline{A}x \leq \underline{b} \text{ and } \overline{A}x = \overline{b}. \tag{4.8}$$

Proof. Suppose that x is a special control right solution of $\mathbb{A}x = \mathbb{b}$. Let $i \in \{1, 2, \ldots, m\}$. Then

$$(\mathbb{A}x)_i \supseteq \mathbb{b}_i \text{ and } (\overline{A}x)_i = \overline{b}_i. \tag{4.9}$$

The statement (**??**) implies

$$[(\underline{A}x)_i, (\overline{A}x)_i] = (\mathbb{A}x)_i \supseteq (\mathbb{b})_i = [\underline{b}_i, \overline{b}_i].$$

Thus,

$$(\underline{A}x)_i \leq \underline{b}_i \leq \overline{b}_i = (\overline{A}x)_i.$$

Therefore, (**??**) holds for all $i \in \{1, 2, \cdots, m\}$.

Conversely, let x satisfy the condition (**??**). Then

$$\underline{A}x \leq \underline{b} \tag{4.10}$$

and

$$\overline{A}x = \overline{b}. \tag{4.11}$$

Let $i \in \{1, 2, \cdots, m\}$. From (**??**) and (**??**), we have

$$(\underline{A}x)_i \leq \underline{b}_i \leq \overline{b}_i = (\overline{A}x)_i.$$

Then

$$[(\underline{A}x)_i, (\overline{A}x)_i] = (\mathbb{A}x)_i \supseteq (\mathbb{b})_i = [\underline{b}_i, \overline{b}_i].$$

Therefore, x satisfying (**??**) is a special control right solution of $\mathbb{A}x = \mathbb{b}$. The proof is completed. □

We are also able to transform (??) to

$$\overline{A}x^{+} - \underline{A}x^{-} = \overline{b} \tag{4.12}$$

and

$$\underline{A}x^{+} - \overline{A}x^{-} \leq \underline{b}, \tag{4.13}$$

in the same fashion as the approach to the systems (??) and (??). Then we can form Inequality (??) and Equation (??) into the linear equations system as follows:

$$\begin{bmatrix} \overline{A} & -\underline{A} & 0_m \\ \underline{A} & -\overline{A} & I_m \end{bmatrix} \begin{bmatrix} x^{+} \\ x^{-} \\ s \end{bmatrix} = \begin{bmatrix} \overline{b} \\ \underline{b} \end{bmatrix}, \tag{4.14}$$

where I_m is the $m \times m$ identity matrix, 0_m is the $m \times m$ zero matrix and s is the vector of the surplus variables. Therefore, the linear equations system (??) has at least one solution which satisfy $x^{+}, x^{-} \geq 0, x^{+} \cdot x^{-} = 0$ and $s \geq 0$ if and only if there exist at least one special control right solution of $\mathbb{A}x = \mathbb{b}$.

Moreover, by the same concept of Lemma ??, the condition that $Rank(\begin{bmatrix} \overline{A} & \underline{A} \end{bmatrix}) = m$ confirms that the linear equations system (??) has at least one solution. Besides, if the statement that $Rank(\underline{A}) = m$ or $Rank(\overline{A}) = m$ is true for this system (??), then we can also verify the existence of its solution.

CHAPTER V

AN APPLICATION OF SPECIAL TOLERANCE LEFT SOLUTION IN A COURSE ASSIGNMENT PROBLEM

In this chapter, we present an application of special tolerance left solution in a course assignment problem with an interval workload constraint.

5.1 Review of the course assignment problem with interval workload constraint

The course assignment problem is an optimization problem that assigns the subjects for the instructors. In [?], authors divided the teaching preference values into 6 ranks shown in Table ??.

Table 5.1: The teaching preference rank descriptions.

Rank description	Rank	Value of rank
The most preferable subject to teach	1	1
A preferable subject to teach	2	0.95
A subject is able to teach	3	0.9
A non preferable subject but is able to teach	4	0.85
A non preferable subject and unwilling to teach	5	0.8
A subject is unable to teach	6	0

Let m and n be the positive integers. We define $I = \{1, 2, \ldots, n\}$ and $J = \{1, 2, \ldots, m\}$ as the instructors and the subjects index sets, respectively. The parameters and decision variables are presented as follows:

- x_{ij} : Decision variable

$$x_{ij} = \begin{cases} 1, & \text{the } i^{\text{th}} \text{ instructor is assigned to the } j^{\text{th}} \text{ subject} \\ 0, & \text{otherwise.} \end{cases}$$

- c_{ij} : A value of rank of the j^{th} subject corresponding to the preference of the i^{th} instructor, (see Table **??**).
- $[\underline{a}_j, \overline{a}_j]$: An interval workload of the j^{th} subject.
- $[\underline{b}_i, \overline{b}_i]$: An interval requested workload of the i^{th} instructor.

Next, the standard restrictions for a course assignment problem are presented as follows:

- No more than 3 subjects for each instructor:

$$\sum_{j=1}^{m} x_{ij} \leq 3, \quad \forall i \in I.$$

- Only one instructor for each subject:

$$\sum_{i=1}^{n} x_{ij} = 1, \quad \forall j \in J.$$

- Interval workload constraint:

$$\sum_{j=1}^{m} [\underline{a}_j, \overline{a}_j]\, x_{ij} = [\underline{b}_i, \overline{b}_i]\,, \quad \forall i \in I. \tag{5.1}$$

The interval workload constraint shows an association of the amount of instructor's assigned and requested workload. From the exact values of the number of student's enrollments and the instructor's requested workload, we can calculate the instructor's assigned workload in order to confirm that all courses are assigned to instructors. However, for the interval forms of number of student's enrollments and the instructor's requested workload, there are many semantics of the workload constraint according to a specific interpretation of a modeler such as in Leela-apiradee et al. [?] which presented the interval workload constraint with the concept of a tolerance-localized solution. In the next section, we present an appropriate situation for the special tolerance left concept in the interval workload constraint of some instructors.

5.2 Special tolerance left solution for the course assignment problem with interval workload constraint

Let the interval $[\underline{a}_j, \overline{a}_j]$ be the workload range for the j^{th} subject, which happens when the number of student's enrollments of the j^{th} subject is not yet a precise value at the time of modeling. Since if the number of students in the j^{th} subject is not more than 50 students, then the j^{th} subject is calculated 3 workload per credit and it is increased by 0.5 workload per credit for each 50 excessive students. Therefore, if it is not certain how many students will enroll in the 3 credit hours class j that normally has $45-55$ students, that class j will have $[\underline{a}_j, \overline{a}_j] = [3, 4.5]$. Some i^{th} instructor offer the interval requested workload as $[\underline{b}_i, \overline{b}_i]$ which he/she can accept, due to the imprecise subject workload. The value $\underline{b}_i$ is defined from the ideal workload that the instructor genuinely desires to obtain, while the value $\overline{b}_i$ is the maximum workload that the instructors can accept. In order to receive a suitable amount of workload $\sum_{j\in J}[\underline{a}_j, \overline{a}_j]x_{ij}$ should be a subset of $[\underline{b}_i, \overline{b}_i]$, for each i^{th} instructor. Moreover, the condition

$$\sum_{j\in J} \underline{a}_j x_{ij} = \underline{b}_i,$$

which shows the equality of the lower bound of the assigned workload and the ideal workload of the i^{th} instructor, can confirm that the real assigned workload is in an interval requested workload, $[\underline{b}_i, \overline{b}_i]$. Then the instructors have the opportunity to receive the amount of workload as their ideal workload. Hence, the semantics of a special tolerance left solution is appropriate for the interval workload constraint of the course assignment problem. However, for some instructors who do not worry to receive the amount of workload that more than his/her ideal workload, the concept of tolerance solution can be used to work with the condition that the lower bound of the assigned workload, $\sum_{j\in J} \underline{a}_j x_{ij}$, is very close to the ideal workload $\underline{b}_i$.

Let I_1 and I_2 be the sets of instructors who are treated by the special tolerance left and tolerance concepts in the interval requested workload constraint, respectively, where $|I_1| = n_1$, $|I_2| = n_2$ and $n_1+n_2 = n$. For the set of instructors in I_1, we apply Theorem **??** to transform the interval workload Constraint (**??**) according to the semantics of special tolerance left solution as follows:

$$\sum_{j=1}^{m} \underline{a}_j x_{ij} = \underline{b}_i, \forall i \in I_1$$
$$\sum_{j=1}^{m} \overline{a}_j x_{ij} \le \overline{b}_i, \forall i \in I_1.$$

Next, we use Theorem **??** in Constraint (**??**) to transform the interval workload constraint for instructors in the set I_2 as follows:

$$\begin{aligned} &\sum_{j=1}^{m} \underline{a}_j x_{ij} \ge \underline{b}_i, \forall i \in I_2 \\ &\sum_{j=1}^{m} \overline{a}_j x_{ij} \le \overline{b}_i, \forall i \in I_2. \end{aligned} \tag{5.2}$$

In order to penalize the difference of the assigned workload $\sum_{j=1}^{m} \underline{a}_j x_{ij}$ and the requested workload $\underline{b}_i$, we define δ_i as the difference between them and use the constraint

$$\sum_{j=1}^{m} \underline{a}_j x_{ij} - \delta_i = \underline{b}_i, \forall i \in I_2$$

in our mathematical model of a course assignment problem to replace Constraint (**??**).

In our model, we concerns about the teaching preference of the instructors and attempt to reduce the difference of the amount of assigned and requested workloads.

Therefore, the objective function is stated as

$$\max \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}x_{ij} - \sum_{k=1}^{n_2} p_k\delta_k,$$

where p_k is a penalty term for δ_k for all $k \in I_2$.

Hence, the mathematical model for our course assignment model with special tolerance left and tolerance workload constraints can be written as follows:

$$\begin{aligned}
\max \ & \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}x_{ij} - \sum_{k=1}^{n_2} p_k\delta_k \\
\text{s.t.} \ & \sum_{j=1}^{m} x_{ij} \leq 3, \quad \forall i \in I \\
& \sum_{i=1}^{n} x_{ij} = 1, \quad \forall j \in J \\
& \sum_{j=1}^{m} \underline{a}_j x_{ij} = \underline{b}_i, \forall i \in I_1 \\
& \sum_{j=1}^{m} \overline{a}_j x_{ij} \leq \overline{b}_i, \forall i \in I_1 \\
& \sum_{j=1}^{m} \underline{a}_j x_{ij} - \delta_i = \underline{b}_i, \forall i \in I_2 \\
& \sum_{j=1}^{m} \overline{a}_j x_{ij} \leq \overline{b}_i, \forall i \in I_2 \\
& x_{ij} \in \{0, 1\}.
\end{aligned}$$

The model may not have a feasible region in general even if $Rank(\begin{bmatrix}\overline{A} & \underline{A}\end{bmatrix}) = m$, due to the other constraints in the model and the fact that the decision variables are binary variables. Moreover, if we use the same penalty terms $p_k = p$ for all k in I_2, then $\sum_{i\in I_2} \delta_i$ becomes a fixed value $\sum_{j\in J} \underline{a}_j - \sum_{i\in I} \underline{b}_i$ for any set I_2. The explanation is as follows.

$$
\begin{aligned}
\delta_i &= \sum_{j=1}^{m} \underline{a}_j x_{ij} - \underline{b}_i, \forall i \in I_2 \\
\implies \sum_{i\in I_2} \delta_i &= \sum_{i\in I_2} (\sum_{j=1}^{m} \underline{a}_j x_{ij} - \underline{b}_i) \\
&= \sum_{i\in I_2} (\sum_{j=1}^{m} \underline{a}_j x_{ij}) - \sum_{i\in I_2} \underline{b}_i \\
&= (\sum_{i\in I} (\sum_{j=1}^{m} \underline{a}_j x_{ij}) - \sum_{i\in I_1} (\sum_{j=1}^{m} \underline{a}_j x_{ij})) - (\sum_{i\in I} \underline{b}_i - \sum_{i\in I_1} \underline{b}_i) \\
&= \sum_{i\in I} (\sum_{j=1}^{m} \underline{a}_j x_{ij}) - \sum_{i\in I} \underline{b}_i - (\sum_{i\in I_1} (\sum_{j=1}^{m} \underline{a}_j x_{ij}) - \sum_{i\in I_1} \underline{b}_i) \\
&= \sum_{i\in I} (\sum_{j=1}^{m} \underline{a}_j x_{ij}) - \sum_{i\in I} \underline{b}_i ; \sum_{i\in I} (\sum_{j=1}^{m} \underline{a}_j x_{ij}) - \sum_{i\in I} \underline{b}_i = 0 \\
&= \sum_{j=1}^{m} \underline{a}_j - \sum_{i\in I} \underline{b}_i .
\end{aligned}
$$

Therefore, the model is actually trying to maximize the total preference, in this case. However, the constraint $\sum_{j=1}^{m} \underline{a}_j x_{ij} - \delta_i = \underline{b}_i, \forall i \in I_2$, is still good to be added in the model to check the difference δ_i of i^{th} instructor.

5.3 Results and discussion

In this work, we use our mathematical model for the course assignment problem to work with the data from the second semester in 2018 of the Department of Mathematics and Computer Science, faculty of Science, Chulalongkorn University, which has 61 instructors with the total of 117 subjects.

We use the instructor's requested workload data from the department as the requested workload $\underline{b}_i$ of i^{th} instructor. We define the set of instructors with the special tolerance left treatment I_1 as $I_1 = \{2, 3, 8, 19, 25, 32, 34\}$. It refers to the set of instructors who prefer to obtain their teaching workload be their ideal workload, $\underline{b}_i$. From the department data of the subject workload, the minimum workload per a subject is equal to 9. For the upper bound, $\bar{b}_i$, we create two cases as follows:

(a) $\bar{b}_i$ is set to be 9 more workload than $\underline{b}_i$, for i^{th} instructor,

(b) $\bar{b}_i$ is set to be 18 more workload than $\underline{b}_i$, for i^{th} instructor.

Then, each instructor who is in I_2 may receive one or two more assigned subjects on top of his/her requested workload. The result in the aspect of number of subjects with instructors' preference rank, is provided in Table **??**. We also report the result in terms of the difference δ_i in both cases (a) and (b), for i^{th} instructor as shown in Table **??**. The sum of total differences δ_i in both cases equal to 246.91, when we use the penalty $p_i = 1$. From the results of the both aspects, there is a significant reduction in the subjects with preference rank 6 when $\bar{b}_i$ has 18 more workload than the instructor's requested workload, but also affects the increasing of the difference δ_i of some instructors. Therefore, these results could be a good information for the head of the department. For example, if the the head of the department prefers to get a better preference rank, the head of the department should ask some instructors to increase the amount of his/her requested workload, or open some new teaching positions.

Table 5.2: Number of subjects for each preference rank using two cases of $\bar{b}_i$ with $I_1 = \{2, 3, 8, 19, 25, 32, 34\}$.

Rank	Number of subjects	
	case (a)	case (b)
1	80	81
2	17	20
3	1	2
4	0	0
5	1	3
6	18	11

We expand the set of instructors with the special tolerance left treatment into $I_1 = \{1, 2, 3, 8, 10, 19, 25, 27, 30, 32, 33, 34, 51, 53, 59, 60, 61\}$, and get the result in the aspect of the number of subjects with instructor's preference rank, provided in Table **??**. We also report the result in terms of the difference δ_i in both cases (a) and (b), for i^{th} instructor as shown in Table **??** where the sum of total differences δ_i in both cases also equal to

Table 5.3: The difference δ_i by using $\bar{b}_i$ in case (a) and (b) with $I_1 = \{2, 3, 8, 19, 25, 32, 34\}$.

i	$\delta_{i(a)}$	$\delta_{i(b)}$		i	$\delta_{i(a)}$	$\delta_{i(b)}$		i	$\delta_{i(a)}$	$\delta_{i(b)}$		i	$\delta_{i(a)}$	$\delta_{i(b)}$
1	4.5	18		17	3	0		32	0	0		47	3.25	1.75
2	0	0		18	4.5	2.5		33	1.5	1.5		48	5.25	5.25
3	0	0		19	0	0		34	0	0		49	0.25	9.25
4	4.5	1.5		20	5.25	0.75		35	7	1		50	7.25	7.25
5	6.4	15.4		21	4.82	4.07		36	9	0		51	3	0
6	6.52	11.02		22	2.22	2.22		37	6	4		52	8.85	8.85
7	8.02	8.02		23	3.25	12.25		38	3.03	13.53		53	5.5	0.5
8	0	0		24	1.9	1.9		39	5.25	14.25		54	3.55	3.55
9	4.8	9.3		25	0	0		40	4.75	4.75		55	2.9	2.9
10	2.5	2.5		26	6.4	6.4		41	3	4.5		56	5.53	5.53
11	1.5	1.5		27	6	3		42	2.25	2.25		57	8.5	1
12	4.75	0.25		28	6.53	6.53		43	2.77	0.52		58	6.4	7.9
13	3.5	7		29	6.8	0.3		44	1	1		59	0	0
14	7.5	1.5		30	6	6		45	5	5		60	0	0
15	5.3	0.8		31	6.75	6.75		46	6.77	5.27		61	0	0
16	6.15	6.15												

$\delta_{i(a)}$: the difference δ_i of the model using $\bar{b}_i$ as in case (a) for i^{th} instructor and
$\delta_{i(b)}$: the difference δ_i of the model using $\bar{b}_i$ as in case (b) for i^{th} instructor.

246.91, as expected. These results confirm that when the size of I_1 is larger, there are a significant a reduction in the number of subjects with preference rank 1 and increment in the number of subjects with preference rank 6, and the number of instructors whose $\delta_i = 0$ increases. Therefore, the head of the department should consider the requirements in order to adjust the model accordingly. If the head of the department wants to obtain a better preference rank, he/she may consider to reduce the number of the instructors with the special tolerance left treatment. Or if the head of the department wants to balance the workload of every instructor, it may consider to add the upper bound of δ_i in the model.

Table 5.4: Number of subjects for each preference rank using two cases of $\overline{b}_i$, with $I_1 = \{1,2,3,8,10,19,25,27,30,32,33,34,51,53,59,60,61\}$.

Rank	Number of subjects	
	case (a)	case (b)
1	74	79
2	15	16
3	2	1
4	1	1
5	1	0
6	24	20

Table 5.5: The difference δ_i by using $\overline{b}_i$ in case (a) and (b) with $I_1 = \{1,2,3,8,10,19,25,27,30,32,33,34,51,53,59,60,61\}$.

i	$\delta_{i(a)}$	$\delta_{i(b)}$		i	$\delta_{i(a)}$	$\delta_{i(b)}$		i	$\delta_{i(a)}$	$\delta_{i(b)}$		i	$\delta_{i(a)}$	$\delta_{i(b)}$
1	0	0		17	0	0		32	0	0		47	3.25	1.75
2	0	0		18	6.5	6.5		33	0	0		48	5.25	5.25
3	0	0		19	0	0		34	0	0		49	0.25	9.25
4	7.5	1.5		20	5.25	0.75		35	7	4		50	7.25	7.25
5	6.4	15.4		21	7.07	11.57		36	9	9		51	0	0
6	4.27	13.27		22	2.22	2.22		37	5.5	11		52	8.85	8.85
7	8.02	8.02		23	3.25	12.25		38	5.28	14.28		53	0	0
8	0	0		24	8.65	1.9		39	5.25	14.25		54	3.55	3.55
9	4.8	9.3		25	0	0		40	4.75	4.75		55	2.9	2.9
10	0	0		26	6.4	6.4		41	9	0		56	5.53	5.53
11	1.5	1.5		27	0	0		42	6.75	2.25		57	7	7
12	4.75	4.75		28	5.03	2.03		43	7.27	0.52		58	6.4	1.9
13	5.5	2.5		29	6.8	2.3		44	5.5	1		59	0	0
14	7.5	0.5		30	0	0		45	5	9.5		60	0	0
15	5.3	0.8		31	6.75	6.75		46	6.77	6.77		61	0	0
16	6.15	6.15												

$\delta_{i(a)}$: the difference δ_i of the model using $\overline{b}_i$ as in case (a) for i^{th} instructor and
$\delta_{i(b)}$: the difference δ_i of the model using $\overline{b}_i$ as in case (b) for i^{th} instructor.

CHAPTER VI

CONCLUSION

This work concerned the situations that some boundaries of intervals are important, leading to the special left and special right concepts. We combine these concepts together with the semantics of tolerance and control solutions in order to present the new types of solutions to $\mathbb{A}x = \mathbb{b}$, called special tolerance left, special tolerance right, special control left and special control right solutions. The characteristics and the existence of these new solution types were provided in Chapters 3 and 4.

In Chapter 5, we determined the course assignment problem with an interval workload constraint to work with the concept of special tolerance left solution. We used the information from the second semester in 2018 of the Department of Mathematics and Computer Science, faculty of Science, Chulalongkorn University. From the results, setting i^{th} instructor's assigned workload $\overline{b}_i$ to be more than the instructor's requested workload $\underline{b}_i$ by 18 workload increases the difference δ_i of some instructors, but it causes the reduction in the number of subjects with preference rank 6 while the number of subjects with preference rank 1 increases. The head of the department can use this information modifying an appropriate model. For example, if the head of the department wants to assign subjects to instructors with a better preference rank, they should increase the amount of the upper bound, $\overline{b}_i$, of interval requested workload or reduce the number of the instructors with the special tolerance left treatment.

APPENDICES

In these appendices, there are the data from the second semester in 2018 of the Department of Mathematics and Computer Science, faculty of Science, Chulalongkorn University, and the CPLEX code of our mathematical model for the course assignment problem along with the Python code of finding the instructor's preference rank and the original mathematical model for the course assignment problem.

APPENDIX A :

code thesis.jpg code thesis.bb

Figure 1: All data and source code.

APPENDIX B : the original mathematical model for the course assignment problem without idea of interval linear equations system is presented as follows:

$$\max \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}x_{ij} - M_1\sum_{i=1}^{n}\alpha_i - M_2\sum_{i=1}^{n}\beta_i$$

$$\text{s.t.} \quad \sum_{j=1}^{m} x_{ij} \leq 3, \;\; \forall i \in I$$

$$\sum_{i=1}^{n} x_{ij} = 1, \;\; \forall j \in J$$

$$\sum_{j=1}^{m} a_j x_{ij} - \alpha_i + \beta_i = b_i - d_i, \forall i \in I$$

$$\alpha_i, \beta_i \geq 0, \forall i \in I$$

where α_i and β_i are the extra and residual workloads of the i^{th} instructor, respectively, and M_1 and M_2 are the large positive penalty values of α_i and β_i, respectively.

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