คำตอบเจนัสจากเกจซูเปอร์กราวิตี $N = 4$ ในสี่มิติ

นายธิษณะ อัศวโสวรรณ

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
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JANUS SOLUTIONS FROM FOUR-DIMENSIONAL $N = 4$ GAUGED SUPERGRAVITY

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งานวิจัยชิ้นนี้ได้ทำการศึกษาคำตอบเจนัสจากเกจซูเปอร์กราวิตี $N = 4$ ในสี่มิติที่มีเกจกรุ๊ปเป็น $SO(4)\times SO(4)$ เราพบว่ามีคำตอบเจนัสที่มีสมมาตรยิ่งยวด $N = 2$ และ $N = 1$ สำหรับเกจกรุ๊ป $SO(4)\times SO(4)$ ที่ถูกแยกเป็น $SO(3)\times SO(3)\times SO(3)\times SO(3)$ จะมีดีฟอร์เมชั่นพารามิเตอร์ในกรุ๊ป $SO(3)$ ทั้งสี่ตัว ได้แก่ α_0, α, β_1 และ β_2 คำตอบที่มีสมมาตรยิ่งยวด $N = 2$ ที่ถูกพบในชุดของสนามสเกลาร์ที่มีสมมาตร $SO(2)\times SO(2)\times SO(2)\times SO(2)$ จะเป็นผลเฉลยที่ผ่านจุด AdS_4 ที่มีสมมาตรยิ่งยวด $N = 4$ และสมมาตร $SO(4)\times SO(4)$ ในขณะที่คำตอบที่มีสมมาตรยิ่งยวด $N = 1$ ไม่เพียงแต่ให้ผลเฉลยที่ผ่านจุดวิกฤต AdS_4 แบบชัดแจ้งเท่านั้นแต่ยังให้ผลเฉลยที่ผ่านจุดวิกฤต AdS_4 แบบไม่ชัดแจ้งอีกด้วย จุดวิกฤตแบบไม่ชัดแจ้งดังกล่าวที่มีสมมาตร $SO(3)_{diag}\times SO(3)\times SO(3)$ จะถูกพบในกรณีที่ $\beta_1 = 0$ และจุดวิกฤต AdS_4 แบบไม่ชัดแจ้งที่มีสมมาตร $SO(3)\times SO(3)_{diag}\times SO(3)$ จะถูกพบในกรณีที่ $\beta_1 = \frac{\pi}{2}$ คำอธิบายทางทฤษฎีสนามที่สอดคล้องกับคำตอบข้างต้นคือ คำตอบที่มีสมมาตรยิ่งยวด $N = 2$ จะสอดคล้องกับความบกพร่องคอนฟอร์มอลที่มีสมมาตรยิ่งยวด $N = (2, 0)$ หรือ $N = (0, 2)$ ในทฤษฎีสนามซูเปอร์คอนฟอร์มอลที่มีสมมาตรยิ่งยวด $N = 4$ และมีสมมาตร $SO(4)\times SO(4)$ ในขณะที่คำตอบที่มีสมมาตรยิ่งยวด $N = 1$ จะสามารถอธิบายความบกพร่องคอนฟอร์มอลที่มีสมมาตรยิ่งยวด $N = (1, 0)$ หรือ $N = (0, 1)$ ในทฤษฎีสนามซูเปอร์คอนฟอร์มอลที่สอดคล้องกับจุดวิกฤต AdS_4 แบบไม่ชัดแจ้งที่มีสมมาตรต่างกันทั้งสองแบบขึ้นอยู่กับค่า β_1

ภาควิชา	ฟิสิกส์	ลายมือชื่อนิสิต	....
สาขาวิชา	ฟิสิกส์	ลายมือชื่อ อ.ที่ปรึกษาหลัก	....
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We find new classes of Janus solutions that preserve $N = 2$ and $N = 1$ supersymmetries in four-dimensional $N = 4$ gauged supergravity with $SO(4)\times SO(4)$ gauge group. The $SO(4)\times SO(4) \sim SO(3)\times SO(3)\times SO(3)\times SO(3)$ includes four deformation parameters $\alpha_0, \alpha, \beta_1$ and β_2 for each $SO(3)$ group. The $N = 2$ solutions found from $SO(2)\times SO(2)\times SO(2)\times SO(2)$ truncation interpolate between $SO(4)\times SO(4)$ $N = 4$ AdS_4 critical points while the $N = 1$ solutions connect not only trivial AdS_4 critical point but also non-trivial ones. These non-trivial $N = 4$ AdS_4 critical points preserve $SO(3)_{diag} \times SO(3) \times SO(3)$ symmetry, for $\beta_1 = 0$, and $SO(3)\times SO(3)_{diag}\times SO(3)$ symmetry, for $\beta_1 = \frac{\pi}{2}$. In the dual theories, the $N = 2$ solutions correspond to $N = 4$ superconformal field theory $(SCFT)$ with $SO(4)\times SO(4)$ symmetry in the presence of $N = (2, 0)$ or $N = (0, 2)$ conformal defects while the $N = 1$ solutions also holographically describe $N = (1, 0)$ or $N = (0, 1)$ conformal defects within the $N = 4$ $SCFT$ dual to non-trivial AdS_4 critical points depending on the values of β_1.

Department	: Physics	Student's Signature	....
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CHAPTER I

INTRODUCTION

The ultimate goal of high-energy physics is to construct the theory that unifies all four fundamental forces into one theory called the theory of everything. The first step had been achieved by the standard model that describes elementary particles named the standard model. This is quite an effective model since electro-weak and strong interaction can be clarified via one theory. However, the weakest point of this is that gravity is not included. Hence, it cannot still be the theory of everything for that reason. Besides, in the quantum gravity sense, the coupling constant from the theory defined by $\kappa = \sqrt{8\pi G}$ corresponds to UV divergences in the Feynman diagrams and cannot be renormalized. In this theory, the UV divergence cannot be canceled by counter terms. Gravity is also distinguished from the other three fundamental forces by its geometrical nature. Another contrast between gravity and other fundamental forces is the hierarchy problem as an enormous gap of energy between electro-weak, clearer depicted by 100 GeV, and quantum gravity giving approximately 10^{19} GeV.

The problems above are solved mainly by the emergence of supersymmetry, a symmetry that unifies bosons and fermions. With the applications of supersymmetry, UV divergences that seem problematic at first can even become softer. Another benefit of having supersymmetry is that with the help of supersymmetry, the three-fundamental forces excluding gravity can be unified at a certain scale, called grand unification.

After the introduction of supersymmetry, see [?, ?, ?, ?], there was an attempt to construct a theory including gravity with supersymmetry. The gravity theory is generally described in terms of curved spacetime. It turns out that supersymmetry in a curved spacetime must be promoted to a local symmetry. When combining all symmetries, including supersymmetry that this system could have, the Poincare group is extended to be a bigger group called super-Poincare, and the theory of curved spacetime with the addition of supersymmetry is called supergravity, see [?, ?, ?, ?, ?, ?, ?, ?, ?, ?]. Supergravity is such an effective tool in constructing the theory of supersymmetric interaction. However, the big drawback of all supersymmetric theories is that superpartners of particles have never been observed, possibly leading to the question that if they really exist in the universe. Therefore, the reasonable assumption that can answer this question is that supersymmetry may be spontaneously broken in our universe. In other words, our spacetime is not a supersymmetric vacuum.

In string theory which represents as a one-dimensional object instead of a point particle whose length is $l_s = \sqrt{\alpha'}$ and the tension $T \sim l_s$, elementary particles arise from the mode of oscillation of the string, see [?, ?, ?, ?, ?]. A strongly compelling motivation of this theory is the existence of graviton that does not emerge from any theories but is effortlessly generated by the closed string, while the open string also encompasses the gauge field. In addition, the UV cut-off is also provided by the length of string in the theory. For this reason, superstring theory can be formulated as a quantum theory combining all fundamental interactions with gravity. The effective field theory of ten-dimension supergravity which is found by taking string length to zero or, equivalently, the string's tension close to infinity, where the string is now seen as a particle, is described by supergravity theories. Further development of string theory shows that the different five superstring theories might originate from an eleven-dimensional theory called M-theory, with all string theories related to each other by dualities. Despite the precise form of M-theory not being clearly known, its effective theory is given by eleven-dimensional supergravity.

Although string or M-theory is the candidate theory that might unify all four-fundamental forces, it is needed to be described in ten or eleven dimensions. This does not match the reality since our universe is four-dimensional spacetime and leads to the question that how the other six or seven dimensions come from. However, those six and seven dimensions can be describable due to the idea of Kaluza-Klein reduction. The method is to compactify extra dimensions into a compact internal manifold. Therefore, ten or eleven-dimensional spacetime can be written as

$$M = M_{d=4} \times M_{int} \tag{1.1}$$

where $M_{d=4}$ is our non-compact four-dimensional manifold and M_{int} is other compact manifold with $D-d$ dimensions where D is 10 or 11. With this method of compactification, massless fields in the string theory become massless and massive fields in four-dimensional spacetime. Those massless fields are field contents of supergravity in four dimensions. It can also be said that the string theory in ten dimensions can be reduced to supergravity in four dimensions. Nevertheless, supergravity is not close to realistic model as it cannot provide cosmological constant. Gauged supergravity is a supergravity with non-abelian gauge symmetries and can give rise to cosmological constant. Besides, the introduction of D-brane in string theory at low energy limit can lead to the effective theory in the form of gauged supergravity.

D-brane or sometime called Dp-brane, standing for Dirichlet brane, is the object where open string can end that extends in p spatial dimensions. With the existence of D-brane, open string can have various orientations. It can not only start and end on the same brane, but also start from one brane and end with another brane. Without D-brane, there is only open string with Neumann boundary conditions. Spectrum of open string are richer when the D-brane is present as there can be open string with Dirichlet boundary conditions or even mixed boundary conditions. D-brane is a dynamical object that can interact with open and closed strings. The action that includes the dynamics of D-brane can give rise to gauged supergravity at low energy limit. One of the most well-known examples of D-brane that leads to gauged supergravity is coincident N D3-branes in IIB string theory in ten dimensions. At low energy limit, the string theory with coincident N D3-brane can be described by supergravity on $AdS_5 \times S^5$ where AdS_5 is the five-dimensional spacetime. The theory is effectively reduced to $SO(6)$ gauged supergravity in five dimensions with $SO(6)$ corresponding to the isometry of S^5.

To gauge supergravity theory, subgroup G_0 of G, the global symmetry of the scalar manifold that includes all scalars in theory, is promoted to be a local symmetry. During the gauging procedure, vector fields in ungauged theory are coupled to other fields that are charged under the gauge symmetry. The gauged supergravity with the local symmetry group G_0 corresponds to the internal manifold with an isometry group G_0. For example, in the first construction of this procedure in [?], $SO(8)$ gauged supergravity can be described by the compactification of eleven-dimensional supergravity on S^7 $SO(8)$ group. The structure of this gauging is summarized in Figure 1.1, see [?, ?, ?] for more details.

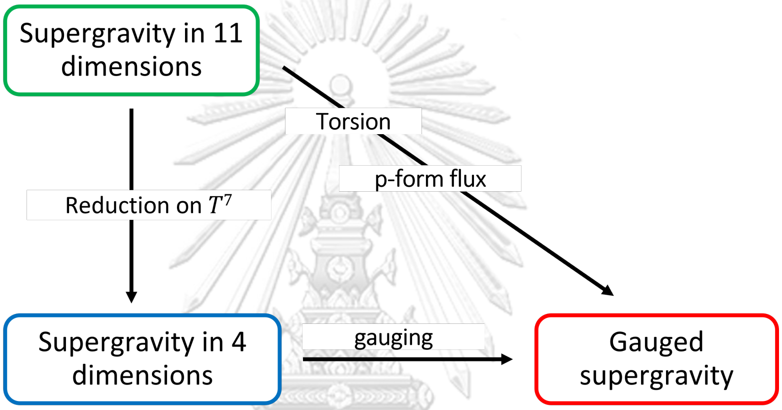


Figure 1.1: A journey of gauge thoery in supergravity

An advent of the AdS/CFT correspondence proposed by Maldacena in [?], see also [?, ?], is the first bridge where quantum theory and gravity can be conjecturally linked to each other in a consistent manner. The correspondence claims that conformal field theory in the limit of large N in d dimensions is dual to the compactification of supergravity on $(d+1)$-dimensional AdS space with a compact internal manifold. The most well-known correspondence is four-dimensional $N = 4$ super-Yang-Mills (SYM) with $SU(N)$ gauge symmetry and type IIB supergravity on $AdS_5 \times S^5$. This correspondence or duality leads to various applications in several fields of studies, ranging from statistical physics to high-energy physics. For example, in quantum field theory, a technique called renormalization group (RG) has been introduced to solve the problem of infinities in the calculation. The idea leads to the introduction of the β function describing the relation between coupling constant g and energy scale μ. The β function at some g^* such that

$\beta(g^*) = 0$ is called a conformal fixed point that can be deformed by some operator $\mathcal{O}_\Delta$ to another conformal point. In the context of the AdS/CFT correspondence, the RG flow of between two CFTs is described by solutions in gauged supergravity in the form of flat domain wall. The metric ($d+1$ dimensions) takes the form as

$$ds^2 = e^{2A(r)}(\eta_{\mu\nu}dx^\mu dx^\nu) + dr^2, \tag{1.2}$$

where $A(r)$ is a warp factor that becomes a linear function to reproduce AdS_{d+1} space. Previous works on domain walls can be found in [?, ?, ?, ?, ?, ?] followed by the studies on BPS flat domain wall in [?, ?, ?, ?, ?, ?, ?, ?, ?].

The BPS curved domain wall solutions in [?, ?, ?], later led to another holographic solution called Janus that describes a conformal defect within conformal field theory. This kind of solution can be obtained through AdS_d-sliced domain wall

$$ds^2 = e^{2A(r)}(ds^2_{AdS_d}) + dr^2 \tag{1.3}$$

that preserves some amount of conformal symmetry $SO(2, d-1)$ corresponding to the conformal symmetry on the defect instead of Poincare symmetry on a flat domain wall. In this report, we will focus mainly on Janus solutions.

The first Janus solutions are not supersymmetric solutions, see [?, ?]. The solutions are solved from an ansatz of AdS-slice domain wall. The field theory interpretation dual to this solution are further described in [?] and correlation functions in terms of holographic description are given in [?]. After that, for decades, supersymmetric Janus solutions are found in gauged supergravities with different gauge groups and dimensions, see [?, ?]. Our scope of study is to find Janus solutions from $N = 4$ gauged supergravity in four dimensions. The first Janus solutions from $N = 4$ gauged supergravity has been found by compactification of eleven-dimensional supergravity on tri-sasakian manifold studied in [?] where Janus solutions interpolate between singular geometries. The next study is $N = 4$ gauged supergravity with $ISO(3) \times ISO(3)$ gauge symmetry which admits only one AdS_4 critical points, see [?]. $N = 4$ $SO(4) \times SO(4)$ gauged supergravity gives Janus solutions interpolating between trivial $N = 4$ AdS_4 critical points. With such studies above, no Janus solutions interpolate between other vacua apart from trivial ones.

Our work eventually provides Janus solutions interpolating between non-trivial critical points. By the application of symplectic deformation [?], more deformations parameters allow us to find more general structures and vacua as earlier studies on symplectic deformation of $SO(8)$ gauged supergravity are found in [?, ?, ?, ?]. We find Janus solutions preserving $N = 2$ and $N = 1$ supersymmetries [?]. Despite $N = 2$ solutions from $SO(2)\times SO(2)\times SO(2)\times SO(2)$ truncation giving only trivial critical points, we find that these solutions are more general than the solutions found in [?]. Much more interesting solutions are found in $N = 1$ supersymmetry since we finally find Janus solutions that interpolate between non-trivial vacua.

This thesis is organized as follows. The main purpose of chapter II is to provide a gist of supersymmetry in the construction of field contents in each multiplet with different numbers of supercharges and provide the algebra corresponding to supersymmetry that will be promoted to local symmetry in chapter III to build up supergravity. One of the simplified reviews on supersymmetry is [?]. The main feature of chapter III is to generalize supergravity with $N > 2$ and relevant symmetries for other fields with spin different from zero. Once an ungauged theory is known, it is much more exciting to make the theory become gauged supergravity in chapter IV, responsible for describing the string theory compactified on different manifolds at low energy, see [?] a review and [?] for more details. Moving to chapter V, the AdS/CFT duality, see [?, ?, ?], is clarified, and some applications such as RG-flow found from flat domain wall metric are given. By making a bit alteration of the flat domain walls to be AdS-sliced domain walls, the Janus solutions are described in chapter VI. Besides, our new Janus solutions are also provided in this chapter. Finally, we review recent Janus-related works from [?, ?] and comment on some possible future works in the last chapter.

CHAPTER II

SUPERSYMMETRY

Supersymmetry, the biggest symmetry of spacetime, is beneficial in solving many theories. Some problems, especially in High-Energy physics, could not be solved without supersymmetry. Supersymmetry claims that there is a superpartner particle with different statistics for each particle, uniting between bosons and fermions.

In this chapter, some details of supersymmetry will be briefly discussed, starting with a bit of history to give motivation and mainly focusing on the extension of the Poincare group with the addition of supersymmetry named super-Poincare and the algebra behind it, see [**?**, **?**].

2.1 History of supersymmetry

With the advent of supersymmetry, it had been developed throughout history. Important development will be given consecutively.

- In 1967, Coleman and Mandula proposed the “No-go theorem” that claims the most generally possible symmetry can be explained by S-matrix being the direct product between Poincare’ and internal symmetry.

- In 1971, Poincare’ algebra was extended by including spinor generator Q_α, claimed by Golfand Likhtman.

- In 1971, Applying supersymmetry in two-dimensional string theory was succeeded by Ramond, Neveu-Schwarz, Gervais, and Sakita.

- In 1974, a complete field theory including supersymmetry in four dimensions was created by Wess and Zumino.

- In 1975, Hagg, Lopuszanski, and Sohnius presented a generalized No-go theorem, adding both spinor generator and its Hermitian generator in Poincare' algebra.

2.2 Lie group and Lie algebra

Lie group is a group that contains elements as smooth parameters. Each element of a group can be described by a set of a finite number of parameters $\{\alpha^a, \alpha^b, ..., \alpha^N\}$ where N is a dimension of a group. Geometrically, because of the smooth property of elements of a group, the Lie group can also be seen as a manifold. α^a then is seen as a manifold coordinate.

Normally, the group identity can be written with parameter $\alpha^a = 0$ where $a = 1, 2, ..., N$ as

$$e = g(0, ..., 0) \tag{2.1}$$

Since the neighbourhood of identity element denoted as 1 changes continuously, we can find this neighbourhood by considering infinitesimal Taylor expansion around $g(0, ..., 0)$ as

$$g(d\alpha) = g(0) + d\alpha^a \frac{\partial g}{\partial \alpha^a}|_{\alpha^a=0} \tag{2.2}$$

$$= 1 + id\alpha^a T_a \tag{2.3}$$

where T_a is a generator of the group defined by

$$T_a = -i\frac{\partial g}{\partial \alpha^a}|_{\alpha^a=0} \tag{2.4}$$

The elements far away from identity element can be found by

$$g(\alpha) = \lim_{n\to\infty}(1 + i\frac{\alpha^a}{n}T_a)^n = e^{i\alpha^a T_a} \tag{2.5}$$

The action between two elements of the group can be found by

$$g(\alpha)g(\beta) = g(\gamma) \tag{2.6}$$

$$e^{i\alpha^a T_a} e^{i\beta^b T_b} = e^{i\gamma^a T_a} \tag{2.7}$$

Take ln to both sides of the equation to get

$$i\gamma^a T_a = \ln(1 + e^{i\alpha^a T_a} e^{i\beta^b T_b} - 1) \tag{2.8}$$

With Taylor expansion of $\ln(1 + K)$ where K is

$$\begin{aligned} K &= e^{i\alpha^a T_a} e^{i\beta^b T_b} - 1 \\ &= (1 + i\alpha^a T_a - \frac{1}{2}(\alpha^a T_a)^2 + ...)(1 + i\beta^b T_b - \frac{1}{2}(\beta^b T_b)^2 + ...) - 1 \\ &= i\alpha^a T_a + i\beta^b T_b - \alpha^a T_a \beta^b T_b - \frac{1}{2}(\alpha^a T_a)^2 - \frac{1}{2}(\beta^b T_b)^2 + ... \end{aligned} \tag{2.9}$$

that leads to

$$\begin{aligned} i\gamma^a T_a &= K - \frac{1}{2}K^2 + ... \\ &= i\alpha^a T_a + i\beta^b T_b - \alpha^a T_a \beta^b T_b - \frac{1}{2}(\alpha^a T_a)^2 - \frac{1}{2}(\beta^b T_b)^2 + \frac{1}{2}(\alpha^a T_a + \beta^b T_b)^2 + ... \\ &= i\alpha^a T_a + i\beta^b T_b - \frac{1}{2}[\alpha^a T_a, \beta^b T_b] + ... \end{aligned} \tag{2.10}$$

which gives

$$[\alpha^a T_a, \beta^b T_b] = -2i(\gamma^c - \alpha^c - \beta^c)T_c = i\delta^c T_c \tag{2.11}$$

where δ^c can be redefined as

$$\delta^c = \alpha^a \beta^b f_{ab}{}^c \tag{2.12}$$

Placing this δ^c back in **??** to get

$$[\alpha^a T_a, \beta^b T_b] = i\alpha^a \beta^b f_{ab}{}^c \tag{2.13}$$

$$\alpha^a \beta^b [T_a, T_b] = i\alpha^a \beta^b f_{ab}{}^c \tag{2.14}$$

$$[T_a, T_b] = i f_{ab}{}^c T_c \tag{2.15}$$

This is called Lie algebra as it tells about the structure of the group from structure constant through the commutator between its generator that will depend on which the group is. It is really important in Physics since Physic must face symmetry inevitably and a symmetry can be identified from determining the Lie algebra relevant to that symmetry.

2.3 Graded algebra

In this section, we want to identify the symmetry called supersymmetry. However, since Lie algebra is not enough to indicate algebra for supersymmetry, the Poincare algebra abiding by Lie algebra structure must be extended in accordance with covering more symmetries. The concept of graded Lie algebra is then necessarily introduced. A grade of a generator can be defined by

$$[T_a, T_b\} = T_a T_b - (-1)^{\eta_a \eta_b} T_a T_b = f_{ab}{}^c T_c \tag{2.16}$$

where $\eta_a = 0, 1$ is a grade of T_a which is in a group $\mathbb{Z}_2$.

The product of two fermionic generators will result in bosonic genertaors due to their grades equal to zero. This generally show that

$$[B, B'] = B", \quad [B, F] = F' \quad \{F, F'\} = B" \tag{2.17}$$

where B and F are a bosonic and fermionic generator. Hagg-Lopuszanski-Sohnius theorem proposed that possible fermionic generators must be in spinor representation $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ of Lorentz group. This can be proved by considering the product of fermionic generator in representation $(j, j+\frac{1}{2})$ for fermionic generator $F^i, i = 1, ..., N$ and its conjugate $F^{i\dagger}$ in representation $(j+\frac{1}{2}, j)$. The product can be shown as the following

$$\begin{aligned} \{F^i, F^{j\dagger}\}: &\quad (2j+\frac{1}{2}, 2j+\frac{1}{2}) \\ \{F^i, F^j\}: &\quad (2j, 2j+1) \\ \{F^{i\dagger}, F^{j\dagger}\}: &\quad (2j+1, 2j) \end{aligned} \tag{2.18}$$

The result must be in bosonic generator under Poincare group that has generators $J^{\mu\nu}$ (a Lorentz generator), P^μ (a momentum generator), and t_a (a generator of internal symmetry) in representation $(1,0)\oplus(0,1)$, $(\frac{1}{2}, \frac{1}{2})$ and $(0,0)$ respectively. More importantly, fermionic generators with $j > 0$ cannot be closed the algebra $\{F, F\}$ to give bosonic generators in Poincare' group. Therefore, the only possible fermionic gernerators must have $j = 0$.

Let Q_{ai} and $\bar{Q}^i_{\dot{a}} = (Q_{ai})^\dagger$ be fermionic generators in $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ respectively. These generators are usually called supercharges. The indices $i = 1, 2, ..., N$ refer to a number of generators while $a, b = 1, 2$ and $\dot{a}, \dot{b} = 1, 2$ are the spinor indices and their conjugate respectively.

Generators Q_{ai} and $\bar{Q}^i_{\dot{a}}$ are spinors under Lorentz group that have commutation with $J^{\mu\nu}$ as

$$[Q_{ai}, J_{\mu\nu}] = i(\sigma_{\mu\nu})_a^b Q_{bi}, \quad [\bar{Q}^{\dot{a}i}, J_{\mu\nu}] = i(\bar{\sigma}_{\mu\nu})^{\dot{a}}_{\dot{b}} \bar{Q}^{\dot{b}i} \tag{2.19}$$

where $\sigma_{\mu\nu}$ is a Lorentz generator in Weyl spinor representation which is defined by

$$\sigma_{\mu\nu} = \frac{1}{4}(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu), \quad \bar{\sigma}_{\mu\nu} = \frac{1}{4}(\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu) \tag{2.20}$$

where $\sigma_\mu = (-\mathbb{I}, \sigma_i)$, $\bar{\sigma}_\mu = (\mathbb{I}, \sigma_i)$ and σ_i is a 2×2 Pauli matrices where $i = 1, 2, 3$. Other algebras including both of commutators and anti-commutators are found by super-Jacobi and considering their representations.

The result of $\{Q_{ai}, \bar{Q}^j_{\dot{a}}\}$ must be in $[(\frac{1}{2}, 0)\otimes(0, \frac{1}{2})]_S = (\frac{1}{2}, \frac{1}{2})$. This shows that the product is a generator P^μ.

$$\{Q_{ai}, \bar{Q}^j_{\dot{a}}\} = -C^j_i \sigma_{\mu a\dot{a}} P^\mu. \tag{2.21}$$

Negative sign is from $\sigma_\mu = (\mathbf{I}, \sigma^i)$.

Due to the result of $\{Q_{ai}, \bar{Q}^j_{\dot{a}}\}$ being hermitain, C^j_i must be hermitian matrix and can be diagonalized. Moreover, $\{Q_{ai}, \bar{Q}^j_{\dot{a}}\}$ is always positive definite. Hence, Q_{ai} and $\bar{Q}^j_{\dot{a}}$ can have new definitions as $Q'_{ai} = \dfrac{Q_{ai}}{\sqrt{2c_i}}$. Therefore, the commutator can be rewritten as

$$\{Q_{ai}, \bar{Q}^j_{\dot{a}}\} = -\frac{1}{2}\delta^j_i \sigma_{\mu a\dot{a}} P^\mu \tag{2.22}$$

Consider the commutator

$$[Q_{ai}, P^\mu] = c_{ij}\sigma^\mu_{a\dot{a}}\bar{Q}^{\dot{a}j} \tag{2.23}$$

The product of antisymmetric tensor $[(\frac{1}{2}, 0) \otimes (\frac{1}{2}, \frac{1}{2})]_A = (0, \frac{1}{2})$ is represented by $\bar{Q}^{\dot{a}j}$. Its conjugation is therefore shown by

$$[\bar{Q}^{\bar{a}i}, P^\mu] = -(c_{ij})^\dagger \bar{\sigma}^{\mu\dot{a}a} Q_{aj} \tag{2.24}$$

From super-Jacobi identity,

$$[P^\mu, [P^\nu, Q_{ai}]] + [P^\nu, [Q_{ai}, P^\mu]] + [Q_{ai}, [P^\mu, P^\nu]] = 0 \tag{2.25}$$

With $[P^\mu, P^\nu] = 0$, the equation **??** will be true when $cc^\dagger = 0$ because $\sigma^\mu\bar{\sigma}^\nu \neq 0$ so

$$[P^\mu, Q_{ai}] = [P^\mu, \bar{Q}^i_{\dot{a}}] = 0 \tag{2.26}$$

$\{Q_{ai}, Q_{bj}\}$ is found by considering representation $(\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0) = (0, 0) \oplus (1, 0)$. Thereby, the result will be

$$\{Q_{ai}, Q_{bj}\} = -\frac{1}{2}\epsilon_{ab} Z^{ij} + \frac{1}{2}\sigma^{\mu\nu}_{ab} Y^{ij}_{\mu\nu} \tag{2.27}$$

where ϵ_{ab} is the Levi-civita tensor, $\sigma^{\mu\nu}$ is a Lorentz generator in the Weyl spinor representation, $Z^{ij} = -Z^{ji}$ is a scalar field and $Y^{ij}_{\mu\nu} = Y^{ji}_{\mu\nu} = -Y^{ij}_{\nu\mu}$ is antisymmetric tensor.

With $[P^\mu, Q_{ai}] = [P^\mu, P^\nu] = 0$ and super-Jacobi identity

$$[P^\mu, \{Q_{ai}, Q_{bj}\}] - \{Q_{bj}, [P^\mu, Q_{ai}]\} + \{Q_{ai}, [Q_{bj}, P^\mu]\} = 0, \tag{2.28}$$

it can show that $[P^\mu, \{Q_{ai}, Q_{bj}\}] = 0$. This will result in $Y^{ij}_{\mu\nu} = 0$ due to $[P^\mu, \sigma^{\nu\rho}] \neq 0$.

A generator Z_{ij} is a scalar under Lorentz group so Z_{ij} must be in the form of internal symmetry T_A as

$$Z_{ij} = a^A_{ij} T_A \tag{2.29}$$

where T_A satisfies Lie algebra

$$[T_A, T_B] = f_{AB}{}^C T_C. \tag{2.30}$$

Suppose Q_{ai} and $\bar{Q}^i_{\dot{a}}$ transform under internal symmetry

$$[Q_{ai}, T_A] = (S_A)_i{}^j Q_{aj}, \quad [T_A, \bar{Q}^i_{\dot{a}}] = (S{*}^A)^i{}_j \bar{Q}^j_{\dot{a}} \tag{2.31}$$

where $(S^A)^j_i$ is a generator T_A in the representation of a supercharge.

The equation **??** and $[P^\mu, T_A] = 0$ and super-Jacobi

$$[T_A, \{Q_{ai}, \bar{Q}^j_{\dot{a}}\}] + \{Q_{ai}, [\bar{Q}^j_{\dot{a}}, T_A]\} - \{\bar{Q}^j_{\dot{a}}, [T_A, Q_{ai}]\} = 0 \tag{2.32}$$

will give $S^{A\dagger} = S^A$ and super-Jacobi identity

$$[T_A, \{Q_{aj}, Q_{bj}\} + \{Q_{ai,}[Q_{bj}, T_A]\} - \{Q_{bj}, [T_A, Q_{ai}]\} = 0 \tag{2.33}$$

will give rise to

$$[T_A, Z_{ij}] = (S_A)^k_i Z_{ik} - (S_A)^k_j Z_{ik}. \tag{2.34}$$

Using **??** and **??** will show that

$$[Q_{ai}\{Q_{bj},\bar{Q}^k_{\dot{c}}\}]+[Q_{bj}\{\bar{Q}^k_{\dot{c}},Q_{ai}\}]+[\bar{Q}^k_{\dot{c}},\{Q_{ai},Q_{bj}\}]=0. \tag{2.35}$$

leading to $[\bar{Q}^k_{\dot{a}},Z_{ij}]=[Q_{ak},Z_{ij}]=0$. Apart from these, it can also be shown

$$[Z_{ik},Z_{kl}]=\epsilon^{ab}[\{Q_{ai},Q_{bj}\},Z_{kl}]=0 \tag{2.36}$$

or $a^A_{kl}[Z_{ij},T_A]=0$ which shows $[Z_{ij},T_A]=0$ for any $a^A_{kl}\neq 0$. Because of Z_{ij} commuting with all generators, Z_{ij} will be central charge that seems like no interation on this charge.

In conclusion, all algebras of supersymmtery are shown by

$$\begin{gathered}
[P^\rho,J^{\mu\nu}]=i(\eta^{\mu\rho}P^\nu-\eta^{\nu\rho}P^\nu),\quad [P^\nu,P^\nu]=0\\
[J^{\mu\nu},J^{\rho\sigma}]=-i(J^{\mu\sigma}\eta^{\nu\rho}-J^{\nu\sigma}\eta^{\mu\rho}+J^{\nu\rho}\eta^{\mu\sigma}-J^{\mu\rho}\eta^{\nu\sigma})\\
[P^\mu,Q_{ai}]=[P^\mu,\bar{Q}^i_{\dot{a}}]=0\quad \{Q_{ai},\bar{Q}^i_{\dot{a}}\}=-\frac{1}{2}\delta^j_i\sigma_{\mu a\dot{a}}P^\mu\\
[Q_{ai},J_{\mu\nu}]=i(\sigma_{\mu\nu})_a{}^bQ_{bi},\quad [\bar{Q}^{\dot{a}i},J_{\mu\nu}]=i(\bar{\sigma}^{\dot{a}})_{\dot{b}}\bar{Q}^{\dot{b}i}\qquad .\\
\{Q_{ai},Q_{bj}\}=-\frac{1}{2}\epsilon_{ab}Z_{ij},\quad \{\bar{Q}^i_{\dot{a}},\bar{Q}^j_{\dot{b}}\}=-\frac{1}{2}\epsilon_{\dot{a}\dot{b}}Z^{ij}\\
[Q_{ai},T_A]=(S_A)_i{}^jQ_{aj},\quad [\bar{Q}^i_{\dot{a}},T_A]=-(S^{*A})^i{}_j\dot{Q}^j_{\dot{a}}\\
Z_{ij}=a^A_{ij}T_A,\quad [T_A,T_B]=f_{AB}{}^CT_C
\end{gathered} \tag{2.37}$$

Also, there is a generator that does not commute with supercharge Q_{ai} called R-symmetry. Let O_A be a generator of this R-symmetry

$$[O_A,Q_{ai}]=(U_A)_i{}^jQ_{aj},\quad [O_A,Q^i_{\dot{a}}]=(U_A)^i{}_jQ^j_{\dot{a}}. \tag{2.38}$$

where $(U_Q)^i{}_j$ is a conjugate of $(U_A)_i{}^j$.

Consider super-Jacobi identity

$$[[O_A,O_B],Q_{ai}]+[[O_B,Q_{ai}],O_A]+[[Q_{ai},O_A],O_B]=0 \tag{2.39}$$

will give

$$[U_A, U_B] = -f_{AB}{}^C U_C \tag{2.40}$$

It shows that $-(U_A)_i{}^j$ is the representation of R-symmetry. By Super-Jacobi

$$\{[O_A, Q_{ai}], \bar{Q}^j_{\dot{b}}\} + \{[O_A, \bar{Q}^j_{\dot{b}}], Q_{ai}\} + [\{Q_{ai}, \bar{Q}^j_{\dot{b}}\}, O_A] = 0 \tag{2.41}$$

and $[P^\mu, O_A] = 0$ lead to

$$-\frac{1}{2}\sigma^\mu_{a\dot{b}} P_\mu [(U_A)_i{}^k \delta^j_k + (U_A)^j{}_k \delta^k_i] = 0 \tag{2.42}$$

or the other word

$$(U_A)_i{}^j = -(U_A)^j{}_i = -((U_A)_j{}^i)^*. \tag{2.43}$$

which shows $(U_A)_i{}^j$ is an anti-hermitian matrix. Thus, R-symmetry of supersymmetry with $4N$ supercharges is $U(N)$

2.3.1 Massless Representation

This representation refers to a massless particle in which $P^2 = 0$ and the momentum is chosen to be $k^\mu = (E, 0, 0, E)$

Replacing k^μ in the equation **??**, the result will be

$$\{Q_{ai}, Q^j_{\dot{b}}\} = \delta^j_i E \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.44}$$

Central mass is not valid for massless particle. The rest of anticommutator will be

$$\{Q_{ai}, Q_{bj}\} = \{\bar{Q}^i_{\dot{a}}, \bar{Q}_{\dot{b}^j}\} = 0. \tag{2.45}$$

The equation **??** shows that

$$\{Q_{2i}, \bar{Q}^j_{\dot{2}}\} = E\delta^j_i \tag{2.46}$$

The equation **??** gives

$$[J, Q_{ai}] = \frac{1}{2}\sigma_a^b Q_{bi} \tag{2.47}$$

and its conjugate is

$$[J, \bar{Q}^i_{\dot{a}}] = -\frac{1}{2}\sigma^{\dot{b}}_{\dot{a}} Q^i_{\dot{b}}. \tag{2.48}$$

Focusing only on z-direction, the equation **??** and **??** become

$$[J_3, Q_{2i}] = -\frac{1}{2}Q_{2i} \quad \text{and} \quad [J_3, \bar{Q}^i_{\dot{2}}] = \frac{1}{2}\bar{Q}^i_{\dot{2}}. \tag{2.49}$$

For simplicity, redefining $\hat{a}_i = \dfrac{Q_{2i}}{\sqrt{E}}$ and $\hat{a}_i^\dagger = \dfrac{\bar{Q}^i_{\dot{2}}}{\sqrt{E}}$, a new algebra is rewritten as

$$\{\hat{a}_i, \hat{a}_j^\dagger\} = \delta_{ij} \tag{2.50}$$

which gives the same sense of fermionic harmonic oscillator. Beginning with the lowest state with the minimal helicity k, h_{min}, the algebra can be defined by

$$\hat{a}_i |k, h_0\rangle = 0, \quad J_3 |k, h_0\rangle = h_0 |k, h_0\rangle . \tag{2.51}$$

Other higher states are found by raising the lowest state by $\hat{a}_i^\dagger$ as

$$\begin{aligned}
|k, h_0 + \frac{1}{2}; i\rangle &= \hat{a}_i^\dagger |k, h_0\rangle \\
|k, h_0 + 1; i, j\rangle &= \hat{a}_i^\dagger \hat{a}_j^\dagger |k, h_0\rangle \\
&\vdots \\
|k, h_0 + \frac{n}{2}; i_1, ..., i_n\rangle &= \hat{a}_{i_1}^\dagger ... \hat{a}_{i_n}^\dagger |k, h_0\rangle \\
&\vdots \\
|k, h_0 + \frac{N}{2}; i_1, ..., i_N\rangle &= \hat{a}_{i_1}^\dagger ... \hat{a}_{i_N}^\dagger |k, h_0\rangle
\end{aligned} \tag{2.52}$$

A state $\hat{a}^\dagger_{i_1}...\hat{a}^\dagger_{i_N}\,|k,h_0\rangle$ has a number of possible states as

$$\begin{pmatrix} N \\ n \end{pmatrix} = \frac{N!}{n!(N-n)!} \tag{2.53}$$

A total number of states in massless representation is

$$D(N) = \sum_{n=0}^{N} \begin{pmatrix} N \\ n \end{pmatrix} = (1+1)^N = 2^N \tag{2.54}$$

In Lorentz symmetry, discrete symmetry called CPT symmetry says that all states with helicity h must have states with helicity $-h$.

Quantum field theory under Lorentz symmetry allows particles to have helicity h where $-2 < h < 2$ to exist. Eventually, $h_{max} - h_{min} \leq 4$. From supposing $h_{min} = h_0$ and $h_{max} = h_0 + \dfrac{N}{2}$,

$$\begin{aligned} \frac{N}{2} &\leq 4 \\ N &\leq 8 \end{aligned} \tag{2.55}$$

This shows that in four dimensions, the possible highest supersymmetry is $N = 8$, which consists of 32 supercharges. In a different dimension, a number of supercharges remain the same while an amount of supersymmetries can be altered.

For example, the simplest case $N = 1$, chiral multiplet or scalar multiplet with the lowest-helicity state $h_0 = -\frac{1}{2}$ has possible states as

$$|k, -\frac{1}{2}\rangle\,, \quad |k, 0\rangle = \hat{a}^\dagger\,|k, -\frac{1}{2}\rangle\,. \tag{2.56}$$

which obviously have no CPT symmetry. According to CPT symmetry, $(0, \frac{1}{2})-$helicity state must be included and the opposite-helicity state of $|k,0\rangle$ is also $|k,0\rangle$. Hence, all states in this multiplet are

$$|k,-\frac{1}{2}\rangle\,, \quad |k,0\rangle\,, \quad |k,0\rangle\,, \quad |k,\frac{1}{2}\rangle \tag{2.57}$$

where $|k,\pm\frac{1}{2}\rangle$ is described by Weyl spinor while two $|k,0\rangle$ are described by two scalars.

Another example is the multiplet with $h_0 = -1$ called vector multiplet or gauge multiplet. By the same idea, all states are therefore shown by

$$(|-1\rangle\,, |-\frac{1}{2}\rangle) \oplus (|\frac{1}{2}\rangle\,, |1\rangle) \tag{2.58}$$

Supergravity multiplet with $h_0 = -2$ that will be the main multiplet to find holographic solutions in gauged supergravity will have states as

$$(|-2\rangle\,, |-\frac{3}{2}\rangle) \oplus (|\frac{3}{2}\rangle\,, |2\rangle) \tag{2.59}$$

where state $|\pm 2\rangle$ is a graviton while state $|\pm\frac{3}{2}\rangle$ is a gravitino, a superpartner of a graviton.

As the process of finding field contents in each multiplet shown above, repeating the same process with different N will give the field contents as shown at the Table 2.1.

Table 2.1: Field contents for each multiplet for $1 \leq N \leq 8$ in four dimensions

N	s_{max}	$s=2$	$s=3/2$	$s=1$	$s=1/2$	$s=0$
$N=1$	2	1	1			
	3/2		1	1		
	1			1	1	
	1/2				1	$1+1$
$N=2$	2	1	2	1		
	3/2		1	2	1	
	1			1	2	$1+1$
	1/2				2	$2+2$
$N=3$	2	1	3	3	1	
	3/2		1	3	3	$1+1$
	1				$3+1$	$3+3$
$N=4$	2	1	4	6	4	$1+1$
	3/2		1	4	$6+1$	$4+4$
	1			1	4	6
$N=5$	2	1	5	10	$10+1$	$5+5$
	3/2		1	$5+1$	$10+5$	$10+10$
$N=6$	2	1	6	$15+1$	$20+6$	$15+15$
	3/2		1	6	15	20
$N=7$	2	1	$7+1$	$21+7$	$35+21$	$35+35$
$N=8$	2	1	8	28	56	70

2.4 Supersymmetry transformations and algebra

In quantum field theory, the state of a particle can be constructed by a field operator acting on a vacuum state. In this section, we begin such construction from a vacuum state and raise the spin of a particle by a supercharge operator that leads to a spinor field operator.

Let's suppose Φ is the operator that can construct a particle with momentum p^μ and spin j as

$$|p, j\rangle = \Phi |0\rangle . \tag{2.60}$$

In a vacuum, supercharge will annihilate the vacuum state as

$$Q_{ai} |0\rangle = 0, \qquad \bar{Q}_{\dot{a}i} |0\rangle = 0. \tag{2.61}$$

Let's Z is a field operator of $|p, 0\rangle$ that can be built up from $|0\rangle$ as

$$|p, 0\rangle = Z |0\rangle \tag{2.62}$$

$Q_{\dot{a}}$ that annihilates $|p, 0\rangle$ can be written as

$$\bar{Q}_{\dot{a}} |p, 0\rangle = \bar{Q}_{\dot{a}} Z |0\rangle = 0. \tag{2.63}$$

This shows that Z commutes with $\bar{Q}_{\dot{a}}$ written as

$$[Z, \bar{Q}_{\dot{a}}] = 0. \tag{2.64}$$

Super-Jacobi identity

$$\{[Z, Q_a], \bar{Q}_{\dot{a}}\} - \{[\bar{Q}_{\dot{a}}, Z], Q_a\} + [\{Q_a, \bar{Q}_{\dot{a}}\}, Z] = 0, \tag{2.65}$$

leads to

$$[\{Q_a, \bar{Q}_{\dot{a}}\}, Z] = -\frac{1}{2}\sigma^\mu_{a\dot{a}}\partial_\mu Z = 0. \tag{2.66}$$

$|p, \frac{1}{2}\rangle$ is brought up by Q_a from the state $|p, 0\rangle$ as

$$Q_a |p, 0\rangle = Q_a Z |0\rangle = |p, \frac{1}{2}\rangle . \tag{2.67}$$

Now, let's define spinor field operator χ_a to construct spin-$\frac{1}{2}$ particle

$$\chi_a |0\rangle = |p, \frac{1}{2}\rangle , \tag{2.68}$$

or written as

$$\chi_a = [Q_a, Z]. \tag{2.69}$$

Then, consider super-Jacobi identity

$$\{Q_a, [Q_b, Z]\} - \{Q_b, [Z, Q_a]\} + [Z, \{Q_a, Q_b\}] = 0, \tag{2.70}$$

and

$$\{Q_a, Q_b\} = 0, \tag{2.71}$$

lead to

$$\{Q_a, \chi_b\} = \{Q_b, [Z, Q_a]\} = -\{Q_b, \chi_a\}. \tag{2.72}$$

This is obviously seen that $\{Q_a, \chi_b\}$ has anti-symmetric property under switching indices a and b. Thereby, this can be written as

$$\{Q_a, \chi_b\} = \epsilon_{ab} F. \tag{2.73}$$

To find $[Q_c, F]$, one may begin by

$$\epsilon_{ab}[Q_c, F] = [Q_c, \{Q_a, \chi_b\} \tag{2.74}$$

$$= -[Q_a, \{\chi_b, Q_c\}] - [\chi_b, \{Q_c, Q_a\}] \tag{2.75}$$

$$= -\epsilon_{cb}[Q_a, F] \tag{2.76}$$

and make a contraction with ϵ^{ab} to get

$$[Q_c, F] = 0. \tag{2.77}$$

Another commutator is $[\bar{Q}_{\dot{a}}, F]$ that can be found by similar way as

$$\epsilon_{ab}[\bar{Q}_{\dot{a}}, F] = [\bar{Q}_{\dot{a}}, \{Q_a, \chi_b\}] \tag{2.78}$$

$$= -[Q_a, \{\chi_b, \bar{Q}_{\dot{a}}\}] - [\chi_b, \{\bar{Q}_{\dot{a}}, Q_a\}] \tag{2.79}$$

$$= \frac{1}{2}\sigma^{\mu}_{b\dot{a}}[Q_a, \partial_\mu Z] + \frac{1}{2}\sigma^{\mu}_{a\dot{a}}[\chi_b, P_\mu] \tag{2.80}$$

$$= \frac{1}{2}\sigma^{\mu}_{b\dot{a}} - \frac{1}{2}\sigma^{\mu}_{a\dot{a}}\partial_\mu \chi_b, \tag{2.81}$$

contracted by ϵ^{ab} that results in

$$[\bar{Q}_{\dot{a}}, F] = -\frac{1}{2}\partial_\mu \chi^a \sigma^{\mu}_{a\dot{a}}. \tag{2.82}$$

This can be shown that all fields can close the algebra by considering the action of supercharges, which means (Z, χ_a, F) are in the same multiplet.

For any fields Φ, supersymmetry parameters and supercharges can lead to thier transformation as

$$\delta\Phi = [\bar{\epsilon}_{\dot{a}}\bar{Q}^{\dot{a}} + \epsilon^a Q_a, \Phi]. \tag{2.83}$$

From this relation, supersymmetry of the multiplet above can be concluded as

$$\delta Z = [\bar{\epsilon}_{\dot{a}}\bar{Q}^{\dot{a}} + \epsilon^a Q_a, Z] = \epsilon^a [Q_a, Z] = \epsilon^a \chi_a, \tag{2.84}$$

$$\delta \chi_a = \epsilon^c \{Q_c, \chi_a\} - \bar{\epsilon}^{\dot{a}} \{\bar{Q}_{\dot{a}}, \chi_a\} = F\epsilon_a + \frac{1}{2}\sigma^{\mu}_{a\dot{a}}\bar{\epsilon}^{\dot{a}}\partial_\mu Z, \tag{2.85}$$

$$\delta F = -\bar{\epsilon}^{\dot{a}}[\bar{Q}_{\dot{a}}, F] = \frac{1}{2}\bar{\epsilon}^{\dot{a}}\partial_\mu \chi^a \sigma^{\mu}_{a\dot{a}}. \tag{2.86}$$

The content in this chapter shows precisely that the anti-commutator of supercharges can generate momentum. This can be proved by considering supersymmetry transformations in the form of algebra, including supersymmetry parameters as

$$[\delta_1, \delta_2]\Phi = [\bar{\epsilon}_1 Q, [\bar{\epsilon}_2 Q, \Phi]] - [\bar{\epsilon}_2 Q, [\bar{\epsilon}_1 Q, \Phi]]. \tag{2.87}$$

With super-Jacobi identity

$$[\bar{\epsilon}_1 Q, [\bar{\epsilon}_2 Q, \Phi]] + [\bar{\epsilon}_2 Q, [\Phi, \bar{\epsilon}_1 Q]] + [\Phi, [\bar{\epsilon}_1 Q, \bar{\epsilon}_2 Q]], \tag{2.88}$$

and $\bar{\epsilon}_2 Q = Q\epsilon_2$ will give

$$[[\delta_1, \delta_2]\Phi] = [\bar{\epsilon}_1^{\alpha} Q_\alpha, \bar{Q}^b \epsilon_{2b}, \Phi] = -\frac{1}{2}\bar{\epsilon}_1 c^\mu \epsilon_2 [P_\mu, \Phi] = -\frac{1}{2}\bar{\epsilon}_1 c^\mu \epsilon_2 \partial_\mu \Phi \tag{2.89}$$

As known that the algebra can point out the symmetry of theory, this algebra given above will be the first step in construction of supergravity due to the constraint that supergravity must admit supersymmetry as its local symmetry which will be explained in the next chapter.

CHAPTER III

SUPERGRAVITY

The previous chapter provides the procedure to find what field content should be included in each multiplet. However, the theory cannot be described on the system with curved space due to the addition of graviton, a particle that can cause gravity. To broaden supersymmetry into gravitational theory, supersymmetry must play a role as local symmetry. This theory that supersymmetry is promoted to be locally invariant is called supergravity.

It is also said that supergravity is the gravitational theory having supersymmetry as gauge symmetry that gives the algebra

$$[\bar{\epsilon}_1(x)Q, \bar{\epsilon}_2(x)Q] = -\frac{1}{2}\bar{\epsilon}_1(x)\gamma^\mu\epsilon_2(x)\partial_\mu\Phi(x). \tag{3.1}$$

Compared to the previous chapter, the difference of the algebra is spotted clearly that supersymmetry parameters and a field are function of spacetime that is the indication of local symmetry or gauge symmetry.

3.1 Fermionic fields in curved spacetime

Before encountering supergravity, behaviors of fermions in curved spacetime must be clarified.

Fermions which, in this case, is a spinor that has no symmetry under diffeomorphism, but rather has local Lorentz transformation (LLT), their forms can be written as

$$\Phi'(x) = e^{-\frac{1}{2}\lambda^{ab}M_{ab}}\Phi(x) \tag{3.2}$$

where M_{ab} is a Lorentz generator in a representation of $\Phi(x)$. It should be noted that this equation can also describe not only a fermionic field but also other fields. The difference for such other fields is that M_{ab} will be a Lorentz generator in a representation of those fields. The covariant derivative is defined by

$$D_\mu \Phi = \partial_\mu \Phi + \frac{1}{2}\omega_\mu^{ab} M_{ab} \Phi. \tag{3.3}$$

For a spinor, it is transformed as

$$\psi'(x) = e^{-\frac{1}{4}\omega^{ab}\gamma_{ab}}\psi(x) \tag{3.4}$$

where its covariant derivative is shown by

$$D_\mu \psi = \partial_\mu \psi + \frac{1}{2}\omega_\mu^{ab}\gamma_{ab}\psi. \tag{3.5}$$

3.2 Torsion

Since supergravity is the theory of gravity coupled to fermionic fields, it generates a term of torsion inevitably. Unlike the theory of general relativity, Christoffel symbol having symmetry under switching indices as $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$ cannot give a birth to torsion. Torsion tensor is defined by

$$T^a = de^a + \omega^a_b \wedge e^b \tag{3.6}$$

which has a component on coordinate basis as

$$T^a_{\mu\nu} = 2e_{[\mu}e^a_{\nu]} + 2\omega^{ab}_{[\mu}e_{\nu]b} \tag{3.7}$$

or veilbein basis as

$$T_{abc} = \Omega_{abc} + \omega_{bac} - \omega_{cab} \tag{3.8}$$

where $T^a_{bc} = T^a_{\mu\nu}e^\mu_b e^\nu_c$, $\omega_{abc} = e^\mu_a \omega_{\mu bc}$ and $\Omega^a_{bc} = 2e^\mu_b e^\nu_a \partial_{[\mu}e^a_{\nu]}$ are anholonomy coefficients. Switching indices can give a relation

$$\omega_{abc} = \omega(e)_{abc} + K_{abc} \tag{3.9}$$

where $\omega(e)$ is the function of veilbein without torsion and K_{abc} is a contorsion tensor defined by

$$\omega(e)_{abc} = \frac{1}{2}(\Omega_{abc} - \Omega_{bca} + \Omega_{cab}) \tag{3.10}$$

$$K_{a[bc]} = -\frac{1}{2}(T_{[ab]c} - T_{[bc]a} + T_{[ca]b}). \tag{3.11}$$

With the existence of torsion, connection $\Gamma^\rho_{\mu\nu}$ will change into the form

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) - K_{\mu\nu}{}^\rho. \tag{3.12}$$

Antisymmetrizing $\Gamma^\rho_{\mu\nu}$ can give

$$\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu} = K_{\nu\mu}{}^\rho - K_{\mu\nu}{}^\rho = T_{\mu\nu}{}^\rho. \tag{3.13}$$

Spinor connection also allows to give curvature tensor in the form of two-form as

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \frac{1}{2}R_{\mu\nu}{}^a{}_b dx^\mu \wedge dx^\nu = \frac{1}{2}R_{cd}{}^a{}_b e^c \wedge e^d. \tag{3.14}$$

Besides, Bianchi identity is changed to

$$\nabla_{[\mu}R_{\nu\rho]}{}^{\sigma\tau} = -T_{[\mu\nu}{}^\lambda R_{\rho]\lambda}{}^{\sigma\tau}, \tag{3.15}$$

but its covariant derivative remains the same as

$$D_{[\mu}R_{\nu\rho]}{}^{ab} = 0 \tag{3.16}$$

Since D_μ always involves connection, their commutaotr can give the curvature tensor as

$$[D_\mu, D_\nu]\Phi = \frac{1}{2}R_{\mu\nu}{}^{ab}M_{ab}\Phi \tag{3.17}$$

where M_{ab} is a Lorent generator in the appropriate representation of Φ. Applied to vector and spinor field,

$$[D_\mu, D_\nu]V^a = R_{\mu\nu}{}^{a}{}_{b}V^b \tag{3.18}$$

$$[D_\mu, D_\nu]\psi = \frac{1}{4}R_{\mu\nu}{}^{ab}\gamma_{ab}\psi. \tag{3.19}$$

3.3 $N = 1$ Supergravity

Starting with the simplest model of supergravity, since the supergravity is the theory of gravity, which acts as a particle named graviton, that has supersymmetry, its multiplet called supergravity multiplet must comprise a particle with spin $\frac{3}{2}$ which is later named as gravitino. Gravitino $\psi_{\alpha\mu}$ where α is the spinor index and μ is the spacetime index is conventionally written by ψ_μ.

To construct gravitino, the first step may begin by considering the product under Lorentz group as

$$(\frac{1}{2}, \frac{1}{2}) \otimes [(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)] = (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \oplus (\frac{1}{2}, 1) \oplus (1, \frac{1}{2}). \tag{3.20}$$

It is clearly seen that the product can give a particle spin $\frac{3}{2}$ via representation $(\frac{1}{2}, 1) \otimes (1, \frac{1}{2})$. $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ can be traced out by gamma-traceless condition $\gamma^\mu\psi_\mu = 0$. Accordingly, the product $(\frac{1}{2}, \frac{1}{2}) \otimes [(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)]$ altogether with $\gamma^\mu\psi_\mu = 0$ will perfectly describe $\frac{3}{2}$-spin particle whic is called "Rarita-Schwinger" field.

ψ_μ can as usual other gauge fields generally transforms as

$$\psi_\mu(x) \to \psi'_\mu(x) = \psi_\mu(x) + \partial_\mu\epsilon(x) \tag{3.21}$$

where $\epsilon(x)$ is a spinor parameter.

The action of Rarita-Schwinger field is represented by

$$S = -\int d^4x \bar{\psi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \psi_\rho \tag{3.22}$$

which gives a field equation

$$\gamma^{\mu\nu\rho} \partial_\nu \psi_\rho = 0. \tag{3.23}$$

Thus, the supergravity multiplet consisting of graviton e^a_μ and gravitino ψ_μ has the action given by

$$S = \frac{1}{2\kappa^2} \int d^4x e[e^{a\mu} e^{b\nu} R_{\mu\nu ab}(\omega) - \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho] \tag{3.24}$$

where the first term refers to Einstein-Hilbert action and the second term is the action of gravitino or Rarita-Scwinger field.

Gravitino is transformed covariantly as

$$D_\mu \psi_\nu = \partial_\mu \psi_\nu + \frac{1}{4} \omega^{ab}_\mu \gamma_{ab} \psi_\nu. \tag{3.25}$$

The action is invariant under supersymmetry transformation

$$\delta e^a_\mu = \frac{1}{2} \bar{\epsilon} \gamma^a \psi_\mu \quad \text{and} \quad \delta \psi_\mu = D_\mu \epsilon \tag{3.26}$$

where

$$D_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{4} \omega^{ab}_\mu \gamma_{ab} \epsilon. \tag{3.27}$$

3.4 General structure of supergravity with $N > 2$

Ungauged supergravity with $N = 1$ supersymmetry can be seen obviously that its Lagrangian and supersymmetry transformations of fermionic fields can be shown explicitly. However, difficulty probably arises when it comes to the theory with $N > 2$. In this section, a general structure of supergravity with $N > 2$ will be clarified, see [?,?,?,?,?,?,?].

Regarding to a growing numbers of supercharges, supersymmetries are adequate to indicate a general structure clearly. For $N > 2$ the Lagrangian of bosonic fields generally has a form as

$$e^{-1}\mathcal{L}_B = \frac{R}{2} - \frac{1}{2}G_{st}\partial_\mu\phi^s\partial^\mu\phi^t + \frac{1}{4}I_{\Lambda\Sigma}(\phi)F^{\Lambda}_{\mu\nu}F^{\Sigma\mu\nu} + \frac{1}{8}e^{-1}R_{\Lambda\Sigma}(\phi)\epsilon^{\mu\nu\rho\sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma}_{\rho\sigma} \tag{3.28}$$

where e is a vielbein. $R_{\Lambda\Sigma}$ and $I_{\Lambda\Sigma}$ are real and imaginary part of matrix $\mathcal{N}_{\Lambda\Sigma}$ respectively. The indices $s, t = 1, ..., n_s$ represents all n_s scalar fields. The index $\Lambda, \Sigma = 1, ..., n_v$ indicates a number of all vector fields.

From equation **??**, G_{st}, a metric of a scalar manifold, clearly indicates an information of symmetries in the theory. As supergravity with $N > 2$ contains sufficient supersymmetries to give a precise geometrical structure of scalar manifold, this scalars manifold described as homogeneous symmetric space in the form of G/H show that G is the isometry of the manifold and H is a subgroup of G.

3.4.1 Scalar manifold

Due to the rich symmetry of the structure on $N > 2$ supergravity, various scalar can be described by scalar manifold where each scalar is the coordinate of this manifold. This manifold has an isometry as $g \in G$ acting on ϕ^s generates transformation $\phi^s \to \phi^{s'}(\phi) = g \circ \phi^s$ where $\circ$ represents action of the g on ϕ^s which leave G_{st} invariant under isometry

$$G_{\bar{s}\bar{t}}(\phi'(\phi))\frac{\partial\phi'^{s'}}{\partial\phi^s}\frac{\partial\phi'^{t'}}{\partial\phi^t} = G_{st}(\phi). \tag{3.29}$$

Since holonomy group H is responsible for parallel transportation on a closed trajectory, the connection will be entwined with this group. H group can be described in supergravity by

$$H = H_R \times H_m \tag{3.30}$$

where $H_R = U(N)$ is R-symmetry for $N < 8$ and $H_R = SU(8)$ for only $N = 8$ and H_m is compact group relevant to matter field. Generally, $H = H_R$ for $N > 4$ due to the absence of matter multiplet.

Manifold $\mathcal{M} = G/H$ is described by members of a group G in the form $L(x)$. The transformation of $L(x)$ under g (member) G is found by multiplying with g on the left of $L(x)$ while the transformation under H is to multiply on the right as

$$L(x) \to L'(x) = gL(x)h(x) \tag{3.31}$$

By the choice of arbitrary local gauge symmetry under $h(x)$, $L(x)$ can be written in the form of $L(\phi(x))$ called coset representation Lie algebra of G and H given by $\mathfrak{g}$ and $\mathfrak{h}$ can be described in the form of coset space as

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}. \tag{3.32}$$

where $\mathfrak{h}$ and $\mathfrak{t}$ are an algebra of H and a complement. Homogeneous manifold gives that

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{t}] \subset \mathfrak{t}, \quad [\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{h} \oplus \mathfrak{t} \tag{3.33}$$

This concludes that generator in $\mathfrak{h}$ and $\mathfrak{t}$ will be compact and non-compact generator respectively.

A construction of $L(\phi)$ by a generator is called parameterization. Two ways of parameterizations are described by solvable parameterization as

$$L = e^{\phi^r T_r} \tag{3.34}$$

as the first example where T_r $r = 1, ..., n_s$ is a generator of G_s while the second is unitary parameterization as

$$L = e^{\phi^s Y_s} \tag{3.35}$$

where Y_s is a basis vector of $\mathfrak{t}$

Geometrical structure of $\mathcal{M}$ is described by left-invariant 1-form defined by

$$\Omega = L^{-1} dL \tag{3.36}$$

which satisfies Maurer-Cartan equation

$$d\Omega + \Omega \wedge \Omega = 0 \tag{3.37}$$

Due to $\Omega \in \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}$, Ω then can be decomposed into $P \in \mathfrak{t}$ and $Q \in \mathfrak{h}$ as

$$\Omega = P + Q \tag{3.38}$$

which is also written in the form of coordinate basis

$$\Omega_r d\phi^r = P_r d\phi^r + Q_r d\phi^r \tag{3.39}$$

This equation changed the form under the transformation of $L(\phi)$ will consequently show that

$$\Omega(g \circ \phi) = h^{-1} L^{-1}(\phi) g^{-1} d(gL(\phi)h) = h^{-1} L^{-1}(\phi) dL(\phi) h + h^{-1} dh \tag{3.40}$$

A global transformation under G results in $dg = 0$. The value $h^{-1}dh \in \mathfrak{h}$ projected to subspace h and t leads to

$$\begin{aligned} P(g \circ \phi) &= h^{-1} P h \\ Q(g \circ \phi) &= h^{-1} dh + h^{-1} dh. \end{aligned} \tag{3.41}$$

This obviously describes that P transforms linearly and Q transforms as a gauge connection called composite connection.

P can be written in the form of basis $Y_{\hat{s}}$ where $\hat{s}, \hat{t}, ...$ are the tangent space indices as

$$P = P^{\hat{s}} Y_{\hat{s}} \tag{3.42}$$

and $P^{\hat{s}}$ on the coordinate basis is

$$P^{\hat{s}} = P^{\hat{s}}_s d\phi^s. \tag{3.43}$$

From the equation **??**, left invariant 1-form veilbein $P^{\hat{s}}$ transforms under G as

$$P^{\hat{s}}(g \circ \phi) = h^{\hat{s}}_{\hat{t}} P^{\hat{t}} \tag{3.44}$$

Covariant derivative of L can be expressed together with Q as

$$DL = dL - LQ = LP, \quad L^{-1}DL = L^{-1}dL - Q = P. \tag{3.45}$$

This veilbein P satisfies veilbein's postulate as the same as veilbein in spacetime as

$$DP = dP + Q \wedge P + P \wedge Q = 0. \tag{3.46}$$

2-form curvature of manifold $\mathcal{M}$ can be found by

$$R(Q) = dQ + Q \wedge Q = -P \wedge P \tag{3.47}$$

The component of this curvature have its components

$$R(Q) = \frac{1}{2} R_{rs} d\phi^r \wedge d\phi^s \tag{3.48}$$

where $R_{rs} = -[P_r, P_s] \in \mathfrak{h}$.

For any fields $\Phi(x)$ on $\mathcal{M}$, the covariant derivative can be defined as

$$D_r\Phi = \partial_r\Phi + Q_r \circ \Phi \tag{3.49}$$

where $Q \circ \Phi$ is the operation of Q acting on the representation that Φ lives in. The derivative of D_r satisfies Ricci identity

$$[D_r, D_s]\Phi = R_{rs} \circ \Phi \tag{3.50}$$

The metric that is invariant under H is defined by the basis $Y_{\hat{s}}$

$$\eta_{\hat{s}\hat{t}} = kTr(Y_{\hat{s}}Y_{\hat{t}}) \tag{3.51}$$

where k is a positive constant depending on the representation of $Y_{\hat{s}}$ leading to the metric on $\mathcal{M}$ as

$$ds^2 = G_{st}d\phi^s d\phi^t = P_{\hat{s}}^{\hat{s}} P_{\hat{t}}^{\hat{t}} \eta_{\hat{s}\hat{t}} d\phi^s d\phi^t = kTr(PP). \tag{3.52}$$

The eq. **??** will show that the metric is invariant under G.

$$ds^2(g \circ \phi) = ds^2(\phi) \tag{3.53}$$

The equation **??** can also be used to write the Lagrangian of scalar field as

$$\mathcal{L}_{scalar} = \frac{1}{2}eG_{st}\partial_\mu\phi^s\partial^\mu\phi^t = \frac{1}{2}ekTr(P_\mu P^\mu) \tag{3.54}$$

3.4.2 Electric-magnetic duality and vector fields

A field strength tensor $F^\Lambda_{\mu\nu}$ can be used to define its dual tensor called magnetic dual tensor

$$G_{\Lambda\mu\nu} = -\epsilon_{\mu\nu\rho\sigma}\frac{\partial\mathcal{L}}{\partial F^\Lambda_{\mu\nu}} = R_{\Lambda\Sigma}F^\Sigma_{\mu\nu} - I_{\Lambda\Sigma} * F^\Sigma_{\mu\nu} \tag{3.55}$$

where

$$*F^{\Lambda}_{\mu\nu} = \frac{1}{2} e\epsilon_{\mu\nu\rho\sigma} F^{\Lambda\rho\sigma} \tag{3.56}$$

The Bianchi identity will bring up equation

$$\nabla^{\mu}(*G_{\Lambda\mu\nu}) = 0, \quad \nabla^{\mu}(*F^{\Lambda}_{\mu\nu}) = 0 \tag{3.57}$$

The definition **??** helps to write $*F^{\Lambda}$ in the form of F^{Λ} and G^{Λ} as

$$*F^{\Lambda} = I^{\Lambda\Sigma}(R_{\Sigma\Gamma}F^{\Gamma} - G_{\Sigma}) \tag{3.58}$$

where $I^{\Lambda\Sigma}$ is the inverse of matrix $I_{\Lambda\Sigma}$.

Duality of G_{Λ} in the equation **??** gives rise to

$$*G_{\Lambda} = (RI^{-1}R + I)_{\Lambda\Sigma}F^{\Sigma} - (RI^{-1})_{\Lambda}{}^{\Sigma}G_{\Sigma} \tag{3.59}$$

Combining F^{Λ} and G_{Λ} will form the vector $2n_{\nu}$ dimensions

$$\mathcal{G}^{M} = \begin{pmatrix} F^{\Lambda} \\ G_{\Lambda} \end{pmatrix} \tag{3.60}$$

This is used to write field equation and Bianchi identity as

$$d\mathcal{G}^{M} = 0 \tag{3.61}$$

and $\mathcal{G}$ can relate to its duality as

$$*\mathcal{G} = -\mathbb{C}\mathbb{M}(\phi)\mathcal{G} \tag{3.62}$$

where $\mathbb{C}$ is a symplectic matrix

$$\mathbb{C} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_{\mathbf{n}_\nu} \\ -\mathbf{I}_{\mathbf{n}_\nu} & \mathbf{0} \end{pmatrix} \tag{3.63}$$

where $\mathbf{I}_{\mathbf{n}_\nu}$ is the $n_\nu \times n_\nu$ identity matrix and $\mathbb{M}$ has components as

$$\mathbb{M}_{MN} = \begin{pmatrix} (RI^{-1}R + I)_{\Lambda\Sigma} & -(RI^{-1})^{\Gamma}_{\Lambda} \\ -(I^{-1}R)^{\Delta}_{\Sigma} & I^{\Delta\Gamma} \end{pmatrix} \tag{3.64}$$

and $\mathbb{M}$ is also symplectic matrix

$$\mathbb{M}\mathbb{C}\mathbb{M} = \mathbb{C} \tag{3.65}$$

These all aforementioned ingredients help to write field equation and Einstein equation under the symmetry of duality

$$\mathcal{D}_\mu \partial^\mu \phi^s = \frac{1}{8} G^{st} \mathcal{G}^T_{\mu\nu} \partial_t \mathbb{M} \mathcal{G}^{\mu\nu} \tag{3.66}$$

$$R_{\mu\nu} = G_{rs} \partial_\mu \phi^r \partial_\nu \phi^s + \frac{1}{2} \mathcal{G}^T_{\mu\rho} \mathbb{M} \mathcal{G}_\nu{}^\rho \tag{3.67}$$

3.4.3 Global symmetry

For supergravity $N > 1$, G symmetry is extended to be a symmetry of field equation. It can be said that every transformation $\phi \to g \circ \phi$ has $2n_\nu \times 2n_\nu$ matrix $R_\nu[g]$ that give a transformation

$$\mathcal{G}^M = R_\nu[g]^M{}_N \mathcal{G}^N \tag{3.68}$$

where $R_\nu[g]^M N$ is $g \in G$ in a representation of vector and its hodge duality.

The explicit form of $R_\nu[g]^M_N$ can be expressed by

$$R_\nu[g]^M{}_N = \begin{pmatrix} A[g]^\Lambda{}_\Sigma & B[g]^{\Lambda\Sigma} \\ C[g]_{\Lambda\Sigma} & D[g]_\Lambda{}^\Sigma \end{pmatrix} \tag{3.69}$$

$*F^\Lambda$ found by duality transformation of F^Λ will give constraints to $R_\nu[g]^M_N$ as

1. $R_\nu[g]^M_N$ must be symplectic matrix as

$$R_\nu[g]^T \mathbb{C} R_\nu[g] = \mathbb{C} \tag{3.70}$$

2. $R_\nu[g]$ leads $\mathbb{M}$ to transform as

$$\mathbb{M}(g \circ \phi) = (R_\nu[g]^{-1})^T \mathbb{M}(\phi) R_\nu[g]^{-1} \tag{3.71}$$

Because of the invariance of **??**, matrix $\mathcal{N} = R_{\Lambda\Sigma} + iI_{\Lambda\Sigma}$ must transform under $R_\nu[g]$ as

$$\mathcal{N}(g \circ \phi) = \frac{C[g] + D[g]\mathcal{N}(\phi)}{A[g] + B[g]\mathcal{N}(\phi)} \tag{3.72}$$

Duality of R_ν can be also defined as $R_{\nu *} = (R_\nu^{-1})^T$ and the constraint **??** leads $(R_\nu^{-1})^T$ to

$$(R_\nu[g]^{-1})^T = \mathbb{C} R_\nu[g] \mathbb{C}, \quad R_{\nu *}[g]_M{}^N = \mathbb{C}_{MP} R_\nu[g]^P{}_Q \mathbb{C}^{NQ} \tag{3.73}$$

Matrix $\mathbb{M}$ can be transformed by simplectic matrix E as

$$\mathbb{M}' = E \mathbb{M} E^T \tag{3.74}$$

where $E \in Sp(2n_\nu, \mathbb{R})$. However, this transformation gives redundancy of matrix $\mathbb{M}$. Matrix E that gives different frames is

$$E \in GL(n_\nu, \mathbb{R}) \setminus Sp(2n_\nu, \mathbb{R}) / R_{\nu^*}[G]. \tag{3.75}$$

In general, symmetry of duality is not action's symmetry, but the symmetry of field equations and Bianchi identity. From equation **??**, if $B[g]^{\Lambda\Sigma} \neq 0$, the equation is rewritten to

$$F^{\Lambda'} = A[g]^\Lambda{}_\Sigma F^\Sigma + B[g]^{\Lambda\Sigma} G_\Sigma. \tag{3.76}$$

However, $R_\nu[g]$ for $g \in G_e$ generally has the form as

$$R_\nu[g]^M{}_N = \begin{pmatrix} A[g]^\Lambda{}_\Sigma & 0 \\ C[g]_{\Lambda\Sigma} & (A[g]^{-1})^T{}_\Lambda{}^\Sigma \end{pmatrix}. \tag{3.77}$$

where $B[g]^{\Lambda\Sigma} = 0$ and $D = (A^{-1})^T$. $R_\nu[g]$ can also be transformed by matrix $S^N{}_{\bar{N}} \in Sp(2n_\nu, \mathbb{R})/U(n_\nu)$ as $\tilde{R}_\nu[h] = S^{-1} R_\nu S$ that satisfies

$$\tilde{R}_\nu[h]^T \tilde{R}_\nu[h] = \mathbf{I} \tag{3.78}$$

Let's define coset representative in $\tilde{R}_\nu$ as

$$\tilde{L}^M_{\bar{N}} = R_\nu[L]^M{}_N S^N{}_{\bar{N}} \tag{3.79}$$

that leads to

$$R_\nu[g]\tilde{L}(\phi) = \tilde{L}(g \circ \phi)\tilde{R}_\nu[h] \tag{3.80}$$

for all $g \in G$ and $h(\phi, g) \in H$ where indices $M, N, ... = 1, ..., 2n_\nu$ and $\bar{M}, \bar{N}, ..., = 1, ..., 2n_\nu$ indicate the transformation under G and H respectively

$\tilde{L}$ can be used to write $\mathbb{M}$ as

$$\mathbb{M}_{MN} = \mathbb{C}_{MP}\tilde{L}^P_{\bar{L}}\tilde{L}^R_{\bar{L}}\mathbb{C}_{RN} \tag{3.81}$$

With symplectic properties on $R_\nu[g]$ and orthogonal condition of $\tilde{R}_\nu[h]$ can give

$$\mathbb{M}(g \circ \phi) = (R_\nu[g]^{-1})^T \mathbb{M}(\phi) R_\nu[g]^{-1} \tag{3.82}$$

where $\mathbb{M}$ is invariant under H and can be used to write Lagrangian density of scalar as

$$\mathcal{L}_{scal} = \frac{1}{8} ekTr[(\mathbb{M}^{-1}\partial_\mu \mathbb{M})(\mathbb{M}^{-1}\partial^\mu \mathbb{M})] \tag{3.83}$$

Thereby, bosons will transform under G as

$$\delta\phi^s = \Lambda^a k_a^s \tag{3.84}$$

$$\delta\mathbb{M} = \Lambda^a k_a^s \partial_s \mathbb{M} = \Lambda^a (R_{\nu *}[t_a]\mathbb{M} + \mathbb{M} R_{\nu *}[t_a]^T) \tag{3.85}$$

$$\delta\mathcal{G}^M_{\mu\nu} = -\Lambda^a (t_a)_N{}^M \mathcal{G}^N_{\mu\nu} \tag{3.86}$$

where $(t_a)_M{}^M$ holds symplectic condition as

$$(t_a)_N{}^N \mathbb{C}_{NP} = (t_a)_P{}^N \mathbb{C}_{NM} \tag{3.87}$$

3.5 Fermionic field

For $N > 2$ supergravity, spinors live only in gravity and vector multiplets, cannot transform under G, but transform locally under holonomy group H. Different supersymmetries will give different fermions. $A, B, ... = 1, ..., N$ represent the fundamental representation of $H_R = U(N)$ indices for $3 \leq N \leq 6$ and $H_R = SU(8)$ and $i, j = 1, ..., n$ are indices of fundamental representation of $H_m = SU(n)$ for $N = 3$ and $H_m = SO(n)$ for $N = 4$. The spinors that have no index are singlet while spinors with indices ABC are antisymmetric rank-3 tensor of H_R representation. Fermion $(\psi_{\mu A}, \chi_{ABC}, \lambda_{Ai})$ have positive chirality as

$$\gamma_5\psi_{\mu A} = \psi_{\mu A}, \quad \gamma_5\chi_{ABC} = \chi_{ABC}, \quad \gamma_5\lambda_{Ai} = \lambda_{Ai} \tag{3.88}$$

while their conjugate $(\psi^A_\mu, \chi^{ABC}, \lambda^A_i)$ have negative chirality

$$\gamma_5\psi^A_\mu = -\psi^A_\mu, \quad \gamma_5\chi^{ABC} = -\chi^{ABC}, \quad \gamma_5\lambda^A_i = -\lambda^A_i \tag{3.89}$$

For $N = 3, 5, 6$, additional spinors are $\lambda_{ABCi} = \lambda_i \epsilon_{ABC}, \chi, \chi^A$ which have negative chirality as

$$\gamma_5\lambda_i = -\lambda_i, \quad \gamma_5\chi = -\chi, \quad \gamma_5\chi^A = -\chi^A. \tag{3.90}$$

Apart from these spinors, the remaining spinors will be all incuded as

$$\lambda_I = (\chi_{ABC}, \lambda_{Ai}), \quad \gamma_5 \lambda_I = \lambda_I. \tag{3.91}$$

It is interesting to see that bosons transform under G, but not for H while fermions transform under H, but not for G. These transformations under G and H are similar to transformations under GCT and LLT under spacetime.

Due to fermions transforming under H, like under LLT in spacetime, there are appropriate composite connections to describe covariant derivative as

$$\mathcal{D}_\mu \psi = D_\mu \psi + Q \circ \psi \tag{3.92}$$

where D_μ is covariant derivative of spacetime and $Q \circ \psi$ is an action of connection Q in representation of ψ. This will lead to kinetic term of fermionic fields

$$\mathcal{L}_f = i\epsilon^{\mu\nu\rho\sigma}(\bar{\psi}^A_\mu \gamma_\nu \mathcal{D}_\rho \psi_{A\sigma} - \bar{\psi}_{A\mu}\gamma_\nu \mathcal{D}_\rho \psi^A_\sigma) - \frac{1}{2}e(\bar{\lambda}^I \gamma^\mu \mathcal{D}_\mu \lambda_I + \bar{\lambda}_I \gamma^\mu \mathcal{D}_\mu \lambda^I) \tag{3.93}$$

where second term is written explicitly as

$$\begin{aligned} \frac{1}{2}e(\bar{\lambda}^I \gamma^\mu \mathcal{D}_\mu \lambda_I + \bar{\lambda}_I \gamma^\mu \mathcal{D}_\mu \lambda^I) &= -\frac{1}{12}e(\bar{\chi}^{ABC}\gamma^\mu \mathcal{D}_\mu \chi_{ABC} + \bar{\chi}_{ABC}\gamma^\mu \mathcal{D}_\mu \chi^{ABC}) \\ &\quad - \frac{1}{2}e(\bar{\lambda}^{Ai}\gamma^\mu \mathcal{D}_\mu \lambda_{Ai} + \bar{\lambda}_{Ai}\gamma^\mu \mathcal{D}_\mu \lambda^{Ai}) \end{aligned} \tag{3.94}$$

This also takes the same form for additional fermions found in $N = 3, 5, 6$.

3.6 Completed Lagrangian of $N > 2$ supergravity

Using Cayley matrix as the following,it is more convenient to write the interaction between bosons and fermions

$$A^{\bar{M}}{}_{\bar{N}} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{I} & i\mathbf{I} \\ \mathbf{I} & -i\mathbf{I} \end{pmatrix}. \tag{3.95}$$

This is used to transform

$$R_\nu = A\tilde{R}_\nu A^\dagger \tag{3.96}$$

into complex form.

Composite connection from $Q = A\tilde{R}_\nu[Q]A^\dagger$ will then be rewritten as

$$Q^{\bar{M}}{}_{\bar{N}} = \begin{pmatrix} Q^{\bar{\Lambda}}{}_{\bar{\Sigma}} & 0 \\ 0 & Q_{\bar{\Lambda}}{}^{\bar{\Sigma}} \end{pmatrix} \tag{3.97}$$

where indices $\bar{\Lambda}, \bar{\Sigma}$ can be separated into $A, B, ...$ representing indices of H_R and i, j are indices of H_m and submatrix

$$Q^{\bar{\Lambda}}{}_{\bar{\Sigma}} = \begin{pmatrix} Q^{AB}{}_{CD} & 0 \\ 0 & Q^i{}_j \end{pmatrix}, \qquad Q_{\bar{\Lambda}}{}^{\bar{\Sigma}} = \begin{pmatrix} Q_{AB}{}^{CD} & 0 \\ 0 & Q_i{}^j \end{pmatrix} \tag{3.98}$$

where $Q^{AB}{}_{CD}$ and $Q^i{}_j$ are connections of H_R and H_m respectively.

These gain benefits in writing covariant derivative of $\psi_{A\mu}$, χ_{ABC} and λ_{Ai} as

$$\begin{aligned} \mathcal{D}_\mu \psi_{A\nu} &= \partial_\mu \psi_{A\nu} - \Gamma^\rho_{\mu\nu}\psi_{A\rho} + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\psi_{A\nu} + Q^B_{\mu A}\psi_{B\nu} \\ \mathcal{D}_\mu \chi_{ABC} &= \partial_\mu \chi_{ABC} + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\chi_{ABC} + 3Q_{\mu[A}{}^D\chi_{BC]D} \\ \mathcal{D}_\mu \lambda_{Ai} &= \partial_\mu \lambda_{Ai} + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\lambda_{Ai} + Q_{\mu A}{}^B\lambda_{Bi} + Q^j_{\mu i}\lambda_{Aj} \end{aligned} \tag{3.99}$$

This matrix also helps write veilbein of scalar manifold as $P = A\tilde{R}_\nu[P]A^\dagger$

$$P^{\bar{M}}{}_{\bar{N}} = \begin{pmatrix} 0 & P^{\bar{\Lambda}\bar{\Sigma}} \\ P_{\bar{\Lambda}\bar{\Sigma}} & 0 \end{pmatrix} \tag{3.100}$$

where

$$P^{\bar{\Lambda}\bar{\Sigma}} = \begin{pmatrix} P^{ABCD} & P^{ABj} \\ P^{iCD} & P^{ij} \end{pmatrix} \tag{3.101}$$

and

$$P_{\bar{\Lambda}\bar{\Sigma}} = \begin{pmatrix} P_{ABCD} & P_{ABj} \\ P_{iCD} & P_{ij} \end{pmatrix}. \tag{3.102}$$

Supergravity with different supersymmetries has different components of $P_{\bar{\Lambda}\bar{\Sigma}}$ which will be used to write $\mathcal{L}$ of scalar seperately as the followings. For $N > 4$,

$$e^{-1}\mathcal{L}_{scalar} = \frac{1}{48} P_{\mu}^{ABCD} P^{\mu}_{ABCD}. \tag{3.103}$$

For $N = 6$ and $N = 5$,

$$e^{-1}\mathcal{L}_{scalar} = \frac{1}{24} P_{\mu}^{ABCD} P^{\mu}_{ABCD}. \tag{3.104}$$

For $N = 4$,

$$e^{-1}\mathcal{L}_{scalar} = \frac{1}{24} P_{\mu}^{ABCD} P^{\mu}_{ABCD} + \frac{1}{4} P_{\mu}^{iAB} P^{\mu}_{AB}. \tag{3.105}$$

For $N = 3$,

$$e^{-1}\mathcal{L}_{scalar} = \frac{1}{2} P_{\mu}^{iAB} P^{\mu}_{iAB}. \tag{3.106}$$

By the transformation of $\tilde{L}(\phi)$ under H, $\tilde{L}(\phi)$ can be redefined as $\mathbb{L}(\phi) = \tilde{L}(\phi)A^{\dagger}$ which has components as

$$\mathbb{L} = \begin{pmatrix} f^{\Lambda}{}_{AB} & f^{\Lambda}{}_{i} & \bar{f}^{\Lambda AB} & \bar{f}^{\Lambda}{}_{i} \\ h_{\Lambda AB} & h_{\Lambda i} & \bar{h}_{\Lambda}^{AB} & \bar{h}_{\Lambda}^{i} \end{pmatrix} \tag{3.107}$$

transforming under G and H as

$$R_{\nu}[g]L(\phi) = L(g \circ \phi)R_{\nu}[h] \tag{3.108}$$

and symplectic condition of $\mathbb{L}$ leads to

$$\mathbb{L}^{\dagger}\mathbb{C}\mathbb{L} = \hat{\mathbb{C}} \tag{3.109}$$

where $\hat{\mathbb{C}} = A\mathbb{C}A^\dagger$. Complex form is extended to left-invariant one-form as

$$\Omega^c = A\tilde{R}_\nu[\Omega]A^\dagger = L^{-1}dL = \mathcal{P} + \mathcal{Q} \tag{3.110}$$

$\mathbb{M}$ can also be written as

$$\mathbb{M} = \mathbb{C}\mathbb{L}\mathbb{L}^\dagger\mathbb{C} \tag{3.111}$$

and in the form of matrix $\mathbf{f} = (f^\Lambda{}_{AB})$ and $\mathbf{h} = (h_{\Lambda AB}, h_{\Lambda i})$ as

$$\mathbb{M} = \begin{pmatrix} -2\mathbf{h}\mathbf{h}^\dagger & 2\mathbf{h}\mathbf{f}^\dagger + i\mathbf{I} \\ 2\mathbf{f}\mathbf{h}^\dagger - i\mathbf{I} & -2\mathbf{f}\mathbf{f}^\dagger \end{pmatrix}. \tag{3.112}$$

Compared to **??**, I and R can be written as

$$I = -\frac{1}{2}(\mathbf{f}^{-1})^\dagger\mathbf{f}^{-1}, \qquad R = \frac{1}{2}(2\mathbf{h} + i(\mathbf{f}^{-1})^\dagger)\mathbf{f}^{-1} \tag{3.113}$$

To couple vectors with fermions, antisymmetric tensor $O^{\bar{M}}_{\mu\nu} = (O^{\bar{\Lambda}}_{\mu\nu}, O_{\bar{\Lambda}\mu\nu})$ that transforms under H as $O'_{\mu\nu} = \mathbb{R}_\nu[h]O_{\mu\nu}$ in the form of bilinear of fermion is introduced to wrtie duality in the equation **??** as

$$*\mathcal{G} = -\mathbb{C}\mathbb{M}(\mathcal{G} + \mathbb{L}O). \tag{3.114}$$

This gives a definition of composite field strength tensor as

$$\mathbb{F}_{\mu\nu} = -L^\dagger C\mathcal{G}_{\mu\nu} \tag{3.115}$$

having components

$$\mathbb{F}^{\bar{M}}_{\mu\nu} = (F^{\bar{\Lambda}}_{\mu\nu}, F_{\bar{\Lambda}\mu\nu}) = -(\mathbb{L}^*)^N{}_{\bar{M}}\mathbb{C}_{NP}\mathcal{G}^P_{\mu\nu} \tag{3.116}$$

With symplectic condition of $R_\nu[g]$, $\mathbb{F}_{\mu\nu}$ can apparently be shown to transform under H.

Let's define self-dual tensor and anti-self dual tensor by

$$F^{\pm}_{\mu\nu} = \frac{1}{2}(F_{\mu\nu\pm} + i * F_{\mu\nu}) \tag{3.117}$$

where

$$i * F^{\pm} = \pm F^{\pm}_{\mu\nu}. \tag{3.118}$$

Therefore, component of self-dual and anti-self-dual tensor will become

$$\mathbb{F}^{\pm}_{\mu\nu} = -\mathbb{L}^{\dagger}\mathbb{C}\mathcal{G}^{\pm}_{\mu\nu}. \tag{3.119}$$

Also, with symplectic condition of $\mathbb{L}$, it demonstrates that

$$O^{-}_{\bar{\Lambda}\mu\nu} = O^{+\bar{\Lambda}}{}_{\mu\nu} = 0. \tag{3.120}$$

Thus, $\mathbb{F}^{\pm}_{\mu\nu}$ will have components as

$$\mathbb{F}^{+}_{\mu\nu} = (F^{+\,AB}_{\mu\nu}, F^{+i}_{\mu\nu}, \frac{i}{2}O^{+}_{AB\mu\nu}, \frac{1}{2}O^{+}_{i\mu\nu}) \tag{3.121}$$

and

$$\mathbb{F}^{-}_{\mu\nu} = (-\frac{1}{2}O^{-AB}_{\mu\nu}, -\frac{i}{2}O^{-i}_{\mu\nu}, F^{-}_{AB\mu\nu}, F^{-}_{i\mu\nu}). \tag{3.122}$$

Beside, using equation **??**, components of $G^{\pm}_{\Lambda}$ are given by

$$G^{+}_{\Lambda} = \mathcal{N}_{\Lambda\Sigma}F^{+\Sigma} + iI_{\Lambda\Sigma}\bar{f}^{\Sigma\bar{\Gamma}}O_{\bar{\Gamma}} \tag{3.123}$$

and

$$G^{-}_{\Lambda} = \bar{N}_{\Lambda\Sigma}F^{-\Sigma} - iI_{\Lambda\Sigma}f^{\Sigma}_{\bar{\Gamma}}O^{\bar{\Gamma}} \tag{3.124}$$

By using definition of $G^{\pm}_{\Lambda\mu\nu}$ in the form

$$G^{\pm}_{\Lambda\mu\nu} = \pm\frac{2i}{e}\frac{\partial\mathcal{L}}{\partial F^{\pm\Lambda\mu\nu}}, \tag{3.125}$$

Lagrangian density of vector fields becomes

$$
\begin{aligned}
e^{-1}\mathcal{L}_{vector} = & \frac{i}{4}(\bar{\mathcal{N}}_{\Lambda\Sigma}F^{-\Lambda}_{\mu\nu}F^{-\Sigma\mu\nu} - \mathcal{N}_{\Lambda\Sigma}F^{+\Lambda}_{\mu\nu}F^{+\Sigma\mu\nu}) \\
& + \frac{1}{2}(F^{+\Lambda\mu\nu}I_{\Lambda\Sigma}\bar{f}^{\Sigma\bar{\Gamma}}O_{\bar{\Gamma}\mu\nu} + F^{-\Lambda\mu\nu}I_{\Lambda\Sigma}f^{\Sigma}{}_{\bar{\Gamma}}O^{\bar{\Gamma}}_{\mu\nu}).
\end{aligned}
\tag{3.126}
$$

The second term called Pauli term represents interaction between fermions and $F^{\pm\Lambda}_{\mu\nu}$ tensor. where λ_I, $O_{\bar{\Lambda}}$ is shown by

$$
O_{AB\mu\nu} = 2\bar{\psi}_{A\rho}\gamma^{[\rho}\gamma_{\mu\nu}\gamma^{\sigma]}\psi_{B\sigma} + C_{AB,C}{}^{I}\bar{\psi}^{C}_{\rho}\gamma_{\mu\nu}\gamma^{\rho}\lambda_I + C_{AB,IJ}\bar{\lambda}^{I}\gamma_{\mu\nu}\lambda^{J} \tag{3.127}
$$

and

$$
O_{i\mu\nu} = C_{i,A}{}^{I}\bar{\psi}^{A}_{\rho}\gamma_{\mu\nu}\gamma^{\rho}\lambda_I + C_{i,IJ}\bar{\lambda}^{I}\gamma_{\mu\nu}\lambda^{J} \tag{3.128}
$$

where $C_{AB,C}{}^{I}$, $C_{AB,IJ}$, $C_{i,A}{}^{I}$ and $C_{i,IJ}$ are coefficient tensor depending on different supergravities.

From all ingredients mentioned above, it is sufficient tto write general Lagrangian of $N > 2$ supergravity.

$$
\begin{aligned}
e^{-1}\mathcal{L} = & \frac{1}{2}R - \frac{1}{2}ekTr(P_\mu P^\mu) + \frac{i}{4}(\bar{\mathcal{N}}_{\Lambda\Sigma}F^{-\Lambda}_{\mu\nu}F^{-\Sigma\mu\nu} - \mathcal{N}_{\Lambda\Sigma}F^{+\Lambda}_{\mu\nu}F^{-\Sigma\mu\nu}) \\
& + ie^{-1}\epsilon^{\mu\nu\rho\sigma}(\bar{\psi}^{A}_{\mu}\gamma_\nu\mathcal{D}_\rho\psi_{A\sigma} - \bar{\psi}_{A\mu}\gamma_\nu\mathcal{D}_\rho\psi^{A}_{\sigma}) - \frac{1}{2}\bar{\lambda}^{I}\gamma^{\mu}\mathcal{D}_\mu\lambda_I + \bar{\lambda}_I\gamma^{\mu}\mathcal{D}_\mu\lambda^{I} \\
& + \frac{1}{2}(F^{+\Lambda\mu\nu}I_{\Lambda\Sigma}\bar{f}^{\Sigma\bar{\Gamma}}O_{\bar{\Gamma}\mu\nu} + F^{-\Lambda\mu\nu}I_{\Lambda\Sigma}f^{\Sigma}{}_{\bar{\Sigma}}O^{\bar{\Gamma}}_{\mu\nu}) \\
& + \bar{\lambda}^{I}\gamma^{\mu}\gamma^{\nu}\psi^{B}_{\mu}\partial_\nu\phi^{s}P_{sIB} + \bar{\lambda}^{I}\gamma^{\mu}\gamma^{\nu}\psi_{B\mu}\partial_\nu\phi^{s}P^{IB}_{s}
\end{aligned}
\tag{3.129}
$$

The last term indicates interaction between scalar and fermionic fields.

This Lagrangian has supersymmetry transformations as follows

$$\begin{aligned}
\delta e^a_\mu &= \epsilon^A \gamma^a \psi_{A\mu} + \bar{\epsilon}_A \gamma^a \psi^A_\mu \\
\delta A^\Lambda_\mu &= \mathbb{L}^A{}_{\bar{M}} O^{\bar{M}}_\mu = \frac{1}{2} f^\Lambda{}_{AB} O^{AB}_\mu + f^\Lambda{}_i O^i_\mu + h.c. \\
P_s^{ABCD} \delta\phi^s &= \Sigma^{ABCD}, \quad P_s^{iAB} \delta\phi^s = \Sigma^{iAB} \\
\delta\psi_{A\mu} &= \mathcal{D}_\mu \epsilon_A + \frac{i}{8} F^-_{\rho\sigma AB} \gamma^{\rho\sigma} \gamma_\mu \epsilon^B \\
\delta\chi_{ABC} &= P_{sABCD} \partial_\mu \phi^s \gamma^\mu \epsilon^D + \frac{3}{4} i F^-_{\mu\nu[AB} \gamma^{\mu\nu} \epsilon_{C]} \\
\delta\lambda_{Ai} &= P_{siAB} \partial_\mu \phi^s \gamma^\mu \epsilon^B + \frac{1}{4} i F^-_{\mu\nu i} \gamma^{\mu\nu} \epsilon_A
\end{aligned} \tag{3.130}$$

For $N = 3, 5, 6$, there are additional fermions which transform under supersymmetry as follows.

For $N = 3$,

$$\delta\lambda_i = \frac{1}{2} P_{siAB} \partial_\mu \phi^s \gamma^\mu \epsilon_C \epsilon^{ABC}. \tag{3.131}$$

For $N = 5$,

$$\delta\chi = \frac{1}{24} \epsilon^{ABCDE} P_{sABCD} \partial_\mu \phi^s \gamma^\mu \epsilon_E. \tag{3.132}$$

For $N = 6$,

$$\delta\chi_F = \frac{1}{24} \epsilon_{FABCDE} P_s^{ABCD} \partial_\mu \phi^s \gamma^\mu \epsilon^E + \frac{i}{4} \tilde{F}^-_{\mu\nu} \gamma^{\mu\nu} \epsilon_F. \tag{3.133}$$

CHAPTER IV

GAUGED SUPERGRAVITY

This chapter gives a general review on gauging used in gauged supergravity by embedding tensor formalism. The early section tells about the gauging procedure leading to a change in a structure of supergravity given in chapter III. At the end of the chapter, $N = 4$ gauged supergravity are provided since our scope of the research is to find the Janus solutions in $N = 4$ gauged supergravity.

4.1 Gauging procedure

Gauging in supergravity is promoting a subgroup G_0 of G to be a local symmetry. In the context of Kaluza-Klein reduction, local gauge symmetry encodes the information of the internal manifold that leads to more realistic model because of the presence of scalar potential. To gauge the theory with G_0, the Lagrangian is required to be locally invariant under G_0 symmetry. The first condition to consider in gauging is a number n_ν of vector fields must be sufficient to gauge as

$$\dim(G_0) \leq n_\nu. \tag{4.1}$$

Beginning with gauge connection defined by

$$\Omega_{g\mu} = gA_\mu^{\hat{\Lambda}} X_{\hat{\Lambda}} \tag{4.2}$$

where g is an coupling constant and $X_{\hat{\Lambda}}$ is a generator of G_0 which satisfies

$$[X_{\hat{\Lambda}}, X_{\hat{\Sigma}}] = f_{\hat{\Lambda}\hat{\Sigma}}{}^{\hat{\Gamma}} X_{\hat{\Gamma}} \tag{4.3}$$

To make G_0 be a closed group, $f_{\hat{\Lambda}\hat{\Sigma}}{}^{\hat{\Gamma}} X_{\hat{\Gamma}}$ must satisfy Jacobi identity

$$f_{[\hat{\Lambda}\hat{\Sigma}}{}^{\hat{\Gamma}} f_{\hat{\Delta}]\hat{\Gamma}}{}^{\hat{\Pi}} = 0. \tag{4.4}$$

$X_{\hat{\Lambda}}$ can be written in symplectic matrix under R_ν representation as

$$X_{\hat{\Lambda}}{}^{\hat{M}}{}_{\hat{N}} = R_\nu[X_{\hat{\Lambda}}]^{\hat{M}}{}_{\hat{N}} = \begin{pmatrix} X_{\hat{\Lambda}}{}^{\hat{\Sigma}}{}_{\hat{\Gamma}} & 0 \\ -X_{\hat{\Lambda}\hat{\Sigma}\hat{\Gamma}} & X_{\hat{\Lambda}\hat{\Sigma}}{}^{\hat{\Gamma}} \end{pmatrix} \tag{4.5}$$

Due to symplectic condition of $X_{\hat{\Lambda}}{}^{\hat{M}}{}_{\hat{N}}$, it gives a relation

$$X_{\hat{\Lambda}}{}^{\hat{\Sigma}}{}_{\hat{\Gamma}} = -X_{\hat{\Lambda}\hat{\Gamma}}{}^{\hat{\Sigma}} \tag{4.6}$$

compared to $\delta F^{\hat{\Lambda}} = \xi^{\hat{\Gamma}} f^{\hat{\Lambda}}_{\hat{\Gamma}\hat{\Sigma}} F^{\hat{\Sigma}}$ and $\delta \mathcal{G}^{\hat{M}} = \xi^{\hat{\Lambda}} (X_{\hat{\Lambda}})^{\hat{M}}_{\hat{N}} \mathcal{G}^{\hat{N}}$, it determines that

$$f_{\hat{\Gamma}\hat{\Sigma}}{}^{\hat{\Lambda}} = -X_{\hat{\Gamma}\hat{\Sigma}}{}^{\hat{\Lambda}} \tag{4.7}$$

which rewrite the equation **??** to

$$[X_{\hat{\Lambda}}, X_{\hat{\Sigma}}] = -X_{\hat{\Lambda}\hat{\Sigma}}{}^{\hat{\Gamma}} X_{\hat{\Gamma}}. \tag{4.8}$$

From this algebra, a generator further shows additional relation that

$$X_{(\hat{\Gamma}\hat{\Sigma})}{}^{\hat{\Lambda}} = 0 \tag{4.9}$$

called quadratic constraint and for $X_{\hat{\Lambda}\hat{\Gamma}\hat{\Sigma}} \neq 0$, symmetric under Pecci-Quin,

$$X_{(\hat{\Lambda}\hat{\Gamma}\hat{\Sigma})} = 0. \tag{4.10}$$

These two additional constraints are called together as linear constraint.

Transforming under gauge connection under $g(x) \in G_0 \subset G$ is described by

$$\Omega'_g = g(x)\Omega_g g^{-1}(x) + dg(x)g^{-1}(x). \tag{4.11}$$

The connections generate 2-form curvature tensor $R(\Omega) = F^{\hat{\Lambda}} X_{\hat{\Lambda}}$ defined by

$$R(\Omega_g) = \frac{1}{g}(d\Omega_g - \Omega_g \wedge \Omega_g). \tag{4.12}$$

Components of $F^{\hat{\Lambda}}_{\mu\nu}$ are found by

$$F^{\hat{\Lambda}}_{\mu\nu} = \partial_\mu A^{\hat{\Lambda}}_\nu - \partial_\nu A^{\hat{\Lambda}}_\mu + gX_{\hat{\Gamma}}\Sigma^{\hat{\Lambda}} A^{\hat{\Gamma}}_\mu A^{\hat{\Sigma}}_\nu \tag{4.13}$$

Total covariant derivative including gauge connection is defined by

$$\nabla_\mu = \mathcal{D}_\mu - gA^{\hat{\Lambda}} X_{\hat{\Lambda}.} \tag{4.14}$$

This derivative can be used to find $F^{\hat{\Lambda}}_{\mu\nu}$ as

$$[\nabla_\mu, \nabla_\nu] = -gF^{\hat{\Lambda}}_{\mu\nu} X_{\hat{\Lambda}} + ... \tag{4.15}$$

where ... represents a curvature tensor of spacetime and curvature tensor $R(Q)$ in the scalar manifold. Resulting from gauging that obviously relates to a new connection, left-invariant one-form can be newly defined as

$$\hat{\Omega}_\mu = L^{-1}\nabla_\mu L = L^{-1}(\partial_\mu - gA^{\hat{\Lambda}}_\mu X_{\hat{\Lambda}})L. \tag{4.16}$$

With relation $L^{-1}dL = P + Q$, $\hat{P}_\mu$ and $\hat{Q}_\mu$ are redefined as

$$\hat{P}_\mu = P_\mu - gA^{\hat{\Lambda}}_\mu P_{\hat{\Lambda}} \quad \text{and} \quad \hat{Q}_\mu = Q_\mu - gA^{\hat{\Lambda}}_\mu Q_{\hat{\Lambda}} \tag{4.17}$$

projected to $\mathfrak{k}$ and $\mathfrak{h}$, it describes

$$P_{\hat{\Lambda}} = L^{-1}X_{\hat{\Lambda}}L|_{\mathfrak{k}} \quad \text{and} \quad Q_{\hat{\Lambda}} = L^{-1}X_{\hat{\Lambda}}L|_{\mathfrak{h}} \tag{4.18}$$

transformed under gauging as

$$\hat{P}(g\circ\phi) = h^{-1}\hat{P}h \quad \text{and} \quad \hat{Q}(g(x)\circ\phi) = h^{-1}\hat{Q}h + h^{-1}dh. \tag{4.19}$$

This also satisfies Maurer-Cartan equation for gauged form

$$d\hat{\Omega} + \hat{\Omega}\wedge\hat{\Omega} = -gL^{-1}R(\Omega_g)L \tag{4.20}$$

projected on subspace $\mathfrak{k}$ and $\mathfrak{h}$, it turns out that

$$\begin{aligned} \mathcal{D}\hat{P} = d\hat{P} + \hat{Q}\wedge\hat{P} + \hat{P}\wedge\hat{Q} = -gF^{\hat{\Lambda}}P_{\hat{\Lambda}} \\ \hat{R}(\hat{Q}) = d\hat{Q} + \hat{Q}\wedge\hat{Q} = -\hat{P}\wedge\hat{P} - gF^{\hat{\Lambda}}Q_{\hat{\Lambda}}. \end{aligned} \tag{4.21}$$

Furthermore, affected by this gauging, fermionic fields will have total covariant derivative as

$$\nabla_\mu\psi = D_\mu\psi + \hat{Q}\circ\psi \tag{4.22}$$

In symplectic frame, embedding tensor is shown by $\Theta_{\hat{\Lambda}}{}^{\sigma}$ responsible for projecting Lie algebra g_e of G_e, whose generator is t_σ, on the algebra g_0 on gauge symmetry G_0 as

$$X_{\hat{\Lambda}} = \Theta_{\hat{\Lambda}}{}^{\sigma}t_\sigma \tag{4.23}$$

where $\Theta_{\hat{\Lambda}}{}^{\sigma}$ is in the product $n_\nu \otimes \mathrm{adj}(G_e)$ and $\hat{\Lambda} = 1, 2, 3, ..., n_\nu$ and $\sigma = 1, 2, ..., dimG_e$.

X_M in a representation R_{ν^*} is written in the form

$$X_{MN}{}^{P} = R_{\nu^*}[X_M]_N{}^P = \Theta_M{}^a t_a N^P \tag{4.24}$$

which is transformed under symplectic frame of the action as

$$X_{MN}{}^{P} = (E^{-1})_{M}{}^{\hat{M}}(E^{-1})_{N}{}^{\hat{N}}X_{\hat{M}\hat{N}}{}^{\hat{P}}E_{\hat{P}}{}^{P} \tag{4.25}$$

4.2 Lagrangian density of gauged supergravity

To write complete Lagrangian density, many terms in ungauged theory have to be adjusted which will be shed the light on in this section.

Let's firstly consider kinetic term of fermionic fields. For gravitino,

$$\mathcal{L}_{\psi_{A\mu}} = -e\bar{\psi}^{A}_{\mu}\gamma^{\mu\nu\rho}\nabla_{\nu}\psi_{A\rho} + h.c. \tag{4.26}$$

whose supersymmetry transformation is given by

$$\delta\psi_{A\mu} = \nabla_{\mu}\epsilon_{A} + ... \tag{4.27}$$

This has an effecton its lagrangian density clarified by

$$\delta\mathcal{L}_{\psi_{A\mu}} = -2e\bar{\psi}^{A}_{\mu}\gamma^{\mu\nu\rho}\nabla_{\nu}\nabla_{\rho}\epsilon_{A} + ... = ge\bar{\psi}^{A}_{\mu}\gamma^{\mu\nu\rho}F^{\hat{\Lambda}}_{\nu\rho}Q^{B}_{\hat{\Lambda}A}\epsilon_{B} + ... \tag{4.28}$$

For kinetic term of λ^{I}

$$\mathcal{L}_{\lambda^{I}} = -\frac{1}{2}e\bar{\lambda}_{I}\gamma^{\mu}\nabla_{\mu}\lambda^{I} + ... \tag{4.29}$$

whose supersymmetry transformation is

$$\delta\lambda^{I} = \hat{P}^{AI}_{\mu}\gamma^{\mu}\epsilon_{A} + ... \tag{4.30}$$

which gives an outcome

$$\delta\mathcal{L}_{\lambda^I} = -e\bar{\lambda}_I\gamma^\mu\gamma^\nu\nabla_\mu\hat{P}_\nu^{AI}\epsilon_A + ... = \frac{1}{2}ge\bar{\lambda}_I\gamma^{\mu\nu}F_{\mu\nu}^{\hat{\Lambda}}P_{\hat{\Lambda}}^{AI}\epsilon_A + ... \tag{4.31}$$

These show that additional terms apart form which represented by ungauged supergravity are

$$\begin{aligned} \delta\mathcal{L}_{\psi_{A\mu}} &\sim g\bar{\psi}_\mu^A\gamma^{\mu\nu\rho}F_{\nu\rho}^{\hat{\Lambda}}(L^{-1}X_{\hat{\Lambda}}L|_{\mathfrak{h}})_A{}^B\epsilon_B + ... \\ \delta\mathcal{L}_{\lambda^I} &\sim g\bar{\lambda}_I\gamma^{\mu\nu\rho}F_{\mu\nu}^{\hat{\Lambda}}(L^{-1}X_{\hat{\Lambda}}L|_{\mathfrak{k}})^{IA}\epsilon_A + ... \end{aligned}. \tag{4.32}$$

By writing $F^{\bar{\Lambda}}L^{-1}X_{\bar{\Lambda}L}$ in covariant form under G

$$F^{\bar{\Lambda}}L^{-1}X_{\bar{\Lambda}}L = F^{\bar{\Lambda}}E_{\bar{\Lambda}}{}^M L^{-1}X_M L = \mathcal{G}^M L^{-1}X_M L. \tag{4.33}$$

From this relation, a new tensor called T-tensor can be introduced as

$$T_{\bar{M}} = \mathbb{L}_{\bar{M}}{}^N L^{-1}X_N L \tag{4.34}$$

where $\mathbb{L}_{\bar{M}}{}^N = (\mathbb{L}^T)^N{}_{\bar{M}}$. Components of $T_{\bar{M}} = L_{\bar{M}}{}^N L^{-1}X_N L$ on the complex basis is shown by

$$T_{\bar{M}\bar{N}}{}^{\bar{P}} = \mathbb{L}_{\bar{M}}{}^M L_{\bar{N}}{}^N X_{MN}{}^P(\mathbb{L}^{-1})_P{}^{\bar{P}}. \tag{4.35}$$

With the relation $Q_{\hat{\Lambda}} = E_{\hat{\Lambda}}{}^M Q_M$ and $P_{\bar{\Lambda}} = E_{\hat{\Lambda}}{}^M P_M$, $T_{\bar{M}}$ can be seen in the form of Q_M and P_M

$$T_{\bar{M}} = \mathbb{L}_{\bar{M}}{}^M(P_M + Q_M). \tag{4.36}$$

To preserve supersymmetry, additional terms are needed to be introduced. One of them is called Yakawa term defined by

$$e^{-1}\mathcal{L}_{Yukawa} = g(-2\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu^B S_{AB} + \bar{\lambda}^I\gamma^\mu\psi_{A\mu}N_I{}^A + \bar{\lambda}^I\lambda^J M_{IJ}) + h.c. \tag{4.37}$$

This term plays a role as mass-like term of fermion. T-tensor can also be written in the form $S_{AB} = S_{BA}$, ${N_I}^A$ and M_{IJ} and their conjugate are

$$S^{AB} = (S_{AB})^*, \quad {N^I}_A = ({N_I}^A)^*, \quad M^{IJ} = (M_{IJ})^*. \tag{4.38}$$

Apart from Yukawa term, supersymmetry transformation must be fixed in order to preserve supersymmetry as well.

$$\begin{aligned} \delta\psi_{A\mu} &= \nabla_\mu \epsilon_A - gS_{AB}\gamma_\mu \epsilon^B + ... \\ \delta\lambda_I &= \hat{P}^A_{\mu I}\gamma^\mu \epsilon_A + g{N_I}^A \epsilon_A + ... \end{aligned} \tag{4.39}$$

The additional terms involve S_{AB}, ${N_I}^A$ and M_{IJ} which are known as fermion-shift matrix.

According to new supersymmetry variations with additional fermion-shift term, varying Yukawa term will create second order term of g. This will be cancled by a term called "scalar potential" that play a role like potential energy for gauged supergravity system. It is defined by

$$V(\phi) = \frac{1}{N} g^2 ({N_I}^A {N^I}_A - 12 S^{AB} S_{AB}) \tag{4.40}$$

In summary, a construction of gauged supergravity has been motivated by beginning with an ungauged theory with replacing previous covariant derivative into total covariant derivative including gauge connection. In order to preserve supersymmetry, Yukawa term and scalar potential must be introduced as

$$\begin{aligned} e^{-1}\mathcal{L}_{gauged} = {} & e^{-1}\mathcal{L}_{ungauged}(\partial \to \nabla, dA \to dA + A \wedge A) \\ & + g(-2\bar{\psi}_\mu \gamma^{\mu\nu}\psi^B_\nu S_{AB} + \bar{\lambda}^I \gamma^\mu \psi_{A\mu} {N_I}^A + \bar{\lambda}^I \lambda^J M_{IJ}) + h.c. \\ & - V(\phi) \end{aligned} \tag{4.41}$$

The lagrangian above is invariant under supersymmetry transformation as follows

$$
\begin{aligned}
\delta e^a_\mu &= \epsilon^A \gamma^a \psi_{A\mu} + \bar{\epsilon}_A \gamma^a \psi^A_\mu \\
\delta A^\Lambda_\mu &= \mathbb{L}^A{}_{\bar{M}} O^{\bar{M}}_\mu = \frac{1}{2} f^\Lambda{}_{AB} O^{AB}_\mu + f^\Lambda{}_i O^i_\mu + h.c. \\
P^{ABCD}_s \delta\phi^s &= \Sigma^{ABCD}, \quad P^{iAB}_s \delta\phi^s = \Sigma^{iAB} \\
\delta\psi_{A\mu} &= \mathcal{D}_\mu \epsilon_A + \frac{i}{8} F^-_{\rho\sigma AB} \gamma^{\rho\sigma} \gamma_\mu \epsilon^B - g S_{AB} \epsilon^B \\
\delta\chi_{ABC} &= P_{sABCD} \partial_\mu \phi^s \gamma^\mu \epsilon^D + \frac{3}{4} i F^-_{\mu\nu[AB} \gamma^{\mu\nu} \epsilon_{C]} + g N_{ABC}{}^D \epsilon_D \\
\delta\lambda_{Ai} &= P_{siAB} \partial_\mu \phi^s \gamma^\mu \epsilon^B + \frac{1}{4} i F^-_{\mu\nu i} \gamma^{\mu\nu} \epsilon_A + g N_{iA}{}^B \epsilon_B
\end{aligned}
\quad . \tag{4.42}
$$

4.3 A general structure of $N = 4$ gauged supergravity

Our scope of study is to find Janus solutions in $N = 4$ gauged supergravity. It is much more convenient in finding such solutions to have a general structure of $N = 4$ gauged supergravity as being our tool. Generally, $N = 4$ gauged supergravity is coupled to vector multiplets that lead to consisting of two multiplets, two of which are gravity multiplet and vector multiplet.

The gravity multiplet has fields content

$$
(e^{\hat{\mu}}_\mu, \psi^i_\mu, A^m_\mu, \chi^i, \tau) \tag{4.43}
$$

while vector multiplet provides

$$
(A^a_\mu, \lambda^{ia}, \phi^{ma}) \tag{4.44}
$$

The scalar field τ is a complex scalar that contains dilaton ϕ and axion χ parametrized by $SL(2,\mathbb{R})$. The indices $\mu, \nu, ... = 0,1,2,3$ and $\hat{\mu}, \hat{\nu}, ... = 0,1,2,3$ describe spacetime and tanget space respectively. Fundamental representations of $SO(6)_R$ and $SU(4)_R$ R-symmetry can be indicated to indices $m, n = 1, ..., 6$ and $i, j = 1,2,3,4$ respectively.

In vector multiplet, $SO(6,n)/SO(6) \times SO(6)$ coset is used to descibes $6n$ scalar field ϕ^{ma} where indices $a, b = 1, ..., n$.

For fermionic fields, they are determined by fundamental representation of $SU(4)_R \sim SO(6)_R$ abiding by chirality projections as

$$\gamma_5 \psi^i_\mu = \psi^i_\mu, \quad \gamma_5 \chi^i = -\chi^i, \quad \gamma_5 \lambda^i = \lambda^i \tag{4.45}$$

and its conjugation

$$\gamma_5 \psi_{\mu i} = -\psi_{\mu i}, \quad \gamma_5 \chi_i = \chi_i, \quad \gamma_5 \lambda_i = -\lambda_i \tag{4.46}$$

Complex scalar τ contain dilaton and axion as the form

$$\tau = \chi + ie^{\phi} \tag{4.47}$$

written in $SL(2,\mathbb{R})/SO(2)$ coset as

$$\mathcal{V}_\alpha = e^{\phi/2} \begin{pmatrix} \chi + ie^{\phi} \\ 1 \end{pmatrix}. \tag{4.48}$$

$6n$ scalars in vector multiplet ϕ^{ma} is described by

$$\mathcal{V}_M{}^A = (\mathcal{V}_M{}^m, \mathcal{V}_M{}^a) \tag{4.49}$$

satifying

$$\eta_{MN} = -\mathcal{V}_M{}^m \mathcal{V}_N{}^m + \mathcal{V}_M{}^a \mathcal{V}_N{}^a \tag{4.50}$$

where $\eta_{MN} = diag(-1,-1,-1,-1,-1,-1,1,...,1)$ is a metric of $SO(6,n)$ and index $A = (m,a)$ is from separating $SO(6) \times SO(n)$ index.

The bosonic lagrangian of $N = 4$ gauged supergravity can be written by

$$e^{-1}\mathcal{L} = \frac{1}{2}R + \frac{1}{16}\partial_\mu M_{MN}\partial^\mu M^{MN} - \frac{1}{4(Im\tau)^2}\partial_\mu \tau \partial^\mu \tau^* - V \tag{4.51}$$

where $e = \sqrt{-g}$ is the determinant of the veibein matrix. The scalar potential of this equation can be found by

$$\begin{aligned} V = \frac{1}{16}[f_{\alpha MNP}f_{\beta QRS}M^{\alpha\beta}[\frac{1}{3}M^{MQ}M^{NR}M^{PS} + (\frac{2}{3}\eta^{MQ} - M^{MQ})\eta^{NR}\eta^{PS}] \\ -\frac{4}{9}f_{\alpha MNP}f_{\beta QRS}\epsilon^{\alpha\beta}M^{MNPQRS}]. \end{aligned} \tag{4.52}$$

The embedding tensor, which is earlier studied in [?, ?, ?, ?], $\xi^{\alpha M}$ and $f_{\alpha MNP}$, which play a role in gauging, are clearly contributed in scalar potential and due to the emergence of supersymmetric AdS_4 vacua, $\xi^{\alpha M}$ is compulsorily required to be zero, see [?]. Hence, in gauging method, the embedding tensor $f_{\alpha MNP}$ will only be perfomed.

The matrix M_{MN}, the inverse of matrix M^{MN}, is written as

$$M_{MN} = \mathcal{V}_M{}^m\mathcal{V}_N{}^m + \mathcal{V}_M{}^a\mathcal{V}_N{}^a. \tag{4.53}$$

The M^{MNPQRS} tensor is found by

$$M_{MNPQRS} = \epsilon_{mnpqrs}\mathcal{V}_M{}^m\mathcal{V}_N{}^n\mathcal{V}_P{}^p\mathcal{V}_Q{}^q\mathcal{V}_R{}^r\mathcal{V}_S{}^s \tag{4.54}$$

which is raised the indices by η^{MN}.

The $M^{\alpha\beta}$ matrix is calculated by taking the inverse of matrix $M_{\alpha\beta}$ shown by

$$M_{\alpha\beta} = Re(\mathcal{V}_\alpha\mathcal{V}^*_\beta) \tag{4.55}$$

Supersymmetry variations of this theory are

$$\delta\psi^i_\mu = 2D_\mu\epsilon^i - \frac{2}{3}A^{ij}_1\gamma_\mu\epsilon_j \tag{4.56}$$

$$\delta\chi^i = -\epsilon^{\alpha\beta}\mathcal{V}_\alpha D_\mu\mathcal{V}_\beta\gamma^\mu\epsilon^i - \frac{4}{3}iA^{ij}_2\epsilon_j \tag{4.57}$$

$$\delta\lambda^i_a = 2i\mathcal{V}_a{}^M D_\mu\mathcal{V}_M{}^{ij}\gamma^\mu\epsilon_j - 2iA_{2aj}{}^i\epsilon^j. \tag{4.58}$$

The definition of relavant fermion shift matrices can be given by

$$A_1^{ij} = \epsilon^{\alpha\beta}(\mathcal{V}_\alpha)^*\mathcal{V}_{kl}{}^M\mathcal{V}_N{}^{ik}\mathcal{V}_P{}^{jl}f_{\beta M}^{NP} \tag{4.59}$$

$$A_2^{ij} = \epsilon^{\alpha\beta}\mathcal{V}_\alpha\mathcal{V}_{kl}{}^M\mathcal{V}_N{}^{ik}\mathcal{V}_P{}^{jl}f_{\beta M}^{NP} \tag{4.60}$$

$$A_{2ai}^{j} = \epsilon^{\alpha\beta}\mathcal{V}_\alpha\mathcal{V}_a{}^M\mathcal{V}_{ik}{}^N\mathcal{V}_P{}^{jk}f_{\beta MN}{}^P. \tag{4.61}$$

The matrix $\mathcal{V}_M{}^{ij}$ and $\mathcal{V}_{ij}{}^M$ can be found by the 't Hooft symbols as

$$\mathcal{V}_M{}^{ij} = \frac{1}{2}\mathcal{V}_M{}^m G_m^{ij} \tag{4.62}$$

$$\mathcal{V}_{ij}{}^M = -\frac{1}{2}\mathcal{V}_m{}^M (G_m^{ij})^* \tag{4.63}$$

where G_m^{ij} obeys that

$$G_{mij} = (G_m^{ij})^* = \frac{1}{2}\epsilon_{ijkl}G_m^{kl} \tag{4.64}$$

and the explicit form of matrix G_m^{ij} can be written by

$$\begin{gathered}
G_1^{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad G_2^{ij} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\
G_3^{ij} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad G_4^{ij} = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \\
G_5^{ij} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \qquad G_6^{ij} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}
\end{gathered} \tag{4.65}$$

All of this structure will be used in finding Janus solutions in chapter VI.

CHAPTER V

ANTI-DE SITTER SPACE AND CONFORMAL FIELD THEORY

AdS/CFT correspondence, firstly proposed by Maldacena since 1997, mainly plays a crucial role in making the calculation in quantum theory more easily than ever before. Due to tedious results of calculations from quantum field theory, they are difficult to give analytic explanations, most of which rather show numerically. Analytical solutions are much more better in describing things in physics. However, luckily, the calculation from quantum field theory can be converted to gravity theory, which usually provides solutions analytically, through AdS/CFT correspondence.

This chapter will give the first introduction of AdS_5/CFT_4 duality, originated from two perspectives of string theory. The following content, compelling to give better understanding of AdS/CFT, is the identical isometry between conformal field theory (CFT) and anti-de sitter space (AdS). In the end, general princaiple of AdS_{d+1}/CFT_d will be given.

5.1 Two perspectives of string theory

AdS/CFT correspondence is firstly originated from describing the dynamics of N D3-branes from closed and open string perspectives. The end of this section will show that due to the same dynamics of N D3-branes, two perspectives, compulsorily equivalent to each other, give the duality between $N = 4$ Super Yang-Mills theory and $AdS_5 \times S^5$.

5.1.1 Open string perspective

In this perspective, N coincident D3-branes coupled to the string with coupling constant $g_s \ll 1$ regardless of massive sates show that an effective field theory will become four-dimensional gauged theory with $U(N)$ gauge symmetry where the coupling constant is $g_s N$.

In ten dimensions, the supersymmetries halfly preserved by D3-brane give massless state of the string representing $N = 4$ field theory on world volume of D3-brane. This action of D3-brane is shown by

$$S = S_{closed} + S_{open} + S_{int} \tag{5.1}$$

where S_{closed}, S_{open} and S_{int} are the actions of closed, open string and interaction between these two types of string respectively.

With the limit of, $E\sqrt{\alpha'} \ll 1$, the action will be reduced to

$$S_{closed} = \frac{1}{2\kappa^2} \int d^{10}x\sqrt{-g}e^{-2\phi}(R + 4\partial_M R\partial^M R + \partial_M \phi \partial^M \phi + ...). \tag{5.2}$$

In the limit $g_s \ll 1$ that κ becomes small value, the metric g_{MN} can be expanded as

$$g_{MN} = \eta_{MN} + \kappa h_{MN}. \tag{5.3}$$

Then, the action will approximately turns into

$$S_{closed} \sim \frac{1}{2} \int d^{10}x(\partial_M h \partial^M h + \partial_M \phi \partial^M \phi + ...), \tag{5.4}$$

where h_{MN} is the perturbation term of metric g_{MN} from η_{MN} as $g_{MN} = \eta_{MN} + \kappa h_{MN}$ and ϕ is the scalar field of the theory.

S_{open} and S_{int} found by the Dirac-Born-Infel action are shown by

$$S_{DBI} = -\frac{1}{(2\pi)^3 \alpha'^2 g_s} \int d^4x e^{-\phi} Tr\sqrt{-det(g_{\mu\nu} + B_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})}, \tag{5.5}$$

The scalar $\phi^i(x^\mu)$ relates to x^{i+3}

$$x^{i+3} = 2\pi\alpha'\phi^i \tag{5.6}$$

along with the pullback $g_{\mu\nu} = P[g]_{\mu\nu}$

$$P[g]_{\mu\nu} = g_{\mu\nu} + (2\pi\alpha')(\partial_\mu\phi^i g_{i+3,\nu} + g_{\mu,j+3}\partial_\nu\phi^j) + (2\pi\alpha')^2 g_{i+3,j+3}\partial_\mu\phi^i\partial_\nu\phi^j \tag{5.7}$$

and expanding $e^{-\phi} = 1 - \phi + ...$ and $g_{MN} = \eta_{MN} + \kappa h_{MN}$ can separate free fields and interaction term as

$$S_{open} = -\frac{1}{2\pi g_s}\int d^4x[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i + O(\alpha')] \tag{5.8}$$

$$S_{int} = -\frac{1}{8\pi g_s}\int d^4x\phi F_{\mu\nu}F^{\mu\nu} + ... \tag{5.9}$$

Vectors and scalars lie on adjoint representation of $U(N)$, will give non-abelian gauge theory with $U(N)$ gauge symmetry. Thus, vector and scalar fields will be written in $U(N)$ representation as

$$A_\mu = A_\mu^a T^a, \qquad \phi^i = \phi^{ia}T^a \tag{5.10}$$

where T^a is a generator of the group. Therefore, kinetic term of gauge fields will change to $F^a_{\mu\nu}F^{a\mu\nu}$ and the covariant derivative of scalar field will be made an adjustment to

$$D_\mu\phi^i = \partial_\mu\phi^i + g[A_\mu, \phi^i] \tag{5.11}$$

Besides, scalar potential is added

$$V = \frac{1}{2\pi g_s}\sum_{i,j} Tr[\phi^i, \phi^j]^2 \tag{5.12}$$

Changing the scale of ϕ to $\sqrt{2}\kappa\phi$, S_{int} can be estimated as

$$S_{int} \sim \int d^4x\kappa\phi F^2 \tag{5.13}$$

This interaction term will become 0 at the $\alpha' \to 0$ since $\kappa \sim \alpha'^2 \to 0$. To conclude, with this limit, there is no interaction term so only S_{open} and S_{closed} remain. Also, at this limit S_{open} will be reduced to

$$S_{open} = -\frac{1}{{g_{YM}}^2}\int d^4x[\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i + O(\alpha')]. \tag{5.14}$$

which reveals the bosonic sectors of $N = 4$ SYM where $g^2_{YM} = 2\pi g_s$ while the closed string consequently represents supergravity in ten-dimensional flat spacetime.

5.1.2 Closed string

Coupled to N D3-branes with $g_sN \to \infty$, open string cannot be described in this limit, but closed string showing that the brane is a charged object of RR field coupled to field from IIB supergravity give an insight of ten-dimensional curved space where fields from closed string live.

Supergravity in D-branes will give solutions as soliton. Dp-brane, which generally halfly preserves supersymmetries, has Poincare' symmetry in $p+1$ dimensions and give rotation symmetry in $9-p$ dimensions. The action can be shown by

$$S = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}[e^{2\phi}(R + 4\partial_M\phi\partial^M\phi - \frac{1}{2(p+2)!}F_{M_1\ldots M_{p+2}}F^{M_1\ldots M_{p+2}})] \tag{5.15}$$

For $p = 3$, the theory will give solutions of D3-brane as

$$ds^2 = H^{-\frac{1}{2}}\eta_{\mu\nu}dx^\mu dx^\nu + H^{\frac{1}{2}}\delta_{ij}dx^i dx^j \tag{5.16}$$

$$e^\phi = g_s^2, \quad H = 1 + \frac{L^4}{r^4}, \quad L^4 = 4\pi g_s N\alpha'^2 \tag{5.17}$$

$$C_{(4)} = (H(r)^{-1} - 1)dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \tag{5.18}$$

where $H(r)$ is a function of asymptotic flatness written as

$$H(r) = 1 + (\frac{L}{r})^{7-p} \tag{5.19}$$

The thing needed to consider is to analyse ds^2 where $r \gg L$ and $r \ll L$. For $r \gg L$, ds^2 become flat spacetime's while $r \ll L$ show that

$$ds^2 = \frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dr^2 + L^2 d\Omega_5{}^2 \tag{5.20}$$

This is the product of metrices of five-dimensional anti-de sitter and five-sphere with radius L. This limit is called near-horizontal limit. Therefore, at low-energy limit, closed string perspective gives the existence of closed string living in flat spacetime and near horizon.

5.2 AdS/CFT correspondence

From the previous section, two perspectives of string has a mutual component, IIB supergravity in ten-dimensional spacetime. These two must be equivalent according to describing the same system, the dynamics of N D3-branes. Thus the remaining components, $N = 4$ SYM with $SU(N)$ gaguge symmetry and IIB supergravity in $AdS_5 \times S^5$ spacetime from open and closed strings perspectives respectively, rudimentarily dual to each other. $N = 4$ SYM is the field theory with conformal symmetry, also known as superconformal field theory. This equivalence seems likely to give the holography of AdS and CFT, usually known as AdS/CFT duality.

This chapter will show that apart from $N = 4$ SYM equivalent to IIB supergravity in $AdS_5 \times S^5$ spacetime, anti-de sitter space also gives the identical isometry as conformal field theory.

5.2.1 Conformal field theory

Conformal field theory is the symmetry that does not change the angle between vector V^μ and U^μ in d-dimensional space defined as

$$\frac{U.V}{(U.U)(V.V)} = \frac{U^\mu V_\mu}{U^\nu U_\nu V^\rho V_\rho} \tag{5.21}$$

This will give the metric transformation as

$$g'_{\mu\nu}(x') = e^{2\omega(x)} g_{\mu\nu}(x) \tag{5.22}$$

An operator of infinitesimal conformal transformation is shown by

$$U(a, \omega, \lambda, b) = \mathbf{I} + a_\mu P^\mu + \frac{1}{2}\omega_{\mu\nu} J^{\mu\nu} + \lambda D + b_\mu K^\mu \tag{5.23}$$

where P^μ and $J^{\mu\nu}$ are usual generators in Poincare group while D and K^μ are responsible for scale ransformation and special conformal transformation. Scale transformation is defined as

$$x^{\mu'} = \lambda x^\mu \tag{5.24}$$

and special transformation is given by

$$x^{\mu'} = \frac{x^\mu + b^\mu x^2}{1 + 2b.x + b^2 x^2} \tag{5.25}$$

In accordance with closed group, the generators $P^\mu, J^{\mu\nu}, D$ and K^μ can form conformal algebra as the following

$$\begin{aligned} [J_{\mu\nu}, J_{\rho\sigma}] &= 4\eta_{[\mu[\rho} J_{\sigma]\nu]}, \qquad [P_\mu, J_{\nu\rho}] = 2\eta_{\mu[\nu} P_{\rho]} \\ [K_\mu, J_{\nu\rho}] &= 2\eta_{\mu[\nu} K_{\rho]}, \qquad [P_\mu, K_\nu] = 2(\eta_{\mu\nu} D + J_{\mu\nu}) \\ [D, P_\mu] &= P_\mu, \qquad [D, K_\mu] = -K_\mu \end{aligned} \tag{5.26}$$

Resulting from redefinition of the generators as

$$\begin{aligned} &J_{\mu\nu} = J_{\mu\nu}, \qquad J_{d,d+1} = D \\ &J_{\mu d} = \frac{1}{2}(K_\mu - P_\mu), \qquad J_{\mu,d+1} = \frac{1}{2}(P_\mu + K_\mu), \end{aligned} \tag{5.27}$$

it can be rewritten as

$$[J_{ab}, J_{cd}] = 4\eta_{[a[c}J_{d]b]} \tag{5.28}$$

Because of the above equation describing an algebra of $SO(2, d)$ group, this conformal group in d dimensionsis isomorphic to $SO(2, d)$ group.

5.2.2 Anti-de sitter spacetime

Anti-de Sitter spacetime, the most symmetrical spacetime, is a hyperboloid spacetime with negative curvature

AdS_{d+1} can be defined on $R^{2,d}$ with signature $(-, +, +, ..., +, -)$. Let Y^A, $A = 0, 1, ..., d, d+1$ be rhe coordinate. The surface of this space is described by

$$Y^A Y^B \eta_{AB} = -(Y^0)^2 - (Y^{d+1})^2 + \sum_{i=1}^{d}(Y^i)^2 = -L^2 \tag{5.29}$$

which corresponds to metric

$$ds^2 = \eta_{AB} dY^A dY^B \tag{5.30}$$

It obviously shows that its isometry is $SO(2, d)$ as the same as in conformal field theory.

Metric of AdS_{d+1} can be written in different coordinate systems, one of which can be found by fixing $Y^A \to (x^0, x^i, u) = (x^\alpha, u), i = 1, 2, ..., d-1$ as

$$\begin{aligned} &Y^0 = Lux^0, \quad Y^i = Lux^i \\ &Y^d = \frac{1}{2u}[u^2(L^2 - x^2) - 1], \quad Y^{d+1} = \frac{1}{2u}[u^2(L^2 + x^2) + 1] \end{aligned} \tag{5.31}$$

where $x^2 = (-x^0)^2 + \sum_{i=1}^{d-1} = \eta_{\alpha\beta}$ is a Minkowski metric in d dimensions.

Replacing Y^A in **??** gives

$$ds^2 = L^2[\frac{du^2}{u^2} + u^2\eta_{\alpha\beta}dx^\alpha dx^\beta]. \tag{5.32}$$

Chinging u to $\frac{1}{z}$ will provide other coordinates as

$$ds^2 = \frac{L^2}{z^2}(\eta_{\alpha\beta}dx^\alpha dx^\beta + dz^2) \tag{5.33}$$

where (x^α, z) is Poincare patch coordinates.

The other one is found by transforming

$$e^{\frac{r}{L}} = \frac{L}{z} \tag{5.34}$$

resulting in

$$ds^2 = e^{\frac{2r}{L}}\eta_{\alpha\beta}dx^\alpha dx^\beta + dr^2 \tag{5.35}$$

which is widely used in finding holographic solutions.

Curvature tensor of the AdS_{d+1} space is shown by

$$R_{\mu\nu\rho\sigma} = -\frac{1}{L^2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \tag{5.36}$$

where L is the radius of curvature of AdS_{d+1} that also leads to Ricci tensor and Ricci scalar as

$$R_{\mu\nu} = -\frac{1}{L^2}dg_{\mu\nu}, \quad R = -\frac{1}{L^2}(d+1)d \tag{5.37}$$

5.2.3 AdS_5/CFT_4 duality

$AdS_5 \times S^5$ gives an isometry $SO(2,4) \times SO(6)$ where $SO(2,4)$ is an isometry of AdS_5 and $SO(6)$ is an isometry of S^5. It is clearly seen that the isometry of AdS_5 correspond to conformal symmetry of $N = 4$ SYM theory and isometry of S^5 is dual to

$SU(4)$ R-symmetry. To conclude, symmetries between these two theories can be matched as $SO(2,4)$ is responsible for isometry of AdS_5 and CFT_4 in the SYM theory while $SO(6)$ is dual to $SU(4)$ R-symmetry of the SYM. By $N = 4$ SYM, all symmetries in the theory can form a supergroup denoted as $SU(2,2|4)$ where $SO(2,4) \times SO(6) \sim SU(2,2) \times SU(4)$ that $2,2$ in $SU(2,2|4)$ is from $SU(2,2)$ while 4 in $SU(2,2|4)$ is from $SU(4)$.

Duality between $AdS_5 \times S^5$ and $N = 4$ SYM theory then implies that there is always map one-to-one from a field in $AdS_5 \times S^5$ to an operator in $N = 4$ SYM theory and the fields and operators must lie in $SU(2,2|4)$ representation.

5.3 AdS in $(d+1)$ and CFT in d dimensions

From the previous section, $AdS_5 \times S^5$ and $N = 4$ SYM theory demonstrate that there is always a map one-to-one between fields in $AdS_5 \times S^5$ and operators in $N = 4$ SYM. This idea is broaden to apply in different dimensions as fields in $AdS_{d+1} \times M^{D-d-1}$ dual to superconformal field theory in d dimensions called AdS_{d+1}/CFT_d

5.3.1 Correlation function

Correlation function in quantum theory is linked to physical qualtity found by

$$< \phi(x_1)\phi(x_2)...\phi(x_n) > \tag{5.38}$$

This is the expression of n-points function. It is normally calculated by path integral as

$$< \phi(x_1)\phi(x_2)...\phi(x_n) >= \mathcal{N} \int \mathcal{D}\phi(x_1)...\phi(x_n)e^{iS} \tag{5.39}$$

where S is an action of the system and $\mathcal{N}$ is the normalization constant. It can be also found by generating function by the definition

$$Z[J] =< e^{i\int d^dxJ(x)\phi(x)} >= \mathcal{N} \int \mathcal{D}e^{iS+i\int d^dxJ(x)\phi(x)} \tag{5.40}$$

Using generating function, correlation is in the form

$$\langle \phi(x_1)\phi(x_2)...\phi(x_n)\rangle = (i)^n \frac{\delta^n Z[J]}{\delta J(x_1)...\delta J(x_n)} \tag{5.41}$$

where now $J(x)$ is called source.

Generally, generating functional $Z[J]$ is precise to independent fields with order 2 in the action, but for the action involving interaction terms, the generating function is needed to adjust its form a little as

$$Z[J] = \mathcal{N}\int \mathcal{D}\phi e^{iS_0 + i\int d^dx J(x)\phi(x) + i\int d^dx g\phi(x)^m} \tag{5.42}$$

where S_0 is the action of independent fields.

AdS/CFT proposes that correlation function of operators in superconformal field theory

$$< \mathcal{O}(x_1)...\mathcal{O}(x_n) > \tag{5.43}$$

can be found by gravity theory of AdS_{d+1} by generating function

$$Z[\Phi_{(0)}] = < e^{\int d^dx \Phi_{(0)}(x)\mathcal{O}(x)} >_{\mathrm{CFT}} \tag{5.44}$$

where $\Phi_{(0)}(x)$ is the field $\Phi(z,x)$ at the boundary of AdS_{d+1}.

The above equation shows that the operator $\mathcal{O}(x)$ has dimensions Δ dual to $\Phi(z,x)$ that has dimensions $d-\Delta$ because the total dimensions of $\mathcal{O}(x)$ and $\Phi_{(0)}(x)$ must be d. Therefore, $\Phi(z,x)$ must be in the form $\Phi(z,x) \sim z^{d-\Delta}\Phi_{(0)}(x)$. At $z \to 0$, AdS/CFT will give

$$Z_{\mathrm{CFT}} = Z_{\mathrm{string}}\big|_{lim_{z\to 0}\Phi(z,x)z^{\Delta-d}=\Phi_{(0)}(x)} \tag{5.45}$$

At the limit of small α' and g_s, it can be estimated that

$$Z_{\mathrm{string}}\big|_{lim_{z\to 0}\Phi(z,x)z^{\Delta-d}=\Phi_{(0)}(x)} = e^{iS_{\mathrm{supergravity}}}\big|_{lim_{z\to 0}\Phi(z,x)z^{\Delta-d}=\Phi_{(0)}(x)} \tag{5.46}$$

where $S_{\text{supergravity}}$ is the action of supergravity. Duality of AdS/CFT at low energy can be written as

$$\langle e^{\int d^dx \Phi_{(0)}(x)\mathcal{O}(x)} \rangle_{\text{CFT}} = e^{iS_{\text{supergravity}}}\big|_{lim_{z\to 0}\Phi(z,x)z^{\Delta-d}=\Phi_{(0)}(x)} \tag{5.47}$$

From this result, it is said that correlation function of superconformal field theory is found by the action on mass-shell of supergravity.

5.3.2 A map between operators and fields

Let's consider the metric of AdS_{d+1}

$$ds^2 = g_{mn}dx^m dx^n = \frac{L^2}{z^2}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2) \tag{5.48}$$

where L is a radius of AdS_{d+1}. The action of scalar field ϕ is

$$S = \frac{1}{2}\int dz d^dx \sqrt{-g}(g^{mn}\partial_m\phi\partial_n\phi + m^2\phi^2) \tag{5.49}$$

which gives the Klein-Gordon's equation

$$\Box_g - m^2\phi = \frac{1}{\sqrt{-g}}\partial_m(\sqrt{-g}g^{mn}\partial_n\phi) - m^2\phi \tag{5.50}$$

For the AdS_{d+1}, This equation can be written as

$$\Box_g(AdS_{d+1}) = \frac{1}{L^2}[z^2\partial_z^2 - (d-1)z\partial_z + z^2\eta_{\mu\nu}\partial^\mu\partial^\nu] \tag{5.51}$$

$\eta_{\mu\nu}\partial^\mu\partial^\nu$ gives a solution of wave equation in d dimensions. Besides, $\phi(z,x)$ can be written in the form of Fourier's transformation

$$\phi(z,x) = \int \frac{d^dp}{(2\pi)^d}e^{ip^\mu x_\mu}\phi(z,p) \tag{5.52}$$

Let's consider the solution of $\phi(z,x)$ at $z \to 0$ by supposing $\phi(z,x)$ as

$$\phi(z,x) = e^{ip^\mu x_\mu}\phi(z) \tag{5.53}$$

Klein-Gordon's equation in the equation **??** will give

$$z^2\partial_z^2\phi(z) - (d-1)z\partial_z\phi(z) - (m^2L^2 + p^2z^2)\phi(z) = 0 \tag{5.54}$$

where $p^2 = p^\mu p_\mu = p^\mu p^\nu \eta_{\mu\nu}$

In the limit $z \to 0$, there are two independent solutions written as

$$\phi_\pm(z) = z^{\Delta_\pm} \tag{5.55}$$

where $\Delta_\pm$ are the solutions of equation

$$m^2L^2 = \Delta(\Delta - d) \tag{5.56}$$

exactly written as

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2L^2} \tag{5.57}$$

From the equation, it is obviously seen that Δ_+ will be greater than Δ_- and $\Delta_+ + \Delta_- = d$. $\phi_+(z)$ and $\phi_-(z)$ usually are known as normalizable and non-normalizable solution respectively. $\Delta_\pm$ can be defined only when

$$m^2L^2 \geq -\frac{d^2}{4} \tag{5.58}$$

The minimum of this limit is $\Delta_{min} = -\dfrac{d^2}{4}$ called Breitenlohner-Freedman (BF) bound. This shows the bound that allows which scalars to exist.

5.4 Holographic renormalization group

The calculation in quantum field theory usually encounter infinity which is difficult to explain physically. There is a technique called renormalization playing a role in canceling out this unexplainable infinity. The procedure involves making an alteration of dimensions on parameter such as coupling constant or masses of fields. Function β defined by

$$\beta = \mu \frac{\partial g}{\partial \mu}, \tag{5.59}$$

is responsible for describing the relation between coupling constant g and scale of energy μ. To have a value of g depending on scale of energy, it means that in some certain scale of energy, infinity may be avoidable due to the minimized $g(\mu)$. More importantly, at some g^* that gives $\beta(g^*)$, coupling constant no longer depends on scale of energy at this point. Due to the invariance of coupling constant under changing energy scale, the theory that remains the same no matter the size of energy scale changed resembles conformal field theory that has a symmetry under scaling transformation. This point where $\beta(g^*) = 0$ is accordingly called conformal fixed point or critical point.

The conformal field theory at the fixed point can be deformed to quantum field theory, which has no conformal symmetry, by operator $\mathcal{O}_\Delta$ with dimension Δ as

$$S_{\mathrm{QFT}} = S_{\mathrm{CFT}} + \int d^d x \phi_0(x) \mathcal{O}_\Delta(x), \tag{5.60}$$

where $\phi_0(x)$ is a source of perturbation. Since this deformation can describe the flow from conformal fixed points, it is possible that the flow can start from one conformal point at high energy (UV level) and terminate at the other conformal points at low energy (IR level) if that space has more than one critical points. This process is called renormalization group that determines the deformation from one critical point to the other along the energy scale μ.

5.4.1 Asymptotically anti-de Sitter space

The AdS_{d+1}/CFT_d correspondence describes the duality between string theory compactified on $AdS_{d+1} \times M^{D-d-1}$ and quantum field theory on d dimensions. At low energy, the string theory in $AdS_{d+1} \times M^{D-d-1}$ can be depicted as gauged supergravity in $d+1$ dimensions where the internal manifold M^{D-d-1} can be seen as a local symmetry or the identical gauge group promoted to be locally invariant in the theory. The purpose of this duality is to illustrate the RG-flow from a conformal point to one another. An appropriate space to describe the flow is that the space AdS_{d+1} that must be a little adjusted to be asymptotically AdS_{d+1} written in short as $AAdS_{d+1}$. Their boundaries reproduce AdS_{d+1} space holographically describing the flow of deformation of CFT_d. For $AAdS_{d+1}$ providing more AdS_{d+1} critical points, there are possibilities in explaining RG-flow from conformal UV to IR point. This kind of solutions is called holographic RG-flow.

5.4.1.1 Domain wall metric

To obtain the holographic RG-flow solutions, we must find an appropriate $AAdS$ space. One of this kind of space is domain wall taken in the form of metric as

$$ds^2 = e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + dr^2. \tag{5.61}$$

We must not forget that this metric must reproduce AdS_{d+1} at the boundaries. Compared to AdS_{d+1} metric

$$ds^2 = e^{\frac{2r}{L}}\eta_{\mu\nu}dx^\mu dx^\nu + dr^2, \tag{5.62}$$

Condition of reproducing AdS_{d+1} is $A(r)$ must be a linear function as $A(r \to \infty) = kr$, where k is a constant that determines the radius of AdS_{d+1} as

$$kr = \frac{r}{L}, \tag{5.63}$$

$$L = \frac{1}{k}. \tag{5.64}$$

It is noticeable that changing energy scale μ in quantum field theory corresponds to changing coordinate r in gravity side. The holographic β function can be written by

$$\beta^s = \frac{d\phi^s}{dr}. \tag{5.65}$$

5.4.2 Vacua of gauged supergravity

A vacua in gauged supergravity with Lorentz symmetry preserved provides a classical background where only non-vanishing scalar fields appear. In other words, apart from scalar fields, other fields vanish. At this point, scalar fields become constant denoted by ϕ_0^s which can be found by

$$\frac{\partial V}{\partial \phi^s}\Big|_{\phi_0} = 0, \tag{5.66}$$

where $V(\phi_0)$ is a scalar potential at the vacua. The vacua at $\phi = \phi_0$ can also preserve some amount of supersymmetry. At vacuum, other fields including fermions vanish as seen that the vacuum state $|0\rangle$ acted by supercharge, the state will be annihilated. This can be written in the form of supersymmetry variation as

$$\delta f(x) = \langle 0| \left[\bar{\epsilon}Q, \hat{f}(x)\right] |0\rangle = 0, \tag{5.67}$$

where $f(x)$ is a fermionic field and $\hat{f}(x)$ is a fermionic field operator. This can be concluded that supersymmetry transformations of fermionic fields are zero is the condition to find supersymmetric vacua.

CHAPTER VI

JANUS SOLUTIONS

With a little of adjustment from flat to AdS-sliced domain wall metric, this can describe defect or interface of conformal field theory. The metric is given by

$$ds^2_{AdS_{d+1}} = e^{2A} ds^2_{AdS_d} + dr^2, \tag{6.1}$$

which is clearly seen that $\eta_{\mu\nu} dx^\mu dx^\nu$ replaced by $ds^2_{AdS_d}$ that preserves $SO(2, d-1)$ symmetry corresponding to the symmetry of the conformal defect. Conformal defect has benefits in many aspects in physics, ranging from statistical physics to high-energy particle. In the context of AdS/CFT correspondence, conformal defect can be also describe in the gravity side as a solution called Janus. Non-supersymmetric Janus was first introduced in [?], found on an ansatz of AdS-sliced domain wall from the theory of IIB supergravity. After that there were several publications clarifying the essence of Janus solutions. The description in equivalent field theory was given in [?] and correlation function was calculated holographically in [?]. Due to the clearer of Janus's features, a large number of Janus solutions has been researching continuously, most of which can preserve supersymmetry and turn the solutions supersymmetric.

For deeply diving in the motivation, a little brief of original Janus will be given.

6.1 Original Janus solutions

The first step is to consider the deformation on AdS_5 space on the ansatz

$$ds^2 = f(\mu)(d\mu^2 + ds^2_{AdS_4}) + ds^2_{S^5} \tag{6.2}$$

with assuming that ϕ depend only on μ

$$\phi = \phi(\mu) \tag{6.3}$$

and the field

$$F_5 = 2f(\mu)^{\frac{5}{2}} d\mu \wedge \omega_{AdS_4} + 2\omega_{S^5} \tag{6.4}$$

where ω_{AdS_4} and ω_{S^5} are volume forms of AdS_4 and S^5 respectively.

The equation of motion and Bianchi identity in IIB supergravity are found by

$$R_{\alpha\beta} - \frac{1}{2}\partial_\alpha\phi\partial_\beta\phi - \frac{1}{4}F^2_{\alpha\beta} = 0 \tag{6.5}$$

$$\partial_\alpha(\sqrt{g}g^{\alpha\beta}\partial_\beta\phi) = 0 \tag{6.6}$$

$$*F_5 = F_5 \tag{6.7}$$

$$dF_5 = 0 \tag{6.8}$$

These equations can be solved together and get

$$\phi'(\mu) = \frac{c_0}{f^{3/2}(\mu)} \tag{6.9}$$

$$f'f' = 4f^3 - 4f^2 + \frac{c_0^2}{6}\frac{1}{f}. \tag{6.10}$$

This may be depicted that the potential in the function of f allows zero-energy particle to move only in the region at f_{min} to ∞ as Figure 6.1.

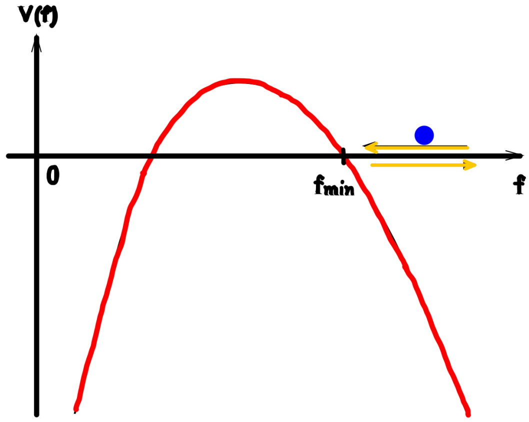


Figure 6.1: Potential plotted in function of f

Then, the equation **??** is solved that

$$\mu = \int_{f_{min}}^{f} \frac{d\tilde{f}}{2\sqrt{\tilde{f}^3 - \tilde{f}^2 + \frac{c_0^2}{24}\frac{1}{\tilde{f}}}}. \tag{6.11}$$

As the equation **??**, the solution of $\phi(\mu)$ can be found by the integration with μ as

$$\phi(\mu) - \phi(-\mu) = \int_{-\mu_0}^{\mu_0} \frac{c_0 d\mu}{f^{3/2}(\mu)}. \tag{6.12}$$

Together with the **??**, the boundary of integral can be changed to corresponding region of f as

$$\phi(\mu) - \phi(-\mu) = 2\int_{f_{min}}^{\infty} \frac{c_0 df}{2f^{3/2}\sqrt{f^3 - f^2 + \frac{c_0^2}{24}\frac{1}{f}}} \tag{6.13}$$

The solution of $\phi(\mu)$ is shown numerically by Figure 6.2 with different c_0. The important feature here of this solution is dilaton will take the constant at the maximum and minimum of μ.

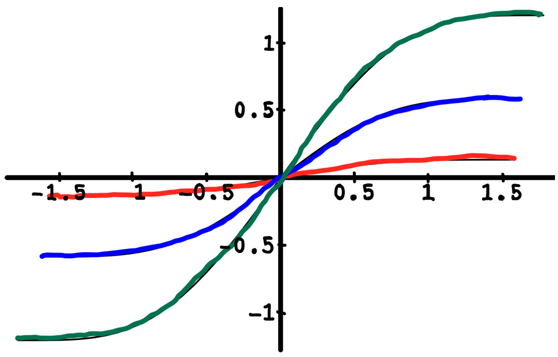


Figure 6.2: Profile of dilaton field with different c_0

With adopting the Poincare patch of the slice of AdS_4, a picture of corresponding conformal mapping is two half-spaces with different gauge coupling constants on each space attached each other at the wedge which breaks $SO(2,4)$ symmetry of the space into $SO(2,3)$ symmtery of the wedge.

From the dictionary of AdS/CFT, the dual field theory is $N = 4$ SYM theory in four dimensions with the boundary of two half spaces where the gauge coupling constant on each space corresponding to dilaton field that takes a constant at $\pm\mu$. The relation between constant dilaton at the boundary and gauge coupling constant of SYM theory can be given by

$$\mu = +\mu_0, \quad \frac{g_{YM}^2}{4\pi} = e^{\phi_0^+}, \qquad \mu = -\mu_0, \quad \frac{g_{YM}^2}{4\pi} = e^{\phi_0^-}. \tag{6.14}$$

6.2 Procedure to solve BPS equations

Instead of finding the solution by solving Einstein equation that depict non-supersymmteric Janus's configuration, supersymmteric Janus solutions can be found by BPS equations to consider whether there are ways to preserve supersymmtries from determining Killing spinors.

In four dimensions, chiral projection can be applied for Majorana spinors

$$\epsilon^i = \frac{1}{2}(1+\gamma_5)\epsilon^i_M, \quad \epsilon = \frac{1}{2}(1-\gamma_5)\epsilon^i_M \tag{6.15}$$

$$\bar{\epsilon}^i = \frac{1}{2}\bar{\epsilon}^i_M(1+\gamma_5), \quad \bar{\epsilon}_i = \frac{1}{2}\bar{\epsilon}^i_M(1-\gamma_5) \tag{6.16}$$

where ϵ^i_M are Majorana spinors and γ_5 is purely imaginary.

The process starts with considering supersymmetry transformations in $N = 4$ gauged supergravity in chapter IV written by

$$\delta\psi^i_\mu = 2D_\mu\epsilon^i + \frac{2}{3}A_1^{ij}\gamma_\mu\epsilon_j \tag{6.17}$$

$$\delta\chi^i = -\epsilon^{\alpha\beta}\mathcal{V}_\alpha D_\mu\mathcal{V}_\beta\gamma^\mu\epsilon^i - \frac{4}{3}iA_2^{ij}\epsilon_j \tag{6.18}$$

$$\delta\lambda^i_a = 2i\mathcal{V}_a{}^M D_\mu\mathcal{V}_M{}^{ij}\gamma^\mu\epsilon_j - 2iA_{2aj}{}^i\epsilon^j. \tag{6.19}$$

and AdS_3-sliced domain wall ansatz

$$ds^2 = e^{2A(r)}(e^{2\rho/l}dx^2_{1,1} + d\rho^2) + dr^2 \tag{6.20}$$

To solve BPS equations together with the ansatz, the scalars will depend radially only on r to ensure that scalars are invariant under $SO(2,2)$ symmetry of the AdS_3 and the projector is needed to be imposed as

$$\gamma_r\epsilon^i = M\epsilon_i, \quad \gamma_r\epsilon_i = M^*\epsilon^i \tag{6.21}$$

where $MM^* = 1$. This means M can be written as

$$M = e^{i\Lambda} \tag{6.22}$$

where Λ is a real phase. Let's define

$$\epsilon = e^{i\Lambda/2}\varepsilon. \tag{6.23}$$

This leads to

$$\gamma_r \varepsilon^i = \varepsilon_i, \quad \gamma_r \varepsilon_i = \varepsilon^i. \tag{6.24}$$

Replacing the projector along with the ansatz in the variation of gravitino, the equation turns into

$$(A'\gamma_r + \frac{1}{l}e^{-A}\gamma_\rho)\epsilon_i + \mathcal{W}\epsilon^i = 0. \tag{6.25}$$

The equation can also become

$$(A')^2 = -\frac{1}{l^2}e^{-2A} + \mathcal{W}^2. \tag{6.26}$$

by making complex conjugation. It should be noted that $\mathcal{W}^2 = W^2$ where W is a real superpotential. According to **??** and **??** that should compatibly get along together, one can see that an appropriate projection of γ^ρ should be

$$\gamma_\rho \epsilon^i = i\kappa e^{i\Lambda}\epsilon_i, \quad \longrightarrow \quad \gamma_\rho \varepsilon^i = i\kappa \varepsilon_i \tag{6.27}$$

The compatibility needs

$$\kappa^2 = 1. \tag{6.28}$$

All of this set of equations will give

$$(A' + \frac{i\kappa}{l}e^{-A})e^{i\Lambda} = \mathcal{W}. \tag{6.29}$$

Now, the materials above along with other two supersymmetry transformations of remaining fields as

$$\delta\chi^i = -\epsilon^{\alpha\beta}\mathcal{V}_\alpha D_\mu \mathcal{V}_\beta \gamma^\mu \epsilon^i - \frac{4}{3}iA_2^{ij}\epsilon_j, \tag{6.30}$$

$$\delta\lambda_a^i = 2i\mathcal{V}_a{}^M D_\mu \mathcal{V}_M{}^{ij}\gamma^\mu \epsilon_j - 2iA_{2aj}{}^i\epsilon^j, \tag{6.31}$$

are well prepared to find supersymmetric Janus solutions and will be used later to find new classes of Janus solutions in our work.

6.3 A general behavior of Janus solutions

This subsection provides general behavior of Janus solutions and slightly illustrates the difference between Janus solutions and holographic RG-flow.

6.3.1 Turning point and analysis on critical points

First of all, to analyze the behavior on Janus solutions, we must show the general from of their BPS equation after making the calculation in AdS_3-sliced domain wall ansatz that can be shown generally by

$$\phi_i' = \text{terms of } \frac{A'}{W}\frac{\partial W}{\partial \phi_i} + \text{terms of } \frac{\kappa e^{-A}}{l}\frac{\partial W}{\partial \phi_i} \tag{6.32}$$

where W is a real superpotential and derivative of of the warp factor $A'(r)$ is given by

$$A'^2 = W^2 - \frac{e^{-2A}}{l^2} \tag{6.33}$$

where ϕ_i is a scalar field related to the theory. It should be noted that in RG-flow solutions, there is no terms of $\frac{\kappa e^{-A}}{l}\frac{\partial W}{\partial \phi_i}$ appearing in the BPS equations and $A' = W$ for warp factor derivative. This is interestingly noticeable that above equations will be accordingly recovered to BPS equations for holographic RG-flow where $l \to \pm\infty$ which is reasonable as ansatz of AdS-sliced domain wall is reduced to flat domain wall at the condition of huge radius of sliced AdS.

Despite the addition of $\frac{\kappa e^{-A}}{l}\frac{\partial W}{\partial \phi_i}$ of Janus-type BPS equations different from RG-flow's, their analysis and implication on critical points are the same. Since critical points normally emerge at $r \to \pm\, \infty$, $A(r)$ will be come linear function at the points and $e^{A(r)}$ is enormously growing. Thus, $\frac{\kappa e^{-A}}{l}\frac{\partial W}{\partial \phi_i}$ can be suppressed and now our BPS equations for Janus are the same as holographic RG-flow's.

For details on critical points, as known that critical points are normally found at $r \to \pm\infty$ that reproduces the space into AdS space, one may calculate the radius of this

AdS by the behavior of warp factor A from A' equations. Beginning with

$$A'^2 = W^2 - \frac{e^{-2A}}{l^2}, \tag{6.34}$$

at large r, $\frac{e^{-2A}}{l^2}$ will become as small as it can be neglected. Moreover, in general, W takes a constant at critical points which is denoted by W_0. The equation thus turns into

$$A'^2 = {W_0}^2, \tag{6.35}$$

$$A' = \pm W_0, \tag{6.36}$$

$$\frac{dA}{dr} = \pm W_0, \tag{6.37}$$

$$A = \pm W_0 r. \tag{6.38}$$

The scalar potential V can be generally written in the form of W as

$$V = k_1 G^{ab} \frac{\partial W}{\partial \Phi^a} \frac{\partial W}{\partial \Phi^b} - k_2 W^2. \tag{6.39}$$

At $r \to \pm\infty$, W becoming constant W_0 leads to

$$V_0 = -k_2 W_0^2, \tag{6.40}$$

$$W_0 = \sqrt{-\frac{V_0}{k_2}}, \tag{6.41}$$

where V_0 is $V(r)$ at the boundary. Moving back to the ansatz of AdS_3-sliced domain wall,

$$ds^2 = e^{2A}(e^{2\rho/l} dx_{1,1}^2 + d\rho^2) + dr^2, \tag{6.42}$$

$$= e^{2(A+\rho/l)} dx_{1,1}^2 + e^{2A} d\rho^2 + dr^2. \tag{6.43}$$

For very large r, A will dominate ρ/l in $e^{2(A+\rho/l)}$ and the metric can become

$$ds^2 = e^{2A}dx_{1,1}^2 + e^{2A}d\rho^2 + dr^2, \tag{6.44}$$

$$= e^{2A}(dx_{1,1}^2 + d\rho^2) + dr^2, \tag{6.45}$$

$$= e^{2A}(\eta_{\mu\nu}dx^\mu dx^\nu) + dr^2. \tag{6.46}$$

Placing $A = \pm W_0 r$ into **??** will lead to

$$ds^2 = e^{\pm 2W_0 r}(\eta_{\mu\nu}dx^\mu dx^\nu) + dr^2. \tag{6.47}$$

Compared to AdS_4 ansatz

$$ds^2 = e^{\frac{2r}{L}}(\eta_{\mu\nu}dx^\mu dx^\nu) + dr^2, \tag{6.48}$$

it is obvious to see that

$$L = \frac{1}{W_0}, \tag{6.49}$$

or in terms of V_0

$$L = \sqrt{-\frac{k_2}{V_0}}. \tag{6.50}$$

The sign $\pm$ in $A = \pm W_0$ is just the indication where the boundary of r is whether it is $r \to +\infty$ or $r \to -\infty$.

To sum up, at critical points where $r \to \pm\infty$, the space can be clearly seen as AdS_4 and the radius of this AdS_4 is calculated by $L = \frac{1}{W_0} = \sqrt{-\frac{k_2}{V_0}}$.

Also, one special feature on Janus's behavior that should be put the emphasis on is that $A'^2 = W^2 - \frac{e^{-2A}}{l^2}$ gives a possibility to have a turning point $A' = 0$ where we can shift the coordinate r to $r = 0$ so that the value of $A(r)$ at this turning point can be denoted by $A(0)$ found by

$$A'(0) = \sqrt{W^2 - \frac{e^{-2A}}{l^2}} = 0. \tag{6.51}$$

Obviously, the value under the square root must be zero so

$$\frac{e^{-2A(0)}}{l^2} = W^2(0), \tag{6.52}$$

$$A(0) = -\frac{1}{2}\ln l^2 W^2(0). \tag{6.53}$$

The existence of a turning point found in Janus solutions lead to distinguished feature from RG-flow as the Janus solutions can possibly start and move back to the same critical point. Unlike holographic RG-flow, such solutions will interpolate between a critical point and the other critical point or a critical point to a singularity.

6.3.2 Holographic description on Janus solutions

For holographic description, the AdS_4 at the boundary corresponds to $SCFT_3$. The flow of Janus solutions can be holographically described as two-dimensional conformal defect that interpolates between a critical point to the other one that we should not forget that by the behavior of Janus solutions mentioned earlier, these points can be the same. For more details, AdS_4 and $SCFT_3$ correspond to each other because of having the same $SO(2,3)$ symmetry. During the flow of the solutions, $SO(2,3)$ symmetry will be broken to $SO(2,2)$ symmetry that describes two-dimensional conformal defect in $SCFT_3$

6.4 New supersymmetric Janus solution

Supersymmteric Janus solution has been studying with various gauge groups, dimensions and supersymmetries, some of which can be found in [?, ?]. Our work aimed to find new classes of Janus solution from four-dimensional gauged supergravity. After trying many ways, we found $SO(4) \times SO(4)$ gauge group and application of symplectic deformations can provide us new supersymmetric Janus solutions. These Janus solutions have $N = 1$ and $N = 2$

supersymmetries.

6.4.1 $SO(4) \times SO(4)$ gauge symmetry deformed by free parameters

We begin to find the solutions in $N = 4$ gauged supergravity coupled to $n = 6$ vector multiplets where its general structure is provided in chapter IV. Then, using symplectic deformation in [?], $SO(4) \times SO(4)$ gauge group can be deformed with deformation parameters α_0, α, β_1 and β_2, corresponding to electric-magnetic phases in each $SO(3)$ decomposed from $SO(4) \times SO(4)$ gauge group, written as

$$f_{+\hat{m}\hat{n}\hat{p}} = -g_0 \cos \alpha_0 \epsilon_{\hat{m}\hat{n}\hat{p}}, \quad f_{-\hat{m}\hat{n}\hat{p}} = g_0 \sin \alpha_0 \epsilon_{\hat{m}\hat{n}\hat{p}} \tag{6.54}$$

$$f_{+\tilde{m}\tilde{n}\tilde{p}} = g \cos \alpha \epsilon_{\tilde{m}\tilde{n}\tilde{p}}, \quad f_{-\tilde{m}\tilde{n}\tilde{p}} = g \sin \alpha_0 \epsilon_{\tilde{m}\tilde{n}\tilde{p}} \tag{6.55}$$

$$f_{+\hat{a}\hat{b}\hat{c}} = h_1 \cos \beta_1 \epsilon_{\hat{a}\hat{b}\hat{c}} \quad f_{-\hat{a}\hat{b}\hat{c}} = h_1 \sin \beta_1 \epsilon_{\hat{a}\hat{b}\hat{c}} \tag{6.56}$$

$$f_{+\tilde{a}\tilde{b}\tilde{c}} = h_2 \cos \beta_2 \epsilon_{\tilde{a}\tilde{b}\tilde{c}}, \quad f_{-\tilde{a}\tilde{b}\tilde{c}} = h_2 \sin \beta_2 \epsilon_{\tilde{a}\tilde{b}\tilde{c}} \tag{6.57}$$

where g_0, g, h_1 and h_2 are gauge coupling constant in four $SO(3)$ group and indices $M = (\hat{m}, \tilde{m}, \hat{a}, \tilde{a})$, for $\hat{m}, \tilde{m}, \hat{a}, \tilde{a} = 1, 2, 3$ corresponds to $SO(6, 6)$ in fundamental representation. This kind of components of embedding tensor are provided in [?] and rewritten in the conventions of [?]. In our work, for simplicity, we can set $\alpha_0 = 0$ by the transformation of global symmetry of $SL(2, \mathbb{R}) \times SO(6, n)$ and $\alpha = \frac{\pi}{2}$ due to providing an equivalent theory for any $\alpha > 0$.

6.4.2 $N = 2$ Janus solutions

$N = 2$ solutions can be found by the truncation of scalars in $SO(4) \times SO(4)$ gauge group into $SO(2) \times SO(2) \times SO(2) \times SO(2)$ whose is the subgroup of $SO(4) \times SO(4)$. We can find this coset representative by starting with $SO(6, 6)$ generators in fundamental representation as

$$(t_{MN})_P{}^Q = 2\delta^Q_{[M} \eta_{N]P} \tag{6.58}$$

whose non-compact generator is written by

$$Y_{ma} = t_{m,a+6}. \tag{6.59}$$

We can use this non-compact generator to build up the coset representative of $SO(2) \times SO(2) \times SO(2) \times SO(2)$ as

$$\mathcal{V} = e^{\phi_1 Y_{33}} e^{\phi_2 Y_{36}} e^{\phi_3 Y_{63}} e^{\phi_4 Y_{66}} \tag{6.60}$$

The next step is to get AdS_3-slice domain walls ansatz as

$$ds^2 = e^{2A(r)}(e^{\frac{2\rho}{l}} dx^2_{1,1} + d\rho^2) + dr^2 \tag{6.61}$$

where l is a radius of AdS_3 and $dx^2_{1,1} = \eta_{\alpha\beta} dx^\alpha dx^\beta$, α, $\beta = 0, 1$ represents flat Minkowski space. Then, we will perform the same process as previously mentioned and get the equations

$$A'^2 = W^2 - \frac{1}{l^2} e^{-2A} \tag{6.62}$$

$$\epsilon^{\hat{i}} = e^{\rho/2l} \tilde{\epsilon}^{\hat{i}} \tag{6.63}$$

where, now, W in our case is $W = |\mathcal{W}|$ that can be found by

$$\mathcal{W} = \frac{2}{3} \hat{\alpha} \tag{6.64}$$

where $\hat{\alpha}$ is the eigenvalue of A_1^{ij}. In this case, we found that A_1^{ij} is represented by

$$A_1^{ij} = \mathrm{diag}(\mathcal{A}_-, \mathcal{A}_+, \mathcal{A}_+, \mathcal{A}_-) \tag{6.65}$$

6.4.2.1 Consideration on broken supersymmetries

Since different eigenvalues of A_1^{ij} lead to different superpotentials, it might not be able to solve the BPS equations. To avoid this inconsistency, an amount of supersymmetries must be broken or some scalar fields must be truncated out. By breaking supersymmetry with no truncation of scalar fields, there are two possibilities of choosing superpotential whether it is $\mathcal{W}_+ = \frac{2}{3}\mathcal{A}_+$ or $\mathcal{W}_- = \frac{2}{3}\mathcal{A}_-$. These two choices correspond to $N = 2$ where details will be clarified through variation of gravitino field $\delta\psi_\mu^i = D_\mu\epsilon^i - \frac{2}{3}A_1^{ij}\gamma_\mu\epsilon_j$ below

$$\delta\psi_\mu^1 = D_\mu\epsilon^1 - \frac{2}{3}A_1^{11}\gamma_\mu\epsilon_1 \tag{6.66}$$

$$\delta\psi_\mu^2 = D_\mu\epsilon^2 - \frac{2}{3}A_1^{22}\gamma_\mu\epsilon_2 \tag{6.67}$$

$$\delta\psi_\mu^3 = D_\mu\epsilon^3 - \frac{2}{3}A_1^{33}\gamma_\mu\epsilon_3 \tag{6.68}$$

$$\delta\psi_\mu^4 = D_\mu\epsilon^4 - \frac{2}{3}A_1^{44}\gamma_\mu\epsilon_4 \tag{6.69}$$

where $A_1^{11} = A_1^{44} = \mathcal{A}_-$ and $A_1^{22} = A_1^{33} = \mathcal{A}_+$.

If we want to choose $\frac{2}{3}\mathcal{A}_-$ to be our superpotential. The question is how we can deal with $\mathcal{W}_+ = \frac{2}{3}\mathcal{A}_+$ that $\mathcal{A}_+$ appears in the eigenvalues of A_1^{ij} that will always come out in $\delta\psi_\mu^i$. The way to ensure that $\mathcal{A}_+$ will not mess up with our calculation is to force ϵ_2 and ϵ_3 to zero as we could see in the equations above that $\delta\psi_\mu^2 = 0$ and $\delta\psi_\mu^3 = 0$. However, since ϵ_2 and ϵ_3 are Killing spinors, vanishing Killing spinors accordingly lead to broken supersymmetries. Thereby, choosing $\mathcal{W}_- = \frac{2}{3}\mathcal{A}_-$ to be our superpotential corresponds to breaking $N = 4$ to $N = 2$ because of two vanishing Killing spinors (ϵ_2 and ϵ_3). We can also repeat the same process for selecting superpotential $\mathcal{W} = \mathcal{W}_+$, but ϵ_1 and ϵ_4 will be broken instead.

For one eigenvalue is chosen, supersymmetries then are broken to $N = 2$. In our calculation, we choose $\mathcal{W}_- = \frac{2}{3}\mathcal{A}_-$. Then, due to some broken supersymmetries, ϵ^2 and

ϵ^3 are zero inevitably and superpotential become

$$
\begin{aligned}
\mathcal{W} &= \mathcal{W}_- \\
&= \frac{1}{2}e^{-\phi/2}[\cosh\phi_4[g\cosh\phi_3(e^{\phi}\sin\alpha + i\cos\alpha) - g_0\sinh\phi_1\sinh\phi_3] \\
&- g_0\cosh\phi_1(\cosh\phi_2 + i\sinh\phi_2\sinh\phi_4) + ig\sin\alpha\cosh\phi_3\cosh\phi_4\chi]
\end{aligned} \tag{6.70}
$$

Besides, for the existence of superpotential, scalar potential can also be written as

$$
\begin{aligned}
V &= -2G^{rs}\frac{\partial W}{\partial\Phi^r}\frac{\partial W}{\partial\Phi^s} - 3W^2 \\
&= -\frac{1}{4}e^{-\phi}[g^2(1+\cos 2\alpha) + 2g_0^2 + 2g^2\sin\alpha\chi(2\cos\alpha + \sin\alpha\chi)] \\
&- \frac{1}{2}e^{\phi}g^2\sin^2\alpha + 2gg_0\sin\alpha\cosh\phi_1\cosh\phi_2\cosh\phi_3\cosh\phi_4,
\end{aligned} \tag{6.71}
$$

where $\Phi^r = (\phi, \chi, \phi_1, \phi_2, \phi_3, \phi_4)$ and G^{rs} is the inverse of the matrix shown below.

6.4.2.2 Lagrangian of kinetic term

Substituting the coset representative $SO(2)\times SO(2)\times SO(2)\times SO(2)$ into the lagrangian of $N = 4$ gauged supergravity, the kinetic term turns into

$$
\begin{aligned}
\mathcal{L}_{kin} &= \frac{1}{2}G_{rs}\Phi^{r\prime}\Phi^{s\prime} \\
&= -\frac{1}{4}(\phi'^2 + e^{-2\phi}\chi'^2) - \frac{1}{16}[6 + \cosh 2(\phi_2 - \phi_3) \\
&+ \cosh 2(\phi_2 + \phi_3) + 2\cosh 2\phi_4(\cosh 2\phi_2\cosh 2\phi_3 - 1)]\phi_1'^2 \\
&- \cosh\phi_2\cosh\phi_4\sinh\phi_3\sinh\phi_4\phi_1'\phi_2' - \cosh\phi_3\cosh\phi_4\sinh\phi_2\sinh\phi_4\phi_1'\phi_3' \\
&+ \sinh\phi_2\sinh\phi_3\phi_1'\phi_4' - \frac{1}{2}\cosh^2\phi_4\phi_2'^2 - \frac{1}{2}\cosh\phi_4\phi_3'^2 - \frac{1}{2}\phi_4'^2.
\end{aligned} \tag{6.72}
$$

From this term, scalar matric G_{rs} can be extracted and shown explicitly by

$$
G^{rs} = \begin{pmatrix} -2 & 0 & \mathbf{0_{1\times 4}} \\ 0 & 2e^{2\phi} & \mathbf{0_{1\times 4}} \\ \mathbf{0_{4\times 1}} & \mathbf{0_{4\times 1}} & \mathcal{G}^{r's'} \end{pmatrix} \tag{6.73}
$$

where $G^{r's'}$ and indices $r', s' = 1, 2, 3, 4$ shown by

$$\mathcal{G}^{r's'} = \begin{pmatrix} A_1 & B_1 & B_2 & B_3 \\ B_1 & A_2 & B_4 & B_5 \\ B_2 & B_4 & A_3 & B_6 \\ B_3 & B_5 & B_6 & A_4 \end{pmatrix}, \tag{6.74}$$

where

$$\begin{aligned} &A_1 = -\operatorname{sech}^2\phi_2 \operatorname{sech}^2\phi_3, \quad A_2 = -\operatorname{sech}^2\phi_3 \operatorname{sech}^2\phi_4 - \tanh^2\phi_3, \\ &A_3 = \operatorname{sech}^2\phi_2 \tanh^2\phi_4 - 1, \quad A_4 = -\frac{1}{2}\operatorname{sech}^2\phi_2 \operatorname{sech}^2\phi_3(1 + \cosh 2\phi_2 \cosh 2\phi_3), \\ &B_1 = \operatorname{sech}\phi_2 \operatorname{sech}\phi_3 \tanh\phi_3 \tanh\phi_4, \quad B_2 = \operatorname{sech}\phi_2 \operatorname{sech}\phi_3 \tanh\phi_2 \tanh\phi_4, \\ &B_3 = -\operatorname{sech}\phi_2 \operatorname{sech}\phi_3 \tanh\phi_2 \tanh\phi_3, \quad B_4 = -\tanh\phi_2 \tanh\phi_3 \tanh^2\phi_4, \\ &B_5 = \tanh\phi_2 \tanh^2\phi_3 \tanh\phi_4, \quad B_6 = \tanh^2\phi_2 \tanh\phi_3 \tanh\phi_4 \end{aligned}$$

6.4.2.3 Critical points in the $N = 2$ solutions

We find the critical points from the procedure given in chapter V. It can show that where $\phi_1 = \phi_2 = \phi_3 = \phi_4 = 0$,

$$\phi = \ln(-\frac{g_0}{g \sin\alpha}), \tag{6.75}$$

and

$$\chi = -\frac{\cos\alpha}{\sin\alpha}. \tag{6.76}$$

can give the critical point. It is obviously seen that dilaton and axion depend on α. Nonetheless, owning to choosing $\alpha = \frac{\pi}{2}$ and $g_0 = -g$, this allows us to shift ϕ and χ to zero as

$$\phi = \ln[-\frac{-g}{g\sin\frac{\pi}{2}}], \tag{6.77}$$

$$= \ln 1, \tag{6.78}$$

$$= 0, \tag{6.79}$$

and

$$\chi = -\frac{\cos\pi/2}{\sin\pi/2}, \tag{6.80}$$

$$= 0. \tag{6.81}$$

Now, all scalars vanish so this critical point of AdS_4 is at the origin of scalar manifold which obviously preserves $N = 4$ supersymmetry with $SO(4)\times SO(4)$ symmetry. As seen that dilaton and axion field can be shifted by choosing $g_0 = -g$ and $\alpha = \frac{\pi}{2}$, this crical point will be shifted to the origin of the scalar manifold $SL(2,\mathbb{R})/SO(2) \times SO(6,6)/SO(6) \times SO(6)$ where all scalar fields are zero. From this point, we can calculate the scalar potential at the vacuum

$$V_0 = -3g^2, \tag{6.82}$$

and convert this into the radius of this AdS_4 as

$$L = \sqrt{-\frac{3}{V_0}} = \frac{1}{g}. \tag{6.83}$$

With the positive constant g, the critical points become supersymmetric $N = 4$ $SO(4)\times SO(4)$ vacuum.

6.4.2.4 BPS equations

From now on, all materials is prepared to find BPS equations that can be found as

$$A'^2 + \frac{1}{l^2}e^{-2A} = W^2 \tag{6.84}$$

$$\phi' = -4\frac{A'}{W}\frac{\partial W}{\partial \phi} - 4e^{\phi}\frac{\kappa e^{-A}}{lW}\frac{\partial W}{\partial \chi} \tag{6.85}$$

$$\chi' = -4e^{2\phi}\frac{A'}{W}\frac{\partial W}{\partial \chi} + 4e^{\phi}\frac{\kappa e^{-A}}{lW}\frac{\partial W}{\partial \phi} \tag{6.86}$$

$$\phi_1' = \mathcal{G}^{1r'}\frac{A'}{W}\frac{\partial W}{\partial \hat{\Phi}^{r'}} - 2\,\mathrm{sech}\,\phi_2\,\mathrm{sech}\,\phi_3\,\mathrm{sech}\,\phi_4\frac{\kappa e^{-A}}{lW}\frac{\partial W}{\partial \phi_3} \tag{6.87}$$

$$\begin{aligned}\phi_2' = \mathcal{G}^{2r'}\frac{A'}{W}\frac{\partial W}{\partial \hat{\Phi}^{r'}} &+ \frac{\kappa e^{-A}}{lW}(2\,\mathrm{sech}\,\psi_4\tanh\phi_3\tanh\phi_4\frac{\partial W}{\partial \phi_3}\\ &- 2\,\mathrm{sech}\,\phi_4\frac{\partial W}{\partial \phi_4})\end{aligned} \tag{6.88}$$

$$\begin{aligned}\phi_3' = \mathcal{G}^{3r'}\frac{A'}{W}\frac{\partial W}{\partial \hat{\Phi}^{r'}} &+ \frac{\kappa e^{-A}}{lW}(2\,\mathrm{sech}\,\phi_2\,\mathrm{sech}\,\phi_3\,\mathrm{sech}\,\phi_4\frac{\partial W}{\partial \phi_1}\\ &- 2\,\mathrm{sech}\,\phi_4\tanh\phi_3\tanh\phi_4 + 2\,\mathrm{sech}\,\phi_4\tanh\phi_2\tanh\phi_3\frac{\partial W}{\partial \phi_2})\end{aligned} \tag{6.89}$$

$$\begin{aligned}\phi_4' = \mathcal{G}^{4r'}\frac{A'}{W}\frac{\partial W}{\partial \hat{\Phi}^{r'}} &+ \frac{\kappa e^{-A}}{lW}(-2\,\mathrm{sech}\,\phi 4\frac{\partial W}{\partial \phi_2}\\ &- 2\,\mathrm{sech}\,\phi_4\tanh\phi_2\tanh\phi_3\frac{\partial W}{\partial \phi_3})\end{aligned} \tag{6.90}$$

Clearly seen, with $l \to \infty$, these solutions is recovered to holographic RG-flow solutions in [?]. Besides, these solutions is much more general as giving Janus solutions in [?] that preserve $SO(2)\times SO(2)\times SO(2)\times SO(3)$ or $SO(2)\times SO(2)\times SO(3)\times SO(2)$ symmetry by trucating scalar ϕ_1 and ϕ_3 or ϕ_2 and ϕ_4 to zero.

These BPS equations can give numerical solutions with different α provided in Figure 6.3. Our solutions are calculated with assuming $g = 1$, $\kappa = 1$, $l = 1$ and $g_0 = -g\sin\alpha$ and for the phase parameters, $\alpha = \frac{\pi}{2}$ and $\alpha_0 = 0$. It should be emphasized again that all $\alpha > 0$ equivalent to $\alpha = \frac{\pi}{2}$. The solutions interpolate between $SO(4)\times SO(4)$ AdS_4 critical point. The corresponding theory from the field theory side is $SO(2) \times SO(2)\times SO(2)\times SO(2)$ two-dimensional defect within three-dimensional $N = 4$ $SCFT$ theory with $SO(4)\times SO(4)$ symmetry. Different values of $\kappa = \pm 1$ lead the defect to

$N=(2,0)$ or $N=(0,2)$ supersymmetry respectively.

6.4.3 $N=1$ Janus solutions

Performing to find Janus solutions on $SO(3)_{\text{diag}}\times SO(3)$ symmetry relies on coset representative

$$\mathcal{V}=e^{\phi_1\hat{Y}_1}e^{\phi_3\hat{Y}_3} \tag{6.91}$$

where $\hat{Y}_1$ and $\hat{Y}_3$ are non compact generator defined by

$$\hat{Y}_1=Y_{11}+Y_{22}+Y_{33}+Y_{44},\quad \hat{Y}_3=Y_{51}+Y_{62}+Y_{73}+Y_{84} \tag{6.92}$$

From these generators, A_1^{ij} will be

$$A_1^{ij}=\text{diag}(\mathcal{A},\mathcal{B},\mathcal{B},\mathcal{B}). \tag{6.93}$$

Choosing $\mathcal{A}$ in our calculation, the superpotential becomes

$$\begin{aligned}\mathcal{W}=&\frac{1}{2}e^{\phi/2}[g\cosh^3\phi_3+h_1\sin\beta_1(i\sinh\phi_1-\cosh\phi_1\sinh\phi_3)^3]\\&+\frac{1}{2}e^{-\phi/2}[g(\cosh\phi_1+i\sinh\phi_1\sinh\phi_3)^3-(\sinh\phi_1+i\cosh\phi_1\sinh\phi_3)^3h_1\cos\beta_1]\\&+\frac{1}{2}e^{-\phi/2}[ig\cosh^3\phi_3+h_1\sin\beta_1(\sinh\phi_1+i\cosh\phi_1\sinh\phi_3)^3]\chi\end{aligned} \tag{6.94}$$

Putting on the emphasis on considering to determine residual supersymmetry, ϵ^2, ϵ^3 and ϵ^4 are required to be zero as admitting consistent BPS equations for choosing $\mathcal{A}$ to be superpotential. Thus, the solutions that will be found from this case must preserve $N=1$ supersymmetry. Superpotential can also be given in terms of W as

$$V=4(\frac{\partial W}{\partial\phi})^2+4e^{2\phi}(\frac{\partial W}{\partial\chi})^2+\frac{2}{3}\operatorname{sech}^2\phi_3(\frac{\partial W}{\partial\phi_1})^2+\frac{2}{3}(\frac{\partial W}{\partial\phi_3})^2-3W^2 \tag{6.95}$$

6.4.3.1 Critical points in $N = 1$ case

This sector can give three supersymmteric AdS_4 critical points which will be entailed below. For simplicity of the calculation for the rest of these solutions, α and g_0 will be substituted by $\pi/2$ and $-g$ respectively. The first critical point is trivial $N = 4$ AdS_4 with $SO(4) \times SO(4)$ symmetry where all scalars vanish. The other two special critical points are emerged from the deformation of the parameter β_1 as

$$
\begin{aligned}
i) \quad & \beta_1 = 0; \qquad \phi_3 = \chi = 0, \quad \phi_1 = \frac{1}{2}\ln(\frac{h_1 + g}{h_1 - g}) \\
& \phi = -\frac{1}{2}\ln(1 - \frac{g^2}{h_1^2}), \quad V_0 = -\frac{3g^2 h_1}{\sqrt{h_1^2 - g^2}}, \quad L = \sqrt{\frac{\sqrt{h_1^2 - g^2}}{g^2 h_1}} \\
ii) \quad & \beta_1 = \pi/2; \qquad \phi_1 = \chi = 0, \quad \phi_3 = \frac{1}{2}\ln(\frac{h_1 + g}{h_1 - g}) \\
& \phi = \frac{1}{2}\ln(1 - \frac{g^2}{h_1^2}), \quad V_0 = -\frac{3g^2 h_1}{\sqrt{h_1^2 - g^2}}, \quad L = \sqrt{\frac{\sqrt{h_1^2 - g^2}}{g^2 h_1}}
\end{aligned}
$$

Critical point i has $SO(3)_{\text{diag}} \times SO(3) \times SO(3)$ symmetry while residual symmetry $SO(3) \times SO(3)_{\text{diag}} \times SO(3)$ is found on critical point ii. These two critical points can be checked that $N = 4$ supersymmtery is preserved resulting from all identical eigenvalues of A_1^{ij} where $\chi = \phi_1 = 0$ or $\chi = \phi_3 = 0$. Since the case $\phi_1 = 0$ or $\phi_3 = 0$ does not provide consistent BPS equations, there is no Janus solutions for this case. Also, $\chi = 0$ along with keeping ϕ_1 and ϕ_3 gives inconsistent BPS equations. Hence, the possible Janus solutions must keep ϕ_1 and ϕ_3 that preserve $N = 1$ supersymmetry due to vanishing ϵ^2, ϵ^3 and ϵ^4 for choosing $\mathcal{A}$ as the superpotential while RG-flow in [?] can provide both $N = 1$ and $N = 4$ solutions.

6.4.3.2 BPS equations

By choosing mentioned superpotential above, BPS equations can be given by

$$\phi' = -4\frac{A'}{W}\frac{\partial W}{\partial \phi} - 4e^{\phi}\frac{\kappa e^{-A}}{lW}\frac{\partial W}{\partial \chi} \tag{6.96}$$

$$\chi' = -4e^{2\phi}\frac{A'}{W}\frac{\partial W}{\partial \phi} + 4e^{\phi}\frac{\kappa e^{-A}}{lW}\frac{\partial W}{\partial \phi} \tag{6.97}$$

$$\phi_1' = -\frac{2}{3}\operatorname{sech}^2\phi_3\frac{A'}{W}\frac{\partial W}{\partial \phi_1} - \frac{2}{3}\operatorname{sech}\phi_3\frac{\kappa e^{-A}}{lW}\frac{\partial W}{\partial \phi_3} \tag{6.98}$$

$$\phi_3' = -\frac{2}{3}\phi_3\frac{A'}{W}\frac{\partial W}{\partial \phi_3} + \frac{2}{3}\operatorname{sech}\phi_3\frac{\kappa e^{-A}}{lW}\frac{\partial W}{\partial \phi_1} \tag{6.99}$$

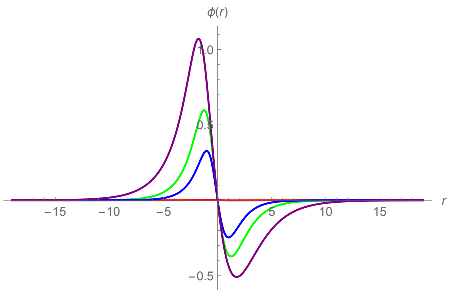
and the metric function is

$$A'^2 + \frac{e^{-2A}}{l^2} = W^2 \tag{6.100}$$

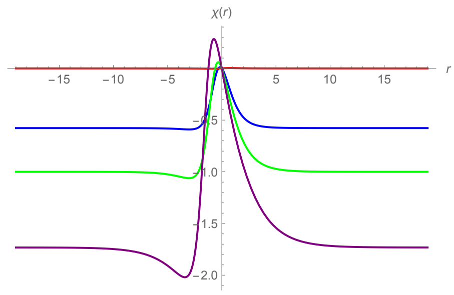
These equations are also reduced to the RG-flow solutions in [?] at $l \to \infty$.

The numerical solutions solved from the equations above will be separated into three sets. The first one is the solutions interpolating between trivial AdS_4 $N = 4$ $SO(4) \times SO(4)$ vacua with different values of β_1 in Figure 6.4 whose dual field theory is $N = 4$ $SCFT_3$ invariant under $SO(4)\times SO(4)$ group with $N = 1$ conformal defects inside. The second set is for β_1 describing the solutions that interpolate between critical points i in Figure 6.5 which has $SO(3)_{\text{diag}}\times SO(3)\times SO(3)$ symmetry. The final set is the solutions that interpolate between critical points ii found at $\beta_1 = \frac{\pi}{2}$ with $SO(3)\times SO(3)_{\text{diag}}\times SO(3)$ symmetry in Figure 6.6. These two latter solutions with non-trivial critical points can be described holographically by $N = 4$ $SCFT_3$ with $SO(3)_{\text{diag}} \times SO(3) \times SO(3)$ and $SO(3)\times SO(3)_{\text{diag}}\times SO(3)$ symmetry respectively with $N = (1, 0)$ or $N = (0, 1)$ conformal defects, depending on the value of κ, included.

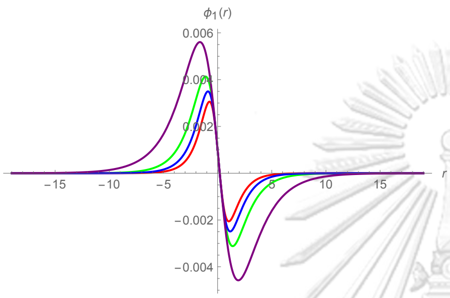
The numerical solutions interpolating between critical points i are represented by pink lines while cyan lines are solutions interpolating between trivial critical points for clear comparison with non-trivial one. Also, the Janus solutions interpolating between critical points ii in yellow lines are compared to purple lines which represent the solutions interpolating between trivial AdS_4 vacua.



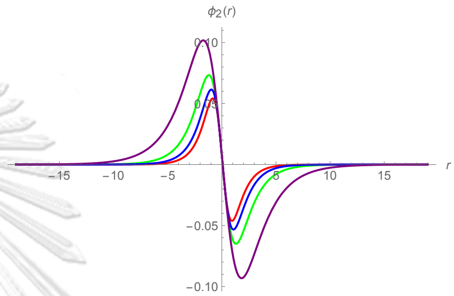
(a) $\phi(r)$ solution



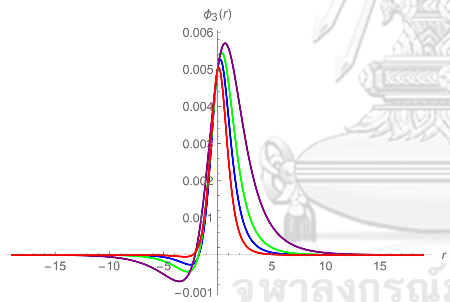
(b) $\chi(r)$ solution



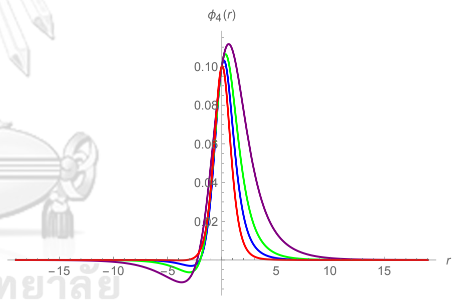
(c) $\phi_1(r)$ solution



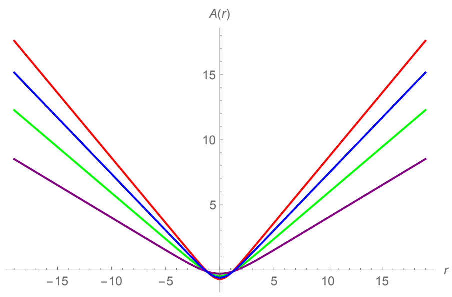
(d) $\phi_2(r)$ solution



(e) $\phi_3(r)$ solution

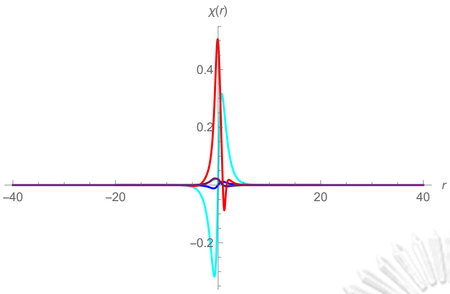


(f) $\phi_4(r)$ solution

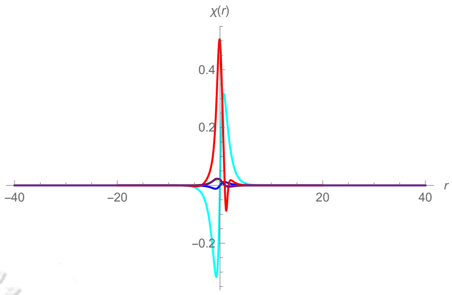


(g) $A(r)$ solution

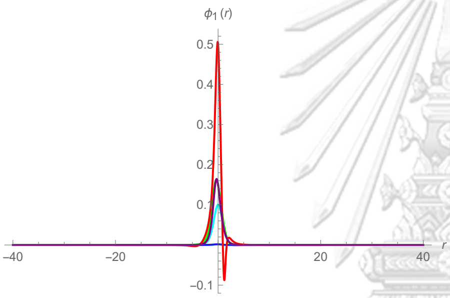
Figure 6.3: The $N = 2$ Janus solutions that interpolate between trivial $N = 4$ $SO(4) \times SO(4)$ AdS_4 critical points with using constant paremeter $\kappa = 1$, $l = 1$, $g = 1$ $g_0 = -g \sin\alpha$ and $\alpha = \frac{\pi}{2}$ (red), $\alpha = \frac{\pi}{3}$ (blue), $\alpha = \frac{\pi}{4}$ (green), $\alpha = \frac{\pi}{6}$ (purple),



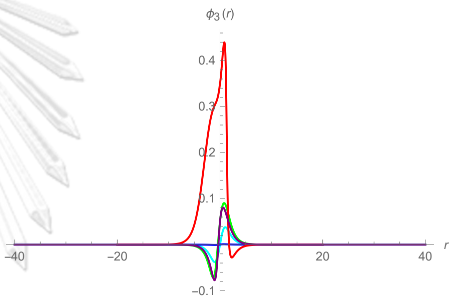
(a) $\phi(r)$ solution



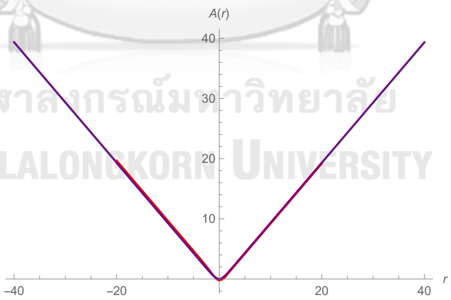
(b) $\chi(r)$ solution



(c) $\phi_1(r)$ solution

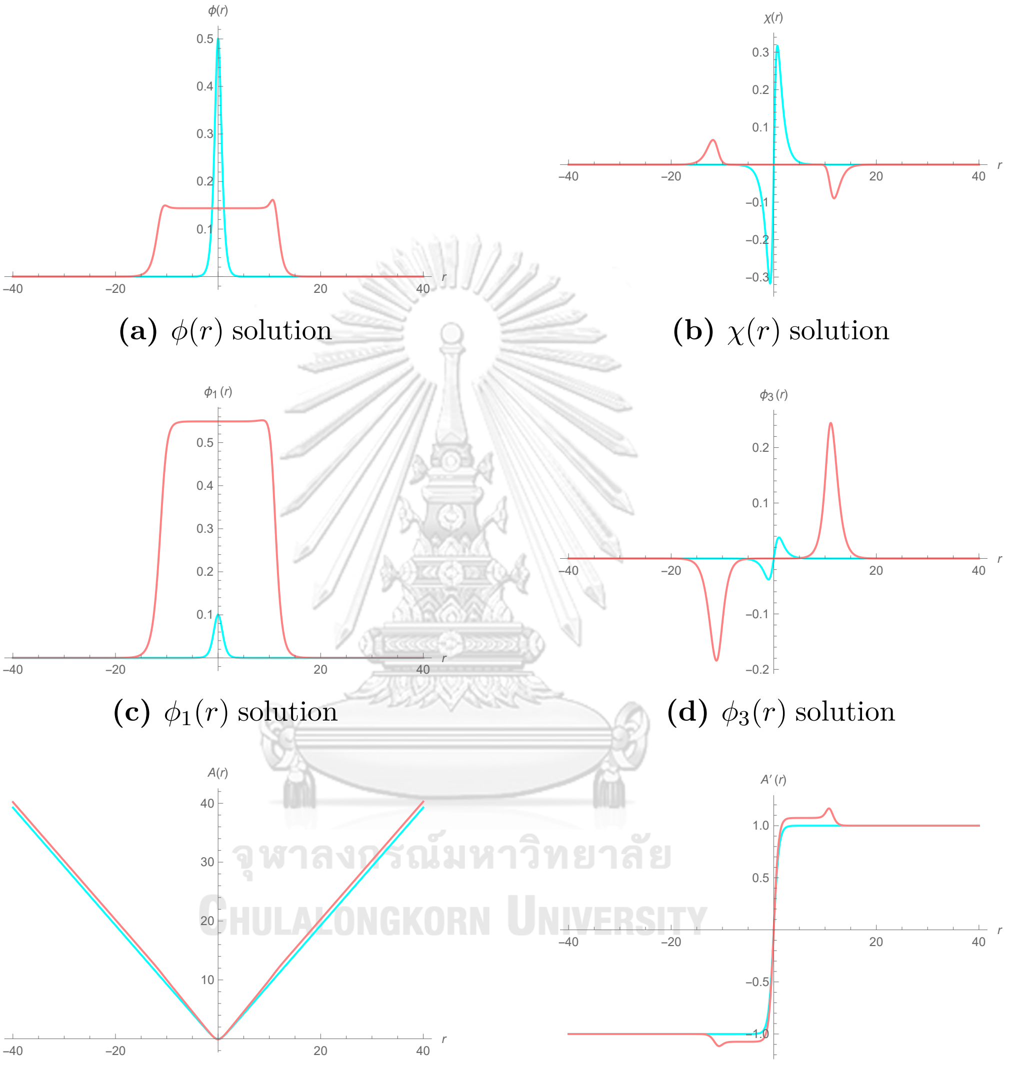


(d) $\phi_3(r)$ solution



(e) $A(r)$ solution

Figure 6.4: The $N = 1$ Janus solutions that interpolate between $SO(4) \times SO(4)$ $N = 4$ AdS_4 critical points with using constant paremeter $\kappa = l = g = 1$ $g_0 = -g$ and $\beta_1 = 0$ (cyan), $\beta_1 = \frac{\pi}{2}$ (purple), $\beta_1 = \frac{\pi}{3}$ (blue), $\beta_1 = \frac{\pi}{4}$ (green), $\beta_1 = \frac{\pi}{6}$ (red)



(a) $\phi(r)$ solution

(b) $\chi(r)$ solution

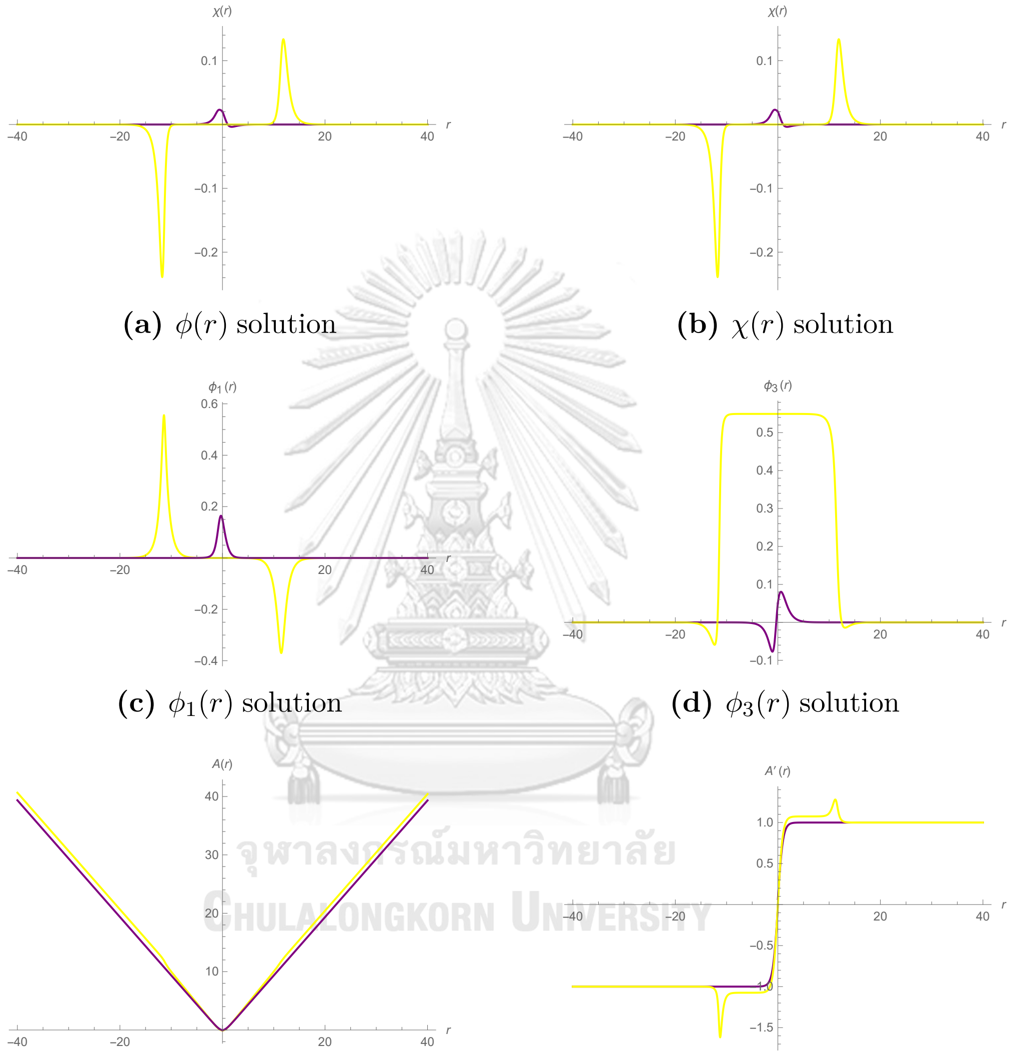
(c) $\phi_1(r)$ solution

(d) $\phi_3(r)$ solution

(e) $A(r)$ solution

(f) $A'(r)$ solution

Figure 6.5: The $N = 1$ Janus solutions (pink) that interpolate between $SO(3)_{\text{diag}} \times SO(3) \times SO(3)$ $N = 4$ AdS_4 critical points i) with using constant paremeters $\kappa = l = g = 1$ and $g_0 = -g$



(a) $\phi(r)$ solution

(b) $\chi(r)$ solution

(c) $\phi_1(r)$ solution

(d) $\phi_3(r)$ solution

(e) $A(r)$ solution

(f) $A'(r)$ solution

Figure 6.6: The $N = 1$ Janus solutions (yellow) that interpolate between $SO(3) \times SO(3)_{\text{diag}} \times SO(3)$ $N = 4$ AdS_4 critical points ii) with using constant paremeters $\kappa = l = g = 1$ and $g_0 = -g$

CHAPTER VII

CONCLUSIONS AND COMMENTS

Our scope of study in this work is to find Janus solutions from four-dimensional $N = 4$ gauged supergravity with $SO(4) \times SO(4)$ gauge symmetry. We found that with symplectic deformation, see [?], applied to $SO(4) \times SO(4) \sim SO(3) \times SO(3) \times SO(3) \times SO(3)$ gauge group, four deformations parameters for each $SO(3)$ group are expected to give richer structure of the theory. The setting of these parameters is summarized below as

$$\begin{array}{ccc} \alpha_0 & \text{for} & SO(3)_1 \\ \alpha & \text{for} & SO(3)_2 \\ \beta_1 & \text{for} & SO(3)_3 \\ \beta_2 & \text{for} & SO(3)_4 \end{array}$$

where $1, 2, 3, 4$ are just labels for the four $SO(3)$ groups and

$$\begin{aligned} &\alpha_0 = 0 \quad \text{because of the transformation of} \quad SL(2,\mathbb{R}) \\ &\alpha = \frac{\pi}{2} \quad \text{because of giving the equivalent theories for any} \quad \alpha > 0 \\ &\beta_1 \quad \text{and} \quad \beta_2 \quad \text{are free parameters.} \end{aligned}$$

The presence of these free parameters leads us to find two classes of Janus solutions comprising $N = 2$ and $N = 1$ supersymmetries.

7.1 $N = 2$ solutions

The $N = 2$ case admits only one critical point at $\phi_1 = \phi_2 = \phi_3 = \phi_4 = 0$ and ϕ, χ can also vanish by fixing $\alpha = \frac{\pi}{2}$ and $g_0 = -g$. Obviously, this critical point is the trivial $N = 4$ AdS_4 critical point with $SO(4) \times SO(4)$ symmetry. Since there are no other critical points found in this case, the $N = 2$ Janus solutions that are discovered in $SO(2) \times SO(2) \times SO(2) \times SO(2)$ sector can interpolate only between AdS_4 trivial critical point. For the holographic description on the field theory side, the solutions correspond to a two-dimensional conformal defect or interface that preserves $N = (2, 0)$ or $N = (0, 2)$ supersymmetries that depends on the value of κ within the $SCFT_3$ with $SO(4) \times SO(4)$ symmetry.

7.2 $N = 1$ solutions

The $N = 1$ case provides much more exciting structure since β_1 appears in the superpotential that give more possibility to have non-trivial critical points. As expected, there are not only trivial critical point of AdS_4 but also two non-trivial critical points, is obtained from $\beta_1 = 0$ and $\beta_1 = \pi/2$ as critical points i and ii respectively. The solutions that can interpolate between trivial AdS_4 critical point correspond to two-dimensional conformal defects preserving $N = 1$ supersymmetry within $N = 4$ $SCFT_3$ with $SO(4) \times SO(4)$ symmetry. Critical point i has $SO(3)_{\text{diag}} \times SO(3) \times SO(3)$ as a residual symmetry while $SO(3) \times SO(3)_{\text{diag}} \times SO(3)$ residual symmetry is preserved for critical point ii. Due to $\chi = \phi_1 = 0$ at critical point ii and $\chi = \phi_3 = 0$ in critical point i giving the same eigenvalues of A_1^{ij}, all supersymmetries are unbroken. Thus, these critical points preserve $N = 4$ supersymmetry. The $N = 1$ solutions that are found from $SO(3) \times SO(3)_{\text{diag}}$ sector with appropriate boundary conditions then can interpolate between these critical points with $\beta_1 = 0$ and $\beta_1 = \pi/2$ for critical point i and ii respectively. This can also be depicted by a holographic picture as the non-trivial critical point i and ii correspond to $SCFT_3$ with $SO(3)_{\text{diag}} \times SO(3) \times SO(3)$ and $SO(3) \times SO(3)_{\text{diag}} \times SO(3)$ symmetry respectively. The solutions that preserve some amount of conformal symmetry are dual to two-dimensional conformal defects with

$N = (1, 0)$ or $N = (0, 1)$ supersymmetries, depending on the value of κ, preserved on the defect.

7.3 Some comments and recent Janus-related works

Since $SO(3)_{\text{diag}}$ in [?] sector involves both β_1 and β_2 parameters, they could provide more vacua and new holographic solutions including Janus solutions owing to different values of those parameters. Another further future study would be the study of a holographic interpretation in dual field theory interms of relevant generators and correlation function. It is also much more compelling to find ways of uplifing these solutions in four dimensions to higher-dimensional theories that might give a possibility to explain conformal defects in string theory.

A few of the recent Janus-related works on finding holographic solutions including Janus solutions are [?] and [?]. These papers show that the holographic solutions including RG-flow and Janus solutions are found from three-dimensional $N = 8$ gauged supergravity with $SO(8)$ gauge symmetry using the embedding tensor formalism that depends on a free parameter α. The α parameter appears in the two copies of the superconformal group $D^1(2, 1; \alpha)$ describing the isometry of $AdS_3 \times S^3 \times S^3 \times S^1$ that admits 16 supercharges, see [?] and [?]. With the choices of α and different truncations of scalar fields, new solutions and vacua are discovered with [?] for the case of $\alpha = 1$ and general α in [?].

It should be highlighted that writing the embedding tensor with free parameter is such a compelling procedure to find new holographic solutions. As seen in the recent publications [?], [?] and our work, all use the same aforementioned technique. It is very interesting that the inclusion of deformation parameters in different gauge groups in gauged supergravity would lead to a richer structure of critical points and holographic solutions that await us to discover.

APPENDIX I

GENERAL RELATIVITY AND GEOMETRY OF SCALAR MANIFOLD

With the structure of general relativity, familiar mathematics applicable in classical theory is not sufficient. For this case, tensor calculus must be introduced. Besides, this advanced subject is a foundation in researching theoretical physics, especially in high-energy physics.

This chapter will give a short review on tensor calculus then move to the emphasis on veilbein formalism whose concepts are developed and adopted in the geometry of the scalar manifold entailed at the end of the chapter

A.1 Tensor calculus in general relativity

Due to the curved space that is not usual flat space, a distance between any two points in the space is no longer given by the Euclid's geometry. Some information encoding the curvature of the space is necessary in measuring how far between those mentioned two points. That information is displayed by the metric $g_{\mu\nu}$ as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{A.1}$$

This is the square of the infinitesimal between any two points in the space.

Quantities in curved manifold can be explained by geometrical object called tensor. Tensor with rank (k, l) can be written by

$$T^{(k,l)} = T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} \partial_{\mu_1} \otimes \dots \otimes \partial_{\mu_1} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_k} \tag{A.2}$$

Since the tensor on the curved manifold living on the tangent space of the manifold must be defined. To find its derivative, connection is introduced to how much the tensor is varied from point to point. Particularly, this is also important in writing covariant derivative which is defined by

$$\begin{aligned}\nabla_\mu T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_k} = \partial_\mu T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_k} + \Gamma^{\mu_1}_{\mu\lambda} T^{\lambda\ldots\mu_k}{}_{\nu_1\ldots\nu_l} + \ldots\Gamma^{\mu_k}_{\mu\lambda} T^{\mu_1\ldots\lambda}{}_{\nu_1\ldots\nu_l} \\ - \Gamma^{\lambda}_{\mu\nu_1} T^{\mu_1\ldots\mu_k}{}_{\lambda\ldots\nu_l} - \ldots - \Gamma^{\lambda}_{\mu\nu_l} T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\lambda}.\end{aligned} \tag{A.3}$$

The connection shown above is named "Christoffel" connection symmetric under switching indices μ and ν, also known as torsion free condition which does not produce torsion, given by

$$\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\nu\mu} \tag{A.4}$$

and the matric compatibility is defined by

$$\nabla_\rho g_{\mu\nu} = 0. \tag{A.5}$$

With two equations above, they can give the connection in the relation with metric as

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}). \tag{A.6}$$

As all the quantities shown above cannot give how much curved of the manifold is, in order to find the curvature of the space, some value must be defined and that quantity is Riemann tensor given by

$$R^{\rho}_{\lambda\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\lambda} - \partial_\nu \Gamma^{\rho}_{\mu\lambda} + \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\lambda} - \Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\mu\lambda} \tag{A.7}$$

Riemann tensor with contraction of the indices can lead to lower-rank tensor which also give the information on the curvature. The results are Ricci tensor and Ricci scalar which are shown respectively as follows

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}, \quad R = R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu}. \tag{A.8}$$

Those definitions which give details on curved manifold altogether with the energy-momentum tensor is a perfect bridge to construct Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{A.9}$$

where the right hand side of the equation is called Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{A.10}$$

which gives the vanishing divergence as

$$\nabla^{\mu} G_{\mu\nu} = 0. \tag{A.11}$$

Also, the divergence of energy-momentum tensor always leads to conservation of energy that is always true in the univers

$$\nabla_{\mu} T^{\mu\nu} = 0. \tag{A.12}$$

A.2 Differential form

$p-$ form A_p on the coordinate basis is defined by

$$A_p = \frac{1}{p!} A_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \tag{A.13}$$

where the basis of $p-$ form is given by

$$dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} = p! = dx^{[\mu_1} \otimes \ldots \otimes dx^{\mu_p]}. \tag{A.14}$$

Higher-rank form can be built by the product of two lower-rank form. For example, the product between $p-$ form A_p and $q-$ form B_q will give $(p+q)-$ form $(A \wedge B)_{p+q}$ form as

$$A \wedge B = \frac{1}{p!q!} A_{\mu_1 ... \mu_p} B_{\nu_1 ... \nu_q} \wedge ... \wedge dx^{\mu_1} \wedge dx^{\mu_p} \wedge dx^{\nu_1} \wedge ... \wedge dx^{\nu_q} \tag{A.15}$$

whose component is

$$(A_p \wedge B_q)_{\mu_1 ... \mu_p \nu_1 ... \nu_q} = \frac{(p+q)!}{p!q!} A_{[\mu_1 ... \mu_p} B_{\nu_1 ... \nu_q]} \tag{A.16}$$

The wedge product is not symmetric under commutativity. Switching the positions of A_p and B_q are represented by

$$A_p \wedge B_q = (-1)^{pq} B_q \wedge A_p \tag{A.17}$$

$(p+1)-$ form is found by the derivative called "exterior derivative" defined as

$$dA_p = \frac{1}{p!} \partial_{[\mu} A_{\mu_1 ... \mu_p]} dx^{\mu_1} \wedge ... \wedge dx^{\mu_p} \tag{A.18}$$

whose component is shown by

$$(dA_p)_{\mu_1 ... \mu_{p+1}} = (p+1) \partial_{[\mu_1} A_{\mu_2 ... \mu_{p+1}]}. \tag{A.19}$$

Mixing the exterior derivative and wedge product give the relation

$$d(A_p \wedge B_q) = dA_p \wedge B_q + (-1)^p A_p \wedge dB_q. \tag{A.20}$$

More generally, another derivative that change $p-$ form to $(p-1)-$ form by the vector $V = V^\mu \partial_\mu$

$$\begin{aligned} i_V\omega_p = V(\omega) &= \frac{1}{p!}V^\mu \omega_{\mu_1 \dots \mu_p}\partial_\mu(dx^{\mu_1}) \wedge dx^{\mu_2} \wedge \dots dx^{\mu_p} \\ &= \frac{1}{(p-1)!}V^\mu \omega_{\mu\mu_2\dots\mu_p} dx^{\mu_2} \wedge \dots \wedge dx^{\mu_p}. \end{aligned} \tag{A.21}$$

The other construction of different form is constructed by hodge duality which transforms $p-$ form into $(n-p)-$ form where n is the dimension of the manifold

$$*(dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}) = \frac{1}{(n-p)!}\epsilon_{\nu_1 \dots \nu_{n-p}}{}^{\mu_1 \dots \mu_p} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_{n-p}}. \tag{A.22}$$

whose component are shown by

$$(*\omega_p)_{\mu_1 \dots \mu_{n-p}} = \frac{1}{p!}\epsilon_{\mu_1 \dots \mu_{n-p}}{}^{\nu_1 \dots \nu_p}\omega_{\nu_1 \dots \nu_p}. \tag{A.23}$$

With the various constructions of different form, lie derivative can be written using d and i_V as

$$\mathcal{L}_V\omega_p = (di_V + i_V d)\omega_p. \tag{A.24}$$

Integral on the manifold is given by compact formula with the benefit in differential form as

$$\epsilon = \frac{1}{n!}\sqrt{|g|}\epsilon_{\mu_1 \dots \mu_n} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n} = \sqrt{|g|}d^n x \tag{A.25}$$

where

$$d^n x = \frac{1}{n!}\epsilon_{\mu_1 \dots \mu_n} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}. \tag{A.26}$$

By the relation,

$$\epsilon_{\mu_1 \dots \mu_n}\epsilon^{\mu_1 \dots \mu_n} = -n! \tag{A.27}$$

?? can also be written as

$$dx^{\mu_1} \wedge ... \wedge dx^{\mu_n} = -\epsilon^{\mu_1 ... \mu_n} d^n x \tag{A.28}$$

A.3 Veilbein formalism

In more complicated manifold, sometimes, usual metric is rather tedious and difficult to calculate things in the manifold. Fortunately, the brink of vielbein e^a_μ contributes the help in easier calculation. This first is found by

$$g_{\mu\nu}(x) = e^a_\mu(x) e^b_\nu(x) \eta_{\mu\nu} \tag{A.29}$$

which encodes the relation between spacetime on curved manifold and the flatness of tangent space.

Like the introduction of usual Christoffel connection, connection related to tangent space and curved spacetime can be described by the first veilbein postulate as

$$\partial_\mu e^a_\mu - \Gamma^\rho_{\mu\nu} e^a_\rho + \omega_\mu{}^a{}_b e^b_\nu = 0. \tag{A.30}$$

With the switching indices μ and ν from the above equation, torsion tensor can be found as

$$T^a{}_{\mu\nu} = e^a_\rho T^\rho{}_{\mu\nu} = 2\Gamma^\rho_{\mu\nu} e^a_\rho = \partial_{[\mu} e^a_{\nu]} + \omega_{[\mu}{}^a{}_b e^b_{\nu]} \tag{A.31}$$

where 2− form torsion is defined by

$$T^a = \frac{1}{2} T^a{}_{\mu\nu} dx^\mu \wedge dx^\nu = de^a + \omega^a{}_b \wedge e^b. \tag{A.32}$$

This can also lead to Riemann tensor with tangent indices involved as

$$R_{\mu\nu}{}^a{}_b = 2(\partial_\mu \omega^a_{\nu]_b}) + \omega_{[\mu}{}^a{}_c \omega_{[\mu}{}^a{}_c \omega_{\nu]}{}^c{}_b. \tag{A.33}$$

Specially, its 2− form curvature tensor is given by

$$R^a{}_b = \frac{1}{2} R_{\mu\nu}{}^a{}_b dx^\mu \wedge dx^\nu \tag{A.34}$$

which can also be found by the connection as

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \tag{A.35}$$

From the metric compatibility and definition of veilbein, they further provide relation

$$\omega^a{}_c \eta^{bc} = -\omega^b{}_c \eta^{ac} \quad \Longleftrightarrow \quad \omega^a{}_b = -\omega_b{}^a. \tag{A.36}$$

With Biachi identity, it gives that

$$dT^a + \omega^a{}_b \wedge T^b = R^a{}_b \wedge e^b \tag{A.37}$$

$$dR^{ab} + \omega^a{}_c \wedge R^{cb} - \omega^b{}_c \wedge R^{ca} = 0. \tag{A.38}$$

Since fermions must be transformed under the tangential Lorentz group $SL(2, \mathbb{C})$, there are no ways to describe fermions on the curved spacetime without the introduction of veilbein. Let's first review a little of gamma matrix in which fermions must be involved.

The gamma matrices abide by the relation $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$. Each element of γ-matrices are found by

$$\gamma^a = \begin{pmatrix} \mathbf{0} & \sigma^a \\ \bar{\sigma}^a & \mathbf{0} \end{pmatrix}; \quad \sigma^a = (\mathbf{1}, \sigma^I); \quad \bar{\sigma}^a = (\mathbf{1}, -\sigma^I) \quad (I = 1, 2, 3). \tag{A.39}$$

where σ^I is the Pauli matrices as

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.40}$$

With the link between γ^μ and γ^a, they can be shown that $\gamma^\mu(x) = e^\mu_a(x)\gamma^a$ gives

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}(x) \tag{A.41}$$

while γ^5 is defined by

$$\gamma^5 = \frac{i}{4!}\epsilon_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d = \frac{ie}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = \begin{pmatrix} -\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}. \tag{A.42}$$

Let's define

$$\gamma^{a_1...a_k} = \gamma^{[a_1}...\gamma^{a_k]} \tag{A.43}$$

with definitions of γ^5 will hold that

$$\gamma^5\gamma_a = -\frac{i}{3!}\epsilon_{abcd}\gamma^{bcd}; \quad \gamma^5\gamma_{abc} = i\epsilon_{abcd}\gamma^d; \tag{A.44}$$

$$\gamma^5\gamma_{abc} = i\epsilon_{abcd}\gamma^d; \quad \gamma^5\gamma_{abcd} = i\epsilon_{abcd}. \tag{A.45}$$

Complex conjugation of Dirac spinor is defined by

$$\bar{\psi} = \psi^\dagger\gamma^0; \quad \psi_c = C\bar{\psi}^T \tag{A.46}$$

where C is a charge conjugation matrix where $C = -i\gamma^2\gamma^0$ and satisfies

$$C^{-1}\gamma^\mu C = (\gamma^\mu)^T; \quad C = C^* = -C^T = -C^{-1} \tag{A.47}$$

which is used to give that

$$(C\gamma^{a_1...a_k})^T = -(-1)^{k(k+1)/2}C\gamma^{a_1...a_k} \tag{A.48}$$

$$\bar{\chi}_c\gamma^{a_1...a_k}\lambda = (-1)^{k(k+1)/2}\bar{\lambda}_c\gamma^{a_1...a_k}\chi \tag{A.49}$$

$$(\bar{\chi}_c\gamma^{a_1...a_k}\lambda)^* = (-1)^k\bar{\chi}\gamma^{a_1...a_k}\lambda_c \tag{A.50}$$

while Majorana spinor is defined by

$$\psi = \psi_c = C\bar{\psi}^T \tag{A.51}$$

which leads to Fierz identity as

$$\lambda\bar{\chi} = -\frac{1}{4}(\bar{\chi}\lambda) - \frac{1}{4}(\bar{\chi}\gamma^5\lambda)\gamma^5 - \frac{1}{4}(\bar{\chi}\gamma^\mu\lambda)\gamma_\mu + \frac{1}{4}(\bar{\chi}\gamma^5\gamma^\mu\lambda)\gamma^5\gamma_\mu + \frac{1}{8}(\bar{\chi}\gamma^{\mu\nu}\lambda)\gamma_{\mu\nu}. \tag{A.52}$$

Some helpful relations, frequently found, in calculations of $\gamma-$matrices are shown by

$$\gamma_{\mu\nu}\gamma^\rho = 2\gamma_{\mu]}\delta^\rho_{\nu]} = 2\gamma_{[\mu}\delta^\rho_{\nu]} + ie\epsilon_{\mu\nu}{}^{\rho\sigma}\gamma^5\gamma_\sigma \tag{A.53}$$

$$\gamma_{\mu\nu}\gamma^{\rho\sigma} = \gamma_{\mu\nu}{}^{\rho\sigma} - 4\delta^{[\rho}_{[\mu}\gamma_{\nu]}{}^{\sigma]} - 2\delta^{\rho\sigma}_{\mu\nu} \tag{A.54}$$

$$\gamma^{[\rho}\gamma_{\mu\nu}\gamma^{\sigma]} = \gamma_{\mu\nu}{}^{\rho\sigma} + 2\delta^{\rho\sigma}_{\mu\nu} = 2(\delta^{\rho\sigma}_{\mu\nu} + \frac{ie}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}\gamma^5) \tag{A.55}$$

$$\gamma_\rho\gamma^{\mu_1\cdots\mu_k}\gamma^\rho = 2(-1)^k(2-k)\gamma^{\mu_1\cdots\mu_k}. \tag{A.56}$$

Another benefit in hodge duality is giving the definitions of self-dual and anti-self-dual tensor of metric

$$F^\pm_{\mu\nu} = \frac{F_{\mu\nu} \pm i * F_{\mu\nu}}{2} \quad \Longrightarrow \quad {}^*F^\pm_{\mu\nu} = \mp iF^\pm_{\mu\nu}. \tag{A.57}$$

With the presence of $\gamma-$matrices, it can show some interesting relations that

$$F^+_{\mu\nu}\gamma^{\mu\nu}\epsilon_* = F^-_{\mu\nu}\gamma^{\mu\nu}\epsilon^* = 0 \tag{A.58}$$

and

$$F^+_{\mu\nu}\gamma^{\mu\nu}\gamma_\rho\epsilon_* = -4F^+_{\rho\nu}\gamma^\nu\epsilon_*; \quad F^+_{\mu\nu}\gamma^{\mu\nu}\gamma_\rho\epsilon^* = 0 \tag{A.59}$$

$$F^-_{\mu\nu}\gamma^{\mu\nu}\gamma_\rho\epsilon^* = -4F^-_{\rho\nu}\gamma^\nu\epsilon^*; \quad F^-_{\mu\nu}\gamma^{\mu\nu}\gamma_\rho\epsilon_* = 0. \tag{A.60}$$

where $*$ indicates the positive and negative chiralities under $SU(N)$ representation as

$$\gamma^5\epsilon_* = \epsilon_*, \quad \gamma^5\epsilon^* = -\epsilon^*. \tag{A.61}$$

A.4 Geometry on scalar manifold

Since supergravity involve with a large number of scalar fields, sometimes, tedious calculation of things in the theory would have not been possible to complete without the help of symmetry. In order to fix the problem, scalars must be put together on the same Riemannian non-compact manifold called scalar manifold $\mathcal{M}_{\text{scal}}$ where n_s is the dimension of the manifold, real scalar fields are represented through curved indices $s, t, r, ...$ while tangent indices are shown by $\bar{s}, \bar{t}, \bar{r}, ...$ ϕ^s, a scalar field, is now a local coordinate of this manifold with the metric tensor described by $\mathcal{G}_{st}(\phi)$. With the adaptation of curved manifold from the previous section, veilbein one-form $\mathcal{P}^{\bar{s}} = \mathcal{P}_t^{\bar{s}} d\phi^t$ and its dual basis on tangent space $K_{\bar{s}} = \mathcal{P}_{\bar{s}}^t$ are allowed to be defined where $\mathcal{P}_{\bar{s}}^t$ and $\mathcal{P}_t^{\bar{s}}$ are their inverse each other.

$$\mathcal{G}_{st}(\phi) = \mathcal{P}_s{}^{\bar{s}}(\phi)\mathcal{P}_t{}^{\bar{t}}(\phi)\eta_{\bar{s}\bar{t}} \tag{A.62}$$

where $\eta_{\bar{s}\bar{t}}$ is $H-$invariant matrix.

The same as usual is when metric of curved and tangent space are defined, it will be the time for connection that must satisfy the first veilbein postulate as

$$\mathcal{D}_s\mathcal{P}_t{}^{\bar{r}} = \partial_s\mathcal{P}_t{}^{\bar{r}} - \Gamma^r_{st}\mathcal{P}_r{}^{\bar{r}} + Q_s{}^{\bar{r}}{}_{\bar{t}}\mathcal{P}_t{}^{\bar{t}} = 0. \tag{A.63}$$

The next step is to find the rank-2 curvature tensor which is given by

$$R(\mathcal{Q})^{\bar{t}}{}_{\bar{s}} = d\mathcal{D}^{\bar{t}}{}_{\bar{s}} + \mathcal{Q}^{\bar{t}}{}_{\bar{r}} \wedge \mathcal{Q}^{\bar{r}}{}_{\bar{s}} = \frac{1}{2}R_{st}{}^{\bar{t}}{}_{\bar{s}} d\phi^s \wedge d\phi^t. \tag{A.64}$$

It is noticeable through the aforementioned conventions that diffeomorphisms also exists on the manifold as the relation

$$\phi^s \to \phi'^s : \quad \mathcal{G}_{\bar{s}\bar{t}}(\phi'(\phi)) \frac{\partial \phi'^{s'}}{\partial \phi^s} \frac{\partial \phi'^{t'}}{\partial \phi^t} = \mathcal{G}_{st}(\phi). \tag{A.65}$$

Due to the property of a transitive action of the manifold $\mathcal{M}_{scal}$, any point P in the manifold are invariant under the action of group H which is the subgroup pf G. One can fix any other point O to P by

$$\forall g \in G \quad \to \quad \exists P \in \mathcal{M}_{scal} : \quad g \cdot O = P \tag{A.66}$$

where $g \in G$, $P \in \mathcal{M}_{scal}$ and $\cdot$ represents action of G on the manifold.

Since this is not one-to-one action, there are other g' that can send the same O to the same P as

$$g \cdot O = P, \quad g' \cdot O = P \quad \to \quad g^{-1} g' \cdot O = O \to g' \in gH'. \tag{A.67}$$

The $H-$connection $\mathcal{Q}$ can be formulated by considering the commutation

$$[K_s, K_t] = f_{\bar{s}\bar{t}}{}^{I} J_I + f_{\bar{s}\bar{t}}{}^{\bar{r}} K_{\bar{r}} \tag{A.68}$$

where K_s is a basis of $\mathfrak{K}$ and J_I is a basis of $\mathfrak{H}$. To construct the quantity that describes all structures on the manifold, left-invariant one-form on $\mathfrak{g}$ must be introduced as

$$\Omega = L^{-1} dL = \mathcal{P} + \mathcal{Q}' \tag{A.69}$$

where $\mathcal{P}$ on $\mathfrak{K}$ and $\mathcal{Q}'$ on $\mathfrak{H}$ are veilbein and the connection respectively.

With the structure of Maurer-Cartan equation as

$$d\Omega + \Omega \wedge \Omega = 0, \tag{A.70}$$

its projection on $\mathfrak{K}$ is represented by

$$d\mathcal{P}^{\bar{s}} + \mathcal{Q}'^{\bar{s}}{}_{\bar{t}} \wedge \mathcal{P}^{\bar{t}} + \frac{1}{2} f_{\bar{r}\bar{t}}{}^{\bar{s}} \mathcal{P}^{\bar{r}} \wedge \mathcal{P}^{\bar{t}} = 0. \tag{A.71}$$

The need in definition

$$\mathcal{D}\mathcal{P}^{\bar{s}} = d\mathcal{P}^{\bar{s}} + \mathcal{Q}^{\bar{s}}{}_{\bar{t}} \wedge \mathcal{P}^{\bar{t}} = 0 \tag{A.72}$$

leads to the introduction of connection one-form $\mathcal{Q}^{\bar{s}}{}_{\bar{t}}$ as

$$\mathcal{Q}^{\bar{s}}{}_{\bar{t}} = \mathcal{Q}'^{\bar{s}}{}_{\bar{t}} + \Delta\mathcal{Q}^{\bar{s}}{}_{\bar{t}}. \tag{A.73}$$

One can also so write

$$\Delta\mathcal{Q}_{[\bar{r}}{}^{\bar{s}}{}_{\bar{t}]} = \frac{1}{2} f_{\bar{r}\bar{t}}{}^{\bar{s}}. \tag{A.74}$$

Compatibility of metric can give

$$\Delta\mathcal{Q}_{\bar{r}}{}^{\bar{s}}{}_{\bar{t}} = \frac{1}{2}(f_{\bar{r}\bar{t}}{}^{\bar{s}} + f_{\bar{s}'\bar{t}}{}^{\bar{r}} \eta_{\bar{r}'\bar{r}} \eta^{\bar{s}'\bar{s}} + f_{\bar{s}'\bar{r}}{}^{\bar{r}'} \eta_{\bar{r}'\bar{t}} \eta^{\bar{s}'\bar{s}}) \tag{A.75}$$

In differential geometry, lie group on of manifold G_s has an association with lie algebra ξ. One can $\mathcal{M}_{scal} \sim G_s = e^{\xi}$. A generator of ξ T_s, $s = 1, ..., n_s$ obeys the algebra

$$[T_r, T_s] = C_{rs}{}^{t} T_t \tag{A.76}$$

where T_s can be projected on $\mathfrak{K}$ and $\mathfrak{H}$ as

$$T_s = K_{\bar{s}} + J_{\bar{s}}; \quad K_{\bar{s}} \in \mathfrak{K}, \quad J_{\bar{s}} \in \mathfrak{H}. \tag{A.77}$$

Coset representative of $\mathcal{M}_{scal}$ can be defined with the generator T_s and scalar field as

$$L(\phi^s) = \exp(\phi^s T_s) \in G_S \tag{A.78}$$

Decomposed in terms of $K_{\bar{s}}$ and $J_{\bar{s}}$, Ω can be splitted into

$$\Omega = L^{-1}dL = \mathcal{P}^s \mathcal{T}_s = \mathcal{P}^s K_{\bar{s}} + \mathcal{P}^s J_{\bar{s}} = \mathcal{P} + \mathcal{Q}. \tag{A.79}$$

Maurer-Cartan equation can give

$$d\mathcal{P}^s + \frac{1}{2} C_{rt}{}^s \mathcal{P}^r \wedge \mathcal{P}^t = 0. \tag{A.80}$$

The metric on $\mathcal{M}_{scal}$ is resembled with veilbein formalism as it is written as

$$ds^2 = (\Omega, \Omega) = \mathcal{P}^s \mathcal{P}^t \eta_{\bar{r}\bar{s}} = \mathcal{P}^{\bar{s}} \mathcal{P}^{\bar{t}} \eta_{\bar{r}\bar{s}}. \tag{A.81}$$

With $P^s = P^{\bar{s}}$, Livi-Civita connection can be shown by

$$\mathcal{D}\mathcal{P}^{\bar{r}} + \mathcal{Q}^{\bar{r}}{}_{\bar{t}} \wedge \mathcal{P}^{\bar{t}} = 0 \tag{A.82}$$

where $\mathcal{Q}^{\bar{s}}{}_{\bar{t}}$ can be written as

$$\mathcal{Q}^{\bar{s}}{}_{\bar{t}} = \frac{1}{2}(C_{\bar{r}\bar{t}}{}^{\bar{s}} + C_{\bar{s}'\bar{t}}{}^{\bar{r}} \eta_{\bar{r}'\bar{r}} \eta^{\bar{s}'\bar{s}} + C_{\bar{s}'\bar{r}}{}^{\bar{r}'} \eta_{\bar{r}'\bar{t}} \eta^{\bar{s}'\bar{s}}) \mathcal{P}^{\bar{r}} \tag{A.83}$$

According to a general structure of supergravity, the group H takes place in the symmetry under which fermions are invariant. The covariant derivative with the addition of $H-$invariant connection is defined by

$$\mathcal{D}_\mu = \nabla_\mu + \mathcal{Q}_\mu \tag{A.84}$$

which is used to write the covariant derivative of fermionic fields, for example, gravitino field as

$$\mathcal{D}_\mu \psi_{A\nu} = \nabla_\mu \psi_{A\nu} + \mathcal{Q}_{\mu A}{}^B \psi_{B\nu} = \partial_\mu \psi_{A\rho} + \frac{1}{4}\omega_{\mu,ab}\gamma^{ab}\psi_{A\nu} + \mathcal{Q}_{\mu A}{}^B \psi_{B\nu}. \tag{A.85}$$

This idea is extended to the theory with more additions of local symmetry, which in this case is gauge symmetry. The connection $\Omega_{g\mu}$ is called gauge connection which will changed the form of covariant derivative to total covariant derivative with gauge connection included as

$$\mathbb{D}_\mu = \mathcal{D}_\mu + \Omega_{g\mu} \tag{A.86}$$

BIOGRAPHY

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