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HAMILTONIAN DECOMPOSITIONS OF HYPERGRAPHS

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วิทยานิพนธ์ฉบับนี้เราศึกษานิยามที่แตกต่างกัน 4 นิยามของวัฏจักรแฮมิลโตเนียนในไฮเพอร์กราฟ ซึ่งเราเน้นการศึกษาปัญหาเรื่องการมีอยู่ของการแยกแฮมิลโตเนียนของไฮเพอร์กราฟเอกรูปโดยใช้นิยามของวัฏจักรแฮมิลโตเนียนสองแบบคือ “นิยามแบบ KK” และ “นิยามแบบ WJ” สำหรับนิยามแบบ KK เราสร้างการแยกแฮมิลโตเนียนแบบ KK ของไฮเพอร์กราฟเอกรูปบริบูรณ์แบบเวียนเกิด การสร้างของเราใช้การแยกแฮมิลโตเนียนแบบ KK ของไฮเพอร์กราฟเอกรูปบริบูรณ์ $K_t^{(3)}$ และการแยกของกราฟบางชนิดซึ่งเป็นที่รู้จักในการสร้างการแยกแฮมิลโตเนียนแบบ KK ของไฮเพอร์กราฟเอกรูปหลายส่วนบริบูรณ์ $K_{t(n)}^{(3)}$ เมื่อ $t \equiv 4, 8 \pmod{12}$ และ $n \geq 2$ รวมไปถึงการแยกแฮมิลโตเนียนแบบ KK ของไฮเพอร์กราฟเอกรูปหลายส่วนบริบูรณ์ $K_{2t}^{(3)}$ ดังนั้นเราสามารถใช้ผลการศึกษาในปัจจุบันของการแยกแฮมิลโตเนียนแบบ KK ของไฮเพอร์กราฟเอกรูปบริบูรณ์ $K_t^{(3)}$ ในการสร้างการแยกแฮมิลโตเนียนแบบ KK ของไฮเพอร์กราฟเอกรูปบริบูรณ์ $K_t^{(3)}$ และไฮเพอร์กราฟเอกรูปหลายส่วนบริบูรณ์ $K_{t(n)}^{(3)}$ เมื่อ $t = 2^m, 5 \cdot 2^m, 7 \cdot 2^m$, $11 \cdot 2^m$, $m \geq 2$ และ $n \geq 2$ นอกจากนี้เรายังได้นำเสนอการแยกแฮมิลโตเนียนแบบ WJ ของไฮเพอร์กราฟเอกรูปสองส่วนบริบูรณ์ $K_{n,n}^{(4)}$ เมื่อ $n \equiv 1 \pmod{4}$ และ n เป็นจำนวนเฉพาะไว้อีกด้วย

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต ....

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In this dissertation, we first discuss four versions of Hamiltonicity in hypergraphs. We mainly study the existence problem of Hamiltonian decompositions of uniform hypergraphs based on two versions of Hamiltonian cycles, so called "KK-definition" and "WJ-definition". For KK-definition, we create a recursive construction of KK-Hamiltonian decomposition of complete 3-uniform hypergraphs. Our construction method uses a KK-Hamiltonian decomposition of the complete 3-uniform hypergraph, $K_t^{(3)}$, and some well-known graph decompositions to obtain a KK-Hamiltonian decomposition of the complete t-partite 3-uniform hypergraph, $K_{t(n)}^{(3)}$, when $t \equiv 4, 8 \pmod{12}$, $n \geq 2$, as well as a KK-Hamiltonian decomposition of $K_{2t}^{(3)}$. Therefore, together with the current results in literatures, our method provides a KK-Hamiltonian decomposition of the complete 3-uniform hypergraph, $K_t^{(3)}$, and the complete t-partite 3-uniform hypergraph, $K_{t(n)}^{(3)}$, when $t = 2^m, 5 \cdot 2^m, 7 \cdot 2^m$ and $11 \cdot 2^m$ and $m \geq 2$, and $n \geq 2$. Furthermore, we establish a WJ-Hamiltonian decomposition of the complete 4-uniform bipartite hypergraph, $K_{n,n}^{(4)}$, where $n \equiv 1 \pmod 4$ and n is a prime number.

Department : Mathematics and Computer Science Student's Signature ....

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CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER I
INTRODUCTION

The existence of Hamiltonian decompositions of graphs is one of the well-known problems which has been studied widely for a longtime, and the Hamiltonicity has been generalized to hypergraphs in various definitions of Hamiltonian cycles. In this dissertation, we mainly establish a construction of Hamiltonian decompositions of three families of hypergraphs based on two versions of Hamiltonian cycles. The first definitions follows from Katona and Kierstead (KK-definition) defined in [11]. We provide a KK-Hamiltonian decomposition for some complete 3-uniform hypergraphs and some complete multipartite 3-uniform hypergraphs. The second definition follows from Wang and Jirimutu (WJ-definition) defined in [16]. We study a WJ-Hamiltonian decomposition for complete bipartite 4-uniform hypergraphs.

We start the first part of the dissertation by providing all required definitions and notations, a brief history of the problems and overview of this dissertation. Moreover, we give some well-known results on graph decompositions which are the important tools of our construction.

1.1 Definitions and notations

The following notations will be used for the rest of this dissertation. A *hypergraph* $\mathcal{H}$ is an ordered pair $(V(\mathcal{H}), E(\mathcal{H}))$ where $V(\mathcal{H})$ is a finite set of elements and $E(\mathcal{H})$ is a collection of non-empty subsets of $V(\mathcal{H})$. The elements in $V(\mathcal{H})$ and $E(\mathcal{H})$ are called *vertices* and *hyperedges*, respectively. We refer to $V(\mathcal{H})$ and $E(\mathcal{H})$ as the *vertex set* and the *hyperedge set of* $\mathcal{H}$, respectively. If each hyperedge has size k, we say that $\mathcal{H}$ is a *k-uniform hypergraph.* A *Hamiltonian cycle* of hypergraph is defined in several ways which we later provide four definitions of it in Chapter II. A *Hamiltonian decomposition* of $\mathcal{H}$ is a family of Hamiltonian cycles in a hypergraph $\mathcal{H}$, $\{C_1, C_2, \ldots, C_m\}$, where each hyperedge in $\mathcal{H}$ is contained in exactly one cycle in the family.

Our work focuses on two families of hypergraphs as follows. The first one is a *complete k-uniform hypergraph of order n* with $n \geq k$ vertices and $k \geq 2$ on the vertex set V, denoted by $K_n^{(k)}$ or $K_n^{(k)}(V)$, which is a k-uniform hypergraph on $|V| = n$ vertices such that all k-subsets of the vertex set form its hyperedge set. The other one is a *complete t-partite k-uniform hypergraph* on the vertex set $V = V_1 \cup V_2 \cup \cdots \cup V_t$ where $|V_i| = n_i$ for all $i \in \{1, 2, \ldots, t\}$ and $t \geq 2$, denoted by $K^{(k)}_{n_1,n_2,\ldots,n_t}$ or $K^{(k)}_{n_1,n_2,\ldots,n_t}(V_1, V_2, \ldots, V_t)$ which is a k-uniform hypergraph on $|V| = \sum_{i=1}^{t} n_i$ vertices such that

$$E(K^{(k)}_{n_1,n_2,\ldots,n_t}) = \{e : e \subseteq V, |e| = k \text{ and } |e \cap V_i| < k \text{ for } i \in \{1, 2, \ldots, t\}\}.$$

In particular, if $n_i = n$ for all $i \in \{1, 2, \ldots, t\}$, $K^{(k)}_{\underbrace{n, n, \ldots n}_{t}}$ is denoted by $K^{(k)}_{t(n)}$.

1.2 History and overview

The existence problem of Hamiltonian decompositions of hypergraphs is extended from those of graphs. A well-known result for graphs by Walecki [1] in 1892 says that the complete graphs of odd order can be decomposed into Hamiltonian cycles. Furthermore, any complete graph of even order in which a perfect matching is removed also has a Hamiltonian decomposition.

Later on, this existence problem for graphs was extended into various versions for k-uniform hypergraphs depending on the definition of Hamiltonian cycles. We will discuss only four versions of Hamiltonian cycles, namely, Berge-Hamiltonian cycles, KK-Hamiltonian cycles, WJ-Hamiltonian cycles and WX-Hamiltonian cycles. We focus on the results for two families of hypergraphs; complete k-uniform hypergraphs and complete t-partite k-uniform hypergraphs.

The first one was defined by Berge [3] in 1979; a *Berge-Hamiltonian* cycle of hypergraph $\mathcal{H}(V,E)$ is a sequence $v_0, e_0, v_1, e_1, \ldots, v_{n-1}, e_{n-1}, v_n$ of all n vertices in V and some n distinct hyperedges in E, such that the hyperedge e_i contains both v_i and v_{i+1}, where $v_n = v_0$. The study of Berge-Hamiltonian decompositions of complete 3-uniform hypergraphs, $K_n^{(3)}$, was completely solved in 1976 and 1994 by Bermond [4], and Verrall [15], respectively. In 2014, Kuhn and Osthus [12] studied the existence of Berge-Hamiltonian decompositions of complete k-uniform hypergraphs $K_n^{(k)}$ where $4 \leq k \leq n-1$ and $n \geq 30$.

In 1999, a new variation of Hamitonicity for k-uniform hypergraphs was defined by Katona and Kierstead in [11] as follows: a *KK-Hamiltonian cycle* of k-uniform hypergraph $\mathcal{H}(V,E)$ is a cyclic ordering $C = (v_1\ v_2\ \cdots\ v_n)$ of all n elements of V such that each k-tuple of consecutive vertices in C is a hyperedge. A KK-Hamiltonian cycle is a Berge-Hamiltonian cycle, but the other direction is not always true. In 2010, Bailey and Stevens conjectured that a necessary condition for the existence of a KK-Hamiltonian decomposition of $K_n^{(k)}$, which states that $\binom{n}{k}$ is divisible by n, is also sufficient (which we call such an n "feasible").

KK-Hamiltonian decompositions of complete 3-uniform hypergraph, $K_n^{(3)}$, were studied for feasible n by several authors in [2, 13, 18] and [10]. The results were settled for $n \leq 46$, $n \neq 43$, and $n = 2^m$ and $m \geq 2$, and also 4-uniform hypergraph, $K_9^{(4)}$. By using "clique finding" method and "difference pattern" method, Bailey and Stevens [2] in 2010 obtained a KK-Hamiltonian decomposition for $K_9^{(4)}$ and $K_n^{(3)}$ where $n = 7, 8, 10, 11, 16$. Afterwards, Meszka and Rosa [13] modified the "difference pattern" method to solve the problem for $K_n^{(3)}$ where $n \leq 32$. Furthermore, the problem for $K_n^{(3)}$ where $n = 2^m$ and $m \geq 2$ was studied by Xu and Wang [18] in 2002, and for $8 \leq n \leq 46$, $n \neq 13, 19, 25, 31, 43$ was provided by Hong Huo *et al.* [10] (published in Chinese) in 2015. While, for complete t-partite 3-uniform hypergraphs, KK-Hamiltonian decompositions of $K_{t(n)}^{(3)}$ when $t = 2$ and $t = 3$, were completely studied for all n by Xu and Wang [18] in 2002, and Boonklurb *et al.* [6] in 2015, respectively.

In this dissertation, we establish a construction of KK-Hamiltonian decompositions of complete t-partite 3-uniform hypergraph $K^{(3)}_{t(n)}$ where $t \equiv 4, 8 \pmod{12}$ for all positive integer n by using KK-Hamiltonian decompositions of $K^{(3)}_t$. As the matter of the fact that $K^{(3)}_{t(2)} = K^{(3)}_{2t}$, our results provide a recursive construction for a KK-Hamiltonian decomposition of a complete 3-uniform hypergraph $K^{(3)}_{2n}$ from one of $K^{(3)}_n$. Our method solves the existence problem of infinitely many complete t-partite 3-uniform hypergraphs and complete 3-uniform hypergraphs from the initial ones.

On the other hand, hypergraphs have been introduced in database theory in order to model relational database schemes. We mention two definitions of Hamiltonian cycles using a new definition of cycles introduced by Wang and Lee [17] which is defined to suit the structure properties of relational database in 1999 (Definition 3 in Chapter II. In 2001, Wang and Jirimutu adopted the definition of cycles to define a *WJ-Hamiltonian cycle* of k-uniform hypergraph $\mathcal{H}$ with $|V(\mathcal{H})| = n$ that is a $(k-1)$-dimensional cycle of length n. In [16], a WJ-Hamiltonian decomposition of complete bipartite 3-uniform hypergraphs $K^{(3)}_{n,n}$ where n is a prime can be constructed successfully (which is also satisfied KK-definition). This motivates us to construct WJ-Hamiltonian decompositions of complete bipartite 4-uniform hypergraphs $K^{(4)}_{n,n}$ where n is prime.

In 2002, using the new definition of cycles in [17], Wang and Xu [18] also defined *WX-Hamiltonian cycles* of k-uniform hypergraph $\mathcal{H}$ that is a $(k-1)$-dimensional cycle which each vertex of H appears in exactly $k-1$ nodes (common vertices of consecutive hyperedges in a cycle). Then, they provided a WX-Hamiltonian decomposition of $K^{(3)}_{n,n}$ and $K^{(3)}_m$ where $m = 2^n$ and $n \geq 2$ (which is also satisfied KK-definition).

This dissertation is organized as follows. The first chapter is the introduction including definitions, notations and some well-known results of graph decompositions which are important tools in our constructions. Chapter II investigates some properties of these four definitions of Hamiltonian cycles, and also proves that WX-Hamiltonian cycles are KK-Hamiltonian cycles.

Chapter III is devoted to construct a KK-Hamiltonian decomposition of complete t-partite 3-uniform hypergraph $K_{t(n)}^{(3)}$ where $t \equiv 4, 8 \pmod{12}$ for all positive integer n except when $t = 4$ and n is even. Later, Chapters IV provides the construction for $K_{t(n)}^{(3)}$ when $t = 4$ and n is even. Finally, our recursive construction provides the results for a KK-Hamiltonian decomposition of complete 3-uniform hypergraph $K_{2t}^{(3)}$ from one of $K_t^{(3)}$ which will be concluded in Chapters III and VI.

In Chapter V, we establish a WJ-Hamiltonian decomposition of complete bipartite 4-uniform hypergraph $K_{n,n}^{(4)}$ where n is a prime number using properties of its hyperedges. Each WJ-Hamiltonian cycle in our construction is neither KK-Hamiltonian cycle nor Berge-Hamiltonian cycle.

Finally, the last chapter concludes all of our results in the research including some interesting open problems.

1.3 Graphs decompositions

Our constructions of Hamiltonian decompositions use several well-known results of graph decompositions such as 1-factorizations and Hamiltonian decompositions of graphs. A 1-*factor* of a a graph is a 1-regular spanning subgraph. A 1-*factorization* of graph is a decomposition of a graph into 1-factors. As a 2-uniform hypergraph is a graph, we use the usual notations such as K_n for $K_n^{(2)}$ and $K_{n,n}$ for $K_{n,n}^{(2)}$. In 1969, Harary [8] provided that K_n has a 1-factorization only when n is even and $K_{n,n}$ has a 1-factorization for all positive integer n.

Theorem 1.3.1. [8]

(i) *The complete graph K_n has a 1-factorization whenever n is even,*

(ii) *The complete bipartite graph $K_{n,n}$ has a 1-factorization for all positive integer n.*

In this dissertation, we refer to a 1-factor by its edge set. More precisely, if a 1-factor F of $K_{2m}(V)$ where $V = \{1, 2, \ldots, 2m\}$ is written as $\{\{j, f(j)\} : j \in \{1, 2, \ldots, m\}\}$, then the vertex set V is automatically relabeled to be $\{1, 2, \ldots, m, f(1), f(2), \ldots, f(m)\}$. (For example, if $F = \{\{1, 2\}, \{3, 4\}\}$ is a 1-factor of $K_4([4])$, then the vertices $1, 2, 3$ and 4 are relabeled to be $1, f(1), 2$ and $f(2)$, respectively.)

The remaining tools are Hamiltonian decompositions of graphs and directed graphs.

The complete graph K_n can be decomposed into Hamiltonian cycles only when n is odd were proved by Hilton [9] in 1984.

Theorem 1.3.2. [9] *Let $n \in \mathbb{N}$. The complete graph K_n has a Hamiltonian decomposition whenever n is odd.*

Now, let us move to the decomposition of directed graph. We follows the definitions from [7]. A *digraph* D consists of a finite nonempty set $V(D)$ of *vertices* and a set $E(D)$ of ordered pairs of distinct vertices. Each element of E is a *directed edge*. If (u, v) is a directed edge of a digraph, then u is said to be *adjacent to* v and v is *adjacent from* u. A (*directed*) *walk* is a sequence $(u = u_1, u_2, \ldots, u_k = v)$ of vertices of D such that u_i is adjacent to u_{i+1} for all $i \in \{0, 1, \ldots, k-1\}$. A walk is *closed* if $u = v$. A (*directed*) *cycle* is a closed walk of length at least 2 in which no vertex is repeated except for the initial and terminal vertices. A cycle C in D is a *Hamiltonian cycle* if C contains every vertex of D.

A *complete digraph* on n vertices is a digraph in which every pair u, v of distinct vertices is connected by exactly two directed edges (u, v) and (v, u), denoted DK_n. Bermond and Faber [5] showed that a Hamiltonian decomposition of DK_4 and DK_6 do not exist, while Tillson [14] proved that the decompositions of DK_n exist whenever $n \neq 4, 6$.

Theorem 1.3.3. [14] *Let $n \in \mathbb{N}$. The complete digraph DK_n has a Hamiltonian decomposition if and only if $n \neq 4, 6$.*

CHAPTER II
HAMILTONIAN CYCLES OF HYPERGRAPHS

Hamiltonian cycles of hypergraphs are generalized from those of graphs in several ways. We focus on the following four definitions of Hamiltonian cycles;

1. Berge's Definition,
2. Katona and Kierstead's Definition,
3. Wang and Jirimutu's Definition, and
4. Wang and Xu's Definition.

In this chapter, we first give the definitions and examples of Hamiltonian cycles of hypergraphs based on each definition. In Section 2.1, we investigate the relation of these four definitions. In Section 2.2, we recall certain results of the existence of Hamiltonian decompositions of hypergraphs based on each definition.

The first classic one was defined by Berge in 1973. He generalized Hamiltonian cycles in graphs to k-uniform hypergraphs as follows:

Definition 1. [3] A *Berge cycle* is a sequence $(v_0, e_0, v_1, e_1, \ldots, v_{n-1}, e_{n-1}, v_n)$, and $v_0, v_1, \ldots, v_{n-1} \in V(\mathcal{H})$ and $e_0, e_1, \ldots, e_{n-1} \in E(\mathcal{H})$ are distinct elements, such that the hyperedge e_i contains both v_i and v_{i+1} where $v_n = v_0$. A Berge cycle is a *Berge-Hamiltonian cycle* if $\{v_0, v_1, \ldots, v_{n-1}\}$ is the vertex set $V(\mathcal{H})$.

Example 1. Let $K_5^{(3)}$ be the complete hypergraph on the vertex set $V = \{0, 1, 2, 3, 4\}$. Consider

$$C_1 = (0, \{0, 3, 1\}, 1, \{1, 4, 2\}, 2, \{2, 0, 3\}, 3, \{3, 1, 4\}, 4, \{4, 2, 0\}, 0),$$

$$C_2 = (0, \{0, 1, 2\}, 2, \{2, 3, 4\}, 4, \{4, 0, 1\}, 1, \{1, 2, 3\}, 3, \{3, 4, 0\}, 0).$$

Then, $\{C_1, C_2\}$ is a Berge-Hamiltonian decomposition of $K_5^{(3)}$. $\square$

In 1999, Katona and Kierstead [11] provided the notion of a KK-Hamiltonian cycle of k-uniform hypergraphs. This notion also satisfies the previous notion by Berge while the other way is not always true.

Definition 2. [11] Let $\mathcal{H}$ be a k-uniform hypergraph. A *KK-Hamiltonian cycle* is a cyclic ordering of the elements of $V(\mathcal{H})$ such that each k-tuple of consecutive vertices is a hyperedge.

Example 2. Let $K_7^{(3)}$ be the complete hypergraph on the vertex set $V = \{0, 1, 2, 3, 4, 5, 6\}$. Based on KK-definition, the Hamiltonian cycle (1 2 4 6 0 5 3) consists of hyperedges $\{1, 2, 4\}, \{2, 4, 6\}, \{4, 6, 0\}, \{6, 0, 5\}, \{0, 5, 3\}, \{5, 3, 1\}$ and $\{3, 1, 2\}$.

Moreover, the following collection is a Hamiltonian decomposition of $K_7^{(3)}$.

{(1 2 4 6 0 5 3), (1 2 6 3 4 0 5), (1 3 4 5 6 2 0), (1 4 5 2 0 3 6), (1 6 5 3 2 4 0)}.

Next, hypergraphs have been introduced in database theory in order to model relational database schemes. Also in 1999, Wang and Lee [17] introduced a new definition of cycles in hypergraphs to suit the structure properties of relation database In 2001, Wang and Jirimutu [16] adopted the new definition of cycles to define a WJ-Hamiltonian cycle.

Definition 3. Let $C = (e_0, e_1, \ldots, e_{r-1})$ be a sequence of hyperedges of $\mathcal{H}$, $S_i = e_i \cap e_{i+1}$ for $i \in \{0, 1, \ldots, r-1\}$ where indices of the hyperedges are considered in the modulus r. We call S_i a *node* and C a *cycle* with the node sequence $S = (S_0, S_1, \ldots, S_{r-1})$ if the following conditions are satisfied:

($p1$) $e_i \neq e_j$ for $i \neq j$

($p2$) $S_i \neq \varnothing$ for $i \in \{0, 1, \ldots, r-1\}$,

($p3$) $S_i \setminus S_j \neq \varnothing$ for $i \neq j$,

($p4$) for any $i \in \{0, 1, \ldots, r-1\}$ there is no hyperedge $e \in E(\mathcal{H})$ such that

$$S_i \cup S_{i+1} \cup S_{i+2} \subseteq e.$$

C is called a *t-dimensional cycle* of *length* r if $t = \min\{|S_i| : i \in \{0, 1, \ldots, r-1\}\}$.

If $\mathcal{H}$ is a k-uniform hypergraph and $|V(\mathcal{H})| = n$, then any $(k-1)$-dimensional cycle of length n in $\mathcal{H}$ is called a *WJ-Hamiltonian cycle* of $\mathcal{H}$.

Example 3. Let $K^{(4)}_{5,5}$ be the complete bipartite 4-uniform hypergraph on vertex set $V = V_1 \cup V_2$ where $V_1 = \{0, 2, 4, 6, 8\}$ and $V_2 = \{1, 3, 5, 7, 9\}$. Let

$$\begin{aligned} C = (e_0 = \{0,2,1,5\}, e_1 = \{0,2,5,9\}, e_2 = \{0,2,3,9\}, e_3 = \{0,2,3,7\},\\ e_4 = \{0,2,1,7\}, e_5 = \{2,4,1,7\}, e_6 = \{2,4,3,7\}, e_7 = \{2,4,3,9\},\\ e_8 = \{2,4,5,9\}, e_9 = \{2,4,1,5\}). \end{aligned}$$

Then, the sequence of nodes in C is

$$\begin{aligned} (S_0 = \{0,2,5\}, S_1 = \{0,2,9\}, S_2 = \{0,2,3\}, S_3 = \{0,2,7\}, S_4 = \{2,1,7\},\\ S_5 = \{2,4,7\}, S_6 = \{2,4,3\}, S_7 = \{2,4,9\}, S_8 = \{2,4,5\}, S_9 = \{2,1,5\}). \end{aligned}$$

It can be verified directly that C satisfies properties $(p1) - (p4)$ and $|S_i| = 3$ for all i. Thus, C is a WJ-Hamiltonian cycle of $K^{(3)}_{5,5}$. Note that vertices 6 and 8 do not belong to any hyperedges in C, thus, C is not a Berge-Hamiltonian cycle. $\square$

In 2002, using the definition of cycles in Definition 3, Wang and Xu [18] also defined another version of Hamiltonian cycles as follows.

Definition 4. [18] Let H be a k-uniform hypergraph. Then, any $(k-1)$-dimensional cycle in H is called a *WX-Hamiltonian cycle* of H if each vertex of H appears in exactly $k-1$ nodes.

Example 4. Let $K^{(3)}_{3,3}$ be the complete bipartite 3-uniform hypergraph on vertex set $V = V_1 \cup V_2$ where $V_1 = \{0, 2, 4\}$ and $V_2 = \{1, 3, 5\}$.

Figure 2.1 illustrates C_1, C_2 and C_3 of $K^{(3)}_{3,3}(V_1, V_2)$ which can be verified directly that each cycle satisfies properties $(p1) - (p4)$. Also in each cycle, each vertex in V appears in exactly two nodes and all nodes are of size two. Since each hyperedge

	$C_i = (e_1, e_2, e_3, e_4, e_5)$					
i	e_0	e_1	e_2	e_3	e_4	e_5
1	$\{0,1,2\}$	$\{1,2,3\}$	$\{2,3,4\}$	$\{3,4,5\}$	$\{4,5,0\}$	$\{5,0,1\}$
2	$\{0,3,2\}$	$\{3,2,5\}$	$\{2,5,4\}$	$\{5,4,1\}$	$\{4,1,0\}$	$\{1,0,3\}$
3	$\{0,5,2\}$	$\{5,2,1\}$	$\{2,1,4\}$	$\{1,4,3\}$	$\{4,3,0\}$	$\{3,0,5\}$

Figure 2.1: C_1, C_2 and C_3 of $K_{3,3}^{(3)}(V_1, V_2)$.

in $K_{3,3}^{(3)}$ is contained in exactly one cycle in the collection $\{C_1, C_2, C_3\}$, we have that the collection is a Hamiltonian decomposition of $K_{3,3}^{(3)}$.

Moreover, these three cycles are KK-Hamiltonian cycles which can be written as $C_1 = (0\ 1\ 2\ 3\ 4\ 5)$, $C_2 = (0\ 3\ 2\ 5\ 4\ 1)$ and $C_3 = (0\ 5\ 2\ 1\ 4\ 3)$.

2.1 The connection of four Hamiltonicity definitions

These four definitions of Hamiltonian cycles are related to each other as shown in Figure 2.2, and in this section, we will show the examples of cycles to verify this figure.

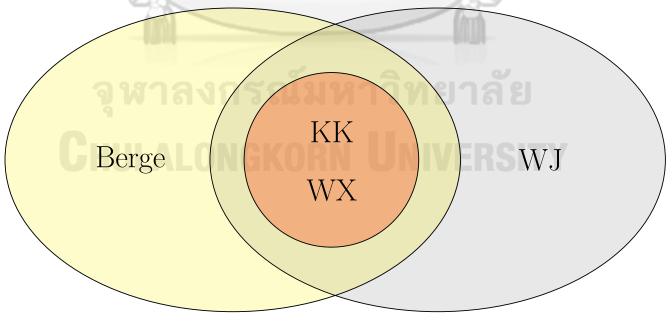

Figure 2.2: The connection of the four definitions of Hamiltonian cycles of hypergraphs.

First, notice that KK-Hamiltonian cycles satisfy the other three definitions. While, Berge-definition and WJ-definition are distinct because of cycles in Example 1 and Example 3. In details, since the Berge-Hamiltonian cycle C of $K_5^{(3)}$ in Example 1 has nodes of size one, C is not a WJ-Hamiltonian cycle. Also, Example

3 illustrates a WJ-Hamiltonian cycle which is not a Berge-Hamiltonian cycle.

The next example shows a Hamiltonian cycle of a k-uniform hypergraph that satisfying both Berge-definition and WJ-definition but it is not a KK-Hamiltonian cycle.

Example 5. Let $K_7^{(4)}$ be the complete 4-uniform hypergraph on vertex set $V = \{1, 2, 3, 4, 5, 6, 7\}$. Let

$$C = (e_0 = \{1, 2, 3, 4\}, e_1 = \{1, 2, 4, 5\}, e_2 = \{1, 2, 5, 6\}, e_3 = \{2, 5, 6, 7\},$$
$$e_4 = \{5, 6, 7, 1\}, e_5 = \{6, 7, 1, 2\}, e_6 = \{7, 1, 2, 3\}).$$

Then, the sequence of nodes in C is

$$(S_0 = \{1, 2, 4\}, S_1 = \{1, 2, 5\}, S_2 = \{2, 5, 6\}, S_3 = \{5, 6, 7\}, S_4 = \{6, 7, 1\},$$
$$S_5 = \{7, 1, 2\}, S_6 = \{1, 2, 3\}).$$

It can be verified directly that C satisfies properties $(p1) - (p4)$ and $|S_i| = 3$ for all i. Thus, C is a WJ-Hamiltonian cycle of $K_7^{(4)}$. Since C can be written as $(3, e_0, 4, e_1, 2, e_2, 5, e_3, 6, e_4, 1, e_5, 7, e_6, 3)$, C is a Berge-Hamiltonian cycle. But the vertex 2 belongs to six hyperedges in C; thus, it is not a KK-Hamiltonian cycle. $\square$

Finally, as Wang and Xu mentioned in [18] that a KK-Hamiltonian cycle is the same as a WX-Hamiltonian cycle without a proof, we will provide the proof in the next theorem.

Theorem 2.1.1. *WX-Hamiltonian cycles and KK-Hamiltonian cycles are the same.*

Proof. Let $\mathcal{H}$ be a k-uniform hypergraph of order n. First, a KK-Hamiltonian cycle of $\mathcal{H}$ satisfies WX-definition since it is a $(k-1)$-dimensional cycle, every nodes are all distinct and each vertex of $\mathcal{H}$ in appears in exactly $k-1$ nodes.

Next, let C be a WX-Hamiltonian cycle of $\mathcal{H}$ and $v \in V(\mathcal{H})$. Then, v appears in exactly $k-1$ nodes in C. It implies that v is contained in at least k hyperedges in C. Since C has n hyperedges each of k vertices, a vertex v appears in exactly k hyperedges.

Suppose that $k-1$ nodes containing v are not consecutive. Then, v is contained in more than k hyperedges in C which is impossible. Hence, $k-1$ nodes and k hyperedges containing v are consecutive.

Let $C = (e_1, e_2, \ldots, e_n)$. Since C is a $(k-1)$-dimensional cycle, e_i and e_{i+k-1} has at least one common vertex. Since there are exactly n collections of k consecutive hyperedges in C, we have that $e_i, e_{i+1}, \ldots, e_{i+k-1}$ have exactly one common vertex. Then, we can order such n vertices in C to be KK-Hamiltonian cycle as follows. Let v_i be the common vertex of k consecutive hyperedges, $e_i, e_{i+1}, \ldots, e_{i+k-1}$. Since e_i and e_{i+k} are disjoint, $v_i \neq v_j$ if and only if $i \neq j$. Hence, C can be written as $C = (v_1\ v_2\ \cdots\ v_n)$. $\square$

2.2 Literature review

The studies of Hamiltonian decompositions of complete uniform hypergraphs and complete uniform multipartite hypergraphs based on four distinct definitions are listed in this section for future references.

First, the study of Berge-Hamiltonian decompositions of complete 3-uniform hypergraphs, $K_n^{(3)}$, was completed by Bermond [5] in 1978 and Verrall [15] in 1994.

Theorem 2.2.1. [5, 15] *There exists a Berge-Hamiltonian decomposition of* $K_n^{(3)}$ *for all* $n \geq 3$.

Then, in 2014, Kuhn and Osthus [12] studied the existence of Berge-Hamiltonian decompositions of $K_n^{(k)}$ for many pairs of n, k where $n \geq k \geq 3$.

Theorem 2.2.2. [12] *There exists a Berge-Hamiltonian decomposition of* $K_n^{(k)}$ *where* $4 \leq k \leq n-1$ *for* $n \geq 30$.

Next, in 2010, Bailey and Stevens [2] conjectured that a necessary condition for the existence of a KK-Hamiltonian decomposition of $K_n^{(k)}$, which is that $\binom{n}{k}$

is divisible by n, is also sufficient (which we call feasible n). When $k = 3$, the necessary condition becomes $n \equiv 1, 2 \pmod 3$.

Several authors has studied KK-Hamiltonian decompositions of $K_n^{(k)}$ for some feasible n as follows. Bailey and Stevens [2] use "clique finding" method and "difference pattern" method, and Meszka and Rosa [13] modified the "difference pattern" method in 2010.

Theorem 2.2.3. [2] *There exists a KK-Hamiltonian decomposition of* $K_9^{(4)}$ *and* $K_n^{(3)}$ *where* $n = 7, 8, 10, 11, 16$.

Theorem 2.2.4. [13] *There exists a KK-Hamiltonian decomposition of* $K_n^{(3)}$ *where* n *is feasible and* $n \leq 32$.

Furthermore, KK-Hamiltonian decompositions (originally, WX-definition) of $K_n^{(3)}$ when $n = 2^m$ and $m \geq 2$ were studied by Xu and Wang [18] in 2002, and when $8 \leq n \leq 46$, $n \neq 13, 19, 25, 31, 43$ was provided by Hong Huo *et al.* [10] (published in Chinese) in 2015.

Theorem 2.2.5. [18] *There exists a KK-Hamiltonian decomposition (and WX-Hamiltonian decomposition) of* $K_n^{(3)}$ *where* $n = 2^m$ *and* $m \geq 2$.

Theorem 2.2.6. [10] *There exists a KK-Hamiltonian decomposition of* $K_n^{(3)}$ *where* n *is feasible,* $8 \leq n \leq 46$ *and* $n \neq 13, 19, 25, 31, 43$.

KK-Hamiltonian decompositions of $K_{t(n)}^{(3)}$ was studied by several authors. In 2001, when $t = 2$, the study of WJ-Hamiltonian decompositions of complete bipartite 3-unifom hypergraphs, $K_{n,n}^{(3)}$, was begun by Wang and Jirimutu [16] for prime n. Then, WX-Hamiltonian decompositions of $K_{n,n}^{(3)}$ was completed for all $n \geq 2$ by Xu and Wang [18] in 2002 based on WX-definition. By constructions, these two results also satisfy KK-definition.

Theorem 2.2.7. [16] *There exists WJ-Hamiltonian decompositions and KK-Hamiltonian decompositions of* $K_{n,n}^{(3)}$ *where* $n \geq 3$ *and* n *is prime.*

Theorem 2.2.8. [18] *There exists WX-Hamiltonian decompositions and KK-Hamiltonian decompositions of* $K_{n,n}^{(3)}$ *for all* $n \geq 3$.

Later on, in 2015, KK-Hamiltonian decompositions of complete tripartite 3-uniform hypergraphs, $K^{(3)}_{n,n,n}$, were completely studied by Boonklurb *et al.* [6].

Theorem 2.2.9. [6] *There exists a KK-Hamiltonian decomposition of* $K^{(3)}_{n,n,n}$ *for all* $n \geq 3$.

CHAPTER III
HAMILTONIAN DECOMPOSITIONS OF COMPLETE MULTIPARTITE 3-UNIFORM HYPERGRAPHS

3.1 Introduction

The Hamitonicity of cycles in hypergraphs in Definition 2 was defined by Katona and Kierstead in 1999 which states that *KK-Hamiltonian cycle* of $\mathcal{H}$ is a cyclic ordering $C = (v_1\ v_2\ \cdots\ v_n)$ of all n elements of V such that k consecutive vertices form a hyperedge in E. In literatures, many authors have studied the existence of KK-Hamiltonian decompositions of complete 3-uniform hypergraphs. Bailey and Stevens [2] also conjectured that a necessary condition for the existence of a Hamiltonian decomposition of $K_n^{(k)}$, which is that $\binom{n}{k}$ is divisible by n, is also sufficient (which we call feasible n). Then, several authors have studied the problem for $K_n^{(3)}$ with feasible n where $n \leq 46$, $n \neq 43$, and $n = 2^m$ and $m \geq 2$ (see [2, 13, 10, 18]). Moreover, the existence problem for complete t-partite 3-uniform hypergraphs $K_{t(n)}^{(3)}$ was completely studied when $t = 2$ and $t = 3$ for all n in [18] and [6], respectively.

The main objective of this chapter is to construct a KK-Hamiltonian decomposition of a complete t-partite 3-uniform hypergraph, $K_{t(n)}^{(3)}$, when $t \equiv 4, 8 \pmod{12}$. Our construction also yeilds a recursive construction of a KK-Hamiltonian decomposition of a complete 3-uniform hypergraph $K_{2t}^{(3)}$ from one of $K_t^{(3)}$. Therefore, together with the current results, we are able to construct a KK-Hamiltonian decomposition of $K_n^{(3)}$ where $n = 2^m, 5 \cdot 2^m, 7 \cdot 2^m$ and $11 \cdot 2^m$ and $m \geq 2$. Remark that unless stated otherwise, Hamiltonian cycles in this chapter always mean KK-Hamiltonian cycles in Definition 2 .

Before starting our construction, we start with the following notations which we use throughout this chapter. As our work focuses on 3-uniform hypergraph $K^{(3)}_{t(n)}$, we first classify hyperedges of $K^{(3)}_{t(n)}$ into two types. Let e be a hyperedge of $K^{(3)}_{t(n)}$, e is called a hyperedge of

Type 1 if e contains at most one vertex from each partite set, or

Type 2, otherwise (that is e contains two vertices from a partite set).

The following notations will be used for the rest of the chapter unless stated otherwise.

n is the size of each partite set,

t is the number of partite sets of our complete multipartite 3-uniform hypergraph,

$\mathcal{T}_i(K^{(3)}_{t(n)})$ is the subhypergraph of $K^{(3)}_{t(n)}$ consisting of all hyperedges of Type i for $i \in \{1, 2\}$,

$V_1 \cup V_2 \cup \cdots \cup V_t$ where $V_i = \{a^i_1, a^i_2, \ldots, a^i_n\}$ for $i \in \{1, 2, \ldots, t\}$ is the vertex set of $K^{(3)}_{t(n)}$.

We represent any Hamiltonian cycle of $K^{(3)}_{t(n)}$ by a cyclic ordering (or a cyclic permutation) of all tn vertices of $K^{(3)}_{t(n)}$. In our construction, we write a Hamiltonian cycle C as $(P_1\ P_2\ \cdots\ P_s)$ if vertices along the cycle C are partitioned into *paths* P_j (a sequence of vertices) along the cycle. On top of that, each hyperedge in C is called

an *inline hyperedge* if it is a hyperedge within a path P_j or,

a *joint hyperedge* if it contains vertices from two consecutive paths.

Definition 5. A Hamiltonian cycle $D = (D(1)\ D(2)\ \cdots\ D(n))$ of a hypergraph of order n on the vertex set $\{1, 2, \ldots, n\}$ is written in *standard form* if $D(1) = 1$ and $D(2) < D(n)$.

Note that we denote the set of integers $\{1, 2, \ldots, n\}$ by $[n]$. Next, a necessary condition for the existence of a Hamiltonian decomposition of $K^{(3)}_{t(n)}$ is given in the following theorem.

Theorem 3.1.1. *If $K_{t(n)}^{(3)}$ has a Hamiltonian decomposition, then* $t \equiv 1, 2 \pmod 3$ *or* $n \equiv 0 \pmod 3$.

Proof. Assume that $t \equiv 0 \pmod 3$, then $3 \nmid t-1$ and $3 \nmid t+1$. Since each Hamiltonian cycle of $K_{t(n)}^{(3)}$ contains tn hyperedges, the number of hyperedges of $K_{t(n)}^{(3)}$ must be divisible by tn. Then, $\frac{1}{tn}\left(\binom{tn}{3} - t\binom{n}{3}\right) = \frac{1}{6}n(t-1)(n(t+1)-3)$ is an integer. It follows that $n \equiv 0 \pmod 3$. Therefore, $t \equiv 1, 2 \pmod 3$ or $n \equiv 0 \pmod 3$ as desired. $\square$

Here, we focus on a construction of a Hamiltonian decomposition of $K_{t(n)}^{(3)}$ when $t \equiv 0 \pmod 4$. Our construction method relies on the existence of Hamiltonian decompositions of complete 3-uniform hypergraphs, $K_t^{(3)}$. Note that a necessary condition for such existence for $K_t^{(3)}$ is $t \equiv 1, 2 \pmod 3$ (see more details in [2]) which is also a part of the necessary condition for $K_{t(n)}^{(3)}$ in Theorem 3.1.1. Therefore, our construction aims to solve the problem for $K_{t(n)}^{(3)}$ when $t \equiv 4, 8 \pmod{12}$ for all positive integer n which is concluded in Theorem A as follows.

Theorem A. (Main theorem) *Let $n \geq 2$ and t be a positive integer such that* $t \equiv 4, 8 \pmod{12}$. *The complete multipartite 3-uniform hypergraphs $K_{t(n)}^{(3)}$ has a Hamiltonian decomposition provided that*

(*i*) $t = 4$ *and n is odd, or*

(*ii*) $t \geq 8$ *and $K_t^{(3)}$ has a Hamiltonian decomposition.*

Our construction will create a collection of Hamiltonain cycles of $K_{t(n)}^{(3)}$ containing only hyperedges of the same type. The subhypergraph $\mathcal{T}_1(K_{t(n)}^{(3)})$ will be decomposed in Section 3.2, while the subhypergraph $\mathcal{T}_2(K_{t(n)}^{(3)})$ will be decomposed into Hamiltonian cycles in Sections 3.3 and 3.4 depending on the parity of n. Besides, Section 3.4.1 is dedicated to establish a Hamiltonian decomposition of complete multigraph $2K_n$ which is an important tool for Section 3.4. Finally, we will prove our main theorem and provide the results for complete hypergraphs in Section 3.5.

3.2 Hamiltonian decomposition of $\mathcal{T}_1(K^{(3)}_{t(n)})$

In this section, we will construct a Hamiltonian decomposition of the subhypergraph $\mathcal{T}_1(K^{(3)}_{t(n)})$ containing all hyperedges of Type 1 when $t \equiv 0 \pmod 4$ in Theorem B. Our construction uses a Hamiltonian decomposition of $K^{(3)}_t$ and a 1-factorization of $K_{n,n}$. Note that a necessary condition for such existence for $K^{(3)}_t$ is $t \equiv 1, 2 \pmod 3$ (see more details in [2]) which is also a part of the condition in Theorem B. To obtain only hyperedges of Type 1, the construction creates each Hamiltonian cycle consisting of n paths of order t; each path contains one vertex from each partite set.

Theorem B. *Let $n, t \in \mathbb{N}$ and $t \equiv 4, 8 \pmod{12}$. The hypergraph $\mathcal{T}_1(K^{(3)}_{t(n)})$ has a Hamiltonian decomposition if $K^{(3)}_t$ has a Hamiltonian decomposition.*

Hence, in this section, we constantly assume that $t \equiv 4, 8 \pmod{12}$ and $K^{(3)}_t([t])$ has a Hamiltonian decomposition $\mathscr{D}$. Let $\mathscr{F}$ be a 1-factorization of the complete bipartite graph $K_{n,n}(X, Y)$ where $X = [n] = Y$, which exists by Theorem 1.3.1. (with some abuse of notation, $X \cup Y$ consists of distinct $2n$ vertices.) Note that $|\mathscr{D}| = \frac{1}{6}(t-1)(t-2)$ and $|\mathscr{F}| = n$. We aim to establish the following collection of cycles in $K^{(3)}_{t(n)}$,

$$\mathscr{C} = \{C_i(D, F) : i \in \{0, 1, \ldots, n-1\}, D \in \mathscr{D} \text{ and } F \in \mathscr{F}\}.$$

The collection $\mathscr{C}$ will contain a total of $|\mathscr{D}||\mathscr{F}| \cdot n = \frac{n^2}{6}(t-1)(t-2)$ cycles.

Let D be any Hamiltonian cycle of $K^{(3)}_t([t])$ in $\mathscr{D}$ and F any 1-factor of $K_{n,n}(X, Y)$ in $\mathscr{F}$, written

$$D = (D(1)\ D(2)\ \cdots\ D(t)) \text{ and } F = \{(j, f(j)) : j \in X, f(j) \in Y, j \in \{1, 2, \ldots, n\}\},$$

which D is written in standard form. As we need to distinguish the partite sets containing vertices in an edge of 1-factor F, we represent each edge in F by an ordered pair instead of a set of vertices.

We will construct n cycles from D and F as follows. For $i \in \{0, 1, \ldots, n-1\}$, define

$$C_i(D,F) = \big(P_1^i(D,F)\ P_2^i(D,F)\ \cdots\ P_n^i(D,F)\big)$$

where, for $j \in \{1, 2, \ldots, n\}$,

$$P_j^i(D,F) = a_{j+i}^{D(1)}\ a_{f(j+i)}^{D(2)}\ a_j^{D(3)}\ a_{f(j)}^{D(4)}\ a_{j+i}^{D(5)}\ a_{f(j+i)}^{D(6)}\ \ldots\ a_{j+i}^{D(t-3)}\ a_{f(j+i)}^{D(t-2)}\ a_j^{D(t-1)}\ a_{f(j)}^{D(t)}.$$

and $j+i$ is considered in the modulus n.

Figure 3.1 illustrates $C_i(D,F)$ in which the j^{th} row represents path $P_j^i(D,G)$ and the m^{th} column consists of vertices from $V_{D(m)}$. In addition, for $r \in \{0, 1, \ldots, \frac{t}{4} - 1\}$, vertices in the $(4r+1)^{th}$ and the $(4r+2)^{th}$ columns are the i^{th}-rotation of those in $C_0(D,F)$, the other columns (indicated by the framed columns) are left the same as $C_0(D,F)$. Since $t \equiv 4, 8 \pmod{12}$, vertices along $C_i(D,F)$ alternate between two fixed vertices and two i^{th}-rotated vertices when compare to $C_0(D,F)$.

$$\begin{array}{cccccccccccccc}
 & \boldsymbol{V_{D(1)}} & \boldsymbol{V_{D(2)}} & \boldsymbol{V_{D(3)}} & \boldsymbol{V_{D(4)}} & \boldsymbol{V_{D(5)}} & \boldsymbol{V_{D(6)}} & \cdots & \boldsymbol{V_{D(t-3)}} & \boldsymbol{V_{D(t-2)}} & \boldsymbol{V_{D(t-1)}} & \boldsymbol{V_{D(t)}} & \\
C_i(D,F) = & (a_{1+i}^{D(1)} & a_{f(1+i)}^{D(2)} & a_1^{D(3)} & a_{f(1)}^{D(4)} & a_{1+i}^{D(5)} & a_{f(1+i)}^{D(6)} & \cdots & a_{1+i}^{D(t-3)} & a_{f(1+i)}^{D(t-2)} & a_1^{D(t-1)} & a_{f(1)}^{D(t)} & \\
 & a_{2+i}^{D(1)} & a_{f(2+i)}^{D(2)} & a_2^{D(3)} & a_{f(2)}^{D(4)} & a_{2+i}^{D(5)} & a_{f(2+i)}^{D(6)} & \cdots & a_{2+i}^{D(t-3)} & a_{f(2+i)}^{D(t-2)} & a_2^{D(t-1)} & a_{f(2)}^{D(t)} & \\
 & \vdots & & \vdots & & \vdots & & \ddots & \vdots & & \vdots & & \\
 & a_{n+i}^{D(1)} & a_{f(n+i)}^{D(2)} & a_n^{D(3)} & a_{f(n)}^{D(4)} & a_{n+i}^{D(5)} & a_{f(n+i)}^{D(6)} & \cdots & a_{n+i}^{D(t-3)} & a_{f(n+i)}^{D(t-2)} & a_n^{D(t-1)} & a_{f(n)}^{D(t)} &)
\end{array}$$

Figure 3.1: Hamiltonian cycle $C_i(D,F)$.

Example 6. Figure 3.2 illustrates the three Hamiltonian cycles $C_0(D,F), C_1(D,F)$ and $C_2(D,F)$ of $\mathcal{T}_1(K_{8(3)}^{(3)})$ constructed from a Hamiltonian cycle $D = (D(1)\ D(2) \cdots D(8))$ and a 1-factor $F = \{(j, f(j)), : j \in \{1,2,3\}\}$. Each vertex a_ℓ^x in the cycle $C_i(D,F)$ is represented by its subscript ℓ. The last column duplicates the first column. The solid lines indicate two consecutive vertices in the same path, while the dash lines indicate two consecutive vertices from different paths.

Next, Lemma 3.2.1 guarantees that certain hyperedges are used to construct cycles in $\mathscr{C}$. For $D \in \mathscr{D}$ and $\mathscr{A} \subseteq \mathscr{F}$, let $E(D, \mathscr{A})$ stand for the collection of

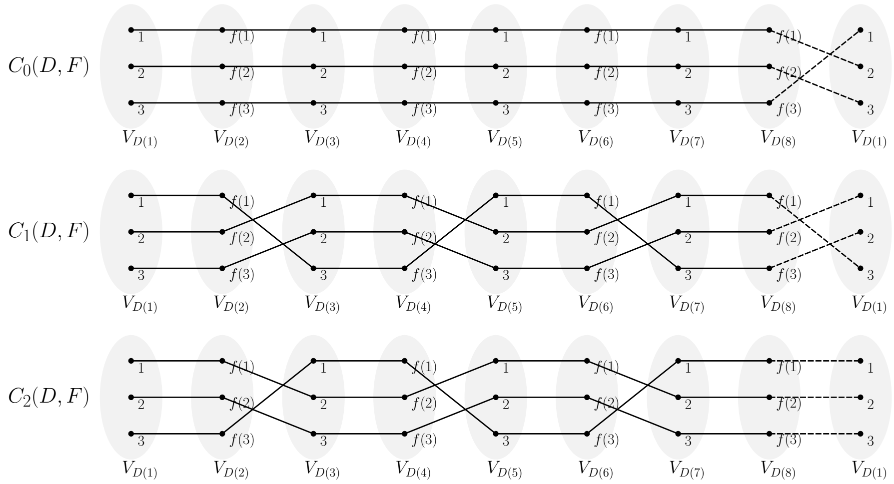

Figure 3.2: $C_0(D,F), C_1(D,F)$ and $C_2(D,F)$ of $\mathcal{T}_1(K^{(3)}_{8(3)})$.

hyperedges of all cycles constructed by D and all $F \in \mathscr{A}$. When $\mathscr{A} = \{F\}$, we write $E(D,F)$ instead. In other words, $E(D,F) = \bigcup_{i=0}^{n-1} E(C_i(D,F))$ and $E(D,\mathscr{F}) = \bigcup_{F\in\mathscr{F}} \bigcup_{i=0}^{n-1} E(C_i(D,F))$.

Lemma 3.2.1. *$E(D,\mathscr{F})$ contains all hyperedges from the collection*

$$A = \{\{a_i^{D(\ell)}, a_j^{D(\ell+1)}, a_k^{D(\ell+2)}\} : i,j,k \in [n], \ell \in [t]\}.$$

Proof. Let $F \in \mathscr{F}$. Let $e_j^i(1), e_j^i(2), \ldots, e_j^i(t)$ be the sequence of t hyperedges along the path $P_j^i(D,F)$ in $C_i(D,F)$, beginning with the two inline hyperedges $e_j^i(1) = \{a_{j+i}^{D(1)}, a_{f(j+i)}^{D(2)}, a_j^{D(3)}\}$, $e_j^i(2) = \{a_{f(j+i)}^{D(2)}, a_j^{D(3)}, a_{f(j)}^{D(4)}\}$ and so on. Note that they are inline hyperedges except the last two hyperedges are joint hyperedges connecting $P_j^i(D,F)$ and $P_{j+1}^i(D,F)$, Then, for $\ell \in [t-2]$,

$$e_j^i(\ell) = \begin{cases} \{a_{j+i}^{D(\ell)}, a_{f(j+i)}^{D(\ell+1)}, a_j^{D(\ell+2)}\}, & \text{if } \ell \equiv 1 \pmod 4, \\ \{a_{f(j+i)}^{D(\ell)}, a_j^{D(\ell+1)}, a_{f(j)}^{D(\ell+2)}\}, & \text{if } \ell \equiv 2 \pmod 4, \\ \{a_j^{D(\ell)}, a_{f(j)}^{D(\ell+1)}, a_{j+i}^{D(\ell+2)}\}, & \text{if } \ell \equiv 3 \pmod 4, \\ \{a_{f(j)}^{D(\ell)}, a_{j+i}^{D(\ell+1)}, a_{f(j+i)}^{D(\ell+2)}\}, & \text{if } \ell \equiv 0 \pmod 4, \end{cases}$$

$e^i_j(t-1) = \{a^{D(t-1)}_j, a^{D(t)}_{f(j)}, a^{D(t+1)}_{j+1+i}\}$ and $e^i_j(t) = \{a^{D(t)}_{f(j)}, a^{D(t+1)}_{j+1+i}, a^{D(t+2)}_{f(j+1+i)}\}$ where the operation $i+j$ in $D(i+j)$ is considered in the modulus t.

First, let ℓ be odd. Then, each hyperedge, $e^i_j(\ell)$, contains two vertices in $V_{D(\ell)}$ and $V_{D(\ell+1)}$ induced by the ordered pair $(j, f(j))$ or $(j+i, f(j+i))$ in the 1-factor F.

Let $B_F = \{\{a^{D(2m-1)}_x, a^{D(2m)}_{f(x)}, a^{D(2m+1)}_y\} \colon x, y \in [n], m \in [\frac{t}{2}]\}$. Then, we have that

$$\begin{aligned}
B_F = \; & \{\{a^{D(\ell)}_{j+i}, a^{D(\ell+1)}_{f(j+i)}, a^{D(\ell+2)}_j\} \colon i \in \{0, 1, \ldots, n-1\}, j \in [n], \ell \in [t-2], \\
& \ell \equiv 1 \pmod 4\} \cup \{\{a^{D(\ell)}_j, a^{D(\ell+1)}_{f(j)}, a^{D(\ell+2)}_{j+i}\} \colon i \in \{0, 1, \ldots, n-1\}, \\
& j \in [n], \ell \in [t-2], \ell \equiv 3 \pmod 4\} \cup \{\{a^{D(t-1)}_j, a^{D(t)}_{f(j)}, a^{D(t+1)}_{j+1+i}\}\} \\
= \; & \{e^i_j(\ell) \colon i \in \{0, 1, \ldots, n-1\}, j \in [n], \ell \in [t], \ell \equiv 1, 3 \pmod 4\} \\
\subseteq \; & E(D, F).
\end{aligned}$$

Now, for hyperedges $e^i_j(\ell)$ in $C_i(D, F)$ when ℓ is even, since each hyperedge contains two vertices in $V_{D(\ell+1)}$ and $V_{D(\ell+2)}$ induced by the ordered pair $(j, f(j))$ or $(j+i, f(j+i))$ in the 1-factor F, we can conclude in the similar way that

$$\begin{aligned}
C_F = \; & \{\{a^{D(2m)}_{f(y)}, a^{D(2m+1)}_x, a^{D(2m+2)}_{f(x)}\} \colon x, y \in [n], m \in [\tfrac{t}{2}]\} \\
= \; & \{e^i_j(\ell) \colon i \in \{0, 1, \ldots, n-1\}, j \in [n], \ell \in [t], \ell \equiv 2, 0 \pmod 4\} \\
\subseteq \; & E(D, F).
\end{aligned}$$

Finally, let $e \in A$, written $e = \{a^{D(\ell)}_u, a^{D(\ell+1)}_v, a^{D(\ell+2)}_w\}$ for some $u, v, w \in [n]$ and $\ell \in [t]$. If ℓ is odd, consider u and v as vertices in two partite sets in $K_{n,n}(X, Y)$. Then, there exists $F' = \{(j, f'(j)) : j \in \{1, 2, \ldots, n\}\} \in \mathscr{F}$ such that $v = f'(u)$. Since $(u, f'(u))$ is an edge in F', $e = \{a^{D(\ell)}_u, a^{D(\ell+1)}_{f'(u)}, a^{D(\ell+2)}_w\} \in B_{F'} \subseteq E(D, F')$. Similarly, if ℓ is even, then there exists $F'' = \{(j, f''(j)) : j \in \{1, 2, \ldots, n\}\} \in \mathscr{F}$ and $z \in [n]$ such that $w = f''(v)$, $u = f''(z)$ and $(v, f''(v))$ is an edge in F''. Thus, $e = \{a^{D(\ell)}_{f''(z)}, a^{D(\ell+1)}_v, a^{D(\ell+2)}_{f''(v)}\} \in C_{F''} \subseteq E(D, F'')$. □

Lemma 3.2.2. *For* $i \in \{0, 1, \ldots, n-1\}$*,* $C_i(D, F)$ *is a Hamiltonian cycle of* $\mathcal{T}_1(K_{t(n)}^{(3)})$.

Proof. A path $P_j^i(D, F)$ in $C_i(D, F)$ contains one vertex from each partite set since D is a Hamiltonian cycle of $K_t^{(3)}([t])$. Note that in $C_0(D, F)$, each path $P_j^0(D, F)$ is determined by an edge $(j, f(j))$ in F. Since F is a 1-factor, all nt vertices in $C_0(D, F)$ are distinct. When compare $P_j^i(D, F)$ to $P_j^0(D, F)$, vertices from the partite sets $V_{D(4m+1)}$ and $V_{D(4m+2)}$ are determined by $(j+i, f(j+i))$ instead of $(j, f(j))$ for $m \in \{1, 2, \ldots, \frac{t}{4} - 1\}$. It means that $C_i(D, F)$ is a Hamiltonian cycle of $K_{t(n)}^{(3)}$ for all i. Moreover, three consecutive vertices in $C_i(D, F)$ always come from three different partite sets. Hence, $C_i(D, F)$ is a Hamiltonian cycle of $\mathcal{T}_1(K_{t(n)}^{(3)})$. $\square$

Now, we are ready to prove Theorem B.

Proof of Theorem B. $\mathscr{C}$ is a collection of Hamiltonian cycles of $\mathcal{T}_1(K_{t(n)}^{(3)})$ by Lemma 3.2.2. It remains to show that $\mathscr{C}$ is a decomposition of $\mathcal{T}_1(K_{t(n)}^{(3)})$. Let $E(\mathscr{C})$ be the set of all hyperedges of all cycles in $\mathscr{C}$. Hence, $E(\mathscr{C})$ contains a total of $\frac{n^3 t}{6}(t-1)(t-2)$ hyperedges of Type 1 (counted repeatedly). Since the number of hyperedges of Type 1 in $K_{t(n)}^{(3)}$ is also $\frac{n^3 t}{6}(t-1)(t-2)$, it suffices to show that each hyperedge of Type 1 in $K_{t(n)}^{(3)}$ is in at least one cycle in $\mathscr{C}$.

Let e be a hyperedge of Type 1 containing vertices from V_p, V_q and V_r, written $e = \{a_u^p, a_v^q, a_w^r\}$ where $u, v, w \in [n]$. Without loss of generality, there exists a unique Hamiltonian cycle $D \in \mathscr{D}$ such that $p = D(\ell), q = D(\ell+1)$ and $r = D(\ell+2)$ for some $\ell \in [t]$. Then, $e = \{a_u^{D(\ell)}, a_v^{D(\ell+1)}, a_w^{D(\ell+2)}\}$. By Lemma 3.2.1, $e \in E(D, \mathscr{F}) \subseteq E(\mathscr{C})$. Therefore, $\mathscr{C}$ is a Hamiltonian decomposition of $\mathcal{T}_1(K_{t(n)}^{(3)})$. $\square$

3.3 Hamiltonian decomposition of $\mathcal{T}_2(K_{t(n)}^{(3)})$ where n is even

In this section, we decompose the subhypergraph $\mathcal{T}_2(K_{t(n)}^{(3)})$ containing all hyperedges of Type 2 when n is even in the following theorem.

Theorem C. *Let $n, t \in \mathbb{N}$. The subhypergraph $\mathcal{T}_2(K^{(3)}_{t(n)})$ has a Hamiltonian decomposition if n and $t \neq 4, 6$ are even.*

Hence, in this section, we constantly assume that n and t are even such that $t \neq 4, 6$. The construction is similar to one in Section 3.2, but here we use a Hamiltonian decomposition of DK_t instead of $K_t^{(3)}$ and a 1-factorization of K_n instead of $K_{n,n}$. To have only hyperedges of Type 2 in the cycles, we will create each cycle comprised with $\frac{n}{2}$ paths of length $2t$, where each path contains two vertices from each partite set.

Let $\mathscr{D}$ be a Hamiltonian decomposition of $DK_t([t])$ and $\mathscr{G}$ be a 1-factorization of the complete graph $K_n([n])$, which exist by Theorem 1.3.3 as $t \neq 4, 6$ and Theorem 1.3.1 as n is even, respectively. We aim to establish the following collection of cycles in $K^{(3)}_{t(n)}$,

$$\mathscr{C} = \{C_i(D, G) : i \in \{0, 1, \ldots, \tfrac{n}{2} - 1\}, D \in \mathscr{D} \text{ and } G \in \mathscr{G}\}.$$

Thus, $\mathscr{C}$ will contain a total of $\frac{n}{2}(n-1)(t-1)$ cycles. Now, let D be any Hamiltonian cycle of $DK_t([t])$ in $\mathscr{D}$ and G any 1-factor of $K_n([n])$ in $\mathscr{G}$, written

$$D = (D(1)\ D(2) \cdots D(t)) \text{ and } G = \{\{j, g(j)\} : j \in \{1, 2, \ldots, \tfrac{n}{2}\}\},$$

where $D(1) = 1$. Consequently, the vertex set $[n]$ of K_n is relabeled according to G to be $\{1, 2, \ldots, \frac{n}{2}, g(1), g(2), \ldots, g(\frac{n}{2})\}$. Thus, all vertices in $V_{D(\ell)}$ are automatically relabeled according to G to be $\{a_1^{D(\ell)}, a_2^{D(\ell)}, \ldots, a_{\frac{n}{2}}^{D(\ell)}, a_{g(1)}^{D(\ell)}, a_{g(2)}^{D(\ell)}, \ldots, a_{g(\frac{n}{2})}^{D(\ell)}\}$ for all $\ell \in [t]$. (For example, if $G = \{\{1, 2\}, \{3, 4\}\}$ is a 1-factor of $K_4([4])$, then the vertices $1, 2, 3$ and 4 could be relabeled to be $1, g(1), 2$ and $g(2)$, respectively.)

We will construct $\frac{n}{2}$ cycles from D and G as follows. For $i \in \{0, 1, \ldots, \frac{n}{2} - 1\}$, define

$$C_i(D, G) = \left(P_1^i(D, G)\ P_2^i(D, G)\ \cdots\ P_{\frac{n}{2}}^i(D, G)\right)$$

where, for $j \in \{1, 2, \ldots, \frac{n}{2}\}$,

$$P^i_j(D,G) = \; a^{D(1)}_{j+i} \;\; a^{D(1)}_{g(j+i)} \;\; a^{D(2)}_{j} \;\; a^{D(2)}_{g(j)} \;\; a^{D(3)}_{j+i} \;\; a^{D(3)}_{g(j+i)} \;\; \ldots \;\; a^{D(t-1)}_{j+i} \;\; a^{D(t-1)}_{g(j+i)} \;\; a^{D(t)}_{j} \;\; a^{D(t)}_{g(j)}$$

and $j+i$ is considered in the modulus $\frac{n}{2}$.

$$\begin{array}{lcccccccccccc}
 & \boldsymbol{V_{D(1)}} & \boldsymbol{V_{D(1)}} & \boldsymbol{V_{D(2)}} & \boldsymbol{V_{D(2)}} & \boldsymbol{V_{D(3)}} & \boldsymbol{V_{D(3)}} & \cdots & \boldsymbol{V_{D(t-1)}} & \boldsymbol{V_{D(t-1)}} & \boldsymbol{V_{D(t)}} & \boldsymbol{V_{D(t)}} & \\
C_i(D,G) = & (\; a^{D(1)}_{1+i} & a^{D(1)}_{g(1+i)} & a^{D(2)}_{1} & a^{D(2)}_{g(1)} & a^{D(3)}_{1+i} & a^{D(3)}_{g(1+i)} & \cdots & a^{D(t-1)}_{1+i} & a^{D(t-1)}_{g(1+i)} & a^{D(t)}_{1} & a^{D(t)}_{g(1)} & \\
 & a^{D(1)}_{2+i} & a^{D(1)}_{g(2+i)} & a^{D(2)}_{2} & a^{D(2)}_{g(1)} & a^{D(3)}_{2+i} & a^{D(3)}_{g(2+i)} & \cdots & a^{D(t-1)}_{2+i} & a^{D(t-1)}_{g(2+i)} & a^{D(t)}_{2} & a^{D(t)}_{g(1)} & \\
 & \vdots & & \vdots & & \vdots & & \ddots & \vdots & & \vdots & & \\
 & a^{D(1)}_{\frac{n}{2}+i} & a^{D(1)}_{g(\frac{n}{2}+i)} & a^{D(2)}_{\frac{n}{2}} & a^{D(2)}_{g(\frac{n}{2})} & a^{D(3)}_{\frac{n}{2}+i} & a^{D(3)}_{g(\frac{n}{2}+i)} & \cdots & a^{D(t-1)}_{\frac{n}{2}+i} & a^{D(t-1)}_{g(\frac{n}{2}+i)} & a^{D(t)}_{\frac{n}{2}} & a^{D(t)}_{g(\frac{n}{2})} &)
\end{array}$$

Figure 3.3: Hamiltonian cycle $C_i(D,G)$.

In the same fashion as in Figure 3.1, Figure 3.3 illustrates $C_i(D,G)$ where the framed columns indicate fixed columns. Note that both $(2m-1)^{th}$ and $(2m)^{th}$ columns consist of vertices from $V_{D(m)}$ for $m \in \{1, 2, \ldots, t\}$. Since t is even, vertices in $C_i(D,G)$ are alternate between two fixed vertices and two i^{th}-rotated vertices.

Example 7. Figure 3.4 illustrates the three Hamiltonian cycles $C_0(D,G), C_1(D,G)$ and $C_2(D,G)$ of $\mathcal{T}_2(K^{(3)}_{8(6)})$ constructed from $D = (D(1)\; D(2) \cdots D(8))$ and 1-factor $G = \{\{j, g(j)\} \colon j \in \{1,2,3\}\}$. Each vertex a^x_ℓ is represented in the figure by its subscript ℓ. The last column duplicates the first column. The solid lines indicate two consecutive vertices in the same path, while the dash lines indicate two consecutive vertices from different paths.

For $D \in \mathscr{D}$ and $G \in \mathscr{G}$, let $E(D,G)$ stand for the collection of hyperedges of all cycles constructed by D and G. In other words, $E(D,G) = \bigcup_{i=0}^{\frac{n}{2}-1} E(C_i(D,G))$.

Lemma 3.3.1. *$E(D,G)$ contains hyperedges in the following two collections.*

(a) $A = \{\{a^{D(m)}_i, a^{D(m)}_{g(i)}, a^{D(m+1)}_j\} \colon i,j \in [\frac{n}{2}], m \in [t]\}$ *and*

(b) $B = \{\{a^{D(m)}_{g(i)}, a^{D(m+1)}_j, a^{D(m+1)}_{g(j)}\} \colon i,j \in [\frac{n}{2}], m \in [t]\}$.

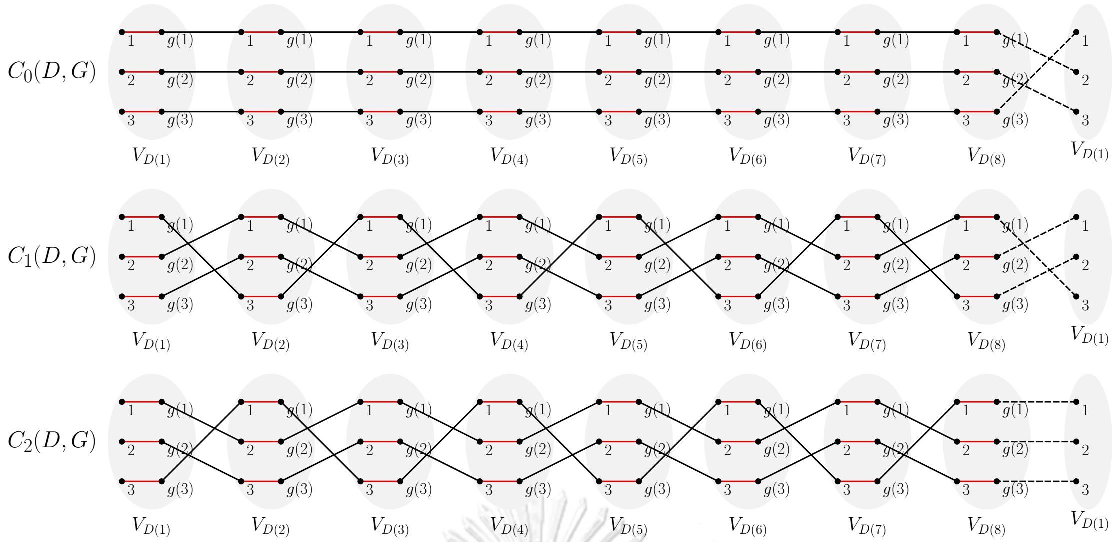

Figure 3.4: $C_0(D,G), C_1(D,G)$ and $C_2(D,G)$ of $\mathcal{T}_2(K^{(3)}_{8(6)})$.

Proof. Let $e^i_j(1), e^i_j(2), \ldots, e^i_j(2t)$ be a sequence of $2t$ hyperedges along the path $P^i_j(D,G)$ in $C_i(D,G)$ beginning with the two inline hyperedges $e^i_j(1) = \{a^{D(1)}_{j+i}, a^{D(1)}_{g(j+i)}, a^{D(2)}_j\}$, $e^i_j(2) = \{a^{D(1)}_{g(j+i)}, a^{D(2)}_j, a^{D(2)}_{g(j)}\}$ and so on. Note that they are inline hyperedges except the last two hyperedges which are joint hyperedges connecting $P^i_j(D,G)$ and $P^i_{j+1}(D,G)$. Then, for $m \in [t-1]$,

$$e^i_j(2m-1) = \begin{cases} \{a^{D(m)}_{j+i}, a^{D(m)}_{g(j+i)}, a^{D(m+1)}_j\}, & \text{if } m \equiv 1 \pmod 2, \\ \{a^{D(m)}_j, a^{D(m)}_{g(j)}, a^{D(m+1)}_{j+i}\}, & \text{if } m \equiv 0 \pmod 2, \end{cases}$$

$$e^i_j(2m) = \begin{cases} \{a^{D(m)}_{g(j+i)}, a^{D(m+1)}_j, a^{D(m+1)}_{g(j)}\}, & \text{if } m \equiv 1 \pmod 2, \\ \{a^{D(m)}_{g(j)}, a^{D(m+1)}_{j+i}, a^{D(m+1)}_{g(j+i)}\}, & \text{if } m \equiv 0 \pmod 2, \end{cases}$$

$$e^i_j(2t-1) = \{a^{D(t)}_j, a^{D(t)}_{g(j)}, a^{D(t+1)}_{j+1+i}\} \text{ and } e^i_j(2t) = \{a^{D(t)}_{g(j)}, a^{D(t+1)}_{j+1+i}, a^{D(t+1)}_{g(j+1+i)}\}.$$

We claim that $\{e^i_j(2m-1) \colon i \in \{0,1,\ldots,\frac{n}{2}-1\}, j \in [\frac{n}{2}], m \in [t]\} = A$ and $\{e^i_j(2m) \colon i \in \{0,1,\ldots,\frac{n}{2}-1\}, j \in [\frac{n}{2}], m \in [t]\} = B$.

First note that for each $m \in [t]$, a hyperedge $e^i_j(2m-1)$ contains two vertices in $D(m)$ induced by an edge $\{j, g(j)\}$ or $\{j+i, g(j+i)\}$ in the 1-factor G. In addition, the other vertex in $e_j^i(2m-1)$ is in $D(m+1)$ with subscript j or $j+i$. Since for each $j \in [\frac{n}{2}]$, $\{j+i \pmod{\frac{n}{2}} : i \in \{0,1,\ldots,\frac{n}{2}-1\}\} = [\frac{n}{2}]$, we have that

$$\{\{a_{j+i}^{D(m)}, a_{g(j+i)}^{D(m)}, a_j^{D(m+1)}\} \colon i \in \{0,1,\ldots,\tfrac{n}{2}-1\}, j \in [\tfrac{n}{2}], m \in [t], m \text{ is odd}\} \cup$$
$$\{\{a_j^{D(m)}, a_{g(j)}^{D(m)}, a_{j+i}^{D(m+1)}\} \colon i \in \{0,1,\ldots,\tfrac{n}{2}-1\}, j \in [\tfrac{n}{2}], m \in [t], m \text{ is even}\}$$

equals to A as claimed.

Similarly, we can show that $\{e_j^i(2m) \colon i \in \{0,1,\ldots,\frac{n}{2}-1\}, j \in [\frac{n}{2}], m \in [t]\} = B$. Since $e_j^i(\ell) \in E(D,G)$ for all $i \in \{0,1,\ldots,\frac{n}{2}-1\}, j \in [\frac{n}{2}]$ and $\ell \in [2t]$, $E(D,G) = A \cup B$. □

Lemma 3.3.2. *For $i \in \{0,1,\ldots,\frac{n}{2}-1\}$, $C_i(D,G)$ is a Hamiltonian cycle of $\mathcal{T}_2(K_{t(n)}^{(3)})$.*

Proof. Each path $P_j^i(D,G)$ in $C_i(D,G)$ contains two vertices from each partite set since D is a Hamiltonian cycle of $DK_t([t])$. In $C_0(D,G)$, each path $P_j^0(D,G)$ is determined by an edge $\{j, g(j)\}$ in G. Since G is a 1-factor, all nt vertices in $C_0(D,G)$ are distinct. When compare $P_j^i(D,G)$ to $P_j^0(D,G)$, vertices from the partite set $V_{D(2m-1)}$ are determined by $\{j+i, g(j+i)\}$ instead of $\{j, g(j)\}$ for $m \in \{1,2,\ldots,\frac{t}{2}\}$. Hence, $C_i(D,G)$ are Hamiltonian cycles for all i. Moreover, three consecutive vertices in $C_i(D,G)$ are from two partite sets. The cycle contains only hyperedges of Type 2. Therefore $C_i(D,G)$ is a Hamiltonian cycle of $\mathcal{T}_2(K_{t(n)}^{(3)})$. □

Now, we are ready to prove our main result of this section.

Proof of Theorem C. $\mathscr{C}$ is a collection of Hamiltonian cycles of $\mathcal{T}_2(K_{t(n)}^{(3)})$ by Lemma 3.3.2. It remains to show that $\mathscr{C}$ is a decomposition of $\mathcal{T}_2(K_{t(n)}^{(3)})$. Let $E(\mathscr{C})$ denote the set of all hyperedges of all cycles in $\mathscr{C}$. Hence, $E(\mathscr{C})$ contains a total of $\frac{n^2t}{2}(n-1)(t-1)$ hyperedges of Type 2 (counted repeatedly). Since the number of hyperedges of Type 2 in $K_{t(n)}^{(3)}$ is also $\frac{n^2t}{2}(n-1)(t-1)$, it suffices to show that each hyperedge of Type 2 in $K_{t(n)}^{(3)}$ is in at least one cycle in $\mathscr{C}$.

Let e be any hyperedge in $\mathcal{T}_2(K_{t(n)}^{(3)})$. Then, $e = \{a_u^p, a_v^p, a_w^q\}$ for some $u, v, w \in [n]$ and $p, q \in [t]$ where $p \neq q$ and $u \neq v$. Since $\mathscr{D}$ is a Hamiltonian decomposition of $DK_t([t])$, there exist exactly two distinct Hamiltonian cycles, say D and D' in $\mathscr{D}$ such that (p, q) and (q, p) are the directed edges in D and D', respectively. Then

$$p = D(r) \text{ and } q = D(r+1), \text{ and}$$

$$p = D'(s+1) \text{ and } q = D'(s).$$

for some $r, s \in [t]$. Since $\mathscr{G}$ is a 1-factorization of $K_n([n])$, there exists unique $G = \{\{j, g(j)\} : j \in \{1, 2, \ldots, \frac{n}{2}\}\} \in \mathscr{G}$ and $c \in [\frac{n}{2}]$ such that without loss of generality, $u = c$ and $v = g(c)$; thus, $\{a_u^p, a_v^p\} = \{a_c^p, a_{g(c)}^p\}$. Consequently, there exists a unique $d \in [\frac{n}{2}]$ such that $w = d$ or $w = g(d)$.

If $w = d$, then e can be written as $\{a_c^{D(r)}, a_{g(c)}^{D(r)}, a_d^{D(r+1)}\}$. By Lemma 3.3.1($a$), $e \in E(D, G) \subseteq E(\mathscr{C})$. If $w = g(d)$, then e can be written as $\{a_c^{D'(s+1)}, a_{g(c)}^{D'(s+1)}, a_{g(d)}^{D'(s)}\}$. By Lemma 3.3.1($b$), $e \in E(D', G) \subseteq E(\mathscr{C})$. Therefore, $\mathscr{C}$ is a Hamiltonian decomposition of $\mathcal{T}_2(K_{t(n)}^{(3)})$. $\square$

3.4 Hamiltonian decomposition of $\mathcal{T}_2(K_{t(n)}^{(3)})$ where n is odd

The last construction will decompose the subhypergraph $\mathcal{T}_2(K_{t(n)}^{(3)})$ containing all hyperedges of Type 2 when n is odd. Similar to Theorem C in Section 3.3, the construction works for an even t. However, the construction uses quite different technique. The following is our main result in this section.

Theorem D. *Let $n, t \in \mathbb{N}$ where $n \geq 3$. The subhypergraph $\mathcal{T}_2(K_{t(n)}^{(3)})$ has a Hamiltonian decomposition if t is even and n is odd.*

Our construction requires a certain decomposition of 2-fold complete graph $2K_t$ which we first construct it in Section 3.4.1.

3.4.1 The Canonical Decompositions of $2K_n$

The 2-*fold complete graph*, $2K_n$ is a multigraph on n vertices with any pair of vertices is joined by exactly two edges. As our construction in Section 3.4 requires a Hamiltonian decomposition of $2K_n$ when n is even with certain properties, we will provide the decomposition along with such properties named the *Canonical Decomposition* $\mathscr{D}$ as follows.

Let n be even integer. Define

$$\mathscr{D} = \{D_\lambda = E_\lambda \cup E_{\frac{n}{2}+\lambda} : \lambda \in \{1, 2, \ldots, n-1\}\}$$

where $E_x = \{\{x, n\}, \{x-i, x+i\} : i \in \{1, 2, \ldots, \frac{n}{2} - 1\}\}$ such that its operation is taken modulo $n-1$.

Example 8. An illustration of D_1 and D_4 in $\mathscr{D}$ when $n = 6$. Then,

$$D_1 = (1\ 6\ 4\ 3\ 5\ 2) \text{ and } D_4 = (1\ 2\ 6\ 4\ 5\ 3)$$

Figure 3.5 illustrates D_1 and D_4 where red pairs and blue pairs are joined by solid and dash lines, respectively.

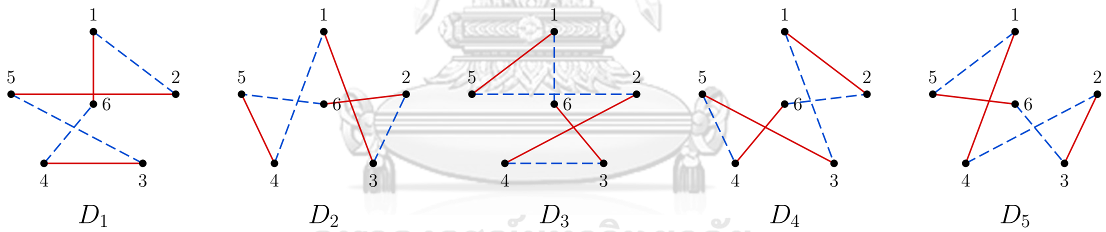

Figure 3.5: D_1, D_2, D_3, D_4 and D_5 in $\mathscr{D}$ when $n = 6$.

Lemma 3.4.1. *$\mathscr{D}$ is a collection of Hamiltonian cycles of the 2-fold complete graph $2K_n([n])$ for all even n.*

Proof. We write $E_\lambda = \{e_0 = \{\lambda, n\}, e_i = \{\lambda - i, \lambda + i\} : i \in \{1, 2, \ldots, \frac{n}{2} - 1\}\}$ and $E_{\frac{n}{2}+\lambda} = \{h_0 = \{\frac{n}{2} + \lambda, n\}, h_i = \{\frac{n}{2} + \lambda - i, \frac{n}{2} + \lambda + i\} : i \in \{1, 2, \ldots, \frac{n}{2} - 1\}\}$. Then we order all edges in D_λ by alternating between edges in $E_{\frac{n}{2}+\lambda}$ and $E_{\frac{n}{2}+\lambda}$ as follows; $e_0, h_{\frac{n}{2}-1}, e_1, h_{\frac{n}{2}-2}, \ldots, e_{\frac{n}{2}-1}, h_0$ which is $\{n, \lambda\}$, $\{\lambda, \lambda+1\}, \{\lambda+1, \lambda-1\}, \{\lambda-1, \lambda+2\}, \ldots, \{\lambda+\frac{n}{2}-1, \lambda-\frac{n}{2}+1\}, \{\lambda+\frac{n}{2}, n\}$. Remark that $\lambda - \frac{n}{2} + 1 \in e_{\frac{n}{2}-1}$ and $\lambda + \frac{n}{2} \in h_0$ are the same because of the operations in the modulus $n-1$. Hence,

$D_\lambda = (n \quad \lambda \quad \lambda+1 \quad \lambda-1 \quad \lambda+2 \quad \cdots \quad \lambda+\frac{n}{2})$ and it is a Hamiltonian cycle of $2K_n([n])$. $\square$

Theorem 3.4.2. *Let n be an even integer. $\mathscr{D}$ is a Hamiltonian decomposition of the 2-fold complete graph $2K_n([n])$.*

Proof. By Lemma 3.4.1, D_λ is a Hamiltonian cycle. Note that $\{E_\lambda : \lambda \in \{1, 2, \ldots, n-1\}\}$ is a 1-factorization of $K_n([n])$. Since each edge of $K_n([n])$ is an edge in two cycles in $\mathscr{D}$ which are D_λ and $D_{\frac{n}{2}+\lambda}$ for some λ, $\mathscr{D}$ is the Hamiltonian decomposition of the 2-fold complete graph $2K_n([n])$. $\square$

By the proof of Lemma 3.4.1, the consecutive edges of a cycle $D_\lambda = (D_\lambda(1)\ D_\lambda(2) \cdots\ D_\lambda(n))$ are alternate between an edge in E_λ and $E_{\frac{n}{2}+\lambda}$. Furthermore, if $\{D_\lambda(1),\ D_\lambda(2)\} \in E_\lambda$ for each $D_\lambda \in \mathscr{D}$, $\mathscr{D}$ is called the *Canonical Decomposition* of $2K_n[n]$. It follows that in any Canonical Hamiltonian cycle D_λ,

$$E_\lambda = \{\{D_\lambda(2j-1),\ D_\lambda(2j)\} \colon j \in \{1, 2, \ldots, \tfrac{n}{2}\}\} \text{ and}$$
$$E_{\frac{n}{2}+\lambda} = \{\{D_\lambda(2j),\ D_\lambda(2j+1)\} \colon j \in \{1, 2, \ldots, \tfrac{n}{2}\}\},$$

and edges in E_λ and $E_{\frac{n}{2}+\lambda}$ will be referred to as *red pairs* and *blue pairs*, respectively. This will be used in our construction in Section 3.4.

Since $\{E_\lambda : \lambda \in \{1, 2, \ldots, n-1\}\}$ and $\{E_{\frac{n}{2}+\lambda} : \lambda \in \{1, 2, \ldots, n-1\}\}$ are 1-factorizations of $K_n([n])$, any pair $\{p, q\}$ in $K_n([n])$ is a red pair once in D_λ and a blue pair once in $D_{\lambda'}$ for some $\lambda, \lambda' \in \{1, 2, \ldots, n-1\}$. Hence, we can conclude this fact in Proposition 3.4.3 for future reference.

Proposition 3.4.3. *Let $n \in \mathbb{N}$ be even and $\mathscr{D}$ the Canonical Decomposition of $2K_n([n])$. For any $p, q \in [n]$ where $p \neq q$, Then, $\{p, q\}$ is a red pair and a blue pair once in $\mathscr{D}$.*

3.4.2 The construction

For the rest of this section, we constantly assume that n is odd, $n \geq 3$ and t is even.

Let $\mathscr{D}$ be the Canonical Decomposition of $2K_t([t])$ and $\mathscr{Q}$ a Hamiltonian decomposition of $K_n([n])$ which always exists by Theorem 3.4.2 as t is even and Theorem 1.3.2 as n is odd, respectively.

We again construct the following collection $\mathscr{C}$ of Hamiltonian cycles of $\mathcal{T}_2(K_{t(n)}^{(3)})$ in which each cycle is composed of $\frac{t}{2}$ paths of length $2n$.

$$\mathscr{C} = \{C_i(D,Q) : i \in \{0,1,\ldots,n-1\}, D \in \mathscr{D} \text{ and } Q \in \mathscr{Q}\}.$$

Thus, $\mathscr{C}$ will contain a total of $\frac{n}{2}(n-1)(t-1)$ cycles. Unlike the two previous subsections, here each path contains vertices from only two partite sets. For each pair of Hamiltonian cycles

$$D = (D(1)\ D(2)\ \cdots\ D(t)) \in \mathscr{D} \text{ and } Q = (Q(1)\ Q(2)\ \cdots\ Q(n)) \in \mathscr{Q}$$

which are written in standard form. Consequently, the vertex set $[n]$ of K_n is relabeled according to Q to be $\{Q(1), Q(2), \ldots, Q(n)\}$. Thus, all vertices in $V_{D(\ell)}$ are automatically relabeled according to Q to be $a_{Q(1)}^{D(\ell)}, a_{Q(2)}^{D(\ell)}, \ldots, a_{Q(n)}^{D(\ell)}$ for all $\ell \in [t]$.

We will construct n Hamiltonian cycles, $C_i(D,Q)$ where $i \in \{0,1,\ldots,n-1\}$, of $\mathcal{T}_2(K_{t(n)}^{(3)})$ in the collection $\mathscr{C}$. For each $i \in \{0,1,\ldots,n-1\}$, $C_i(D,Q)$ consists of $\frac{t}{2}$ paths, written

$$C_i(D,Q) = \left(P_1^i(D,Q)\ \ P_2^i(D,Q)\ \ \ldots\ \ P_{\frac{t}{2}}^i(D,Q)\right)$$

where each path $P_j^i(D,Q)$ consists of $2n$ vertices from $V_{D(2j-1)}$ and $V_{D(2j)}$ ordered as follows. For all $j \in \{1,2,\ldots,\frac{t}{2}\}$,

$$P_j^i(D,Q) = b_n\ \ c_1\ \ b_{n-1}\ \ c_2\ \ \cdots\ \ b_2\ \ c_{n-1}\ \ b_1\ \ c_n,$$

where for $r \in \{1,2,\ldots,n\}$,

$$b_r = a_{Q(r+i)}^{D(x(j,r))}, \quad c_r = a_{Q(r+i)}^{D(\bar{x}(j,r))},$$

$$x(j,r) = \begin{cases} 2j-1, & \text{if } r = n \text{ or } r = \frac{n+1}{2}, \\ 2j, & \text{otherwise and} \end{cases}$$

$$\bar{x}(j,r) \in \{2j-1, 2j\} \setminus \{x(j,r)\}$$

except the case when $n = 3$ and $r = 2$, $x(j,2) = 2j$ and $\bar{x}(j,2) = 2j-1$.

We say that $C_i(D,Q)$ is the i^{th} *rotation* of $C_0(D,Q)$. In other words, $C_0(D,Q)$ is an *initial cycle* which is rotated $n-1$ times to create additional $n-1$ cycles.

Example 9. *An illustration of $C_0(D,Q)$ of $\mathcal{T}_1(K^{(3)}_{8(9)})$ when*

$$D = (D(1)\ D(2) \cdots D(8)) \text{ and } Q = (Q(1)\ Q(2)\ \cdots\ Q(9)).$$

Figure 3.6 shows the Hamiltonian cycle $C_0(D,Q)$ of $\mathcal{T}_1(K^{(3)}_{8(9)})$ constructed from D and Q where each vertex $a^{D(x)}_{Q(\ell)}$ is represented by $Q(\ell)$ in column $V_{D(x)}$. The solid lines join two consecutive vertices in the same path, while the dash lines join two consecutive vertices from different paths. Figure 3.7 shows the values $x(j,r)$ and

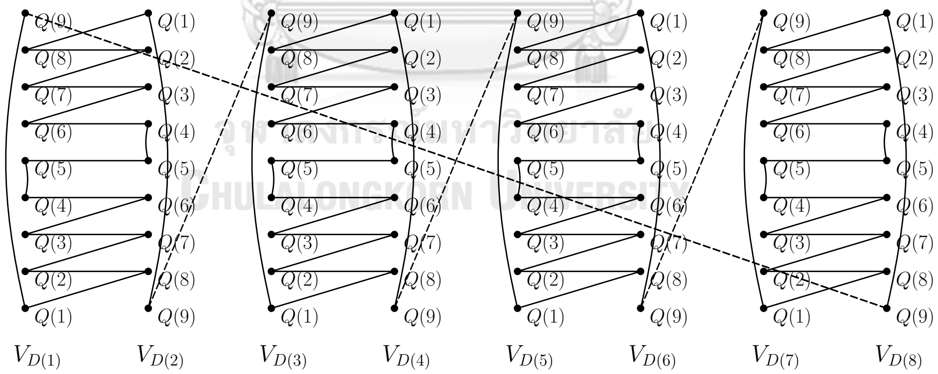

Figure 3.6: $C_0(D,Q)$ of $\mathcal{T}_1(K^{(3)}_{8(9)})$.

$\bar{x}(j,r)$ for the superscripts of vertices b_r and c_r in $P^0_j(D,Q)$, respectively.

From our construction, we have the following two observations of $C_i(D,Q)$ which will later be referred to in Lemma 3.4.7. Given a path $P^i_j(D,Q)$, Observation

	$x(j,r)$ of b_r and $\bar{x}(j,r)$ for c_r																	
b_r, c_r / $P_j^0(D,Q)$	b_9	c_1	b_8	c_2	b_7	c_3	b_6	c_4	b_5	c_5	b_4	c_6	b_3	c_7	b_2	c_8	b_1	c_9
$P_1^0(D,Q)$	1	1	2	1	2	1	2	1	1	2	2	1	2	2	2	1	2	2
$P_2^0(D,Q)$	3	3	4	3	4	3	4	3	3	4	4	3	4	3	4	3	4	4
$P_3^0(D,Q)$	5	5	6	5	6	5	6	5	5	6	6	5	6	5	6	5	6	6
$P_4^0(D,Q)$	7	7	8	7	8	7	8	7	7	8	8	7	8	7	8	7	8	8

Figure 3.7: The values $x(j,r)$ and $\bar{x}(j,r)$ for vertices b_r and c_r, respectively.

1° reveals two partite sets in the path and Observation 2° shows the order of vertices in each partite set along the path.

Observation 1° In each path $P_j^i(D,Q)$, the superscripts $D(x(j,r))$ and $D(\bar{x}(j,r))$ indicate the partite sets in $\{V_{D(2j-1)}, V_{D(2j)}\}$ for b_r and c_r, respectively. All shaded columns and unshaded columns in Figure 3.7 correspond to $V_{D(2j-1)}$ and $V_{D(2j)}$ for $j \in \{1, 2, \ldots, \frac{t}{2}\}$, respectively.

Observation 2° In each path $P_j^i(D,Q)$, the subscripts $Q(r+i)$ for b_r and c_r determines the vertices in the position $r+i$ of the partite set ordered by Q which is invarient for each path in the cycle $C_i(D,Q)$. Furthermore, the subscripts of the shaded entries form an arithmetic sequence with difference 1 in the modulus n, while those of the unshaded entries in Figure 3.7 form an arithmetic sequence with difference -1 in the modulus n. For example, Figure 3.8 illustrate when $i = 0$, the sequence from shaded entries is $9, 1, 2, 3, 4, 5, 6, 7, 8$ and the sequence from unshaded entries is $8, 7, 6, 5, 4, 3, 2, 1, 9$ as shown in .

Lemma 3.4.4. *For $i \in \{0, 1, \ldots, n-1\}$, $C_i(D,Q)$ is a Hamiltonian cycle of $\mathcal{T}_2(K^{(3)}_{t(n)})$.*

$P_j^0(D,Q)$	b_9	c_1	b_8	c_2	b_7	c_3	b_6	c_4	b_5	c_5	b_4	c_6	b_3	c_7	b_2	c_8	b_1	c_9

Figure 3.8: $P_j^0(D,Q)$ with shaded entries from $V_{D(2j-1)}$ and unshaded entries from $V_{D(2j)}$.

Proof. Since Q is a Hamiltonian cycle of $K_n([n])$, for each $j \in \{1, 2, \ldots, \frac{t}{2}\}$, a path $P_j^i(D,Q)$ contains all $2n$ vertices from $V_{D(2j-1)}$ and $V_{D(2j)}$. Since D is a Hamiltonian cycle of $2K_t([t])$, paths $P_j^i(D,Q)$ and $P_\ell^i(D,Q)$ are disjoint for all $j \neq \ell$. Then, $C_i(D,Q)$ is a Hamiltonian cycle. Moreover, any three consecutive vertices in $C_i(D,Q)$ always come from exactly two partite sets. Thus, $C_i(D,Q)$ contains only hyperedges of Type 2 as desired. $\square$

We will next discuss the properties of hyperedges of Type 2 corresponding to $\mathscr{Q}$.

Definition 6. *A hyperedge with $\langle p,q\rangle$-partite sets* is a hyperedge of Type 2 containing two vertices in V_p and one vertex in V_q.

Let e be a hyperedge with $\langle p,q\rangle$-partite sets. Then, $e = \{a_u^p, a_v^p, a_w^q\}$ for some $u \neq v \in [n]$. Consider u and v as vertices of $K_n([n])$, there exists a unique $Q \in \mathscr{Q}$ such that $\{u,v\} \in E(Q)$. As Q is written in standard form, there exists a unique $s \in [n]$ such that $\{a_u^p, a_v^p\} = \{a_{Q(s)}^p, a_{Q(s+1)}^p\}$. Consequently, there exists unique $d \in \{0, 1, \ldots, n-1\}$ such that $a_w^q = a_{Q(s+d)}^q$. Therefore, $e = \{a_{Q(s)}^p, a_{Q(s+1)}^p, a_{Q(s+d)}^q\}$. Hence, given $\mathscr{Q}$, each hyperedge with $\langle p,q\rangle$-partite sets can be written in a unique form; thus, we can define the *delta value* of each hyperedge with $\langle p,q\rangle$-partite sets in our construction as follows.

Definition 7. Let $\mathscr{Q}$ be a Hamiltonian decomposition of $K_n([n])$. For each hyperedge e with $\langle p,q\rangle$-partite sets written $e = \{a_{Q(s)}^p, a_{Q(s+1)}^p, a_{Q(s+d)}^q\}$ where $Q \in \mathscr{Q}$, the *delta value* of e, denoted by $\delta(e)$, is d. It follows that there are n possible delta values in $\{0, 1, \ldots, n-1\}$ denoted by $\mathbb{D}$.

Example 10. Figure 3.9 illustrates the delta values of hyperedges in the cycle $C_0(D,Q)$ in Example 9. Since vertices in this cycle are relabeled according to D

and Q, the delta values of hyperedges are depending on Hamiltonian cycle Q. As each three consecutive vertices along the cycle form a hyperedge, we label its delta value at the middle vertex of each hyperedge.

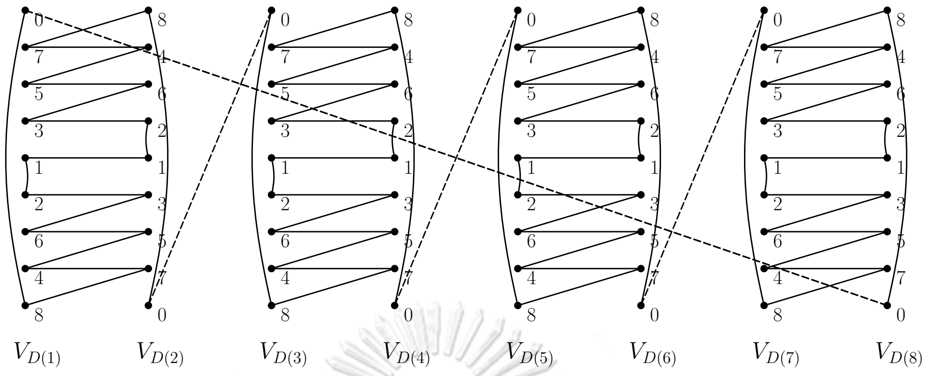

Figure 3.9: Delta values of hyperedges in $C_0(D,Q)$ of $\mathcal{T}_1(K^{(3)}_{8(9)})$.

Remark 3.4.5. *For $i \in \{0,1,\ldots,n-1\}$, the i^{th}-rotation of an initial cycle $C_0(D,Q)$ preserves the delta values of hyperedges in the resulting cycle.*

Lemma 3.4.7 will investigate the delta values of all hyperedges in an initial cycle $C_0(D,Q)$, which yields the result for other cycles $C_i(D,Q)$ in $\mathscr{C}$ by Remark 3.4.5. We begin by observing properties of hyperedges in $P^0_j(D,Q)$ regarding to red pairs and blue pairs in D in the next remark.

For $j \in \{1,2,\ldots,\frac{t}{2}\}$, let $e_j(1), e_j(2), \ldots, e_j(2n-2)$ be $2n-2$ inline hyperedges along the path $P^0_j(D,Q)$ and $e_j(2n-1), e_j(2n)$ be two joint hyperedges connecting $P^0_j(D,Q)$ and $P^0_{j+1}(D,Q)$. Let I_D and J_D be the collections of red pairs and blue pairs in D, respectively. In other words, $I_D = \{\{D(2j-1), D(2j)\} \colon j \in \{1,2,\ldots,\frac{t}{2}\}\}$ and $J_D = \{\{D(2j), D(2j+1)\} : j \in \{1,2,\ldots,\frac{t}{2}\}\}$.

Remark 3.4.6. *The following statements hold.*

(*i*) *For $\ell \in \{1,2,\ldots,2n\}$,*

$$\delta(e_1(\ell)) = \delta(e_2(\ell)) = \cdots = \delta(e_{\frac{t}{2}}(\ell)).$$

(ii) *The set of inline hyperedges in* $P_j^0(D,Q)$ *consists of hyperedges with* $\langle D(2j-1), D(2j)\rangle$*-partite sets and hyperedges with* $\langle D(2j), D(2j-1)\rangle$*-partite sets where* $\{D(2j-1), D(2j)\}$ *is a red pair in* I_D*. Moreover, for* $\ell \in \{1,2,\ldots,2n\}$,

if $e_1(\ell)$ *is with* $\langle D(1), D(2)\rangle$*-partite sets, then* $e_j(\ell)$ *is with* $\langle D(2j-1), D(2j)\rangle$*-partite sets, and*

if $e_1(\ell)$ *is with* $\langle D(2), D(1)\rangle$*-partite sets, then* $e_j(\ell)$ *is with* $\langle D(2j), D(2j-1)\rangle$*-partite sets.*

(iii) *The set of joint hyperedges connecting* $P_j^0(D,Q)$ *and* $P_{j+1}^0(D,Q)$*, consists of*

$e_j(2n-1)$ *with* $\langle D(2j), D(2j+1)\rangle$*-partite sets and*

$e_j(2n)$ *with* $\langle D(2j+1), D(2j)\rangle$*-partite sets*

where $\{D(2j), D(2j+1)\}$ *is a blue pair in* J_D.

Lemma 3.4.7. *The cycle* $C_0(D,Q)$ *consists of the following:*

(i) *for each red pair* $\{x,y\} \in I_D$*, one inline hyperedge with* $\langle x,y\rangle$*-partite sets of delta value* λ*, and one inline hyperedge with* $\langle y,x\rangle$*-partite sets of delta value* λ*, for each* $\lambda \in \mathbb{D} \setminus \{0\}$*, and*

(ii) *for each blue pair* $\{x,y\} \in J_D$*, one joint hyperedge with* $\langle x,y\rangle$*-partite sets of delta value* 0*, and one joint hyperedge in with* $\langle y,x\rangle$*-partite sets of delta value* 0.

Proof. The number of hyperedges in the statements (i) and (ii) are $t(n-1)$ and t, respectively. Since $|E(C_0(D,Q))| = tn$, it remains to show that $C_0(D,Q)$ contains hyperedges in the statements (i) and (ii).

By Remark 3.4.6, it suffices to consider only the first $2n$ hyperedges of $C_0(D,Q)$, $e_j(1), e_j(2), \ldots, e_j(2n)$ when $j=1$. For convenience, rewrite a hyperedge $e_1(\ell)$ as $e(\ell)$. Then, $e(1), e(2), \ldots, e(2n)$ are formed by three consecutive vertices in the sequence of the following $2n+2$ vertices,

$$b_n \;\; c_1 \;\; b_{n-1} \;\; c_2 \;\; \ldots \;\; b_2 \;\; c_{n-1} \;\; b_1 \;\; c_n \;\; b'_n \;\; c'_1.$$

where $P_1^0(D,Q) = b_n\ \ c_1\ \ b_{n-1}\ \ c_2\ \ \ldots\ \ b_2\ \ c_{n-1}\ \ b_1\ \ c_n$ and $P_2^0(D,Q) = b'_n\ \ c'_1\ \ \ldots$

Let $p = D(1)$, $q = D(2)$ and $w = D(3)$.

Claim 1: $\delta(e(2n-1)) = 0 = \delta(e(2n))$. It holds straightforwardly as $e(2n-1) = \{b_1, c_n, b'_n\} = \{a^q_{Q(1)}, a^q_{Q(n)}, a^w_{Q(n)}\}$ and $e(2n) = \{c_n, b'_n, c'_1\} = \{a^q_{Q(n)}, a^w_{Q(n)}, a^w_{Q(1)}\}$.

Therefore, statement (ii) follows from Remark 3.4.6(i), (iii) and Claim 1.

Now we will find the delta values of the remaining $2n-2$ inline hyperedges, $\delta(e(\ell))$ when $\ell \in \{1, 2, \ldots, 2n-2\}$. By the construction of the path $P_1^0(D,Q)$, one can observe that vertices b_r and c_r come from partite sets which are complement to each other in $\{V_p, V_q\}$ (see Figure 3.7). This implies that for $\ell \in \{1, 2, \ldots n-1\}$,

$$e(\ell) \text{ is with } \langle p,q\rangle\text{-partite sets if and only if } e(2n-1-\ell) \text{ is with } \langle q,p\rangle\text{-partite sets,} \tag{3.1}$$

$$e(\ell) \text{ is with } \langle q,p\rangle\text{-partite sets if and only if } e(2n-1-\ell) \text{ is with } \langle p,q\rangle\text{-partite sets.} \tag{3.2}$$

Statements (3.1) and (3.2) together with the fact that the list of subscripts of vertices in path $P_1^0(D,Q)$ is symmetrical about the middle of the path (see Figure 3.8), we have that for $\ell \in \{1, 2, \ldots, n-1\}$,

$$\delta(e(\ell)) = \delta(e(2n-1-\ell)). \tag{3.3}$$

Consequently, it is enough to determine only the delta values of the first $n-1$ hyperedges.

Claim 2: *The delta values of* $e(1), e(2), \ldots, e(n-1)$ *spans the set* $\mathbb{D} \setminus \{0\}$.

We verify separately for $n = 3$ and 5. If $n = 3$, then $\delta(e(1)) = 2$ and $\delta(e(2)) = 1$. If $n = 5$, then $\delta(e(1)) = 4$, $\delta(e(2)) = 3$, $\delta(e(3)) = 2$ and $\delta(e(4)) = 1$.

Now, let $n \geq 7$. First, it can be verified that

$$\delta(e(1)) = n-1, \delta(e(n-1)) = 1 \text{ and } \delta(e(n-2)) = 2.$$

For the remaining cases, note that $\delta(e(2)) = n-2$ and $\delta(e(3)) = 4$, and let $\ell \in \{4, 5, \ldots n-3\}$. By Observations 1° and 2°, if ℓ is even, then $e(\ell - 2) = \{a^p_{Q(r)}, a^q_{Q(s)}, a^p_{Q(r+1)}\}$ and $e(\ell) = \{a^p_{Q(r+1)}, a^q_{Q(s-1)}, a^p_{Q(r+2)}\}$ for some $r, s \in [n]$; thus, $\delta(e(\ell)) = \delta(e(\ell-2)) - 2$. Similarly, if ℓ is odd, then $\delta(e(\ell)) = \delta(e(\ell-2)) + 2$. Thus,

$$\delta(e(\ell)) = \begin{cases} \delta(e(\ell-2)) - 2, & \text{if } \ell \text{ is even,} \\ \delta(e(\ell-2)) + 2, & \text{if } \ell \text{ is odd} \end{cases}$$

where $\delta(e(2)) = n-2$ and $\delta(e(3)) = 4$. That is,

$$\{\delta(e(2)), \delta(e(4)), \ldots, \delta(e(n-5)), \delta(e(n-3))\} = \{n-2, n-4, \ldots, 5, 3\}, \text{ and}$$
$$\{\delta(e(3)), \delta(e(5)), \ldots, \delta(e(n-6)), \delta(e(n-4))\} = \{4, 6, \ldots, n-5, n-3\}.$$

Hence, the claim is completed.

From Claim 2, and statements (3.1), (3.2) and (3.3), we have that for $\lambda \in \mathbb{D} \setminus \{0\}$, and the collection of inline hyperedges $\{e(1), e(2), \ldots, e(2n-2)\}$ contains exactly one hyperedge with $\langle p, q\rangle$-partite sets of delta value λ, and one hyperedge with $\langle q, p\rangle$-partite sets of delta value λ. Therefore, by Remark 3.4.6(ii), the statement (i) is proved. $\square$

Now, we are ready to prove that $\mathscr{C}$ is a Hamiltonian decomposition of $\mathcal{T}_2(K^{(3)}_{t(n)})$. Let $E(\mathscr{C})$ denote the set of all hyperedges of all cycles in $\mathscr{C}$. For $D \in \mathscr{D}$ and $Q \in \mathscr{Q}$, let $E(D, Q)$ stand for the collection of hyperedges of all cycles constructed by D and Q. In other words, $E(D, Q) = \bigcup_{i=0}^{n-1} E(C_i(D, Q))$.

Proof of Theorem D. Since the size of $E(\mathcal{T}_2(K^{(3)}_{t(n)}))$ and $E(\mathscr{C})$ are the same, it is enough to show that each hyperedge of $\mathcal{T}_2(K_{t(n)}^{(3)})$ is contained in at most one cycle in $\mathscr{C}$.

Let $e \in E(\mathscr{C})$ be any hyperedge with $\langle x, y\rangle$-partite sets for some distinct $x, y \in [t]$. Assume that $e \in E(\overline{D}, \overline{Q})$ for some $\overline{D} \in \mathscr{D}$ and $\overline{Q} \in \mathscr{Q}$. There exist a unique $Q \in \mathscr{Q}$, a unique $s \in [n]$ and a unique $d \in \{0, 1, \ldots, n-1\}$ such that

$$e = \{a^x_{Q(s)}, a^x_{Q(s+1)}, a^y_{Q(s+d)}\} \text{ and } \delta(e) = d.$$

Then, $\overline{Q} = Q$. By the property of $\mathscr{D}$ in Proposition 3.4.3, there exist unique $D, D' \in \mathscr{D}$ such that $\{x, y\}$ is a red pair in D and a blue pair in D'. Thus, $\overline{D} = D$ or D'. By Lemma 3.4.7 and the uniqueness of d, a hyperedge e cannot be both an inline hyperedge in $E(D, Q)$ and a joint hyperedge in $E(D', Q)$ at the same time. Thus, we will consider the following two cases depending on d.

Case 1 $d \neq 0$. By Lemma 3.4.7, e is an inline hyperedge in $E(D, Q)$. Then, it suffices to show that inline hyperedges with $\langle x, y\rangle$-partite sets of the same delta value in $E(D, Q)$ are distinct. Now, let $\lambda \in \mathbb{D} \setminus \{0\}$. By Lemma 3.4.7($i$), since $C_0(D, Q)$ has only one hyperedge with $\langle x, y\rangle$-partite sets of delta value λ, such hyperedge can be written as $\{a^x_{Q(m)}, a^x_{Q(m+1)}, a^y_{Q(m+\lambda)}\}$ for a unique m.

For $i \in \{1, 2, \ldots, n-1\}$, since $C_i(D, Q)$ is the i^{th} rotation of $C_0(D, Q)$, and the rotation preserves the delta values of hyperedges, the cycle $C_i(D, Q)$ also contains exactly one hyperedge with $\langle x, y\rangle$-partite sets of delta value λ, namely $\{a^x_{Q(m+i)}, a^x_{Q(m+1+i)}, a^y_{Q(m+\lambda+i)}\}$. Since $\{a^x_{Q(m+j)}, a^x_{Q(m+1+j)}, a^y_{Q(m+\lambda+j)}\} \neq \{a^x_{Q(m+k)}, a^x_{Q(m+1+k)}, a^y_{Q(m+\lambda+k)}\}$ if and only if $j \neq k$, all hyperedges with $\langle x, y\rangle$-partite sets of delta value λ in $E(D, Q)$ are distinct.

Case 2 $d = 0$. By Lemma 3.4.7, e is a joint hyperedge in $E(D', Q)$. Similarly to the proof of Case 1, we can show that joint hyperedges with $\langle x, y\rangle$-partite sets of the delta value 0 in $E(D', Q)$ are distinct.

Hence, by these two cases, each hyperedge of Type 2 of $K_{t(n)}^{(3)}$ is contained in at most one cycle in $\mathscr{C}$. Therefore, $\mathscr{C}$ is a Hamiltonian decomposition of $\mathcal{T}_2(K_{t(n)}^{(3)})$. $\square$

3.5 Conclusions

The results in Sections 3.2-3.4 provide a construction of a Hamiltonian decomposition of $K_{t(n)}^{(3)}$ when $t \equiv 4, 8 \pmod{12}$ excepts when $t = 4$ and n is even. As the assumption in Theorem B includes all requirements in Theorems C and D, we can conclude Theorem A as follows:

Theorem A. (Main theorem) *Let $n \geq 2$ and t be a positive integer such that $t \equiv 4, 8 \pmod{12}$. The complete multipartite 3-uniform hypergraphs $K_{t(n)}^{(3)}$ has a Hamiltonian decomposition provided that*

(i) $t = 4$ and n is odd, or

(ii) $t \geq 8$ and $K_t^{(3)}$ has a Hamiltonian decomposition.

Proof. Let t be a positive integer such that $t \equiv 4, 8 \pmod{12}$. If $t \geq 8$, then assume that $K_t^{(3)}$ has a Hamiltonian decomposition. Then, $\mathcal{T}_1(K_{t(n)}^{(3)})$ has a Hamiltonian decomposition by Theorem B, and $\mathcal{T}_2(K_{t(n)}^{(3)})$ has a Hamiltonian decomposition by Theorems C and D.

However, when $t = 4$, we cannot apply Theorem C to decompose $\mathcal{T}_2(K_{t(n)}^{(3)})$ which n is even. While, Theorems B and D still work (since the hyperedge set of $K_4^{(3)}$ form a Hamiltonian cycle). Therefore, $K_{4(n)}^{(3)}$ has a Hamiltonian decomposition only when n is odd. □

Here, we connect our results to the complete uniform hypergraphs. By a simple but essential fact that $K_{t(2)}^{(3)} = K_{2t}^{(3)}$, together with our main theorem when $n = 2$, we have that the existence problem of Hamiltonian decompositions of complete 3-uniform hypergraphs can be recursively solved as follows.

Theorem 3.5.1. *Let $t \geq 8$ such that $t \equiv 4, 8 \pmod{12}$. If $K_t^{(3)}$ has a Hamiltonian decomposition, so does $K_{2t}^{(3)}$.*

CHAPTER IV
HAMILTONIAN DECOMPOSITIONS OF COMPLETE 4-PARTITE 3-UNIFORM HYPERGRAPHS

4.1 Introduction

The existence problem of KK-Hamiltonian decompositions of complete 3-uniform hypergraphs, $K^{(k)}_{t(n)}$, have been studied when $t = 2$ and $t = 3$ in [18] and [6]. Chapter III establishes a construction of a KK-Hamiltonian decomposition of $K^{(3)}_{t(n)}$ when $t \equiv 4, 8 \pmod{12}$ excepts when $t = 4$ and n is even which we separately construct KK-Hamiltonian decompositions of $\mathcal{T}_1(K^{(3)}_{t(n)})$ and $\mathcal{T}_2(K^{(3)}_{t(n)})$. In details, in Theorem B in Chapter III, we use a KK-Hamiltonian decomposition of $K^{(3)}_4$ to construct one of $\mathcal{T}_1(K^{(3)}_{t(n)})$, and in Theorem C in Chapter III, we use a Hamiltonian decomposition of DK_4 to construct a KK-Hamiltonian decomposition of $\mathcal{T}_2(K^{(3)}_{t(n)})$.

In fact, to construct a KK-Hamiltonian decomposition of $K^{(3)}_{t(n)}$ when $t = 4$ and $n = 2m$, since $K^{(3)}_4$ has trivial Hamiltonian decomposition, the subhypergraph $\mathcal{T}_1(K^{(3)}_{4(2m)})$ can be decomposed into KK-Hamiltonian cycles by Theorem B. While, we cannot provide a KK-Hamiltonian decomposition of the subhypergraph $\mathcal{T}_2(K^{(3)}_{4(2m)})$ by Theorem C since a Hamiltonian decomposition of DK_4 does not exist. Then, we dedicate this chapter to decompose a complete 4-partite 3-uniform hypergraph, $K^{(3)}_{4(2m)}$ into KK-Hamiltonian cycles. The following is our main theorem of the chapter.

Theorem 4.1.1. $K^{(3)}_{4(2m)}$ *has a KK-Hamiltonian decomposition for all positive integer* m.

Thus, Hamiltonian cycles in this chapter always mean KK-Hamiltonian cycles in Definition 2. Also, the notations in this chapter are the same as Section 3.1 in

Chapter III. Hence, to construct a Hamiltonian decomposition of $K^{(3)}_{4(2m)}$ in Theorem 4.1.1, it remains to construct one of the subhypergraph $\mathcal{T}_2(K^{(3)}_{4(2m)})$ containing all hyperedges of Type 2. The construction is revealed in Sections 4.2 and 4.3 depending on the parity of m.

We will construct a collection of Hamiltonian cycles where each cycle consists of two paths of order $4m$. The construction uses the following tools:

the collection of 4-tuples $\mathscr{D} = \{(1,2,3,4),(1,3,4,2),(1,4,2,3)\}$ and

a 1-factorization $\mathscr{F}$ of $K_{2m}([2m])$ which always exists by Theorem 1.3.2.

Now, we aim to establish the following two collections of cycles in $K^{(3)}_{4(2m)}$ which are

$$\mathscr{C} = \{C_t(D,F) : t \in \{0,1,\ldots,m-1\}, D \in \mathscr{D}, \text{ and } F \in \mathscr{F}\} \text{ for odd } m, \text{ and}$$

$$\overline{\mathscr{C}} = \{C_t(D,F), \overline{C}_t(D,F) : t \in \{0,1,\ldots,\tfrac{m}{2}-1\}, D \in \mathscr{D}, \text{ and } F \in \mathscr{F}\} \text{ for even } m.$$

in Sections 4.2 and 4.3, respectively.

Thus, each collection will contain $3m(2m-1)$ cycles. For the construction, let D be any tuple in $\mathscr{D}$, and F any 1-factor of $K_{2m}([2m])$ in $\mathscr{F}$, written

$$D = (p,q,r,s) \text{ and } F = \{\{j, f(j)\} : j \in \{1,2,\ldots,m\}\},$$

consequently, the vertex set $[2m]$ of K_{2m} is relabeled according to F to be $\{1,2,\ldots, m, f(1), f(2), \ldots, f(m)\}$. Thus, all vertices in V_x are automatically relabeled according to F to be $\{a^x_1, a^x_2, \ldots, a^x_m, a^x_{f(1)}, a^x_{f(2)}, \ldots, a^x_{f(m)}\}$ for all $x \in \{p,q,r,s\}$. (For example, if $F = \{\{1,2\},\{3,4\}\}$ is a 1-factor of $K_4([4])$, then the vertices $1,2,3$ and 4 could be relabeled to be $1, f(1), 2$ and $f(2)$, respectively.)

We will construct m Hamiltonian cycles of $\mathcal{T}_2(K^{(3)}_{4(2m)})$ in $\mathscr{C}$ from D and F when m is odd in Section 4.2, namely

$$C_0(D,F),\ C_1(D,F),\ \ldots,\ C_{m-1}(D,F) \text{ and}$$

m Hamiltonian cycles of $\mathcal{T}_2(K^{(3)}_{4(2m)})$ in $\overline{\mathscr{C}}$ from D and F when m is even in Section 4.3, namely

$$C_0(D,F),\ C_1(D,F),\ \ldots,\ C_{\frac{m}{2}-1}(D,F),\ \overline{C}_0(D,F),\ \overline{C}_1(D,F),\ \ldots,\ \overline{C}_{\frac{m}{2}-1}(D,F).$$

Then, we later show that both collections are Hamiltonian decompositions of $\mathcal{T}_2(K^{(3)}_{4(2m)})$ in each section.

4.2 Hamiltonian decomposition of $\mathcal{T}_2(K^{(3)}_{4(2m)})$ where m is odd

Let m be an odd integer. We define $C_t(D,F)$ where $t \in \{0,1,\ldots,m-1\}$ to consist of two paths of order $4m$, written

$$C_t(D,F) = (P_1^t \ \ P_2^t)$$

where for $j \in \{1,2\}$,

$$(x,y) = \begin{cases} (p,q), & \text{if } j=1, \\ (r,s), & \text{if } j=2, \text{ and} \end{cases}$$

$$P_j^t = \begin{array}{cccc} a^x_{1+t} & a^x_{f(1+t)} & a^y_{f(m+t)} & a^y_{m+t} \\ a^x_{2+t} & a^x_{f(2+t)} & a^y_{f(m-1+t)} & a^y_{m-1+t} \\ \vdots & \vdots & \vdots & \vdots \\ a^x_{\frac{m+1}{2}+t} & a^x_{f(\frac{m+1}{2}+t)} & a^y_{f(\frac{m+1}{2}+t)} & a^y_{\frac{m+1}{2}+t} \\ \vdots & \vdots & \vdots & \vdots \\ a^x_{m-1+t} & a^x_{f(m-1+t)} & a^y_{f(2+t)} & a^y_{2+t} \\ a^x_{m+t} & a^x_{f(m+t)} & a^y_{f(1+t)} & a^y_{1+t}, \end{array}$$

We say that $C_t(D,F)$ is the t^{th}*-rotation* of $C_0(D,F)$. In other words, $C_0(D,F)$ is an *initial cycle* which is rotated $m-1$ times to create additional $m-1$ cycles.

Example 11. *An illustration of* $C_0(D,F)$ *which are in the construction of* $K^{(3)}_{4(2m)}$ *when* $m=5$*,* $D=(1,3,4,2)$ *and* $F=\{\{j,f(j)\}: j\in\{1,2,3,4,5\}\}$. In Figure

4.1, each vertex a^x_ℓ in the cycle $C_0(D,F)$ is represented by its subscript ℓ. The solid lines join two consecutive vertices in the same path, while the dash lines join two consecutive vertices from different paths.

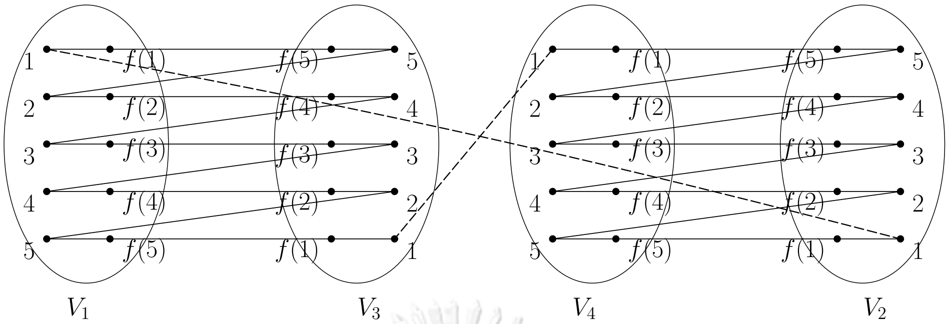

Figure 4.1: $C_0(D,F)$ of $\mathcal{T}_2(K^{(3)}_{4(10)})$.

Lemma 4.2.1. *Let $D \in \mathscr{D}$, $F \in \mathscr{F}$ and $t \in \{0,1,\ldots,m-1\}$. $C_t(D,F)$ is a Hamiltonian cycle of $\mathcal{T}_2(K^{(3)}_{4(2m)})$.*

Proof. Write $D = (p,q,r,s) \in \mathscr{D}$, we have that P^t_1 consists of $4m$ vertices from V_p and V_q and, P^t_2 consists of $4m$ vertices from V_r and V_s. Since (p,q,r,s) is a permutation of $\{1,2,3,4\}$ and F is a 1-factor of K_{2m}, the $8m$ vertices in $C_t(D,F)$ are all distinct. Furthermore, the construction yields that any three consecutive vertices in $C_t(D,F)$ are always from only two partite sets. Therefore, all hyperedges in $C_t(D,F)$ are of Type 2. $\square$

Next, let us observe a certain property of hyperedges in $\mathcal{T}_2(K^{(3)}_{4(2m)})$. Recall that a *hyperedge with $\langle p,q\rangle$-partite sets* stands for a hyperedge of Type 2 containing two vertices in V_p and one vertex in V_q (Definition 6 in Chapter III). Let e be a hyperedge with $\langle x,y\rangle$-partite sets where $x \neq y$, written $e = \{a^x_u, a^x_v, a^y_w\}$. Now, consider u, v as vertices in $K_{2m}([2m])$. Since $\mathscr{F}$ is a 1-factorization of $K_{2m}([2m])$, there exists a unique $F = \{\{j, f(j)\} : j \in \{1,2,\ldots,m\}\} \in \mathscr{F}$ such that $\{u,v\} \in E(F)$. According to F, without loss of generality, there exists a unique $i \in \{1,2,\ldots,m\}$ where u and v are relabeled as i and $f(i)$, respectively. In such vertex set relabeled by F, we also consider w as another vertex. Then, there exists a unique j such that w is relabeled as j or $f(j)$. Thus, e must be one of the followings:

$$\{a_i^x, a_{f(i)}^x, a_j^y\} \text{ or } \{a_i^x, a_{f(i)}^x, a_{f(j)}^y\}.$$

Consequently, given two partite sets in order, we can define the *length* of each hyperedge of Type 2 from such partite sets as follows.

Definition 8. Let $(x, y) \in \{(p, q) \colon p, q \in \{1, 2, 3, 4\}, p \neq q\}$ and e a hyperedge of Type 2 with two partite sets V_x and V_y. Then, there exist a unique $F = \{\{j, f(j)\} : j \in \{1, 2, \ldots, m\}\} \in \mathscr{F}$ and unique $i, j \in \{1, 2, \ldots, m\}$ such that e can be written in one of the following four distinct forms,

$$\{a_i^x, a_{f(i)}^x, a_j^y\}, \quad \{a_i^x, a_{f(i)}^x, a_{f(j)}^y\}, \quad \{a_j^y, a_{f(j)}^y, a_i^x\}, \text{ and } \{a_j^y, a_{f(j)}^y, a_{f(i)}^x\}.$$

Define the *length with respect to* (x, y) of hyperedge e by

$$\mathcal{L}_{(x,y)}(e) = \begin{cases} i - j, & \text{if } e = \{a_i^x, a_{f(i)}^x, a_j^y\} \text{ or } \{a_j^y, a_{f(j)}^y, a_i^x\}, \\ (i-j)', & \text{if } e = \{a_i^x, a_{f(i)}^x, a_{f(j)}^y\} \text{ or } \{a_j^y, a_{f(j)}^y, a_{f(i)}^x\}. \end{cases}$$

where $i - j$ and $(i - j)'$ are considered in the modulus m. Then, there are $2m$ possible lengths in $\{0, 1, \ldots, m-1, 0', 1', \ldots, (m-1)'\}$ denoted by $\mathscr{L}$.

Remark 4.2.2. *Let e be a hyperedge with $\langle x, y\rangle$-partite sets, $x, y \in \{1, 2, 3, 4\}$, $x \neq y$. $\mathcal{L}_{(x,y)}(e) = \ell$ if and only if $\mathcal{L}_{(y,x)}(e) = -\ell$ (in the modulus m). In particular, $\mathcal{L}_{(x,y)}(e)$ and $\mathcal{L}_{(y,x)}(e)$ are both zero $(0$ and $0')$ or both nonzero.*

Moreover, in the construction, as the partite sets of vertices are determined by $D \in \mathscr{D}$, we consider the length of hyperedges in $C_t(D, F)$ according to D as follows.

Definition 9. Let $D = (p, q, r, s) \in \mathscr{D}, F \in \mathscr{F}$ and $e \in C_t(D, F)$. Then, e is a hyperedge of Type 2 with $\langle x, y\rangle$-partite sets or $\langle y, x\rangle$-partite sets for a unique $(x, y) \in \{(p, q), (q, r), (r, s), (s, p)\}$. The *length of a hyperedge* e is $\mathcal{L}_{(x,y)}(e)$.

Example 12. Figure 4.2 illustrates the lengths of hyperedges in the cycle $C_0(D, F)$ in Example 11. As each three consecutive vertices along the cycle form a hyperedge, we label its length at the middle vertex of such hyperedge.

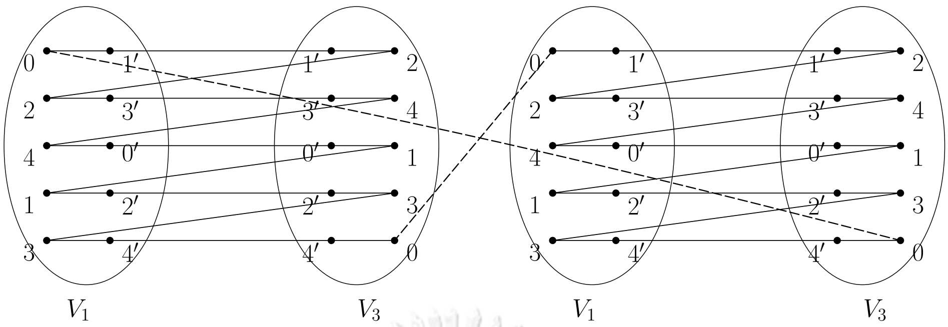

Figure 4.2: Lengths of hyperedges in $C_0(D, F)$ of $\mathcal{T}_2(K^{(3)}_{4(10)})$.

The next lemma discusses the lengths of hyperedges in $C_0(D, F)$, which yields the same result for other cycles in $\mathscr{C}$ as a rotation of an initial cycle preserves the lengths of hyperedges in a new cycle.

Lemma 4.2.3. *Let $D = (p, q, r, s) \in \mathscr{D}, F \in \mathscr{F}$, $I_D = \{(p, q), (r, s)\}$ and $J_D = \{(q, r), (s, p)\}$. The cycle $C_0(D, F)$ consists of the following:*

(i) for $(x, y) \in I_D$, one inline hyperedge with $\langle x, y\rangle$-partite sets of length λ and one inline hyperedge with $\langle y, x\rangle$-partite sets of length λ, for each $\lambda \in \mathscr{L} \setminus \{0\}$,

(ii) for $(x, y) \in J_D$, one joint hyperedge with $\langle x, y\rangle$-partite sets of length 0 and one joint hyperedge with $\langle y, x\rangle$-partite sets of length 0.

Proof. Let $F \in \mathscr{F}$, written $F = \{\{j, f(j)\} : j \in \{1, 2, \ldots, m\}\}$. Let $e_1, e_2, \ldots, e_{8m}$ be $8m$ hyperedges around the cycle $C_0(D, F)$ orderly, beginning with the first four inline hyperedges $e_1 = \{a^p_1,\ a^p_{f(1)}, a^q_{f(m)}\}, e_2 = \{a^p_{f(1)}, a^q_{f(m)}, a^q_m\}, e_3 = \{a^q_{f(m)}, a^q_m, a^p_2\}$, $e_4 = \{a^q_m, a^p_2, a^q_{f(2)}\}$ and so on. Note that $e_{4m-1}, e_{4m}, e_{8m-1}$ and e_{8m} are joint hyperedges while the others $8m - 4$ hyperedges are inline hyperedges.

By our construction, the lengths of inline hyperedges of P^0_1 and P^0_2 have the same spectrum. More precisely, for $\ell \in \{1, 2, \ldots, 4m-2\}$,

$$\mathcal{L}_{(p,q)}(e_\ell) = \mathcal{L}_{(r,s)}(e_{4m+\ell}).$$

For $\ell \in \{4m-1, 4m\}$, e_ℓ and $e_{4m+\ell}$ are joint hyperedges satisfying $\mathcal{L}_{(q,r)}(e_\ell) = \mathcal{L}_{(s,p)}(e_{4m+\ell})$. Then, it suffices to determine the lengths the first $4m$ hyperedges. It is clear that $\mathcal{L}_{(q,r)}(e_{4m-1}) = 0$ and $\mathcal{L}_{(q,r)}(e_{4m}) = 0$, that is, the lengths of joint hyperedges are all 0. Thus, the statement (ii) is proved.

For other hyperedges, Table 4.1 reveals the length of inline hyperedge e_ℓ where $\ell \in \{1, 2, \dots, 4m-2\}$.

e_{4d+k} where $d \in \{0, 1, \dots, m-1\}$ and $4d+k \leq 4m-2$		
k	e_{4d+k}	$\mathcal{L}_{(p,q)}(e_{4d+k})$
1	$\{a^p_{1+d},\ a^p_{f(1+d)},\ a^q_{f(m-d)}\}$	$(1+2d \pmod{m})'$
2	$\{a^p_{f(1+d)},\ a^q_{f(m-d)},\ a^q_{m-d}\}$	$(1+2d \pmod{m})'$
3	$\{a^q_{f(m-d)},\ a^q_{m-d},\ a^p_{2+d}\}$	$2+2d \pmod{m}$
4	$\{a^q_{m-d},\ a^p_{2+d},\ a^p_{f(2+d)}\}$	$2+2d \pmod{m}$

Table 4.1: Lengths of $e_1, e_2, \dots, e_{4m-2}$.

With some abuse of notation, we refer to $(\lambda+2)'$ as $\lambda'+2$. Then, it can be noticed further that the sequence of the lengths of inline hyperedges satisfies a recurrence relation

$$\mathcal{L}_{(p,q)}(e_\ell) = \mathcal{L}_{(p,q)}(e_{\ell-4}) + 2$$

for $\ell \in \{5, 6, \dots, 4m-2\}$ where $\mathcal{L}_{(p,q)}(e_1) = 1'$, $\mathcal{L}_{(p,q)}(e_2) = 1'$, $\mathcal{L}_{(p,q)}(e_3) = 2$, $\mathcal{L}_{(p,q)}(e_4) = 2$.

Now, all inline hyperedges with $\langle p, q\rangle$-partite sets in $C_0(D, F)$ are hyperedges e_ℓ for all $\ell \equiv 0, 1 \pmod{4}$ and $\ell \leq 4m-1$. Since the modulus m is odd, the recurrence relation yields that the lengths of such $2m-1$ inline hyperedges span the set $\mathscr{L} \setminus \{0\}$ (see Tables 4.2 and 4.3). That is,

$$\{\mathcal{L}_{(p,q)}(e_\ell) \colon \ell \equiv 0, 1 \pmod{4}, \ell \in \{1, 2, \dots, 4m-2\}\} = \mathscr{L} \setminus \{0\}.$$

Similarly, inline hyperedges with $\langle q, p\rangle$-partite sets in $C_0(D, F)$ are hyperedges

ℓ	1	5	9	$\cdots$	$\frac{m-3}{2}$	$\frac{m+1}{2}$	$\frac{m+9}{2}$	$\cdots$	$4m-7$	$4m-3$
$\mathcal{L}_{(p,q)}(e_\ell)$	$1'$	$3'$	$5'$	$\cdots$	$(m-2)'$	$0'$	$2'$	$\cdots$	$(m-3)'$	$(m-1)'$

Table 4.2: Lengths of e_ℓ where $\ell \equiv 1 \pmod 4$ and $\ell \leq 4m-2$.

ℓ	4	8	12	$\cdots$	$\frac{m-1}{2}$	$\frac{m+7}{2}$	$\frac{m+15}{2}$	$\cdots$	$4m-8$	$4m-4$
$\mathcal{L}_{(p,q)}(e_\ell)$	2	4	6	$\cdots$	$m-1$	1	3	$\cdots$	$m-4$	$m-2$

Table 4.3: Lengths of e_ℓ where $\ell \equiv 0 \pmod 4$ and $\ell \leq 4m-2$.

e_ℓ for all $\ell \equiv 2, 3 \pmod 4$ and $\ell \leq 4m-2$ which also have lengths spanning the set $\mathscr{L} \setminus \{0\}$ as follows.

$$\{\mathcal{L}_{(p,q)}(e_\ell) \colon \ell \equiv 2, 3 \pmod 4, \ell \in \{1, 2, \ldots, 4m-2\}\} = \mathscr{L} \setminus \{0\}.$$

Hence, for $\lambda \in \mathscr{L} \setminus \{0\}$, $C_0(D, F)$ contains exactly one hyperedge with $\langle p, q\rangle$-partite sets of length λ, and one hyperedge with $\langle q, p\rangle$-partite sets of length λ. Therefore, the statement (i) is proved. $\square$

Now, we are ready to prove that $\mathscr{C}$ is a Hamiltonian decomposition.

Notations

Let $E(\mathscr{C})$ be the set of all hyperedges of all cycles in $\mathscr{C}$.

For $D \in \mathscr{D}$ and $\mathscr{A} \subseteq \mathscr{F}$, let $E(D, \mathscr{A})$ stand for the collection of hyperedges of all cycles in $\mathscr{C}$ constructed by D and all $F \in \mathscr{A}$. When $\mathscr{A} = \{F\}$, we write $E(D, F)$ instead.

In other words, $E(D, \mathscr{F}) = \bigcup_{F \in \mathscr{F}} \bigcup_{i=0}^{m-1} E(C_i(D, F))$ and

$$E(D, F) = \bigcup_{i=0}^{m-1} E(C_i(D, F)).$$

Theorem 4.2.4. *The subhypergraph $\mathcal{T}_2(K_{4(2m)}^{(3)})$ has a Hamiltonian decomposition when m is odd.*

Proof. Let $\mathscr{D} = \{(1,2,3,4),(1,3,4,2),(1,4,2,3)\}$, $\mathscr{F}$ a 1-factorization of K_{2m} and

$$\mathscr{C} = \{C_t(D,F) : t \in \{0,1,\ldots,m-1\}, D \in \mathscr{D} \text{ and } F \in \mathscr{F}\}.$$

By Lemma 4.2.1, $\mathscr{C}$ is a collection of Hamiltionian cycles of $\mathcal{T}_2(K^{(3)}_{4(2m)})$. It remains to show that $\mathscr{C}$ is a decomposition of $\mathcal{T}_2(K^{(3)}_{4(2m)})$.

First, we consider an essential property of $\mathscr{D}$. The following two collections I and J contain ordered pairs induced by $\mathscr{D}$;

$$I = \{(p,q),(r,s) \colon (p,q,r,s) \in \mathscr{D}\} = \{(1,2),(1,3),(1,4),(3,4),(4,2),(2,3)\} \text{ and}$$

$$J = \{(q,r),(s,p) \colon (p,q,r,s) \in \mathscr{D}\} = \{(2,3),(3,4),(4,2),(4,1),(2,1),(3,1)\}.$$

Given $D = (p,q,r,s)$, $E(D,\mathscr{F})$ consists of

(i) inline hyperedges with partite sets V_p and V_q, and with partite sets V_r and V_s, and

(ii) joint hyperedges with partite sets V_q and V_r, and with partite sets V_s and V_p.

Since any pair of elements in $\{1,2,3,4\}$ occurs once in I and once in J, each pair of partite sets is used to construct inline hyperedges once and joint hyperedges once.

Note that the number of hyperedges in $E(\mathscr{C})$ is $24m^2(2m-1)$ counted repeatedly. Since the number of hyperedges of Type 2 in $K^{(3)}_{4(2m)}$ is also $24m^2(2m-1)$, it suffices to show that each hyperedge of Type 2 is contained in at most one cycle in $\mathscr{C}$.

Let $e \in E(\mathscr{C})$ be a hyperedge with $\langle x,y\rangle$-partite sets, say $e = \{a^x_u, a^x_v, a^y_d\}$. Assume that $e \in E(\overline{D},\overline{F})$ for some $\overline{D} \in \mathscr{D}$ and $\overline{F} \in \mathscr{F}$. By the property of $\mathscr{D}$, x and y appear together in both I once and J once. Then, there exist unique $D, D' \in \mathscr{D}$ which pair of partite sets V_x and V_y are used to construct inline hyperedges in $E(D,\mathscr{F})$ and joint hyperedges in $E(D',\mathscr{F})$. Then, $\overline{D} = D$ or D'. Therefore, $e \in E(D,\mathscr{F}) \cup E(D',\mathscr{F})$. Moreover, since $\mathscr{F}$ is a 1-factorization of K_{2m}, there exists a unique $F = \{\{j, f(j)\} : j \in \{1,2,\ldots,m\}\} \in \mathscr{F}$ such that $\{u,v\} \in E(F)$.

Thus, $\overline{F} = F$. Therefore, $e \in E(D, F) \cup E(D', F)$. By Remark 4.2.2, it is enough to consider the following two cases.

Case 1 $\mathcal{L}_{(x,y)}(e)$ *and* $\mathcal{L}_{(y,x)}(e)$ *are not* 0. By Lemma 4.2.3, e is an inline hyperedge in $E(D, F)$. To conclude that e is in at most one cycle, it suffices to show that inline hyperedges with $\langle x, y\rangle$-partite sets of the same length in $E(D, F)$ are distinct. Let $\lambda \in \{1, 2, \ldots, m-1\}$. By Lemma 4.2.3($i$), $C_0(D, F)$ has only one hyperedge of length λ with $\langle x, y\rangle$-partite sets which is $\{a_i^x, a_{f(i)}^x, a_{i-\lambda}^y\}$ for a unique i. For $t \in \{1, 2, \ldots, m-1\}$, since $C_t(D, F)$ is the t^{th}-rotation of $C_0(D, F)$ and the rotation preserves the lengths of hyperedges, the cycle $C_t(D, F)$ also contains exactly one hyperedge of length λ with $\langle x, y\rangle$-partite sets, namely $\{a_{i+t}^p, a_{f(i+t)}^p, a_{i-\lambda+t}^q\}$. Since $\{a_{i+t}^p, a_{f(i+t)}^p, a_{i-\lambda+t}^q\} \neq \{a_{i+w}^p, a_{f(i+w)}^p, a_{i-\lambda+w}^q\}$ if and only if $t \neq w$, all hyperedges of length λ are distinct. Let $\gamma \in \{0', 1', \ldots, (m-1)'\}$. Similarly, by Lemma 4.2.3(i) and the rotation of cycles, all hyperedges of length γ are distinct.

Case 2 $\mathcal{L}_{(x,y)}(e)$ *and* $\mathcal{L}_{(y,x)}(e)$ *are both* 0. By Lemma 4.2.3, e is a joint hyperedge in $E(D', F)$. Similarly, by Lemma 4.2.3(ii) and the rotation of cycles, all hyperedges of length 0 in $E(D', F)$ are distinct.

Hence, each hyperedge is contained in at most one cycle in $\mathscr{C}$. Therefore, $\mathscr{C}$ is a Hamiltonian decomposition of $\mathcal{T}_2(K_{4(2m)}^{(3)})$. $\square$

4.3 Hamiltonian decomposition of $\mathcal{T}_2(K_{4(2m)}^{(3)})$ where m is even

Let m be an even integer, say $m = 2\mu$. We have two initial cycles $C_0(D, F)$ and $\overline{C}_0(D, F)$, each of which is rotated which rotates $\mu - 1$ times to create additional $\mu - 1$ cycles. For $t \in \{0, 1, \ldots, \mu - 1\}$,

$$C_t(D, F) = (P_1^t \ \ P_2^t) \quad \text{and} \quad \overline{C}_t(D, F) = (\overline{P}_1^t \ \ \overline{P}_2^t)$$

where

$$(x,y)=\begin{cases}(p,q), & \text{if } j=1,\\ (r,s), & \text{if } j=2\end{cases}\quad\text{and}$$

$$\begin{array}{lllll}
P_j^t = & a^p_{1+t} & a^p_{f(1+t)} & a^q_{f(2\mu+t)} & a^q_{2\mu+t}\\
& a^p_{2+t} & a^p_{f(2+t)} & a^q_{f(2\mu-1+t)} & a^q_{2\mu-1+t}\\
& \vdots & \vdots & \vdots & \vdots\\
& a^p_{\mu+t} & a^p_{f(\mu+t)} & a^q_{f(\mu+1+t)} & a^q_{\mu+1+t}\\
& a^p_{\mu+1+t} & a^p_{f(\mu+1+t)} & a^q_{f(\mu+t)} & a^q_{\mu+t}\\
& \vdots & \vdots & \vdots & \vdots\\
& a^p_{2\mu-1+t} & a^p_{f(2\mu-1+t)} & a^q_{f(2+t)} & a^q_{2+t}\\
& a^p_{2\mu+t} & a^p_{f(2\mu+t)} & a^q_{f(1+t)} & a^q_{1+t},
\end{array}$$

$$\begin{array}{lllll}
\overline{P}_j^t = & a^p_{f(1+t)} & a^p_{1+t} & a^q_{2\mu+t} & a^q_{f(2\mu+t)}\\
& a^p_{f(2+t)} & a^p_{2+t} & a^q_{2\mu-1+t} & a^q_{f(2\mu-1+t)}\\
& \vdots & \vdots & \vdots & \vdots\\
& a^p_{f(\mu+t)} & a^p_{\mu+t} & a^q_{\mu+1+t} & a^q_{f(\mu+1+t)}\\
& a^p_{f(\mu+1+t)} & a^p_{\mu+1+t} & a^q_{\mu+t} & a^q_{f(\mu+t)}\\
& \vdots & \vdots & \vdots & \vdots\\
& a^p_{f(2\mu-1+t)} & a^p_{2\mu-1+t} & a^q_{2+t} & a^q_{f(2+t)}\\
& a^p_{f(2\mu+t)} & a^p_{2\mu+t} & a^q_{1+t} & a^q_{f(1+t)}.
\end{array}$$

We say that $C_t(D,F)$ and $\overline{C}_t(D,F)$ are the t^{th}-*rotation* of $C_0(D,F)$ and $\overline{C}_0(D,F)$, respectively.

Example 13. *An illustration of the two initial cycles $C_0(D,F)$ and $\overline{C}_0(D,F)$ which are in the construction of a Hamiltonian decomposition of $K^{(3)}_{4(2m)}$ when $m=6$, $D=(1,3,4,2)$ and $F=\{\{j,f(j)\}: j\in\{1,2,3,4,5,6\}\}$.* In the Figures 4.3(a) and 4.4(a), each vertex a^x_ℓ in the initial cycles $C_0(D,F)$ and $\overline{C}_0(D,F)$ is represented by its subscript ℓ. Moreover, Figures 4.3(b) and 4.4(b) illustrate the lengths of hyperedges. As each three consecutive vertices along the cycle form a hyperedge, we label its length at the middle vertex of such hyperedge.

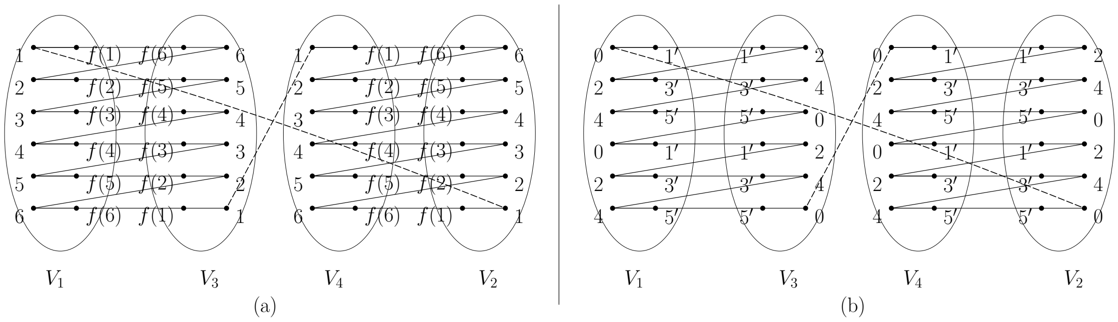

Figure 4.3: (a) $C_0(D,F)$ of $\mathcal{T}_2(K^{(3)}_{4(12)})$ and (b) the lengths of hyperedges in $C_0(D,F)$.

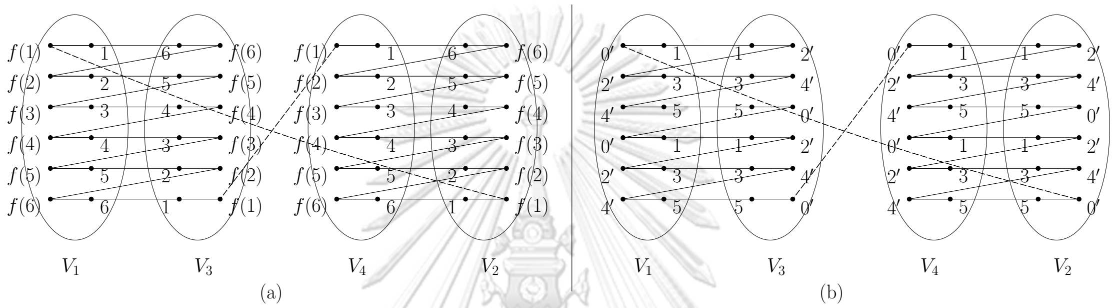

Figure 4.4: (a) $\overline{C}_0(D,F)$ of $\mathcal{T}_2(K^{(3)}_{4(12)})$ and (b) the lengths of hyperedges in $\overline{C}_0(D,F)$.

Now, we will prove that $\overline{\mathscr{C}}$ is a Hamiltonian decomposition.

Notations

For any set A, $2A$ denotes a multi-set containing two repeated elements of each element in A.

Let $E(\overline{\mathscr{C}})$ be the set of all hyperedges of all cycles in $\overline{\mathscr{C}}$.

For $D \in \mathscr{D}$ and $\mathscr{A} \subseteq \mathscr{F}$, let $E(D,\mathscr{A})$ stand for the collection of hyperedges of all cycles in $\overline{\mathscr{C}}$ constructed by D and all $F \in \mathscr{A}$. When $\mathscr{A} = \{F\}$, we write $E(D,F)$ instead.

In other words, $E(D,F) = \bigcup_{t=0}^{\mu-1} E(C_t(D,F)) \cup \bigcup_{t=0}^{\mu-1} E(\overline{C}_t(D,F))$ and

$$E(D,\mathscr{F}) = \bigcup_{F\in\mathscr{F}} \left(\bigcup_{t=0}^{\mu-1} E(C_t(D,F)) \cup \bigcup_{t=0}^{\mu-1} E(\overline{C}_t(D,F))\right).$$

Theorem 4.3.1. *The subhypergraph $\mathcal{T}_2(K^{(3)}_{4(2m)})$ has a Hamiltonian decomposition when m is even.*

Proof. We will define the collection $\overline{\mathscr{C}}$ of Hamiltonian cycles and prove that $\overline{\mathscr{C}}$ is the decomposition of $\mathcal{T}_2(K^{(3)}_{4(2m)})$ by using the lengths of hyperedge.

1. The construction.

Let $m = 2\mu$, $\mathscr{D} = \{(1,2,3,4),(1,3,4,2),(1,4,2,3)\}$, $\mathscr{F}$ a 1-factorization of $K_{2m}([2m])$ and

$$\overline{\mathscr{C}} = \{C_t(D,F), \overline{C}_t(D,F) : t \in \{0,1,\ldots,\mu-1\}, D \in \mathscr{D}, \text{ and } F \in \mathscr{F}\}.$$

Let $D \in \mathscr{D}$ and $F \in \mathscr{F}$, written

$$D = (p,q,r,s) \text{ and } F = \{\{j, f(j)\} : j \in \{1,2,\ldots,m\}\}.$$

Similar to Lemma 4.2.1, the cycles $C_t(D,F)$ and $\overline{C}_t(D,F)$ constructed by D and F are Hamiltonian cycles of $\mathcal{T}_2(K^{(3)}_{4(2m)})$, thus, $\overline{\mathscr{C}}$ is a collection of Hamiltionian cycles of $\mathcal{T}_2(K^{(3)}_{4(2m)})$. It remains to show that $\overline{\mathscr{C}}$ is a decomposition of $\mathcal{T}_2(K^{(3)}_{4(2m)})$.

2. The length of hyperedges.

We write $8m$ hyperedges in $E(C_0(D,F))$ and $8m$ hyperedges in $E(\overline{C}_0(D,F))$ in order around the cycles as $e_1, e_2, \ldots, e_{8m}$, and $\overline{e}_1, \overline{e}_2, \ldots, \overline{e}_{8m}$, respectively, beginning with

$e_1 = \{a^p_1, a^p_{f(1)}, a^q_{f(2\mu)}\}$ and $e_2 = \{a^p_{f(1)}, a^q_{f(2\mu)}, a^q_{2\mu}\}$ and so on, and

$\overline{e}_1 = \{a^p_{f(1)}, a^p_1, a^q_{2\mu}\}$ and $\overline{e}_2 = \{a^p_1, a^q_{2\mu}, a^q_{f(2\mu)}\}$ and so on.

Note that $C_0(D,F)$ is defined exactly the same as in Section 4.2, except even m. Besides, here we rotate $C_0(D,F)$ to construct additional $\frac{m}{2}-1$ cycles instead of $m-1$ cycles. By our construction, for $\ell \in \{1,2,\ldots,4m-2\}$,

$$\mathcal{L}_{(p,q)}(e_\ell) = \mathcal{L}_{(r,s)}(e_{4m+\ell}).$$

For $\ell \in \{4m-1, 4m\}$, e_ℓ and $e_{4m+\ell}$ are joint hyperedges satisfying $\mathcal{L}_{(q,r)}(e_\ell) = \mathcal{L}_{(s,p)}(e_{4m+\ell})$. Then, it suffices to determine the lengths of the first $4m$ hyperedges.

First, we consider the following two observations of hyperedges of the same length.

Observation 1° Since m is even, the lengths of e_ℓ and $e_{2m+\ell}$ are the same for $\ell \in \{1, 2, \dots, 2m\}$. In particular, for $\ell \in \{1, 2, \dots, 2m-2\}$, $e_{2m+\ell}$ and e_ℓ are the μ^{th}-rotation of each other.

Observation 2° Since $e_{2m} = \{a^q_{\mu+1+t}, a^p_{\mu+1+t}, a^p_{f(\mu+1+t)}\}$ and $e_{4m} = \{a^q_{1+t}, a^r_{1+t}, a^r_{f(1+t)}\}$, if $p = r$, then e_{2m} and e_{4m} are the μ^{th}-rotation of each other, so are e_{2m-1} and e_{4m-1}.

Next, we have that e_{2m} is an inline hyperedge of length 0 with $\langle p, q\rangle$-partite sets, and e_{4m} is a joint hyperedge of length 0 with $\langle r, q\rangle$-partite sets.

For $\ell \equiv 0, 1 \pmod 4$, $\ell \leq 4m$ and $\ell \neq 2m, 4m$, e_ℓ is an inline hyperedge with $\langle p, q\rangle$-partite sets in $C_0(D, F)$ of length $\lambda \neq 0$. Tables 4.4 and 4.5 show such lengths.

ℓ	4	8	12	$\dots$	$2m-4$	$2m$	$2m+4$	$2m+8$	$2m+12$	$\dots$	$4m-4$	$4m$
Length of e_ℓ	2	4	6	$\dots$	$m-2$	0	2	4	6	$\dots$	$m-2$	0

Table 4.4: Lengths of e_ℓ where $\ell \equiv 0 \pmod 4$ and $\ell \leq 4m$.

ℓ	1	5	9	$\dots$	$2m-3$	$2m+1$	$2m+5$	$2m+7$	$\dots$	$4m-3$
Length of e_ℓ	$1'$	$3'$	$5'$	$\dots$	$(m-1)'$	$1'$	$3'$	$5'$	$\dots$	$(m-1)'$

Table 4.5: Lengths of e_ℓ where $\ell \equiv 1 \pmod 4$ and $\ell \leq 4m$.

Let $\mathscr{L}_1 = \{2, 4, \dots, m-2\} \cup \{1', 3', \dots, (m-1)'\}$. By Observation 1°, the set of the lengths of $2m-2$ inline hyperedges with $\langle p, q\rangle$-partite sets in $C_0(D, F)$ is

$$\{\mathcal{L}_{(p,q)}(e_\ell) \colon \ell \equiv 0, 1 \pmod 4, \ell \in \{1, 2, \dots, 4m-2\} \smallsetminus \{2m\}\} = 2\mathscr{L}_1.$$

For the lengths of e_ℓ where $\ell \equiv 2, 3 \pmod 4$ and $\ell \leq 4m$, we have the similar results for hyperedges with $\langle q, p\rangle$-partite sets and $\langle q, r\rangle$-partite sets as follows. e_{2m-1} and e_{4m-1} are hyperedges of length 0. By Observation 1°, the set of lengths of $2m-2$ inline hyperedges with $\langle q,p\rangle$-partite sets in $C_0(D,F)$ is

$$\{\mathcal{L}_{(p,q)}(e_\ell)\colon \ell \equiv 2,3 \pmod 4, \ell \in \{1,2,\ldots,4m-2\} \setminus \{2m-1\}\} = 2\mathscr{L}_1.$$

Next, consider the lengths of hyperedges in $\overline{C}_0(D,F)$. Observe that $\overline{C}_t(D,F)$ is a modification of $C_t(D,F)$ by swapping a_i^x and $a_{f(i)}^x$ for all $x \in \{p,q,r,s\}$ and $i \in [m]$. Thus, for $\ell \in \{1,2,\ldots,8m\}$ and $\lambda \in \{0,1,\ldots,m-1\}$,

$$\mathcal{L}(\overline{e}_\ell) = \lambda' \text{ if and only if } \mathcal{L}(e_\ell) = \lambda,$$

$$\mathcal{L}(\overline{e}_\ell) = \lambda \text{ if and only if } \mathcal{L}(e_\ell) = \lambda'.$$

Then, the lengths of all joint hyperedges and inline hyperedges $\overline{e}_{2m-1}, \overline{e}_{2m}, \overline{e}_{2m-1}$ and $\overline{e}_{2m}$ are $0'$. Let $\mathscr{L}_2 = \{2',4',\ldots,(m-2)'\} \cup \{1,3,\ldots,m-1\}$. The remaining $2m-2$ inline hyperedges with $\langle p,q\rangle$-partite sets in $\overline{C}_0(D,F)$ have lengths spanning the multiset $2\mathscr{L}_2$ (see Tables 4.6 and 4.7). Also, $2m-2$ inline hyperedges with $\langle q,p\rangle$-partite sets have lengths spanning the multiset $2\mathscr{L}_2$.

ℓ	4	8	12	$\ldots$	$2m-4$	$2m$	$2m+4$	$2m+8$	$2m+12$	$\ldots$	$4m-4$	$4m$
Length of $\overline{e}_\ell$	$2'$	$4'$	$6'$	$\ldots$	$(m-2)'$	$0'$	$2'$	$4'$	$6'$	$\ldots$	$(m-2)'$	$0'$

Table 4.6: Lengths of $\overline{e}_\ell$ where $\ell \equiv 0 \pmod 4$ and $\ell \leq 4m$.

ℓ	1	5	9	$\ldots$	$2m-3$	$2m+1$	$2m+5$	$2m+7$	$\ldots$	$4m-3$
Length of $\overline{e}_\ell$	$1'$	$3'$	$5'$	$\ldots$	$(m-1)'$	$1'$	$3'$	$5'$	$\ldots$	$(m-1)'$

Table 4.7: Lengths of $\overline{e}_\ell$ where $\ell \equiv 1 \pmod 4$ and $\ell \leq 4m$.

Hence, inline hyperedges with $\langle p,q\rangle$-partite sets (or $\langle q,p\rangle$-partite sets) in both $C_0(D,F)$ and $\overline{C}_0(D,F)$ except those of lengths 0 and $0'$ have lengths spanning the multiset $2\mathscr{L}_1 \cup 2\mathscr{L}_2$. Remark that the multiset $2\mathscr{L}_1 \cup 2\mathscr{L}_2 = 2\mathscr{L} \setminus 2\{0,0'\}$.

Let $(x,y) \in \{(p,q),(q,p),(r,s),(s,r)\}$. In conclusion, we have that for $\lambda_1 \in \mathscr{L}_1$, $C_0(D,F)$ contains exactly two hyperedges of length λ_1 with $\langle x,y\rangle$-partite sets, and for $\lambda_2 \in \mathscr{L}_2$, $\overline{C}_0(D,F)$ contains exactly two hyperedges of length λ_2 with $\langle x,y\rangle$-partite sets.

3. The decomposition.

We will use the lengths of hyperedges in the cycles to prove that $\overline{\mathscr{C}}$ is the decomposition of $\mathcal{T}_2(K^{(3)}_{4(2m)})$. By a similar argument as in Theorem 4.2.4, we count the number of hyperedges in $E(\overline{\mathscr{C}})$ and in $E(\mathcal{T}_2(K^{(3)}_{4(2m)}))$. It suffices to show that any hyperedge of Type 2 with $\langle x,y\rangle$-partite sets is contained in at most one cycle in $E(\overline{\mathscr{C}})$.

Now, let $e \in E(\mathscr{C})$. Assume that $e \in E(\overline{D},\overline{F})$ for some $\overline{D} \in \mathscr{D}$ and $\overline{F} \in \mathscr{F}$. Similarly to the proof of Theorem 4.2.4, the essential property of $\mathscr{D}$ implies that each pair of partite sets is used to construct inline hyperedges once and joint hyperedges once. Then, there exist unique $D', D'' \in \mathscr{D}$ and a unique $F' = \{\{j, f'(j)\} : j \in \{1,2,\ldots,m\}\} \in \mathscr{F}$ such that e is an inline hyperedge in $E(D',F')$ or a joint hyperedge in $E(D'',F')$. Thus, $\overline{F} = F$ and $\overline{D} = D$ or D'. By Remark 4.2.2, it is enough to consider the following two cases.

Case 1 *$\mathcal{L}_{(x,y)}(e)$ and $\mathcal{L}_{(y,x)}(e)$ are not in $\{0,0'\}$.* Then, e is an inline hyperedge in $E(D',F')$. Since hyperedges in the same Hamiltonian cycle are always distinct, to conclude that e is in at most one cycle, we will claim that hyperedges with $\langle x,y\rangle$-partite sets of the same length excepts 0 and $0'$ in $E(D',F')$ are all distinct.

The lengths of hyperedges in $C_0(D',F')$ and hyperedges in $\overline{C}_0(D',F')$ are in $\mathscr{L}_1 \cup \{0\}$ and $\mathscr{L}_2 \cup \{0'\}$, respectively. Since $\mathscr{L}_1 \cup \{0\}$ and $\mathscr{L}_2 \cup \{0'\}$ are disjoint and $\overline{C}_t(D,F)$ is a modification of $C_t(D,F)$, it is enough to show that hyperedges of the same length excepts 0 and $0'$ with $\langle x,y\rangle$-partite sets in $\bigcup_{t=0}^{\mu-1} E(C_t(D',F'))$ are distinct.

Let $\lambda \in \mathscr{L}_1 \setminus \{1',3',\ldots,(m-1)'\}$. Then, $C_0(D',F')$ contains exactly two distinct hyperedges of length λ with $\langle x,y\rangle$-partite sets and By Observation 1°, such two hyperedges are μ-rotation of each other, say $\{a^x_i, a^x_{f(i)}, a^y_{i-\lambda}\}$ and $\{a^x_{i+\mu}, a^x_{f(i+\mu)}, a^y_{i+\mu-\lambda}\}$ for a unique $i \in [m]$. By the proof of Theorem 4.2.4, hyperedges of the same length obtained by the rotation are all distinct. Since we rotate each initial cycle in our construction at most $\mu-1$ times, hyperedges obtained from rotating such two hyperedges are all distinct. Hence, hyperedges with $\langle x,y\rangle$-partite sets of length λ in $\bigcup_{t=0}^{\mu-1} E(C_t(D',F'))$ are all distinct. Similarly, for $\gamma \in \{1',3',\dots,(m-1)'\}$, hyperedges with $\langle x,y\rangle$-partite sets of length γ in $\bigcup_{t=0}^{\mu-1} E(C_t(D',F'))$ are all distinct. Our claim holds.

Case 2 *$\mathcal{L}_{(x,y)}(e)$ and $\mathcal{L}_{(y,x)}(e)$ are in $\{0,0'\}$.* Then, e is an inline hyperedge in $E(D',F')$ or a joint hyperedge in $E(D'',F')$. By Remark 4.2.2, $\mathcal{L}_{(x,y)}(e)$ and $\mathcal{L}_{(y,x)}(e)$ are both 0 or both $0'$.

Suppose that $\mathcal{L}_{(x,y)}(e) = 0 = \mathcal{L}_{(y,x)}(e)$. It is enough to show that hyperedges of the length 0 with $\langle x,y\rangle$-partite sets in $E(D',F') \cup E(D'',F')$ are distinct. By Observation 2°, inline hyperedge of length 0 with $\langle x,y\rangle$-partite sets in $C_0(D',F')$ and a joint hyperedge with $\langle x,y\rangle$-partite sets in $C_0(D'',F')$ are the μ^{th}-rotation of each other. From the same reason in the proof of Case 1, all inline hyperedges and joint hyperedges with $\langle x,y\rangle$-partite sets of length 0 in $\bigcup_{t=0}^{\mu-1} E(C_t(D',F'))$ $\cup \bigcup_{t=0}^{\mu-1} E(C_t(D'',F'))$ are all distinct.

Similarly, if $\mathcal{L}_{(x,y)}(e) = 0' = \mathcal{L}_{(y,x)}(e)$, then we can conclude that hyperedges with $\langle x,y\rangle$-partite sets of length $0'$ in $\bigcup_{t=0}^{\mu-1} E(\overline{C}_t(D',F')) \cup \bigcup_{t=0}^{\mu-1} E(\overline{C}_t(D'',F'))$ are all distinct.

By these two cases, any hyperedges of Type 2 is contained in at most one cycle in $\overline{\mathscr{C}}$, and therefore, $\overline{\mathscr{C}}$ is a Hamiltonian decomposition of $\mathcal{T}_2(K^{(3)}_{4(2m)})$. $\square$

4.4 Conclusion and further remark

The construction of a Hamiltonian decomposition of $\mathcal{T}_2(K^{(3)}_{4(2m)})$ in this chapter uses mainly the following tools:

(1) the collection of 4-tuples $\mathscr{D} = \{(1,2,3,4),(1,3,4,2),(1,4,2,3)\}$ and

(2) a 1-factorization $\mathscr{F}$ of $K_{2m}([2m])$ which always exists by Theorem 1.3.2.

For each cycle in our construction, $\mathscr{D}$ and $\mathscr{F}$ are used to arrange partite sets and vertices in each partite set, respectively.

We actually can extended this technique to create a Hamiltonian decomposition of $\mathcal{T}_2(K_{t(n)}^{(3)})$ when $t > 4$ and $n \equiv 0 \pmod 2$. We generalize the construction in Sections 4.2 and 4.3 by replacing the collection $\mathscr{D}$ with the Canonical Decomposition of $2K_t$. This technique works because the following two properties are the same;

(*i*) the property of the Canonical Decomposition of $2K_4$ in Proposition 3.4.3 and

(*ii*) the property of the collection $\mathscr{D}$ that each pair of partite sets is used to construct inline hyperedges once and joint hyperedges once.

Then, the modified construction gives a Hamiltonian decomposition of $\mathcal{T}_2(K_{t(n)}^{(3)})$ when $t > 4$ and $n \equiv 0 \pmod 2$.

CHAPTER V
HAMILTONIAN DECOMPOSITIONS OF COMPLETE BIPARTITE 4-UNIFORM HYPERGRAPHS

5.1 Introduction

Hypergraphs have been introduced in database theory in order to model relational database schemes. A new definition of cycles in hypergraphs (Definition 3 in Chapter II) was introduced in 1999 by Wang and Lee [17] which defined to suit the structure properties of relation database in computer science. In 2001, Wang and Jirimutu [16] adopted this new definition of cycles to define a WJ-Hamiltonian cycle in Definition 3 (Chapter II) which states as follows. Let $C = (e_0, e_1, \ldots, e_{r-1})$ be a sequence of hyperedges of H and $S_i = e_i \cap e_{i+1}$ for $i \in \{0, 1, \ldots, r-1\}$ where $e_r = e_0$. We call S_i a *node* and C a *cycle* with the node sequence $S = (S_0, S_1, \ldots, S_{r-1})$ if the following conditions are satisfied:

($p1$) $e_i \neq e_j$ for $i \neq j$

($p2$) $S_i \neq \varnothing$ for $i \in \{0, 1, \ldots, r-1\}$,

($p3$) $S_i \setminus S_j \neq \varnothing$ for $i \neq j$,

($p4$) for any $i \in \{0, 1, \ldots, r-1\}$ there is no edge $e \in E(H)$ such that

$$S_i \cup S_{i+1} \cup S_{i+2} \subseteq e.$$

C is called a *t-dimension cycle* of *length* r if $t = \min\{|S_i| : i \in \{0, 1, \ldots, r-1\}\}$.

If $\mathcal{H}$ is a k-uniform hypergraph and $|V(\mathcal{H})| = n$, then any $(k-1)$-dimension cycle of length n in $\mathcal{H}$ is called a *WJ-Hamiltonian* cycle of $\mathcal{H}$.

Remark first that Hamiltonian cycles in this chapter always mean WJ-Hamiltonian cycles. Wang and Jirimutu [16] was studied a Hamiltonian decomposition of complete bipartite 3-uniform hypergraph $K_{n,n}^{(3)}$ where n is prime in 2001. This work motivates us to construct a Hamiltonian decomposition of 4-uniform hypergraph $K_{n,n}^{(4)}$ where n is prime. We focus on bipartite hypergraph $K_{n,n}^{(4)}(V_1, V_2)$ where $V_1 \equiv \mathbb{Z}_n$ and $V_2 \equiv \mathbb{Z}_n$. Moreover, to distinguish the partite sets, we will use a notation of vertices for elements in $\mathbb{Z}_n$ as follows.

$$V_1 = \{\overline{0}, \overline{1}, \ldots, \overline{n-1}\} \text{ and } V_2 = \{0, 1, \ldots, n-1\}.$$

We first consider a necessary condition for the existence of Hamiltonian decomposition of $K_{n,n}^{(4)}$. Note that the number of hyperedges in $K_{n,n}^{(4)}$ is $\binom{2n}{4} - 2\binom{n}{4} = \frac{n^2}{12}(n-1)(7n-11)$.

Lemma 5.1.1. *If $K_{n,n}^{(4)}$ has a Hamiltonian decomposition, then* $n \equiv 0, 1, 5 \pmod 8$.

Proof. If a decomposition exists, the number of hyperedges in $K_{n,n}^{(4)}$ must be divisible by the number of hyperedges of each Hamiltonian cycle which is $2n$. Hence, $\frac{n}{24}(n-1)(7n-11)$ is an integer, which implies that $n \equiv 0, 1, 5 \pmod 8$ as desired. □

Next, we classify hyperedges of 4-uniform hypergraph $K_{n,n}^{(4)}$ into three types depending on the number of members from partite set V_1. In particular, hyperedges of each type can be written as follows,

Type 1 : $\{\overline{a}\ ;\ x, y, z\}$,

Type 2 : $\{\overline{a}, \overline{b}\ ;\ x, y\}$ and

Type 3 : $\{\overline{a}, \overline{b}, \overline{c}\ ;\ x\}$

for some $a, b, c, x, y, z \in \mathbb{Z}_n$. Let $\mathcal{T}_i(K_{n,n}^{(4)})$ denote the subhypergraph of $K_{n,n}^{(4)}$ consisting of all hyperedges of Type i for $i \in \{1, 2, 3\}$. Note that $K_{n,n}^{(4)}) = \cup_{i=1}^{3} \mathcal{T}_i(K_{n,n}^{(4)})$.

First, we partition the hyperedge set of $K_{n,n}^{(4)}$ into collections of hyperedges named *patterns* in Section 5.2. Our construction uses *difference pattern* of hyperedges from the construction of a KK-Hamiltonian decomposition of 3-uniform hypergraph $K_n^{(3)}$ in [2] to create a WJ-Hamiltonian decomposition of 4-uniform hypergraph $K_{n,n}^{(4)}$. We use difference pattern to partition the hyperedge set of $\mathcal{T}_1(K_{n,n}^{(4)})$ and $\mathcal{T}_3(K_{n,n}^{(4)})$ in Subsection 5.2.1. While the hyperedge set of $\mathcal{T}_2(K_{n,n}^{(4)})$ is partitioned by *pair-pattern* in Subsection 5.2.2.

In Section 5.3, we construct two kinds of collections of Hamiltonian cycles which the first one containing hyperedges of the same type and, the other one containing hyperedges of two types. Our method creates a collection of Hamiltonian cycles from two collections of hyperedges which each collection has the same difference pattern or pair-pattern. Finally, we apply these collections to construct a Hamiltonian decomposition of $K_{n,n}^{(4)}$ where n is prime in Section 5.4.

5.2 Pattern of hyperedges

First, we can partition the hyperedges into collections of hyperedges called *patterns* as follows.

Definition 10. Let $e = \{v_1, v_2, \ldots, v_k\} \in E(K_{n,n}^{(4)})$ and

$$\alpha^i(e) = \{v_1 + i, v_2 + i, \ldots, v_k + i\} \text{ for each } i \in \{0, 1, \ldots, n-1\},$$

where $v_j + i$ is considered in the modulus n. The *pattern* of e is the collection $\mathcal{P}(e) = \{e, \alpha(e), \alpha^2(e), \ldots, \alpha^{n-1}(e)\}$.

Lemma 5.2.1. *Let $n \geq 3$ be an odd integer. Then, $|\mathcal{P}(e)| = n$ for any $e \in E(K_{n,n}^{(4)})$.*

Proof. Let $e \in E(K_{n,n}^{(4)})$. Note that $|\mathcal{P}(e)| \leq n$ as each orbit has size at most n. If e is of Type 1 (or Type 3), then all hyperedges in the pattern $\mathcal{P}(e)$ are distinct because they contain a distinct vertex from V_1 (or V_2, respectively).

Assume that e is of Type 2. Write $e =\{\overline{u}, \overline{v}\ ;\ x, y\}$ where $\overline{u}, \overline{v} \in V_1$ and $x, y \in V_2$. Suppose that there exists $i \in \{1, 2 \ldots, n-1\}$ such that $e = \alpha^i(e)$. Then

$x \equiv y+i \pmod{n}$ and $y \equiv x+i \pmod{n}$ which imply that $2i \equiv 0 \pmod{n}$. This is a contradiction as n is an odd number. Thus, each pattern of hyperedges contains exactly n hyperedges. $\square$

Remark 5.2.2. *The following statements hold.*

(*i*) *All hyperedges in any pattern are of the same type.*

(*ii*) $\mathcal{P}(\{\overline{a}\ ;\ b,c,d\}) = \mathcal{P}(\{\overline{0}\ ;\ b-a, c-a, d-a\})$ *and,*
$\mathcal{P}(\{\overline{a},\overline{b}\ ;\ c,d\}) = \mathcal{P}(\{\overline{0}, \overline{b-a}\ ;\ c-a, d-a\})$ *or* $\mathcal{P}(\{\overline{0}, \overline{a-b}\ ;\ c-b, d-b\})$.

Lemma 5.2.3. *Let $n \geq 3$ be an odd integer. Let*

$$\mathscr{A}_1 = \bigcup\nolimits_{\{b,c,d\}\subseteq V_2} \mathcal{P}(\{\overline{0}\ ;\ b,c,d\}),$$

$$\mathscr{A}_2 = \bigcup\nolimits_{a\leq \frac{n}{2}, \{c,d\}\subseteq V_2} \mathcal{P}(\{\overline{0},\overline{a}\ ;\ c,d\}) \text{ and}$$

$$\mathscr{A}_3 = \bigcup\nolimits_{\{\overline{b},\overline{c},\overline{d}\}\subseteq V_1} \mathcal{P}(\{\overline{b},\overline{c},\overline{d}\ ;\ 0\})$$

Then, $\mathscr{A}_i$ is a disjoint union and $E(\mathcal{T}_i(K^{(4)}_{n,n})) = \mathscr{A}_i$ for all $i \in \{1,2,3\}$.

Proof. Obviously, $\mathscr{A}_i \subseteq E(\mathcal{T}_i(K^{(4)}_{n,n}))$. By Remark 5.2.2(*ii*), any hyperedge of Type 1 or Type 3 is contained in $\mathscr{A}_1$ or $\mathscr{A}_3$ respectively. For any hyperedge of Type 2, $\{\overline{a},\overline{b}\ ;\ c,d\}$, is contained in both $\mathcal{P}(\{\overline{0},\overline{b-a}\ ;\ c-a,d-a\})$ and $\mathcal{P}(\{\overline{0},\overline{a-b}\ ;\ c-b,d-b\})$. Since either $a-b \leq \frac{n}{2}$ or $b-a \leq \frac{n}{2}$, $\{\overline{a},\overline{b}\ ;\ c,d\}$ is contained in $\mathscr{A}_2$. Thus, $E(\mathcal{T}_i(K^{(4)}_{n,n})) = \mathscr{A}_i$ for all $i \in \{1,2,3\}$. It remains to show that $\mathscr{A}_i$ is a disjoint union.

Note that $\mathcal{P}(\{\overline{0}\ ;\ b,c,d\}) \neq \mathcal{P}(\{\overline{0}\ ;\ u,v,w\})$ if and only if $\{b,c,d\} \neq \{u,v,w\}$. Thus, $\mathscr{A}_1$ is a disjoint union; so does $|\mathscr{A}_3|$. Now, consider $\mathscr{A}_2$. Let $\{\overline{0},\overline{u}\ ;\ v,w\}$ and $\{\overline{0},\overline{a}\ ;\ b,c\}$ be hyperedges of Type 2 where $a,u \in \{1,2\ldots,\frac{n-1}{2}\}$ and $\bar{a} \neq \bar{u}$. Suppose $\{\overline{0},\overline{a}\ ;\ b,c\} \in \mathcal{P}(\{\overline{0},\overline{u}\ ;\ v,w\})$. Then

$$\{\overline{0},\overline{a}\ ;\ b,c\} = \alpha^i(\{\overline{0},\overline{u}\ ;\ v,w\}) = \{\overline{i},\overline{u+i}\ ;\ v+i,w+i\}$$

for some $i \in \{1,2\ldots,n-1\}$. It implies that $i=a$ and $u+i=0$. Since n is odd, $u=-a=n-a > \frac{n-1}{2}$, which is a contradiction. Therefore, $\mathcal{P}(\{\overline{0},\overline{a}\ ;\ b,c\}\ ) \neq \mathcal{P}($

$\{\overline{0}, \overline{u}\ ;\ v, w\})$ if $\overline{a} \neq \overline{u}$. Besides, $\mathcal{P}(\{\overline{0}, \overline{a}\ ;\ b, c\}) \neq \mathcal{P}(\ \{\overline{0}, \overline{a}\ ;\ v, w\})$ if and only if $\{b, c\} \neq \{v, w\}$. Thus, $\mathscr{A}_2$ is a disjoint union. $\square$

Next, we will group hyperedges in some patterns of hyperedges of each type together depending on some common properties of hyperedges in the patterns. Certain properties of hyperedges of each type will be investigated separately in two subsections.

5.2.1 Hyperedges of Type 1 and Type 3

As each hyperedge of Type 1 contains three vertices from V_2, we will classify hyperedges of Type 1 depending on a property of a triple of vertices from V_2. The property is the *difference pattern* of triple of elements in $\mathbb{Z}_n$ that Bailey and Steven studied in [2] to investigated the existence of a KK-Hamiltonian decomposition of 3-uniform hypergraph $K_n^{(3)}$.

Definition 11. [2] Let $T = \{a, b, c\}$ be a triple of distinct elements of V_1 or V_2. Then, its *difference pattern*, $\pi(T)$ is the equivalence class of ordered triples containing all cyclic rotations of $(b-a, c-b, a-c)$ and $(b-c, a-b, c-a)$ (where the differences are taken modulo n).

We will use above terminology for hyperedges of Type 1 and Type 3 in our 4-uniform hypergraph $K_{n,n}^{(4)}$.

Definition 12. *A difference pattern* of a hyperedge of Type 1 $\{\overline{v}\ ;\ a, b, c\}$ (or hyperedge of Type 3 $\{\overline{a}, \overline{b}, \overline{c}\ ;\ v\}$) is defined by $\pi(\{a, b, c\})$.

Example 14. In $K_{13,13}^{(4)}$, difference patterns of hyperedge of Type 1, $\{\overline{0}\ ;\ 1, 4, 7\}$ and hyperedge of Type 3, $\{\overline{3}, \overline{11}, \overline{5}\ ;\ 2\}$, are

$$\pi(\{1,4,7\}) = \{(3,3,7),(3,7,3),(7,3,3),(10,10,6),(10,6,10),(6,10,10)\} \text{ and}$$
$$\pi(\{\overline{3},\overline{11},\overline{5}\}) = \{(2,6,5),(6,5,2),(5,2,6),(8,7,11),(7,11,8),(11,8,7)\},$$

respectively.

Since three differences sum to zero, if we know that the first two differences are x and y, then the third is $-x-y$. By some abuse of notation, we use $(x, y, -x-y)$ to denote the whole equivalence class that contains it. For convenience in our work, we will represent $\pi(\{a, b, c\})$ where $a, b, c \in \mathbb{Z}_n$ and $a < b < c$ by one of the cyclic rotations of ordered triple $(b-a, c-b, a-c)$. For examples,

$$\pi(\{1,4,7\}) = (3,3,7) \text{ or } (3,7,3) \text{ or } (7,3,3).$$

$$\pi(\{\overline{3}, \overline{11}, \overline{5}\}) = (5,2,6) \text{ or } (2,6,5) \text{ or } (6,5,2).$$

Note that an order triple (x, y, z) represents the class $\{(x, y, z), (y, z, x), (z, x, y), (-y, -x, -z), (-x, -z, -y), (-z, -y, -x)\}$. Thus, from any order triple (x, y, z) we can find its unique class.

Furthermore, when n is not divisible by 3, Bailey and Steven [2] can find the number of triples with the same difference pattern. This number is also the number of patterns of hyperedges of Type 1 (or Type 3) with the same difference pattern in $K^{(4)}_{n,n}$.

Lemma 5.2.4. [2] *Suppose that n is not a multiple of 3. Then, there are exactly n triples of elements in $\mathbb{Z}_n$ with the same difference pattern. Moreover, the number of distinct difference pattern of element of $\mathbb{Z}_n$ is $\frac{1}{n}\binom{n}{3}$.*

However, there are important terminologies of difference pattern from [2] in Definition 13. We also introduce some more terminologies to use in this chapter in Definition 14.

Definition 13. [2] A difference pattern $(x, x, n-2x)$ is called an *isosceles* difference pattern, and the two difference patterns $(x, y, n-x-y)$ and $(y, x, n-x-y)$ is called a *conjugate pair*. Besides, we also say that $(x, y, n-x-y)$ is *conjugate* to $(y, x, n-x-y)$.

Definition 14. A difference pattern $(x, y, n-x-y)$ where $x, y, n-x-y$ are all distinct, is called a *non-isosceles* difference pattern. Furthermore, a hyperedge with isosceles or non-isosceles difference pattern is called *isosceles hyperedge* or *non-isosceles hyperedge*, respectively.

Remark 5.2.5. *The following statements hold.*

(i) *An isosceles difference pattern* $(x, x, n-2x)$ *is conjugate to itself.*

(ii) *If a conjugate pair* $(x, y, n-x-y)$ *and* $(y, x, n-x-y)$ *are two distinct different difference patterns, then* $x, y, n-x-y$ *are all distinct.*

(iii) *Each difference pattern is either isosceles or non-isosceles with a unique conjugate pair.*

It can be noticed that hyperedges of the same pattern also have the same difference pattern. Then, we can group hyperedges in some patterns together by their difference patterns.

Definition 15. $\mathscr{P}(x, y, n-x-y)$ and $\mathscr{P}'(x, y, n-x-y)$ are the collections of all hyperedges of Type 1 and Type 3 of $K_{n,n}^{(4)}$ with difference pattern $(x, y, n-x-y)$, respectively.

Consequently, we can partition $E(\mathcal{T}_1(K_{n,n}^{(4)}))$ depending on their difference pattern in the following remark.

Remark 5.2.6. *Let* n *be an odd integer where* n *is not a multiple of* 3. *Then,*

$$E(\mathcal{T}_1(K_{n,n}^{(4)})) = \left[\bigcupdot_{1\le x\le \frac{n-1}{2}} \mathscr{P}(x, x, n-2x)\right] \bigcupdot \left[\bigcupdot_{\substack{1\le x,y\le \frac{n-1}{2}\\ x\ne y\ne n-x-y\\ x\ne n-x-y}} \mathscr{P}(x, y, n-x-y)\right],$$

where $\mathscr{P}(x, y, n-x-y) = \bigcup_{0\le i\le n-1} \mathcal{P}(\{\overline{0}\ ;\ i, x+i, x+y+i\})$ *and*
$|\mathscr{P}(x, y, n-x-y)| = n^2$.

Proof. Obviously, $\bigcup_{1\le x,y\le \frac{n-1}{2}} \mathscr{P}(x, y, n-x-y) \subseteq E(\mathcal{T}_1(K_{n,n}^{(4)}))$. Let $\mathcal{P}(\{\overline{0}\ ;\ b, c, d\}) \subseteq E(\mathcal{T}_1(K_{n,n}^{(4)}))$ where $b < c < d$. Then, the difference pattern of $\{\overline{0}\ ;\ b, c, d\}$ is

$(c-b, d-c, b-d)$. Without loss of generality, $c-b, d-c \leq \frac{n-1}{2}$. Then, $\mathcal{P}(\{\overline{0}\ ;\ b,c,d\}) \in \bigcup_{1 \leq x,y \leq \frac{n-1}{2}} \mathscr{P}(x,y,n-x-y)$. By Remark 5.2.5 (iii), the union of isosceles patterns and non-isosceles patterns is a disjoint union.

Note that $\mathcal{P}(\{\overline{0}\ ;\ b,c,d\}) \neq \mathcal{P}(\{\overline{0}\ ;\ u,v,w\})$ if and only if $\{b,c,d\} \neq \{u,v,w\}$. Since n is not a multiple of 3, $\{i, x+i, x+y+i\} \neq \{j, x+j, x+y+j\}$ for all $i \neq j$. It follows that

$$\mathcal{P}(\{\overline{0}\ ;\ i, x+i, x+y+i\}) \neq \mathcal{P}(\{\overline{0}\ ;\ j, x+j, x+y+j\}).$$

Therefore, $|\mathscr{P}(x,y,n-x-y)| = n^2$. $\square$

Example 15. The collections of hyperedges of Type 1 of $K_{5,5}^{(4)}$ with difference pattern $(2,2,1)$ and $(1,1,2)$ are the following.

$$\begin{aligned}\mathscr{P}(2,2,1) =& \mathcal{P}(\{\overline{0}\ ;\ 0,2,4\}) \cup \mathcal{P}(\{\overline{0}\ ;\ 1,3,5\}) \cup \mathcal{P}(\{\overline{0}\ ;\ 2,4,1\}) \cup \mathcal{P}(\{\overline{0}\ ;\ 3,0,2\}) \cup \\ & \mathcal{P}(\{\overline{0}\ ;\ 4,1,3\}) \\ \mathscr{P}(1,1,2) =& \mathcal{P}(\{\overline{0}\ ;\ 1,2,3\}) \cup \mathcal{P}(\{\overline{0}\ ;\ 2,3,4\}) \cup \mathcal{P}(\{\overline{0}\ ;\ 3,4,0\}) \cup \mathcal{P}(\{\overline{0}\ ;\ 4,0,1\}) \cup \\ & \mathcal{P}(\{\overline{0}\ ;\ 0,1,2\})\end{aligned}$$

Similarly, patterns of hyperedges of Type 3 are concluded in Remark 5.2.7.

Remark 5.2.7. *Let n be an odd integer where n is not a multiple of 3. Then,*

$$E(\mathcal{T}_3(K_{n,n}^{(4)})) = \left[\bigcupdot_{1 \leq x \leq \frac{n-1}{2}} \mathscr{P}'(x,x,n-2x)\right] \bigcupdot \left[\bigcupdot_{\substack{1 \leq x,y \leq \frac{n-1}{2} \\ x \neq y \neq n-x-y \\ x \neq n-x-y}} \mathscr{P}'(x,y,n-x-y)\right],$$

where $\mathscr{P}'(x,y,n-x-y) = \bigcup_{0 \leq i \leq n-1} \mathcal{P}(\{\overline{i}, \overline{x+i}, \overline{x+y+i}\ ;\ 0\})$ and
$|\mathscr{P}'(x,y,n-x-y)| = n^2$.

5.2.2 Hyperedges of Type 2

Each hyperedge of Type 2 contains a pair of vertices from each partite set. Similar to hyperedge of Type 1, we will classify hyperedges of Type 2 depending on their properties of two pairs of vertices from two partite sets.

Definition 16. Let $\{\overline{u}, \overline{v}\} \subseteq V_1$ (or $\{u, v\} \subseteq V_2$). The *length of a pair* $\{\overline{u}, \overline{v}\}$ (or $\{u, v\}$) are $\min\{v-u, u-v\}$ where $v-u$ and $u-v$ are considered in the modulus n. Remark that since both V_1 and V_2 are $\mathbb{Z}_n$, all possible lengths are in $\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$ denoted by $\mathbb{L}$.

Definition 17. Let $e = \{\overline{x}, \overline{y} \ ; \ u, v\}$ be a hyperedge of Type 2 of $K_{n,n}^{(4)}$. Then, *pair-pattern* of e is an ordered pair (a, b) where a and b are the lengths of $\{x, y\}$ and $\{u, v\}$, respectively.

It can be noticed that hyperedges of the same pattern also have the same pair-pattern. Then, we can group hyperedges in some patterns together by their pair-patterns.

Definition 18. $\tilde{\mathscr{P}}(a, b)$ is the collection of all hyperedges with pair-pattern (a, b) of $K_{n,n}^{(4)}$.

Therefore, we can partition the set of hyperedges of Types 2 depending on its pair-pattern.

Remark 5.2.8. *Let n be an odd integer. The collection of patterns of hyperedges of Type 2,*

$$E(\mathcal{T}_2(K_{n,n}^{(4)})) = \bigcup_{a,b \in \mathbb{L}} \tilde{\mathscr{P}}(a, b)$$

where $\tilde{\mathscr{P}}(a, b) = \bigcup_{1 \le i \le n-1} \mathcal{P}(\{\overline{0}, \overline{a} \ ; \ i, i+b\})$ *and* $|\tilde{\mathscr{P}}(a, b)| = n$.

Proof. Let $\mathcal{P}(\{\overline{0}, \overline{x} \ ; \ y, z\}) \subseteq E(\mathcal{T}_3(K_{n,n}^{(4)}))$, and a and b lengths of $\{0, x\}$ and $\{y, z\}$, respectively. Obviously, $\mathcal{P}(\{\overline{0}, \overline{x} \ ; \ y, z\}) \in \cup_{a,b \in \mathbb{L}} \tilde{\mathscr{P}}(a, b)$.

Note that $\mathcal{P}(\{\overline{0}, \overline{a} \ ; \ b, c\}) \neq \mathcal{P}(\{\overline{0}, \overline{a} \ ; \ v, w\})$ if and only if $\{b, c\} \neq \{v, w\}$. Since n is odd, $\{i, i+b\} \neq \{j, j+b\}$ for all $i \neq j$. Thus, $\mathcal{P}(\{\overline{0}, \overline{a} \ ; \ i, i+b\}) \neq \mathcal{P}(\{\overline{0}, \overline{a} \ ; \ j, j+b\})$ for all $i \neq j$. Therefore, $|\tilde{\mathscr{P}}(a, b)| = n^2$. $\square$

Example 16. The collections of hyperedges of Type 2 in $K_{5,5}^{(4)}$ with pair-patterns $(2,2)$ and $(2,1)$ are the following.

$$\begin{aligned}\tilde{\mathscr{P}}(2,2) =& \mathcal{P}(\{\overline{0},\overline{2}\ ;\ 0,2\}) \cup \mathcal{P}(\{\overline{0},\overline{2}\ ;\ 1,3\}) \cup \mathcal{P}(\{\overline{0},\overline{2}\ ;\ 2,4\}) \cup \mathcal{P}(\{\overline{0},\overline{2}\ ;\ 3,0\}) \cup \\ & \mathcal{P}(\{\overline{0},\overline{2}\ ;\ 4,1\}) \text{ and} \\ \tilde{\mathscr{P}}(2,1) =& \mathcal{P}(\{\overline{0},\overline{2}\ ;\ 0,1\}) \cup \mathcal{P}(\{\overline{0},\overline{2}\ ;\ 1,2\}) \cup \mathcal{P}(\{\overline{0},\overline{2}\ ;\ 2,3\}) \cup \mathcal{P}(\{\overline{0},\overline{2}\ ;\ 3,4\}) \cup \\ & \mathcal{P}(\{\overline{0},\overline{2}\ ;\ 4,0\}).\end{aligned}$$

5.3 Initial cycles

To construct the Hamiltonian cycles in our decomposition, we first create an initial Hamiltonian cycle C with hyperedges from $2n$ distinct patterns. The set of all hyperedges with such $2n$ patterns will be partitioned into n Hamiltonian cycles resulting from rotating C n times.

Definition 19. Let C be a cycle of $K_{n,n}^{(4)}$. Then, the i^{th} *rotation* of C that is the cycle $C+i$ obtained by adding i in the modulus n to each vertex of each hyperedge e in C.

Example 17. Let $C = \{e_1 = \{\overline{0},\overline{1}\ ;\ 0,3\}, e_2 = \{\overline{1}\ ;\ 0,3,1\}, \ldots, e_9 = \{\overline{4},\overline{0}\ ;\ 2,0\}\}$ be a cycle in $K_{5,5}^{(4)}$. Then, the 2^{nd} rotation of C is

$$C+2 = \{e_1' = \{\overline{2},\overline{3}\ ;\ 2,0\}, e_2' = \{\overline{3}\ ;\ 2,0,3\}, \ldots, e_9' = \{\overline{1},\overline{2}\ ;\ 4,2\}\}.$$

Hamiltonian cycles in Lemmas 5.3.2 - 5.3.7 will be our *initial cycles.* Each initial cycle except the last one in Lemma 5.3.7 contains $2n$ hyperedges from $2n$ distinct patterns. While our initial cycle in Lemma 5.3.7 is created by a special construction for the case $n = 5$ which contain $2n$ hyperedges from the same pattern.

To claim that each initial cycle in our constructions is a Hamiltonian cycle, it suffices to show that all nodes are of size three, and all hyperedges and all nodes are distinct. Recall that in $C = (e_0, e_1, \ldots, e_{n-1})$, a node S_i is $e_i \cap e_{i+1}$ for $i \in \{0, 1, \ldots, n-1\}$ where $e_n = e_0$.

Remark 5.3.1. *Let $C = (e_0, e_1, \ldots, e_{n-1})$ be a hyperedge sequence of length n in k-uniform hypergraph. If $e_i \neq e_j$ and $S_i \neq S_j$ for $i \neq j$ and $|S_i| = k-1$ for all $i \in \{0, 1, \ldots, n-1\}$, then C is a Hamiltonian cycle.*

Proof. $(p1)-(p3)$ in Definition 3 are immediately satisfied. Let $i, j \in \{0, 1, \ldots, n-1\}$. Since $|S_i| = k-1$ and $S_i \neq S_j$ for all $i \neq j$, we have $S_i \cup S_{i+1} = e_{i+1}$. Then, $|S_i \cup S_{i+1} \cup S_{i+2}| = |e_{i+1} \cup e_{i+2}| = k+1$ because all hyperedges are distinct. However, each hyperedge contains k vertices, $(p4)$ is followed. Therefore, C is a cycle. Since C is $(k-1)$-dimensional cycle of length n, it is a Hamiltonian cycle. $\square$

In the construction, there are two kinds of initials cycles. The first one is a collection of cycles containing hyperedges of the same type which is constructed in Section 5.3.1 and the other one is a collection of cycles containing hyperedges of two types constructed in Section 5.3.2. For convenience, if $\mathscr{C}$ is a collection of cycles, then we denote $E(\mathscr{C})$ be the set of all hyperedges of all cycles in $\mathscr{C}$.

5.3.1 Initial cycles with hyperedges of the same type

Each Hamiltonian cycle in this section contains only hyperedges of the same type. We begin with a cycle using only hyperedges of Type 2 in Lemma 5.3.2. Then, we establish cycles containing only hyperedges of Type 1 and Type 3 in Lemma 5.3.3 and in Lemma 5.3.4, respectively.

Let $C_{22}(a, b, c)$ denote the cycle of $K_{n,n}^{(4)}$ with $2n$ hyperedges defined by

$$\boxed{\begin{aligned} C_{22}(a,b,c) = \{ \quad e_i &= \{\overline{0}, \overline{a}\ ;\ ci, c(i+1)\}, \\ e_{n+i} &= \{\overline{a}, \overline{a+b}\ ;\ c(n-i), c(n-1-i)\} : i \in \{0, 1, \ldots, n-1\}\}. \end{aligned}}$$

Lemma 5.3.2. *Let $a, b, c \in \mathbb{L}$, $a \neq b$ and $\gcd(c, n) = 1$. Then, $C_{22}(a, b, c)$ is a Hamiltonian cycle of $K_{n,n}^{(4)}$ with $2n$ hyperedges; one from each pattern in $\tilde{\mathscr{P}}(a, c) \cup \tilde{\mathscr{P}}(b, c)$. Moreover, if $\mathscr{C} = \{C_{22}(a, b, c) + i : i \in \{0, 1, \ldots, n-1\}\}$, then $E(\mathscr{C}) = \tilde{\mathscr{P}}(a, c) \cup \tilde{\mathscr{P}}(b, c)$.*

Proof. The nodes of $C_{22}(a,b,c)$ are the following. For $i \in \{0,1,\dots,n-\ 2\ \}$,

$$S_i = \{\overline{a},\overline{0}\ ;\ c(i+1)\},\quad S_{n+i} = \{\overline{a},\overline{b}\ ;\ c(n-1-i)\},$$
$$S_{n-1} = \{\overline{a}\ ;\ 0,c(n-1)\} \text{ and } S_{2n-1} = \{\overline{a}\ ;\ 0,c\}.$$

We have $S_{n-1} \neq S_{2n-1}$ and both of them are different from the other nodes since $\gcd(c,n)=1$ and these two nodes contain only one vertex from V_1. Since $\{\overline{a},\overline{0}\} \neq \{\overline{a},\overline{b}\}$, we have $S_i \neq S_{n+j}$ for all i,j. For $i \neq j$, since $\gcd(c,n)=1$, we have $S_i \neq S_j$ and $S_{n+i} \neq S_{n+j}$. Thus, the nodes of $C_{22}(a,b,c)$ are all distinct.

Next, we will show that all hyperedges in $C_{22}(a,b,c)$ are from distinct patterns which also implies that they are all distinct. Observe that

$$\mathcal{P}(e_i) = \mathcal{P}(\{\overline{0},\overline{a}\ ;\ ci, c(i+1)\}) \text{ and}$$
$$\mathcal{P}(e_{n+i}) = \mathcal{P}(\{\overline{0},\overline{b}\ ;\ c(n-i)-a, c(n-1-i)-a\}).$$

Since $\gcd(c,n)=1$ and $a,b,c \in \mathbb{L}$, we have

$$\bigcup_{1\le i\le n-1} \mathcal{P}(e_i) = \bigcup_{1\le i\le n-1} \mathcal{P}(\{\overline{0},\overline{a}\ ;\ i, i+c\}) = \tilde{\mathscr{P}}(a,c) \text{ and}$$
$$\bigcup_{1\le i\le n-1} \mathcal{P}(e_{n+i}) = \bigcup_{1\le i\le n-1} \mathcal{P}(\{\overline{0},\overline{b}\ ;\ i, i-c\}) = \tilde{\mathscr{P}}(b,c).$$

Hence, $e_0, e_1, \dots, e_{n-1}$ are from distinct patterns in $\tilde{\mathscr{P}}(a,c)$ and

$e_n, e_{n+1}, \dots, e_{2n}$ are from distinct patterns in $\tilde{\mathscr{P}}(b,c)$.

Since $a \neq b$, all hyperedges in $C_{22}(a,b,c)$ are distinct. By Remark 5.3.1, the cycle $C_{22}(a,b,c)$ is a Hamiltonian cycle of $K^{(4)}_{n,n}$. Let $\mathscr{C}$ be the collection of cycles resulting from the rotations of $C_{22}(a,b,c)$, $\mathscr{C} = \{C_{22}(a,b,c)+i : i \in \{0,1,\dots,n-1\}\}$. Therefore, $E(\mathscr{C}) = \tilde{\mathscr{P}}(a,c) \cup \tilde{\mathscr{P}}(b,c)$. □

Example 18. In $K^{(4)}_{5,5}$, Figure 5.1 illustrates 10 hyperedges of Type 2 in $C_{22}(1,2,2)$ with distinct patterns from $\tilde{\mathscr{P}}(1,2) \cup \tilde{\mathscr{P}}(2,2)$ and its nodes.

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9
V_1	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$
	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{3}$
V_2	0	2	4	1	3	0	3	1	4	2
	2	4	1	3	0	3	1	4	2	0

	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
V_1	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$
	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{3}$
V_2	2	4	1	3	0	3	1	4	2	0

Figure 5.1: Hyperedges of Type 2 and nodes in $C_{22}(1,2,2)$ of $K^{(4)}_{5,5}$.

Let $C_{11}(a,b)$ denote the cycle of $K^{(4)}_{n,n}$ with $2n$ hyperedges defined by

$$\boxed{\begin{aligned} C_{11}(a,b) = \{ \quad & e_{2i} = \{\overline{0}\ ;\ bi, bi+a, bi+b\}, \\ & e_{2i+1} = \{\overline{0}\ ;\ bi+a, bi+b, bi+b+a\} : i \in \{0,1,\ldots,n-1\}\}. \end{aligned}}$$

Lemma 5.3.3. *Let $0 < a < b < n$, $2a \neq n$, $2a \neq 2b$ and $\gcd(b,n) = 1$. If $a, b-a$ and $n-b$ are all distinct, then $C_{11}(a,b)$ is a Hamiltonian cycle of $K^{(4)}_{n,n}$ with $2n$ hyperedges; one from each pattern in $\mathscr{P}(a, b-a, n-b) \cup \mathscr{P}(b-a, a, n-b)$. Moreover, if $\mathscr{C} = \{C_{11}(a,b) + i : i \in \{0,1,\ldots,n-1\}\}$, then $E(\mathscr{C}) = \mathscr{P}(a, b-a, n-b) \cup \mathscr{P}(b-a, a, n-b)$.*

Proof. The nodes of $C_{11}(a,b)$ are the following. For $i \in \{0,1,\ldots,n-1\}$,

$$S_{2i} = \{\overline{0}\ ;\ bi+a, bi+b\} \text{ and } S_{2i+1} = \{\overline{0}\ ;\ bi+b, bi+b+a\}.$$

Let $i, j \in \{1,2,\ldots,n-1\}$ such that $i \neq j$. Since $2a \neq n$, we have $\{bi+b, bi+b+a\} \neq \{bj+b, bj+b+a\}$ which implies that $S_{2i+1} \neq S_{2j+1}$. Since $2a \neq 2b$ and $2a \neq b$, we have $\{bi+a, bi+b\} \neq \{bj+a, bj+b\}$ and $\{bi+a, bi+b\} \neq \{bj+b, bj+b+a\}$, respectively. These imply that $S_{2i} \neq S_{2j}$ and $S_{2i} \neq S_{2j+1}$. We also have $S_{2i} \neq S_{2i+1}$ since $0 < b < n$. Thus, the nodes of $C_{11}(a,b)$ are all distinct.

Next, we will show that all hyperedges in $C_{11}(a,b)$ are from distinct patterns which also imply that they are all distinct.

Note that

$$\begin{aligned}\mathcal{P}(e_{2i}) =& \mathcal{P}(\{\overline{0}\ ;\ bi, bi+a, bi+b\}) \text{ and}\\ \mathcal{P}(e_{2i+1}) =& \mathcal{P}(\{\overline{0}\ ;\ bi, bi+b-a, bi+b\}).\end{aligned}$$

Since $\gcd(b, n) = 1$,

$$\bigcup_{1\le i\le n-1} \mathcal{P}(e_{2i}) = \bigcup_{1\le i\le n-1} \mathcal{P}(\ \{\overline{0}\ ;\ i, i+a, i+b\}) = \mathscr{P}(a, b-a, n-b) \text{ and}$$
$$\bigcup_{1\le i\le n-1} \mathcal{P}(e_{2i+1}) = \bigcup_{1\le i\le n-1} \mathcal{P}(\{\overline{0}\ ;\ i, i+b-a, i+b\}) = \mathscr{P}(b-a, a, n-b).$$

Since $a, b-a$ and $n-b$ are all distinct, and $b-a > 0$, difference patterns $(a, b-a, n-b)$ and $(b-a, a, n-b)$ are distinct which this pair is a conjugate pair.

Hence, $e_0, e_2, \ldots, e_{2n-2}$ are from distinct patterns in $\mathscr{P}(a, b-a, n-b)$ and

$e_1, e_3, \ldots, e_{2n-1}$ are from distinct patterns in $\mathscr{P}(b-a, a, n-b)$.

Then, all hyperedges in $C_{22}(a, b, c)$ are distinct. By Remark 5.3.1, the cycle $C_{11}(a, b)$ is a Hamiltonian cycle of $K^{(4)}_{n,n}$. Let $\mathscr{C}$ be the collection of cycles resulting from the rotations of $C_{22}(a, b, c)$, $\mathscr{C} = C_{11}(a, b)$, $\{C_{11}(a, b)+i : i \in \{0, 1, \ldots, n-1\}\}$. Therefore, $E(\mathscr{C}) = \mathscr{P}(a, b-a, n-b) \cup \mathscr{P}(b-a, a, n-b)$. $\square$

Example 19. In $K^{(4)}_{7,7}$, Figure 5.2 illustrates 14 hyperedges of Type 2 in $C_{11}(1, 3)$ with distinct patterns from $\mathscr{P}(1, 2, 4) \cup \mathscr{P}(2, 1, 4)$ and its nodes.

Let $C_{33}(a, b)$ denote the cycle of $K^{(4)}_{n,n}$ with $2n$ hyperedges defined by

$$\boxed{\begin{aligned} C_{33}(a,b) = \{ \quad & e_{2i} = \{\overline{ib}, \overline{ib+a}, \overline{ib+b}\ ;\ 0\},\\ & e_{2i+1} = \{\overline{ib+a}, \overline{ib+b}, \overline{ib+b+a}\ ;\ 0\} : i \in \{0, 1, \ldots, n-1\}\}.\end{aligned}}$$

Lemma 5.3.4. *Let $0 < a < b < n$, $2a \neq n$, $2a \neq 2b$ and $\gcd(b, n) = 1$. If $a, b-a$ and $n-b$ are all distinct, then $C_{33}(a, b)$ is a Hamiltonian cycle of $K^{(4)}_{n,n}$ with $2n$ hyperedges; one from each pattern in $\mathscr{P}(a, b-a, n-b) \cup \mathscr{P}(b-a, a, n-b)$. Moreover, if $\mathscr{C} = \{C_{33}(a, b) + i : i \in \{0, 1, \ldots, n-1\}\}$, then $E(\mathscr{C}) = \mathscr{P}'(a, b-a, n-b) \cup \mathscr{P}'(b-a, a, n-b)$.*

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}
V_1	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
V_2	0	1	3	4	6	0	2	3	5	6	1	2	4	5
	1	3	4	6	0	2	3	5	6	1	2	4	5	0
	3	4	6	0	2	3	5	6	1	2	4	5	0	1

	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}
V_1	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
V_2	1	3	4	6	0	2	3	5	6	1	2	4	5	0
	3	4	6	0	2	3	5	6	1	2	4	5	0	1

Figure 5.2: Hyperedges of Type 1 and nodes in $C_{11}(1,3)$ of $K^{(4)}_{7,7}$.

5.3.2 Initial cycles with hyperedges of two types

We create an initial Hamiltonian cycle $C_{12}(a,b)$ containing hyperedges of Types 1 and 2 in Lemma 5.3.5. Then, define $C_{32}(a,b)$ containing hyperedges of Types 2 and 3 in Lemma 5.3.6. Moreover, for the case that $n = 5$, we have to construct the special initial cycle $C_{12}(a,a)(d)$ with some certain properties of in Lemma 5.3.7.

First, let $C_{12}(a,b)$ denote the cycle of $K^{(4)}_{n,n}$ with $2n$ hyperedges defined by

$$\boxed{\begin{aligned} C_{12}(a,b) = \{ \quad & e_{2i} = \{\overline{ai}\ ;\ b(i-1), bi, b(i+1)\}, \\ & e_{2i+1} = \{\overline{ai}, \overline{a(i+1)}\ ;\ bi, b(i+1)\} : i \in \{0,1,\ldots,n-1\}\}. \end{aligned}}$$

Lemma 5.3.5. *Let $a,b > 0$, $2b \neq n$ and $\gcd(a,n) = \gcd(b-a,n) = 1$. Then, $C_{12}(a,b)$ is a Hamiltonian cycle of $K^{(4)}_{n,n}$ with $2n$ hyperedges; one from each pattern in $\mathscr{P}(b,b,n-2b) \cup \tilde{\mathscr{P}}(a,b)$. Moreover, if $\mathscr{C} = \{C_{12}(a,b)+i : i \in \{0,1,\ldots,n-1\}\}$, then $E(\mathscr{C}) = \mathscr{P}(b,b,n-2b) \cup \tilde{\mathscr{P}}(a,b)$.*

Proof. The nodes of $C_{12}(a,b)$ are the following. For $i \in \{0,1,\ldots,n-1\}$,

$$S_{2i} = \{\overline{ai}\ ;\ bi, b(i+1)\} \text{ and } S_{2i+1} = \{\overline{a(i+1)}\ ;\ bi, b(i+1)\}.$$

For $i \neq j$, since $\gcd(a,n) = 1$, we have $\overline{ai} \neq \overline{aj}$ and $\overline{a(i+1)} \neq \overline{a(j+1)}$ which imply that $S_{2i} \neq S_{2j}$ and $S_{2i+1} \neq S_{2j+1}$, respectively. Since $2b \neq n$, we have

$\{bi, b(i+1)\} \neq \{bj, b(j+1)\}$ which implies that $S_{2i} \neq S_{2j+1}$ for all i, j. Thus, the nodes of $C_{12}(a,b)$ are all distinct.

Next, we will show that all hyperedges in $C_{12}(a,b)$ are from distinct patterns which also implies that they are all distinct. Observe that

$$\mathcal{P}(e_{2i}) = \mathcal{P}(\{\overline{0}\ ;\ i(b-a)-b,\ i(b-a),\ i(b-a)+b\}) \text{ and}$$
$$\mathcal{P}(e_{2i+1}) = \mathcal{P}(\{\overline{0}, \overline{a}\ ;\ \ i(b-a),\ i(b-a)+b\}).$$

Since $\gcd(b-a, n) = 1$, we have

$$\bigcup_{1\le i\le n-1} \mathcal{P}(e_{2i}) = \bigcup_{1\le i\le n-1} \mathcal{P}(\ \{\overline{0}\ ;\ i-b, i, i+b\}) = \mathscr{P}(b,b,n-2b) \text{ and}$$
$$\bigcup_{1\le i\le n-1} \mathcal{P}(e_{2i+1}) = \bigcup_{1\le i\le n-1} \mathcal{P}(\{\overline{0}, \overline{a}\ ;\ i, i+b\}) = \tilde{\mathscr{P}}(a,c).$$

Hence, $e_0, e_2, \ldots, e_{2n-2}$ have distinct patterns in $\mathscr{P}(b,b,n-2b)$ and

$e_1, e_3, \ldots, e_{2n-1}$ have distinct patterns in $\tilde{\mathscr{P}}(a,c)$.

Therefore, $C_{12}(a,b)$ is a Hamiltonian cycle of $K_{n,n}^{(4)}$ by Remark 5.3.1. Moreover, let $\mathscr{C}$ be the collection of cycles resulting from the rotations of $C_{12}(a,b)$, $\mathscr{C} = \{C_{12}(a,b)+i : i \in \{0,1,\ldots,n-1\}\}$. Therefore, $E(\mathscr{C}) = \mathscr{P}(b,b,n-2b) \cup \tilde{\mathscr{P}}(a,b)$. $\square$

Example 20. In $K_{5,5}^{(4)}$, Figure 5.3 illustrates 10 hyperedges of Type 2 in $C_{12}(1,2)$ with distinct patterns from $\mathscr{P}(2,2,1) \cup \tilde{\mathscr{P}}(1,2)$ and its nodes.

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9
V_1	$\overline{0}$	$\overline{0}$	$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{2}$	$\overline{3}$	$\overline{3}$	$\overline{4}$	$\overline{4}$
		$\overline{1}$		$\overline{2}$		$\overline{3}$		$\overline{4}$		$\overline{0}$
V_2	3		0		2		4		1	0
	0	0	2	2	4	4	1	1	3	3
	2	2	4	4	1	1	3	3	0	0

	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
V_1	$\overline{0}$	$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{2}$	$\overline{3}$	$\overline{3}$	$\overline{4}$	$\overline{4}$	$\overline{0}$
V_2	0	0	2	2	4	4	1	1	3	3
	2	2	4	4	1	1	3	3	0	0

Figure 5.3: Hyperedges of Type 1 and Type 2 and nodes in $C_{12}(1,2)$ of $K_{5,5}^{(4)}$.

Next, let $C_{32}(b,a)$ denote the cycle of $K_{n,n}^{(4)}$ with $2n$ hyperedges defined by

$$\boxed{\begin{aligned} C_{32}(b,a) = \{ \quad e_{2i} &= \{\overline{b(i-1)}, \overline{bi}, \overline{b(i+1)} \ ; \ ai\}, \\ e_{2i+1} &= \{\overline{bi}, \overline{b(i+1)} \ ; \ ai, a(i+1)\} : i \in \{0,1,\ldots,n-1\}\}. \end{aligned}}$$

Lemma 5.3.6. *Let $a, b > 0$, $2b \neq n$ and $\gcd(a,n) = \gcd(b-a,n) = 1$. Then, $C_{32}(b,a)$ is a Hamiltonian cycle of $K_{n,n}^{(4)}$ with $2n$ hyperedges; one from each pattern in $\mathscr{P}'(b,b,n-2b) \cup \tilde{\mathscr{P}}(b,a)$. Moreover, if $\mathscr{C} = \{C_{32}(b,a)+i : i \in \{0,1,\ldots,n-1\}\}$, then $E(\mathscr{C}) = \mathscr{P}(b,b,n-2b) \cup \tilde{\mathscr{P}}(b,a)$.*

Now, we will construct the special initial cycle as follows. Let $C_{12}(a,a)(d)$ denote the cycle of $K_{n,n}^{(4)}$ with $2n$ hyperedges defined by

$$\boxed{\begin{aligned} C_{12}(a,a)(d) = \{ \ e_{2i} &= \{\overline{ai} \ ; \ ai+d, a(i+1)+d, a(i+2)+d\}, \\ e_{2i+1} &= \{\overline{ai}, \overline{a(i+1)} \ ; \ a(i+1)+d, a(i+2)+d\} \\ &: i \in \{0,1,\ldots,n-1\}\}. \end{aligned}}$$

Lemma 5.3.7. *Let $a > 0$, $d \geq 0$ and $\gcd(a,n) = 1$. Then, $C_{12}(a,a)$ is a Hamiltonian cycle of $K_{n,n}^{(4)}$ with all $2n$ hyperedges from two patterns $\mathcal{P}(\{\overline{0} \ ; \ d, d+a, d+2a\})$ and $\mathcal{P}(\{\overline{0}, \overline{a} \ ; \ d+a, d+2a\})$. Moreover, if $\mathscr{C} = \{C_{12}(a,a)(d) : d \in \{0,1,\ldots,n-1\}\}$, then $E(\mathscr{C}) = \mathscr{P}(a,a,n-2a) \cup \tilde{\mathscr{P}}(a,a)$.*

Proof. The nodes of $C_{12}(a,b)$ are the following. For $i \in \{0,1,\ldots,n-1\}$,

$$S_{2i} = \{\overline{ai} \ ; \ a(i+1)+d, a(i+2)+d\} \text{ and } S_{2i+1} = \{\overline{a(i+1)} \ ; \ a(i+1)+d, a(i+2)+d\}.$$

For all $i \neq j$, since $\gcd(a,n) = 1$, we have $ai \neq aj$ and $\{a(i+1)+d, a(i+2)+d\} \neq \{a(j+1)+d, a(j+2)+d\}$ which imply that $S_{2i} \neq S_{2j}$, $S_{2i+1} \neq S_{2j+1}$ and $S_{2i} \neq S_{2j+1}$. We also have $S_{2i} \neq S_{2i+1}$ since $\overline{ai} \neq \overline{a(i+1)}$.

Next, we will shows that all hyperedges in $C_{12}(a,a)(d)$ are distinct hyperedges in two patterns. For $i \neq j$, since $\gcd(a,n) = 1$, we have $\overline{ai} \neq \overline{aj}$ and $\{\overline{ai}, \overline{a(i+1)}\} \neq \{\overline{aj}, \overline{a(j+1)}\}$ which yield $e_{2i} \neq e_{2j}$ and $e_{2i+1} \neq e_{2j+1}$. Hence, all hyperedges and all nodes are distinct. Therefore, $C_{12}(a,a)(d)$ is a Hamiltonian cycle of $K_{n,n}^{(4)}$ by

Remark 5.3.1. Observe that

$$\mathcal{P}(e_{2i}) = \mathcal{P}(\{\overline{0}\ ;\ d, d+a, d+2a\}) \text{ and } \mathcal{P}(e_{2i+1}) = \mathcal{P}(\{\overline{0}, \overline{a}\ ;\ d+a, d+2a\})$$

for all $i \in \{0, 1, \ldots, n-1\}$.
Hence, $e_0, e_2, \ldots, e_{2n-2}$ are all n hyperedges in the pattern $\mathcal{P}(\{\overline{0}\ ;\ d, d+a, d+2a\})$ and $e_1, e_3, \ldots, e_{2n-1}$ are all n hyperedges in the pattern $\mathcal{P}(\{\overline{0}, \overline{a}\ ;\ d+a, d+2a\})$.

Consequently, let $\mathscr{C}$ be the collection of cycles $\mathscr{C} = \{C_{12}(a,a)(d) : d \in \{0, 1, \ldots, n-1\}\}$. Therefore, $E(\mathscr{C}) = \mathscr{P}(a, a, n-2a) \cup \tilde{\mathscr{P}}(a, a)$. $\square$

Example 21. In $K_{5,5}^{(4)}$, Figure 5.4 illustrates 10 hyperedges of Types 1 and 2 in $C_{12}(1,1)(3)$ with two patterns $\mathcal{P}(\{\overline{0}\ ;\ 3, 4, 0\})$ and $\mathcal{P}(\{\overline{0}, \overline{1}\ ;\ 4, 0\})$ and nodes.

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9
V_1	$\overline{0}$	$\overline{0}$	$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{2}$	$\overline{3}$	$\overline{3}$	$\overline{4}$	$\overline{4}$
		$\overline{1}$		$\overline{2}$		$\overline{3}$		$\overline{4}$		$\overline{0}$
V_2	3		4		0		1		2	
	4	4	0	0	1	1	2	2	3	3
	0	0	1	1	2	2	3	3	0	0

	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
V_1	$\overline{0}$	$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{2}$	$\overline{3}$	$\overline{3}$	$\overline{4}$	$\overline{4}$	$\overline{0}$
V_2	4	4	0	0	1	1	2	2	3	3
	0	0	1	1	2	2	3	3	0	0

Figure 5.4: Hyperedges of Type 1 and Type 2 in $C_{12}(1,1)(3)$ of $K_{5,5}^{(4)}$ and its nodes.

We summarize Lemmas 5.3.3-5.3.7 in Table 5.1. The table shows the conditions of the parameters to construct each initial cycle along with collections of hyperedges that have the same patterns as hyperedges used to construct each initial cycle.

5.4 Main Theorem

We will construct a Hamiltonian decomposition of $K_{n,n}^{(4)}$ when n is prime and $n \equiv 0, 1, 5 \pmod 8$ using the initial cycles in the previous sections. The construction in the case $n = 5$ is different from the others; thus, we first decompose $K_{5,5}^{(4)}$ into Hamiltonian cycles separately.

Theorem 5.4.1. *There exists a Hamiltonian decomposition of* $K_{5,5}^{(4)}$.

Lemma	Cycles (C)	Conditions	Collection of hyperedges containing hyperedges in C
5.3.2	$C_{22}(a,b,c)$	$a,b,c \in \mathbb{L}, a \neq b$ and $\gcd(c,n)=1$	$\tilde{\mathscr{P}}(a,c) \cup \tilde{\mathscr{P}}(b,c)$
5.3.3	$C_{11}(a,b)$	$0<a<b<n$, $2a \neq n$, $2a \neq 2b$, $\gcd(b,n)=1$, and $a, b-a$ and $n-b$ are all distinct	$\mathscr{P}(a,b-a,n-b) \cup \mathscr{P}(b-a,a,n-b)$
5.3.4	$C_{33}(a,b)$		$\mathscr{P}'(a,b-a,n-b) \cup \mathscr{P}'(b-a,a,n-b)$
5.3.5	$C_{12}(a,b)$	$a,b>0$, $b \neq \frac{n}{2}$ and $\gcd(a,n)=\gcd(b-a,n)=1$	$\mathscr{P}(b,b,n-2b) \cup \tilde{\mathscr{P}}(a,b)$
5.3.6	$C_{32}(a,b)$		$\mathscr{P}'(b,b,n-2b) \cup \tilde{\mathscr{P}}(a,b)$
5.3.7	$C_{12}(a,a)(d)$	$a>0, d \geq 0, gcd(a,n)=1$	$\mathcal{P}(\{\overline{0}\ ;\ d,d+a,d+2a\}) \cup \mathcal{P}(\{\overline{0},\overline{a}\ ;\ d+a,d+2a\})$

Table 5.1: Initial cycles.

Proof. First, observe that in $K^{(4)}_{5,5}$, there are exactly two distinct difference patterns of hyperedges of Type 1 (and Type 3) which are $(1,1,2)$ and $(2,2,1)$. Also, there are exactly four pair-patterns of hyperedge of Type 2 which are $(1,1),(2,2),(1,2)$ and $(2,1)$. Then, we construct collections of cycles,

$$\mathscr{C}_1 = \{C_{32}(1,2)+i : i \in \{0,1,2,3,4\}\}, \mathscr{C}_2 = \{C_{32}(2,1)+i : i \in \{0,1,2,3,4\}\},$$

$$\mathscr{C}_3 = \{C_{12}(1,1)(d) : d \in \{0,1,2,3,4\}\} \text{ and } \mathscr{C}_4 = \{C_{12}(2,2)(d) : d \in \{0,1,2,3,4\}\}$$

By Lemma 5.3.6, $E(\mathscr{C}_1) = \mathscr{P}'(1,1,2) \cup \tilde{\mathscr{P}}(1,2)$ and $E(\mathscr{C}_2) = \mathscr{P}'(2,2,1) \cup \tilde{\mathscr{P}}(2,1)$. By Lemma 5.3.7, $E(\mathscr{C}_3) = \mathscr{P}(1,1,2) \cup \tilde{\mathscr{P}}(1,1)$ and $E(\mathscr{C}_4) = \mathscr{P}(2,2,1) \cup \tilde{\mathscr{P}}(2,2)$.

Let $\mathscr{C} = \mathscr{C}_1 \cup \mathscr{C}_2 \cup \mathscr{C}_3 \cup \mathscr{C}_4$. Therefore, $\mathscr{C}$ is a Hamiltonian decomposition of $K^{(4)}_{5,5}$. □

Next, for feasible prime $n \geq 5$, we will construct a Hamiltonian decomposition of $K^{(4)}_{n,n}$ as follows.

Theorem 5.4.2. *Let n be a prime such that $n \equiv 1 \pmod 4$ and $n \geq 5$. There exists a Hamiltonian decomposition of $K^{(4)}_{n,n}$.*

Proof. When $n = 5$ the statement holds by Theorem 5.4.1. Now, let $n \geq 13$.

In $K^{(4)}_{n,n}$, let $\overline{\mathcal{I}}_1$ (or $\overline{\mathcal{I}}_3$) be the subhypergraph consisting of all non-isosceles hyperedges of Type 1 (or $\mathcal{I}_3$), $\mathcal{I}_1$ (or $\mathcal{I}_3$) the subhypergraph consisting of all isosceles hyperedges of Type 1 (or Type 3). Then, $\mathcal{T}_1(K^{(4)}_{n,n}) = \overline{\mathcal{I}}_1 \cup \mathcal{I}_1$ and $\mathcal{T}_3(K^{(4)}_{n,n}) = \overline{\mathcal{I}}_3 \cup \mathcal{I}_3$.

Note first that

$$K^{(4)}_{5,5} = \overline{\mathcal{I}}_1 \cup \mathcal{I}_1 \cup \mathcal{T}_2(K^{(4)}_{n,n}) \cup \overline{\mathcal{I}}_3 \cup \mathcal{I}_3.$$

Claim 1 : $\overline{\mathcal{I}}_1 \cup \overline{\mathcal{I}}_3$ has a Hamiltonian decomposition.

By Remark 5.2.6, we have that

$$E(\overline{\mathcal{I}}_1) = \bigcup_{\substack{1\le x,y\le \frac{n-1}{2} \\ x\ne y\ne n-x-y}} \mathscr{P}(x,y,n-x-y) \text{ and } E(\overline{\mathcal{I}}_3) = \bigcup_{\substack{1\le x,y\le \frac{n-1}{2} \\ x\ne y\ne n-x-y}} \mathscr{P}'(x,y,n-x-y).$$

Now, let $x,y \in \{1,2,\ldots,\frac{n-1}{2}\}$ where x,y and $n-x-y$ are all distinct. Let

$$\mathscr{C}_1(x,y) = \{C_{11}(x,x+y)+j : j \in \{0,1,\ldots,n-1\}\}.$$

Note that $0 < x < x+y < n$. Since n is a prime number, we have that $2x \ne n$ and $2x \ne 2(x+y)$ in the modulus n, and $\gcd(x+y,n)=1$. Thus, x and $x+y$ satisfy the requirements in Lemma 5.3.3; so, $E(\mathscr{C}_1(x,y)) = \mathscr{P}(x,y,n-x-y) \cup \mathscr{P}(y,x,n-x-y)$. Thus, $\overline{\mathcal{I}}_1$ has a Hamiltonian decomposition.

Similarly, let $\mathscr{C}_2(x,y) = \{C_{33}(x,x+y)+j : j \in \{0,1,\ldots,n-1\}\}$. By Lemma 5.3.4, $E(\mathscr{C}_2(x,y)) = \mathscr{P}(x,y,n-x-y) \cup \mathscr{P}(y,x,n-x-y)$. Therefore, $\overline{\mathcal{I}}_3$ has a Hamiltonian decomposition.

Claim 2 : $\mathcal{I}_1 \cup \mathcal{I}_3 \cup \mathcal{T}_2(K^{(4)}_{n,n})$ has a Hamiltonian decomposition.

By Remark 5.2.6, we have that

$$E(\mathcal{I}_1) = \bigcup_{1\le a\le \frac{n-1}{2}} \mathscr{P}(a,a,n-2a) \text{ and } E(\mathcal{I}_3) = \bigcup_{1\le a\le \frac{n-1}{2}} \mathscr{P}'(a,a,n-2a).$$

First, we will construct a Hamiltonian decomposition of $\mathcal{I}_1 \cup \mathcal{I}_3 \cup \mathcal{I}'_2$ where $\mathcal{I}'_2$ is the subhypergraph consists of hyperedges of Type 2 in

$$\bigcup_{a\in\mathbb{L}} \tilde{\mathscr{P}}(a-1,a) \cup \bigcup_{a\in\mathbb{L}} \tilde{\mathscr{P}}(a-2,a)$$

such that $a-1$ and $a-2$ are consider in the modulus $\frac{n-1}{2}$.

Let $a \in \{1,2,\ldots,\frac{n-1}{2}\}$ and

$$\mathscr{C}_3(a) = \{C_{12}(a-1,a)+j : j \in \{0,1,\ldots,n-1\}\},$$

where $C_{12}(0,1) = C_{12}(\frac{n-1}{2},1)$.

Since n is prime and $0 < a < \frac{n-1}{2}$, we have that $2a \neq n$ and $\gcd(a,n) = 1$. These facts satisfy conditions in Lemma 5.3.5. Then, $E(\mathscr{C}_3(a)) = \tilde{\mathscr{P}}(a-1,a) \cup \mathscr{P}(a,a,n-2a)$.

Similarly, let $\mathscr{C}_4(a) = \{C_{32}(a-2,a)+j : j \in \{0,1,\ldots,n-1\}\}$, where $C_{32}(0,2) = C_{32}(\frac{n-1}{2},2)$ and $C_{32}(-1,1) = C_{32}(\frac{n-1}{2}-1,1)$. By Lemma 5.3.6, $E(\mathscr{C}_4(a)) = \tilde{\mathscr{P}}(a-2,a) \cup \mathscr{P}'(a,a,n-2a)$. Hence, we can decompose $\mathcal{I}_1 \cup \mathcal{I}_3 \cup \mathcal{I}_2'$ into Hamiltonian cycles. It remains to construct a Hamiltonian decomposition of $\mathcal{T}_2(K_{n,n}^{(4)}) \setminus \mathcal{I}_2'$.

By Remark 5.2.8, the hyperedge set of $\mathcal{T}_2(K_{n,n}^{(4)})$ can be written as

$$E(\mathcal{T}_2(K_{n,n}^{(4)})) = \bigcup_{a,b\in\mathbb{L}} \tilde{\mathscr{P}}(a,b) = \bigcup_{a\in\mathbb{L}} \left(\bigcup_{0\le i\le \frac{n-5}{4}} (\tilde{\mathscr{P}}(a+2i,a) \cup \tilde{\mathscr{P}}(a+2i+1,a)) \right)$$

where $a+2i$ and $a+2i+1$ are consider in the modulus $\frac{n-1}{2}$. Note that

$$E(\mathcal{I}_2') = \bigcup_{a\in\mathbb{L}} (\tilde{\mathscr{P}}(a-1,a) \cup \tilde{\mathscr{P}}(a-2,a))$$

and $\tilde{\mathscr{P}}(a-1,a)$ and $\tilde{\mathscr{P}}(a-2,a)$ equal to $\tilde{\mathscr{P}}(a+2i,a)$ and $\tilde{\mathscr{P}}(a+2i+1,a)$ when $i = \frac{n-5}{4}$, respectively. Note that $\frac{n-5}{4}$ is an integer since $n \equiv 1 \pmod 4$.

Finally, the left over of hyperedges of Type 2 in $\mathcal{T}_2(K_{n,n}^{(4)}) \setminus \mathcal{I}_2'$ will be form the

Hamiltonian cycles by Lemma 5.3.2. Let $a \in \mathbb{L}$, $i \in \{0, 1, \ldots, \frac{n-5}{4}\}, i \neq \frac{n-5}{4}$, and

$$\mathscr{C}_5(a) = \{C_{22}(a+2i, a+2i+1, a) + j : j \in \{0, 1, \ldots, n-1\}\}$$

where $a+2i$ and $a+2i+1$ are consider in the modulus $\frac{n-1}{2}$. Since $a+2i, a+2i+1 \in \mathbb{L}$ and $\gcd(a, n) = 1$, by Lemma 5.3.2, all cycles in $E(\mathscr{C}_5(a)) = \tilde{\mathscr{P}}(a+2i, a) \cup \tilde{\mathscr{P}}(a+2i+1, a)$. Hence, our claim is proved.

Therefore, $K_{n,n}^{(4)}$ has a Hamiltonian decomposition. $\square$

CHAPTER VI
CONCLUSIONS AND OPEN PROBLEMS

6.1 Conclusions

In this dissertation, we establish KK-Hamiltonian decompositions and WJ-Hamiltonian decompositions of uniform hypergraphs.

The results of KK-Hamiltonian decompositions are in Chapters III and IV. Theorem A and Theorem 3.5.1 in Chapter III, and Theorem 4.1.1 in Chapter IV can be combined to the following Theorems.

Theorem 6.1.1. *Let $n \geq 2$ and t be a positive integer such that $t \equiv 4, 8 \pmod{12}$. If $K_t^{(3)}$ has a KK-Hamiltonian decomposition, then $K_{t(n)}^{(3)}$ has a KK-Hamiltonian decomposition.*

Theorem 6.1.2. *Let $t \equiv 4, 8 \pmod{12}$. If $K_t^{(3)}$ has a KK-Hamiltonian decomposition, then $K_{2t}^{(3)}$ has a KK-Hamiltonian decomposition.*

Therefore, our construction method in Theorem 6.1.2 yields infinitely many results for $K_{2t}^{(3)}$ from the current results of KK-Hamiltonian decompositions of $K_t^{(3)}$. The studies of the existence problem of KK-Hamiltonian decompositions of $K_t^{(3)}$ were completed for feasible t when $3 \leq t \leq 46$, $t \neq 43$, $t = 2^m$ and $m \geq 2$ in [2, 13, 18, 10] which are collected in Chapter II.

Corollary 6.1.3. *$K_t^{(3)}$ has a Hamiltonian decomposition when $t = 2^m, 5 \cdot 2^m, 7 \cdot 2^m$ and $11 \cdot 2^m$ and $m \geq 2$.*

Proof. Let $m \geq 2$. By Theorems 2.2.4 and 2.2.6, $K_t^{(3)}$ has a Hamiltonian decomposition when $t = 4, 20, 28$ and 44. Therefore our recursive construction in Theorem 6.1.2 confirms that $K_t^{(3)}$ also has a Hamiltonian decomposition when $t = 2^m, 5 \cdot 2^m, 7 \cdot 2^m$ and $11 \cdot 2^m$. $\square$

Thus, the construction in Theorem 6.1.1 together with Corollary 6.1.3 yield infinitely many results for $K^{(3)}_{t(n)}$ as follows.

Corollary 6.1.4. *Let $n \geq 2$. $K^{(3)}_{t(n)}$ has a Hamiltonian decomposition when $t = 2^m, 5 \cdot 2^m, 7 \cdot 2^m$ and $11 \cdot 2^m$ and $m \geq 2$.*

Furthermore, Bailey and Stevens [2] observed that the results for $K^{(k)}_n$ also yield the results for $K^{(n-k)}_n$ due to complementary (see details in [2]). Similarly, since the complement of each hyperedge of $K^{(3)}_{t(n)}$ is not a subset of any partite sets of $K^{(3)}_{t(n)}$, a collection of complement of hyperedges of $K^{(3)}_{t(n)}$ is the hyperedge set of $K^{(tn-3)}_{t(n)}$. Hence the results for $K^{(3)}_{t(n)}$ also yield the results for $K^{(tn-3)}_{t(n)}$ as follows.

Corollary 6.1.5. *$K^{(t-3)}_t$ has a Hamiltonian decomposition when $t = 2^m, 5{\cdot}2^m, 7{\cdot}2^m$ and $11 \cdot 2^m$ and $m \geq 2$.*

Corollary 6.1.6. *Let $n \geq 2$. $K^{(tn-3)}_{t(n)}$ has a Hamiltonian decomposition when $t = 2^m, 5 \cdot 2^m, 7 \cdot 2^m$ and $11 \cdot 2^m$ and $m \geq 2$.*

Finally, in Chapter V, we provide the result for 4-uniform hypergraphs using properties of their hyperedges. We construct a WJ-Hamiltonian decomposition of complete bipartite 4-uniform hypergraph $K^{(3)}_{n,n}$ where n is a prime number which each cycle in the construction is neither Berge-Hamiltonian cycle nor KK-Hamiltonian cycle in the following theorem.

Theorem 6.1.7. *Let n be a prime such that $n \equiv 1 \pmod 4$ and $n \geq 5$. There exists a WJ-Hamiltonian decomposition of $K^{(4)}_{n,n}$.*

6.2 Open problems

Several open problems concerning our work are the following.

1. The existence of a KK-Hamiltonian decomposition of $K^{(3)}_{t(n)}$ when $t \equiv 2 \pmod 4$ and $n \geq 2$. Since Theorems C, D, 4.2 and 4.3 can construct a KK-Hamiltonian decomposition of $\mathcal{T}_2(K^{(3)}_{t(n)})$, it remains to show that $\mathcal{T}_1(K^{(3)}_{t(n)})$ has a Hamiltonian decomposition to complete the problem.

2. The existence of a KK-Hamiltonian decomposition of $K_{t(n)}^{(3)}$ when $t \geq 5$ and $n \equiv 0 \pmod 3$.

3. The existence of a WJ-Hamiltonian decomposition of a complete 4-uniform bipartite hypergraph $K_{n,n}^{(4)}$ for all $n \geq 3$.

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