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VALUATION OF AMERICAN COMMODITY OPTIONS

Mr. Piyapoom Nonsoong

จุฬาลงกรณ์มหาวิทยาลัย
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วิทยานิพนธ์นี้ นำเสนอสูตรการกำหนดราคาตราสารสิทธิเชิงวิเคราะห์สำหรับตราสารสิทธิแบบยุโรปและตราสารสิทธิแบบอเมริกา ซึ่งการเปลี่ยนแปลงราคาของสินทรัพย์เสี่ยงเป็นไปตามกระบวนการสุ่มแบบกลับสู่ค่าเฉลี่ยด้วยพารามิเตอร์ที่ขึ้นกับเวลา โดยกระบวนการนี้ สามารถนำไปปรับใช้เพื่ออธิบายความผันแปรของราคาทั้งแบบตามฤดูกาลและไม่ตามฤดูกาลได้ โดยเฉพาะในตลาดสินค้าโภคภัณฑ์ เช่น สินค้าเกษตร สูตรเหล่านี้ ได้มาจากผลเฉลยของสมการเชิงอนุพันธ์ย่อย แสดงให้เห็นว่ามูลค่าตราสารสิทธิแบบยุโรปและตราสารสิทธิแบบอเมริกาสามารถแบ่งออกเป็นสองส่วน คือ ผลตอบแทนของตราสารสิทธิ ณ เวลาเริ่มต้น และปริพันธ์เวลาตลอดอายุของตราสารสิทธิ ซึ่งขับเคลื่อนโดยพารามิเตอร์ที่ขึ้นกับเวลาและเพิ่มเติมสำหรับตราสารสิทธิแบบอเมริกายังขึ้นอยู่กับค่าขอบเขตการใช้สิทธิที่เหมาะสมด้วย สุดท้าย ผลลัพธ์เชิงตัวเลขและระยะเวลาในการคำนวณซึ่งได้จากสูตรตราสารสิทธิแบบยุโรปภายใต้ฟังก์ชันค่าเฉลี่ยระยะยาวประเภทต่าง ๆ ได้ถูกนำมาเปรียบเทียบกับการจำลองแบบมอนติคาร์โลและสูตรประเภทแบล็คโชลส์ที่ได้จากวิธีทางด้านความน่าจะเป็น นอกจากนี้ ตัวอย่างพฤติกรรมราคาของตราสารสิทธิภายใต้ฟังก์ชันที่ขึ้นกับเวลาเหล่านี้ยังได้ถูกแสดงให้เห็นภาพ

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In this dissertation, we present analytical option pricing formulas for European and American options in which the price dynamics of a risky asset follows a mean-reverting process with time-dependent parameter. The process can be adapted to describe both nonseasonal and seasonal variation in price, especially, in commodity markets such as agricultural commodities. The formulas are derived based on the solutions of partial differential equations showing that the values of both European and American options can be decomposed into two parts: the payoff of the option at initial time and the time-integral over the lifetime of the option, which is driven by the time-dependent parameter, and in addition, by the optimal exercise boundary for the American option. Finally, numerical results and computational time obtained from the European option formula under various kinds of long-run mean functions have been compared with Monte Carlo simulations and Black-Scholes-type formula obtained via probability approach. Moreover, examples of the option price behaviors under these time-dependent functions have been illustrated.

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CHAPTER I

INTRODUCTION

The commodities sector obviously plays an important role for the economy and the development of countries, especially, those depending on farming, producing, manufacturing and exporting commodities. Furthermore, most businesses commonly involve commodity someway. The commodity price risk then is an important factor for economic growth, development, as well as food and energy stability.

There have been several studies involving commodity price models for many decades. In 1976, Black [2] proposed a simple commodity price model satisfying the geometric Brownian motion (GBM) similar to the well-known model for stock price, Black-Scholes model [4]. Since the price behaviors of most commodities are known to exhibit mean reversion, such as agricultures, livestocks, energy and manufactured metal [13], the models for commodity price having mean reversion property were presented, for example, Pilipovic model [26] and one-factor Schwartz (SC) model [27]. For SC model as well as two- and three-factor models, Schwartz showed that they were suitable for the empirical price data of crude oil and copper, and have become standard models for commodities.

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In this study, we consider an extended one-factor Schwartz (ESC) model to describe the mean-reverting commodity price, namely, a mean-reverting process $(S_t)_{t\geq 0}$ under a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ described by the stochastic differential equation (SDE)

$$\mathrm{d}S_t = \kappa(\mu(t) - \ln S_t)S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t, \tag{1.1}$$

where $\kappa > 0$ represents the speed of reversion, $\mu(\cdot)$ represents the long-run mean function, σ represents the volatility of the process and $(W_t)_{t\geq 0}$ is a standard Brownian motion.

The process satisfying model (1.1) generally represents the spot price of asset that exhibit mean reversion with both seasonal and nonseasonal behaviors, especially, commodities and interest rate. In the case of constant long-run mean functions, the model becomes SC model which can describe the price of nonseasonal mean-reverting commodities. For a more general case, the seasonality of commodity prices can be demonstrated by a periodic time-dependent long-run mean function $\mu(\cdot)$ in the model. Furthermore, the model (1.1) was also used for the short-term interest rate called Black-Karasinski model [3] or extended exponential Vasicek model [6], and the logarithm of the process following (1.1) also satisfies the Hull-White interest rate model [14].

To hedge commodity price risks, financial derivatives such as futures and options are used as instruments to prevent fluctuations for market practitioners such as risk managers, investors and farmers. One of the most popular financial derivatives used for hedging the risks from price fluctuations is an option [13], a financial contract which gives the buyer the right, but not the obligation, to buy or sell an underlying asset for a predetermined price before expiration. There are two option's styles being used and investigated most, European and American. European options give the right to exercise only on the expiration date, while American options can be exercised at any time until the expiration date.

To enter a long or short position for an option, a premium or an option value must be determined under a particular assumption known as the arbitrage-free condition. In other words, the price of an option must be fair for both a seller and a buyer. Consequently, the determination of the price for both options is an

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important problem for researchers in the field of economics, including mathematics (see [4] and [22] for European option and [7, 20] and [30] for American option in the case that stock being underlying of option.)

For the case of commodity, there are researches for the option written on futures, which is an option that gives the right, but not the obligation, to enter into a futures contract written on a commodity at a certain price before expiration, since most of the trading commodities takes place in the form of futures contracts (see [1, 2, 8, 9] and [13] for more details). However, since futures price is the expectation of commodity price in the future, that depends on the commodity spot price, it is possible to study the option on commodity directly (see [12] and [26] for example). In addition, the option on futures and the option on commodity are equivalent when their expiration date are the same. For example, Swishchuk [29] studied the option pricing formula when the underlying is commodity price under Pilipovic model.

The fair price of European options is known to be the expected present value of its payoff. To approximate this value, there are some basic methods such as multinomial tree models and Monte Carlo simulations [13]. However, these numerical approximations usually take much time for computations and contain some errors from approximation. To obtain an explicit formula for European option value, the analytical methods such as the probabilistic approach and the partial differential equation (PDE) approach can be used. The most famous formula concerning the valuation of European options on stocks based on a PDE approach was derived by Black and Scholes [4], called Black-Scholes formula, in which the stock prices follow a GBM. Also, a probabilistic approach can be used for this case to obtain the same result [24].

Under model (1.1), the solution in log form satisfying Hull-White model, similar to GBM, has log-normal distribution [28]. One can directly use probabilistic approach via its probability density function to derive a closed-form formula for European option in this case. In this way, the obtained formula is similar to Black-Scholes one, where the more general mean and variance depend on the time to expire; we refer to this resulted formula as Black-Scholes-type (BS-type) formula.

For the case of American options, contrary to European type, the problem becomes more complicated since the exercise time of the contract is a random time. Although the probabilistic approach is effective for European type and the derivation is not complicated, it is not easy to apply and extend the idea to American option, since the exercise time is not exactly known [15]. It turns into the system of PDE involving an exercise boundary varying in time that will be selected to maximize the expected payoff of the option to obtain the premiums. Under the GBM setting, varieties of analytical formulas, including numerical solutions for pricing American options have been proposed by many researchers. For American option on stocks, the reader can see Kim [20], Underwood and Wang [30] and Carr et al. [7] for more details.

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Hence, one powerful method that can handle both option types is PDE approach, where the problem of expectation is transformed into a PDE problem via the Feynman-Kac theorem [24]. Under the model (1.1), the partial differential operator (PDO) involving both European and American options on commodity is

$$\mathcal{L} := \frac{\partial}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2}{\partial S^2} + \kappa(\mu(t) - \ln S)S\frac{\partial}{\partial S} - r.$$

To obtain the formula for both options, one may solve the system of PDEs corresponding to the above PDO, together with some conditions on the option value function directly.

Alternatively, since the process S_t has the Markov property, Nunes [23] showed that the American option price for this case can be decomposed into the sum of the European option value counterpart and the early exercise premium (EEP). Hence, we can obtain the formula of American option value from this decomposition with the solution of the system of PDE for European option value and EEP instead.

In this dissertation, we propose a method for solving the system of PDEs for European option value and EEP in order to derive an analytical pricing formula for both European options and American options on commodity whose prices follow model (1.1). To solve the system of PDEs, we separate the solution into two parts and apply the Fourier transform and the method of characteristic curve to obtain the solutions. Both obtained formulas can be expressed as the sum of the initial payoff of option and the time-integral over the lifetime of option, where the integrand of latter term for American option also depends on the optimal exercise boundary function.

The PDE approach developed in this dissertation has two main advantages: (i) our technique can be easily applied and modified to both European option and American option; and (ii) the decomposition of solution can be used to approximate both option prices using the known initial payoff and the approximation of the integral term.

The rest of this dissertation is organized as follows. Chapter 2 provides some background knowledge and useful results. Chapter 3 presents the details on deriving analytical formulas for pricing European options and American options on commodity under model (1.1). Chapter 4 gives the derivation of BS-type formula and demonstrates numerical results of the European option values computed from the obtained formula under various kinds of long-run mean functions. The comparisons among our results, Monte Carlo simulations and BS-type formula and the behaviors of European option prices are also illustrated in this chapter. The conclusion is given in Chapter 5.

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CHAPTER II

BACKGROUND KNOWLEDGE

In this chapter, we introduce European and American options and the methods for pricing with some examples based on the Black-Scholes model. In addition, some useful definitions, results and mathematical tools for stochastic process and PDEs related to this work are also provided.

This chapter composes of 4 sections: Itô Process, European and American Options and Option Pricing, Option Pricing Methods and Tools for PDEs.

2.1 Itô Process

Itô process is a type of stochastic process based on the Itô stochastic integration with respect to Brownian motion. In financial mathematics, it has been widely used to model a continuous-time price process for describing asset prices such as stock, commodity and interest rate.

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Definition 2.1. [24] An **Itô process** is a stochastic process $(X_t)_{t\geq 0}$ (or simply, X_t) that can be represented by

$$X_t = X_0 + \int_0^t \mu(s, X_s)\, ds + \int_0^t \sigma(s, X_s) dW_s, \tag{2.1}$$

where the term $\int_0^t \sigma(s, X_s) dW_s$ is known as the **Itô integral** with $(W_t)_{t\geq 0}$ denotes a standard Brownian motion and μ and σ are deterministic functions called the **drift** and the **diffusion terms**, respectively.

The equation (2.1) can be written as a **stochastic differential equation** (SDE) in the form

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t. \tag{2.2}$$

The following examples are some simple one-factor models in the form of SDE for Itô processes used for describing stock and commodity prices.

Example 2.2. Let r, μ, κ, δ and σ be constants and S_t be an Itô processes describing asset price at time t.

1. Black-Scholes (BS) model [2] or Black model [4]

$$dS_t = (r - \delta)S_t\, dt + \sigma S_t\, dW_t. \tag{2.3}$$

2. Pilipovic model [26]

$$dS_t = \kappa(\mu - S_t)\, dt + \sigma S_t\, dW_t.$$

3. Schwartz model [27]

$$dS_t = \kappa(\mu - \ln S_t)S_t\, dt + \sigma S_t\, dW_t.$$

An important result for the study of Itô processes is Itô's formula as stated in the following theorem.

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Theorem 2.3 (Itô formula)**.** *[24] Let X_t be an Itô process given by* (2.2)*. Suppose that $f(t,x) \in C^{1,2}([0,\infty) \times \mathbb{R})$. Then, $f(t, X_t)$ is an Itô process and*

$$\begin{aligned} d\, f(t, X_t) = &\left(\frac{\partial f}{\partial t}(t, X_t) + \mu(t, X_t)\frac{\partial f}{\partial x}(t, X_t) + \frac{1}{2}\sigma^2(t, X_t)\frac{\partial^2 f}{\partial x^2}(t, X_t) \right) dt \\ &+ \sigma(t, X_t)\frac{\partial f}{\partial x}(t, X_t)\, dW_t. \end{aligned} \tag{2.4}$$

Example 2.4. Let S_t be an Itô process describe by the BS model (2.3). By applying Itô formula (2.4) to S_t with $f(x) = \ln x$, we have

$$d \ln S_t = \left(r - \delta - \frac{\sigma^2}{2} \right) dt + \sigma\, dW_t. \tag{2.5}$$

2.2 European and American Options and Option Pricing

Option [13] is a financial contract which gives the buyer the right, but not the obligation, to buy or sell an underlying asset for a predetermined price before expiration. The option that gives the right to buy (sell) is called a **call** (**put**) option; the predetermined price for underlying asset is called the **strike price** or the **exercise price**; the last date on which option can be used or **exercised** is called the **expiration date**. The study in this dissertation considers only two styles of options, the European and American: **European** option can be exercised only on the expiration date, while **American** option can be used any time until the expiration date.

Assume that the spot prices of an underlying asset can be described by an Itô process $(S_t)_{t\geq 0}$ on a probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by the process, where $\mathcal{F}_t$ can be considered as information given up to the present time t. Given a strike price K and an expiration date T, let $v(S,t;\phi)$ and $V(S,t;\phi)$ be the values of European and American options, respectively, on the underlying asset price S at time $t \leq T$, where $\phi = -1$ for call option and $\phi = 1$ for put option.

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Assume that the risk-free interest rate is a constant r. Under the risk-neutral probability measure, the fair price of European option [18] is

$$v(S,t;\phi) = e^{-r(T-t)}\mathbb{E}\left[(\phi K - \phi S_T)^+ \mid \mathcal{F}_t\right], \tag{2.6}$$

where $\mathbb{E}[\,\cdot\,|\,\cdot\,]$ denotes the conditional expectation and $(\cdot)^+ = \max(\cdot, 0)$ is the positive part and the fair price of American option [17] is

$$V(S,t;\phi) = \sup_{\tau\in\mathcal{T}^*} \mathbb{E}\left[e^{-r[(T\wedge\tau)-t]}(\phi K - \phi S_{T\wedge\tau})^+ \mid \mathcal{F}_t\right], \tag{2.7}$$

where $T\wedge\tau$ denotes $\min(T,\tau)$ and $\mathcal{T}^*$ is the set of all stopping times for the filtration, taking values in $[t,\infty)$. The terminal payoff of both European option and American option are the same, i.e., $v(S,T;\phi) = V(S,T;\phi) = (\phi K - \phi S)^+$.

From the Markov property of Itô process, the European option value (2.6) and the American option value (2.7) can be written as

$$v(S,t;\phi) = e^{-r(T-t)}\mathbb{E}\left[(\phi K - \phi S_T)^+ \mid S_t = S\right], \tag{2.8}$$

and

$$V(S,t;\phi) = \sup_{\tau \in \mathcal{T}^*} \mathbb{E}\left[e^{-r[(T\wedge\tau)-t]}(\phi K - \phi S_{T\wedge\tau})^+ \mid S_t = S\right], \tag{2.9}$$

respectively.

Note that the domain of the European option value $v(S,t)$ is

$$\{(S,t) \mid 0 \leq S < \infty,\ 0 \leq t \leq T\}.$$

In addition, under the Itô process which is a diffusion process and the condition that the risk-free interest rate is constant, there exists the critical asset spot price $\gamma(t;\phi)$, for each time $t \in [0,T]$, below (above) which the American put (call) option should be early exercised ([7], eqs. (1.2) and (1.3)). The boundary $\gamma(\cdot;\phi)$ is called the **optimal exercise boundary** and is denoted for convenience by $\gamma(\cdot) = \gamma(\cdot;\phi)$. Then, we can write the American option value as $V(S,t;\phi) = V(S,t;\phi,\gamma(t))$ and the domain becomes

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$$\{(S,t) \mid S \geq 0,\ \phi S > \phi\gamma(t),\ 0 \leq t \leq T\}$$

corresponding to the reduction in price domain from the first passage time that the underlying asset prices hit the optimal exercise boundary.

2.3 Option Pricing Methods

In this section, we provide approximation and analytical methods for option pricing which will be applied later.

2.3.1 Monte Carlo Simulation

Monte Carlo (MC) simulation [19] is the approximation technique that uses computational algorithms based on repeatedly random sampling to obtain numerical results in probabilistic problems, in particular, to estimate conditional expected values. Since MC simulation is simple to implement and the result is quite accurate, this method is then one of the standard benchmark approaches for option pricing problem. In this dissertation, we use Euler-Maruyama (EM) scheme [21] to perform MC simulation for European option. Consider an Itô process S_t described by SDE

$$dS_t = \mu(t, S_t)\, dt + \sigma(t, S_t)\, dW_t \tag{2.10}$$

on the time interval $[t_0, T]$ with the initial value $S_{t_0} = S$. The simple EM discretization of (2.10) on $[t_0, T]$ is given by

$$S_{t_i} = S_{t_{i-1}} + \mu(t_{i-1}, S_{t_{i-1}})\, \Delta t + \sigma(t_{i-1}, S_{t_{i-1}})\sqrt{\Delta t}\, Z_{t_i},$$

where a discretization $t_i = t_0 + i\Delta t$ for $i = 1, \ldots, N$ and the time step $\Delta t = \frac{T-t_0}{N}$ for some integer N representing the number of time steps in discretization that is sufficiently large so that $\Delta t < 1$ and Z_t is the standard normal random variable.

For the given number of sample paths used in MC simulation, namely N_p, the approximation of European option value (2.8) is obtained by

$$v(S, t; \phi) \approx \frac{1}{N_p} \sum_{j=1}^{N_p} e^{-r(T-t)}(\phi K - \phi S_T^{(j)})^+,$$

where $S_T^{(j)}$, $j = 1, \ldots, N_p$, denotes the jth estimator of S_T from the simulations.

For American option, MC simulation can be also applied in a similar way but more complicated than European option, together with some modification techniques such as the least-squares approach and the exercise boundary parameterization approach (see [13] and references therein for more details).

2.3.2 Probability Approach

The explicit formula for European option value can be directly obtained via probability approach when the distribution of asset prices is known, especially, the terminal price. For this case, one can derive a formula via the probability density function (PDF) of the terminal price. The following example demonstrates the well-known results called BS formula derived based on the log-normal distribution.

Example 2.5. ([13], p.330) Consider the underlying stock price process S_t satisfying BS model where r denotes the risk-free interest rate and δ is the stock dividend rate. Let $X_t = \ln S_t$. From (2.5) in Example 2.4,

$$d\,X_t = \left(r - \delta - \frac{\sigma^2}{2}\right) dt + \sigma\, dW_t. \tag{2.11}$$

Let the initial price at time $t = 0$ to be a positive number S. The solution of SDE (2.11) is easily obtained by integration over time $[0, T]$ to get

$$X_T = \ln S + \left(r - \delta - \frac{\sigma^2}{2}\right) T + \sigma W_T.$$

This implies that X_T has normal distribution with mean $m := \ln S + \left(r - \delta - \frac{\sigma^2}{2}\right) T$, variance $g := \sigma^2 T$ and the PDF

$$f(x) := \frac{1}{\sqrt{2\pi g}} \exp\left(-\frac{(x-m)^2}{2g}\right).$$

Then, the European option value (2.8) can be computed by

$$v(S, 0; \phi) = e^{-r(T-t)} \int_{-\infty}^{\infty} (\phi K - \phi e^x)^+ f(x)\,\mathrm{d}x.$$

Consider the integral term as

$$K \lim_{n\to-\infty} \int_{\phi n}^{\ln K} f(x)\,\mathrm{d}x - \lim_{n\to-\infty} \int_{\phi n}^{\ln K} e^x f(x)\,\mathrm{d}x.$$

Manipulating the integrand $e^x f(x)$ to be the PDF of the standard normal distribution, $\varphi(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$, and transforming each limit, via a probability measure for standardizing the normal random variable X_T, to be the cumulative distribution function (CDF) of the standard normal random variable,

$$N(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{x^2}{2}}\, dx.$$

The symmetric property

$$N(-a) = 1 - N(a) = \frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} e^{-\frac{x^2}{2}}\, dx \tag{2.12}$$

is also used for the case $\phi = -1$. Finally, the resulted formula is

$$v(S, 0; \phi) = \phi K e^{-rT} N\left[\phi d_1\right] - \phi S e^{-\delta T} N\left[\phi d_2\right], \tag{2.13}$$

where $d_1 = \frac{\ln K - m(T)}{\sqrt{g(T)}}$ and $d_2 = d_1 - \sqrt{g(T)}$.

Remark that BS formula (2.13), when $\delta = 0$, is originally obtained via the solution of PDE.

2.3.3 PDE Approach via Feynman-Kac Formula

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To deal with problem in financial derivatives pricing, especially, complicated derivative such as European and American options, one of the most popular and powerful method is PDE approach that converts the problem from the conditional expectation into the solution of a PDE. The well-known theorem for the transformation is Feynman-Kac formula stated as follow.

Theorem 2.6 (Feynman-Kac formula). *[18] Let S_t be an Itô process given by* (2.10) *where $\mu, \sigma : [0, \infty) \times \mathbb{R}^+ \to \mathbb{R}$, $r, T > 0$ and a function $f \in C(\mathbb{R}^+)$ satisfies the polynomial growth condition*

$$|f(x)| \leq C(1 + x^{\alpha}), \tag{2.14}$$

for some $C > 0$ *and* $\alpha \geq 2$. *Suppose that* $u(t,x) \in C^{1,2}([0,T) \times \mathbb{R}^+)$ *satisfies the PDE*

$$\frac{\partial u}{\partial t} + \mu(t,x)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 u}{\partial x^2} = ru, \quad \text{in } [0,T) \times \mathbb{R}^+, \tag{2.15}$$

$$u(T,x) = f(x), \quad x \in \mathbb{R}^+. \tag{2.16}$$

Then, the unique solution u *can be written as*

$$u(t,x) = e^{-r(T-t)}\mathbb{E}\left[f(S_T) \mid S_t = x\right]$$

on $[0,T) \times \mathbb{R}^+$.

It is easy to see that the polynomial growth condition (2.14) and the terminal condition (2.16) are satisfied for the option payoff function $f(x) = (\phi K - \phi x)^+$. This implies from the theorem that the European option value is the solution of the PDE (2.15).

For the American option value which is an optimal stopping problem, it can be shown to follow the same PDE on another domain associated with a free boundary, namely the optimal exercise boundary function (see [24] (thm.10.4.1 and thm. 12.3.11) for further details). For instance, see [7, 20, 23] and [25] in the case of stock price.

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We give an example applying Feynman-Kac formula for both European and American options as follow.

Example 2.7. ([24], p.296 and [28], p.270) Consider stock price process (2.3) as in Example 2.5 and denote the corresponded PDO by

$$\mathcal{L} := \frac{\partial}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2}{\partial S^2} + (r-\delta)S\frac{\partial}{\partial S} - r.$$

Applying Theorem 2.6 to the process with function $f(x) = (\phi K - \phi x)^+$, the European option value $v(S,t;\phi)$ in (2.8) satisfies the PDE

$$\mathcal{L}v(S,t;\phi) = 0,$$

for $S > 0$ and $0 \leq t < T$, subject to the terminal condition

$$v(S, T; \phi) = (\phi K - \phi S)^+, \quad S > 0.$$

As discussed above, the American option value $V(S, t; \phi, \gamma(t))$ in (2.9) satisfies the PDE

$$\mathcal{L}V(S, t; \phi, \gamma(t)) = 0,$$

for $S > 0,\ \phi S > \phi\gamma(t)$ and $0 \leq t < T$, subject to the terminal condition

$$V(S, T; \phi, \gamma(t)) = (\phi K - \phi S)^+,$$

for $S > 0$ and $\phi S > \phi\gamma(t)$.

To obtain the solution of these PDEs, one can applied some analytical and numerical approaches, for instance, the finite difference method [13].

In this work, we employ analytical approach to derive the pricing formulas for both European and American options. The tools required for obtaining solutions of PDEs are provided in the following section.

2.4 Tools for PDEs

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To derive analytical option pricing formulas, we recall basic mathematical concepts required in this work, such as Fourier transform, the method of characteristic curve and the Dirac delta function.

Definition 2.8. [16] Let $\mathcal{D}(\mathbb{R})$ denotes the space of smooth functions with compact support on $\mathbb{R}$, whose elements are called test functions. The **Dirac delta function** (at origin) is the distribution (or the generalized function) $\delta : \mathcal{D}(\mathbb{R}) \to \mathbb{R}$ defined by

$$\delta[\varphi] = \varphi(0), \quad \varphi \in \mathcal{D}(\mathbb{R}).$$

We denote the Dirac delta function applied to a test function φ using variable x, by $\delta = \delta(x)$.

Proposition 2.9. *[16]*

(i) For $a, c \in \mathbb{R}$ *with* $c \neq 0$, $\delta(cx-a) = \frac{1}{|c|}\delta\left(x - \frac{a}{c}\right)$.

(ii) Let $\phi \in \{1, -1\}$. *The distributional derivative of the indicator function (or the characteristic function) on a set* $\{x \mid \phi x \geq \phi a\}$, $\mathbb{1}_{\{\phi x \geq \phi a\}}$, *is*

$$\frac{d}{dx}\mathbb{1}_{\{\phi x \geq \phi a\}}(x) = \phi\delta(x-a).$$

Definition 2.10. [11] The **Fourier transform** of a complex-valued function $h \in L^1(\mathbb{R})$ is the function $\mathcal{F}[h] : \mathbb{R} \to \mathbb{C}$ defined by

$$\mathcal{F}[h](\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(x)e^{i\xi x}\mathrm{d}x,$$

where $i^2 = -1$. The **inverse Fourier transform** of a complex-valued function $H \in L^1(\mathbb{R})$ is the function $\mathcal{F}^{-1}[H] : \mathbb{R} \to \mathbb{C}$ defined by

$$\mathcal{F}^{-1}[H](x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} H(\xi)e^{-i\xi x}\mathrm{d}\xi.$$

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Note that the Fourier transform of the Dirac delta function is also defined as tempered distributions.

Definition 2.11. [11] For $f, g \in L^1_{loc}(\mathbb{R})$, the **convolution** of f and g is defined by

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)\,dy.$$

For the Dirac delta function, the convolution of δ and $f \in C^\infty(\mathbb{R})$ is the function $\delta * f : \mathbb{R} \to \mathbb{R}$ defined by $(\delta * f)(x) = f(x)$. For convenience, we pretend to write the Dirac delta function as a function and the convolution is simply denoted by $\int_{\mathbb{R}} f(x-y)\delta(y)\,dy = f(x)$.

Proposition 2.12. *[11] Let $\mathcal{S}$ denote the space of smooth and rapidly decreasing functions on $\mathbb{R}$, called the Schwartz space.*

(i) If $h \in \mathcal{S}$ and $\alpha \in \mathbb{N}$, then

$$\mathcal{F}\left[\frac{d^\alpha h}{dx^\alpha}\right] = (-i\xi)^\alpha \mathcal{F}[h]$$

and

$$\mathcal{F}\left[x\frac{dh}{dx}\right] = -\xi\frac{d\mathcal{F}[h]}{d\xi} - \mathcal{F}[h].$$

(ii) (The scaling property) If $G, H \in L^1(\mathbb{R})$ such that $G(\xi) = H(c\xi)$ for some $c \in \mathbb{R}\backslash\{0\}$, then the inverse Fourier transform of G is

$$\mathcal{F}^{-1}[G](x) = \frac{1}{|c|}\mathcal{F}^{-1}[H]\left(\frac{x}{c}\right).$$

(iii) (The convolution property) If $h_1, h_2 \in \mathcal{S}$, then $h_1 * h_2 \in \mathcal{S}$ and

$$\mathcal{F}[h_1 * h_2] = \sqrt{2\pi}\left(\mathcal{F}[h_1]\mathcal{F}[h_2]\right).$$

Hence,

$$\mathcal{F}^{-1}[H_1 H_2] = \frac{1}{\sqrt{2\pi}}\left(\mathcal{F}^{-1}[H_1] * \mathcal{F}^{-1}[H_2]\right),$$

where $H_1 = \mathcal{F}[h_1]$ and $H_2 = \mathcal{F}[h_2]$.

Remark 2.13. [16] Observe that the results of Proposition 2.12 is also applicable for the Dirac delta function as tempered distributions.

Next, we provide an example applying Fourier transform and the method of characteristic curve [10] in transforming the problem of a second-order linear PDE, which will be later applied to our problem, into an ordinary differential equations (ODE).

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Example 2.14. ([10], p.99 and p.188) Consider the initial-valued problem (IVP)

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + u = f, \quad (x,\tau) \in \mathbb{R} \times (0,T], \tag{2.17}$$

$$u(x,0) = 0, \quad x \in \mathbb{R}, \tag{2.18}$$

where f is a known function.

Note that the Fourier transform in x-variable of a function $h(x,\tau)$ is given by

$$H(\xi,\tau) := \mathcal{F}[h](\xi,\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(x,\tau)e^{i\xi x}\mathrm{d}x$$

and from Proposition 2.12 (i),

$$\mathcal{F}\left[\frac{\partial h}{\partial x}\right] = -i\xi H(\xi,\tau), \ \mathcal{F}\left[\frac{\partial^2 h}{\partial x^2}\right] = -\xi^2 H(\xi,\tau) \ \text{ and}$$
$$\mathcal{F}\left[x\frac{\partial h}{\partial x}\right] = -\xi\frac{\partial H(\xi,\tau)}{\partial \xi} - H(\xi,\tau).$$

Using these facts and taking the Fourier transform of (2.17)–(2.18), we obtain a first-order linear PDE

$$\frac{\partial U(\xi,\tau)}{\partial \tau} - \xi\frac{\partial U(\xi,\tau)}{\partial \xi} + \xi^2 U(\xi,\tau) = F(\xi,\tau), \tag{2.19}$$

for $\xi \in \mathbb{R}$ and $0 < \tau \leq T$, with the initial condition

$$U(\xi,0) = \mathcal{F}[u(x,0)] = 0, \tag{2.20}$$

where $F(\xi,\tau) = \mathcal{F}[f](\xi,\tau)$.

To solve the PDE (2.19), we apply the method of characteristic curve by setting a parameter s by $\xi = \xi(s)$, $\tau = \tau(s)$ and $U(\xi,\tau) = U(\xi(s),\tau(s)) = U(s)$ to have the characteristic equations

$$\tau'(s) = s, \quad \xi'(s) = -\xi, \quad \tau'(0) = 0 \ \text{ and } \ \xi(0) = \xi_0 \in \mathbb{R}.$$

Solving the above characteristic equations of ODEs yields $\tau(s) = s$ and $\xi(s) = \xi_0 e^{-s}$. By these settings and the total derivative

$$\frac{\mathrm{d}U(s)}{\mathrm{d}s} = \frac{\partial U(\xi,\tau)}{\partial \tau} - \kappa\xi\frac{\partial U(\xi,\tau)}{\partial \xi},$$

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the IVP (2.19)–(2.20) becomes a first-order linear ODE

$$\frac{dU(s)}{\mathrm{d}s} + (\xi_0 e^{-\kappa s})^2 U(s) = F(\xi_0 e^{-\kappa s}, s), \tag{2.21}$$

with the initial condition

$$U(0) = 0. \tag{2.22}$$

Solving (2.21) subject to (2.22) yields the solution $U(s)$. Then, substituting parameters $s = \tau$ and $\xi_0 = \xi e^{\kappa\tau}$ back, we obtain the solution $U(\xi, \tau) = U(s)$ of the IVP (2.19)–(2.20).

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CHAPTER III

EUROPEAN AND AMERICAN COMMODITY OPTIONS PRICING

Consider a mean-reverting process $(S_t)_{t\geq 0}$ described the prices of underlying commodity, under a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, satisfies the SDE

$$\mathrm{d}S_t = \kappa(\mu(t) - \ln S_t)S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t, \tag{3.1}$$

where $\kappa > 0$ represents the speed of the reversion, $\mu : [0, \infty) \to \mathbb{R}$ represents the long-run mean function of the process, σ represents the volatility and W_t is a standard Brownian motion driven on a filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by the process.

Let $v(S, t; \phi)$ be the value of a European option and $V(S, t; \phi)$ be the value of an American option on the commodity spot price S at time $t \leq T$, with a strike price K and an expiration date T, where $\phi = -1$ for a call option and $\phi = 1$ for a put option. Under the risk-neutral measure $\mathbb{Q}$ and the constant risk-free interest rate $r > 0$, the fair value of the European option is

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$$v(S, t; \phi) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[(\phi K - \phi S_T)^+ \mid S_t = S\right] \tag{3.2}$$

and the fair value of the American option [23] is

$$V(S, t; \phi) = \sup_{\tau \in \mathcal{T}^*} \mathbb{E}^{\mathbb{Q}}\left[e^{-r[(T\wedge\tau)-t]}(\phi K - \phi S_{T\wedge\tau})^+ \mid S_t = S\right], \tag{3.3}$$

where $\mathcal{T}^*$ is the set of all stopping times for the filtration $(\mathcal{F}_t)_{t\geq 0}$, taking values in $[t, \infty)$.

In this chapter, we give analytical representation formulas for pricing European options (3.2) and American options (3.3) on the commodity whose prices exhibit mean reversion with time-dependent parameter following (3.1).

This chapter is organized as follows. Section 3.1 provides the solution of a specific PDE which will be applied in the next sections. An integral representation formula for the European option value (3.2) derived in Section 3.2. In addition, the put-call parity formula for European option is given. Using the result from Section 3.2, an integral representation formula for the American option value (3.3) is proposed in Section 3.3.

3.1 Solution of PDE for Options Pricing

Denote ϕ and $\mathcal{U}$ to be the number and the space, respectively, representing for which function type we are considering. In particular the number $\phi = -1$ for call option and $\phi = 1$ for put option and the space $\mathcal{U} = \mathbb{R}$ for European style and $\mathcal{U} = (-\infty, 0)$ for American style will be applied later. The setting of a specific parabolic PDE and the solution are proposed as follow.

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Lemma 3.1. *Let $\mathcal{U}$ be either a space $(-\infty, 0)$ or $\mathbb{R}$, a number ϕ be either 1 or -1, $r \in \mathbb{R}$ and $K, T, \kappa, \sigma > 0$. Assume that*

(i) $\widetilde{\gamma}(\cdot;\phi) : [0,T] \to (0,\infty)$ *is a differentiable function on time variable* τ*,*

(ii) $\widetilde{\mu} : [0,T] \to \mathbb{R}$ *is integrable,*

(iii) $\widetilde{E}(\cdot,\cdot;\phi) : \mathbb{R} \times [0,T] \to \mathbb{R}$ *is a function on space and time variable* (x,τ) *satisfying*

$$\widehat{\mathcal{L}}\widetilde{E}(x,\tau;\phi) = 0, \qquad \phi x \in \mathcal{U} \text{ and } 0 < \tau \leq T, \tag{3.4}$$

where $\widehat{\mathcal{L}}$ is a PDO defined by

$$\widehat{\mathcal{L}} := \frac{\partial}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \left(-\frac{\sigma^2}{2} + \kappa\widetilde{\mu}(\tau) - \kappa \ln \widetilde{\gamma}(\tau) + \frac{\widetilde{\gamma}'(\tau)}{\widetilde{\gamma}(\tau)} + \kappa x\right)\frac{\partial}{\partial x} + r, \tag{3.5}$$

(iv) $g(\cdot,\cdot;\phi) : \mathbb{R} \times [0,T] \to \mathbb{R}$ *is defined by*

$$g(x,\tau;\phi) = (\phi K - \phi\widetilde{\gamma}(\tau)e^{-x})^+ - \widetilde{E}(x,\tau;\phi) \text{ and} \tag{3.6}$$

(v)

$$f(x,\tau;\phi) = \begin{cases} -\widehat{\mathcal{L}}g(x,\tau;\phi), & \text{for } \phi x \in \mathcal{U} \text{ and } 0 < \tau \le T, \\ 0, & \text{otherwise.} \end{cases} \tag{3.7}$$

Suppose $u(\cdot,\cdot;\phi) : \mathbb{R} \times [0,T] \to \mathbb{R}$ *is the solution to the PDE*

$$\widehat{\mathcal{L}}u(x,\tau;\phi) = f(x,\tau;\phi), \quad x \in \mathbb{R} \text{ and } 0 < \tau \le T, \tag{3.8}$$

subject to the initial condition

$$u(x,0;\phi) = 0, \quad x \in \mathbb{R} \tag{3.9}$$

and the boundary condition

$$\lim_{\phi x \to -\infty} u(x,\tau;\phi) = 0, \quad 0 < \tau \le T. \tag{3.10}$$

Then, the solution u can be written as

$$u(x,\tau;\phi) = \int_0^\tau \int_{-\infty}^\infty f_3(x-z,\tau,\hat{\tau}) f_2(z,\tau,\hat{\tau})\,\mathrm{d}z\,\mathrm{d}\hat{\tau},$$

where

$$\begin{aligned} f_2(z,\tau,\hat{\tau}) = &\frac{\sigma^2 K e^{(r-\kappa)(\hat{\tau}-\tau)}}{2}\delta\Big(z - e^{\kappa(\tau-\hat{\tau})}\ln\frac{\widetilde{\gamma}(\hat{\tau})}{K}\Big) \\ &+ \phi e^{r(\hat{\tau}-\tau)}\Big(-rK + \widetilde{\gamma}(\hat{\tau})e^{-ze^{\kappa(\hat{\tau}-\tau)}}(r - \kappa\widetilde{\mu}(\hat{\tau}) + \kappa\ln\widetilde{\gamma}(\hat{\tau}) \\ &- \kappa z e^{\kappa(\hat{\tau}-\tau)}\Big)\mathbb{1}_{\left\{\phi e^{\kappa(\tau-\hat{\tau})}\ln\frac{\widetilde{\gamma}(\hat{\tau})}{K} \le \phi z,\ \phi z \in \mathcal{U}\right\}}(z,\hat{\tau}) \end{aligned}$$

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and

$$
\begin{aligned}
&f_3(x,\tau,\hat{\tau}) \\
&\quad = \frac{1}{\sqrt{2\pi}}\left(\frac{\sqrt{2\kappa}}{\sigma\sqrt{e^{2\kappa(\tau-\hat{\tau})}-1}}\right) \\
&\qquad \times \exp\left(\frac{-\left(x-\ln\widetilde{\gamma}(\tau)+e^{\kappa(\tau-\hat{\tau})}\ln\widetilde{\gamma}(\hat{\tau})+\frac{\sigma^2}{2\kappa}\left(e^{\kappa(\tau-\hat{\tau})}-1\right)-\varphi(\tau,\hat{\tau})\right)^2}{2\left(\frac{\sigma\sqrt{e^{2\kappa(\tau-\hat{\tau})}-1}}{\sqrt{2\kappa}}\right)^2}\right),
\end{aligned}
$$

with $\varphi(\tau,\hat{\tau}) = \kappa e^{\kappa\tau}\int_{\hat{\tau}}^{\tau}\widetilde{\mu}(w)e^{-\kappa w}\mathrm{d}w.$

Proof. Let $\phi \in \{-1,1\}$. For convenience, we leave the representation ϕ for the type of functions and instead write $\widetilde{\gamma}(\cdot) = \widetilde{\gamma}(\cdot;\phi)$, $\widetilde{E}(\cdot,\cdot) = \widetilde{E}(\cdot,\cdot;\phi)$, $f(\cdot,\cdot) = f(\cdot,\cdot;\phi)$, $g(\cdot,\cdot) = g(\cdot,\cdot;\phi)$, $u(\cdot,\cdot) = u(\cdot,\cdot;\phi)$. This proof is divided into 2 parts: (i) Fourier transform and (ii) inversion.

(i) Fourier transform:

First, we apply the Fourier transform technique to solve (3.8) subject to (3.9)–(3.10). Similar to the method used in Example 2.14, from Proposition 2.12 (i) and by taking the Fourier transform to the PDE (3.8) and the initial condition (3.9), we obtain a first-order linear PDE

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$$
\frac{\partial U(\xi,\tau)}{\partial\tau} - \kappa\xi\frac{\partial U(\xi,\tau)}{\partial\xi} + A(\xi,\tau)U(\xi,\tau) = F(\xi,\tau), \tag{3.11}
$$

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for $\xi \in \mathbb{R}$ and $0 < \tau \le T$, with the initial condition

$$
U(\xi,0) = \mathcal{F}[u(x,0)] = 0,
$$

where $F(\xi,\tau) := \mathcal{F}[f](\xi,\tau)$ and

$$
A(\xi,\tau) = \frac{\sigma^2}{2}\xi^2 + \left(\frac{\sigma^2}{2} - \kappa\widetilde{\mu}(\tau) + \kappa\ln\widetilde{\gamma}(\tau) - \frac{\widetilde{\gamma}'(\tau)}{\widetilde{\gamma}(\tau)}\right)i\xi + r - \kappa. \tag{3.12}
$$

To solve the PDE (3.11), we apply the method of characteristic curve similar to Example 2.14. Then, the PDE (3.11) becomes a first-order linear ODE

$$
\frac{dU(s)}{\mathrm{d}s} + A(\xi_0 e^{-\kappa s}, s)U(s) = F(\xi_0 e^{-\kappa s}, s), \tag{3.13}
$$

with the initial condition

$$U(0) = U(\xi_0, 0) = 0. \tag{3.14}$$

Solving (3.13) subject to (3.14) yields

$$U(s) = e^{-\int_0^s A(\xi_0 e^{-\kappa z}, z)\mathrm{d}z} \int_0^s e^{\int_0^{\hat{\tau}} A(\xi_0 e^{-\kappa z}, z)\mathrm{d}z} F(\xi_0 e^{-\kappa\hat{\tau}}, \hat{\tau})\mathrm{d}\hat{\tau}. \tag{3.15}$$

Substituting $s = \tau$ and $\xi_0 = \xi e^{\kappa\tau}$ back into (3.15), we obtain

$$\begin{aligned}
U(\xi, \tau) &= U(s) \\
&= e^{-\int_0^\tau A(\xi e^{\kappa\tau - \kappa z}, z)\mathrm{d}z} \int_0^\tau e^{\int_0^{\hat{\tau}} A(\xi e^{\kappa\tau-\kappa z}, z)\mathrm{d}z} F(\xi e^{\kappa\tau - \kappa\hat{\tau}}, \hat{\tau})\mathrm{d}\hat{\tau} \\
&= \int_0^\tau e^{B(\xi,\tau,\hat{\tau}) - B(\xi,\tau,\tau)} F(\xi e^{\kappa\tau-\kappa\hat{\tau}}, \hat{\tau})\mathrm{d}\hat{\tau} \\
&= \int_0^\tau F_1(\xi, \tau, \hat{\tau}) F(\xi e^{\kappa(\tau - \hat{\tau})}, \hat{\tau})\, \mathrm{d}\hat{\tau},
\end{aligned} \tag{3.16}$$

where $B(\xi, \tau, y) = \int_0^y A(\xi e^{\kappa\tau - \kappa z}, z)\mathrm{d}z$ and $F_1(\xi, \tau, \hat{\tau}) = e^{\hat{B}(\xi,\tau,\hat{\tau}) - \hat{B}(\xi,\tau,\tau)}$. From (3.12),

$$\begin{aligned}
&B(\xi, \tau, y) \\
&= \int_0^y \left(\frac{\sigma^2}{2}\xi^2 e^{2\kappa\tau - 2\kappa z} + \left(\frac{\sigma^2}{2} - \kappa\widetilde{\mu}(z) + \kappa \ln \widetilde{\gamma}(z) - \frac{\widetilde{\gamma}'(z)}{\widetilde{\gamma}(z)} \right) i\xi e^{\kappa\tau - \kappa z} + r - \kappa \right) \mathrm{d}z \\
&= \frac{\sigma^2}{2}\xi^2 e^{2\kappa\tau} \int_0^y e^{-2\kappa z}\mathrm{d}z + \frac{\sigma^2}{2} i\xi e^{\kappa\tau} \int_0^y e^{-\kappa z}\mathrm{d}z - i\xi e^{\kappa\tau} \int_0^y e^{-\kappa z} \Big(\frac{\widetilde{\gamma}'(z)}{\widetilde{\gamma}(z)} - \kappa \ln \widetilde{\gamma}(z) \Big) \mathrm{d}z \\
&\quad - \kappa i \xi e^{\kappa\tau} \int_0^y \widetilde{\mu}(z) e^{-\kappa z}\mathrm{d}z + (r - \kappa) \int_0^y \mathrm{d}z \\
&= \frac{\sigma^2 e^{2\kappa\tau}(1 - e^{-2\kappa y})}{4\kappa}\xi^2 + (r - \kappa)y \\
&\quad + i\xi e^{\kappa\tau} \left(-e^{-\kappa y} \ln \widetilde{\gamma}(y) + \ln \widetilde{\gamma}(0) - \frac{\sigma^2}{2\kappa}\left(e^{-\kappa y} - 1\right) - \kappa \int_0^y \widetilde{\mu}(z) e^{-\kappa z}\mathrm{d}z \right),
\end{aligned}$$

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and

$$\begin{aligned}
F_1(\xi, \tau, \hat{\tau}) = \exp \Bigg(&\frac{\sigma^2 (1 - e^{2\kappa(\tau - \hat{\tau})})}{4\kappa}\xi^2 + i\xi \Big(\ln \widetilde{\gamma}(\tau) - e^{\kappa(\tau - \hat{\tau})} \ln \widetilde{\gamma}(\hat{\tau}) \\
&- \frac{\sigma^2}{2\kappa}\left(e^{\kappa(\tau - \hat{\tau})} - 1 \right) + \varphi(\tau, \hat{\tau}) \Big) + (r - \kappa)(\hat{\tau} - \tau) \Bigg),
\end{aligned} \tag{3.17}$$

with $\varphi(\tau, \hat{\tau}) = \kappa e^{\kappa\tau} \int_{\hat{\tau}}^{\tau} \widetilde{\mu}(w) e^{-\kappa w}\mathrm{d}w.$

(ii) Inversion:

From Definition 2.11, the scaling property and the convolution property of Fourier transform in Proposition 2.12, we have

$$\begin{aligned}\mathcal{F}^{-1}[H_1(\xi)H_2(c\xi)](x) &= \frac{1}{\sqrt{2\pi}}\left(\mathcal{F}^{-1}[H_1(\xi)] * \mathcal{F}^{-1}[H_2(c\xi)]\right)(x) \\ &= \frac{1}{|c|\sqrt{2\pi}}\int_{-\infty}^{\infty} h_1(x-z)h_2\left(\frac{z}{c}\right)\,\mathrm{d}z \end{aligned} \tag{3.18}$$

for every $c \in \mathbb{R}\backslash\{0\}$, where $h_1 = \mathcal{F}^{-1}[H_1(\xi)]$ and $h_2 = \mathcal{F}^{-1}[H_2(\xi)]$. By applying (3.18) with $c = e^{\kappa(\tau-\hat{\tau})}$ and taking the inverse transform of (3.16),

$$\begin{aligned}u(x,\tau) =& \mathcal{F}^{-1}[U(\xi,\tau)] \\ =& \int_0^{\tau} \mathcal{F}^{-1}\left[F_1(\xi,\tau,\hat{\tau})F(\xi e^{\kappa(\tau-\hat{\tau})},\hat{\tau})\right]\mathrm{d}\hat{\tau} \\ =& \frac{1}{\sqrt{2\pi}}\int_0^{\tau} e^{\kappa(\hat{\tau}-\tau)}\int_{-\infty}^{\infty} f_1(x-z,\tau,\hat{\tau})f(ze^{\kappa(\hat{\tau}-\tau)},\hat{\tau})\,\mathrm{d}z\,\mathrm{d}\hat{\tau}, \end{aligned} \tag{3.19}$$

where $f_1(x,\tau,\hat{\tau}) = \mathcal{F}^{-1}[F_1(\xi,\tau,\hat{\tau})]$ and $f(x,\tau) = \mathcal{F}^{-1}[F(\xi,\tau)]$. From the integral of Gaussian function

$$\int_{-\infty}^{\infty} e^{-\hat{a}y^2+\hat{b}y+\hat{c}}\,\mathrm{d}y = e^{\frac{\hat{b}^2}{4\hat{a}}+\hat{c}}\sqrt{\frac{\pi}{\hat{a}}},$$

(3.17) gives

$$\begin{aligned} & f_1(x,\tau,\hat{\tau}) \\ &= \mathcal{F}^{-1}[F_1(\xi,\tau,\hat{\tau})] \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F_1(\xi,\tau,\hat{\tau})e^{-i\xi x}\mathrm{d}\xi \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\hat{a}\xi^2+\hat{b}\xi+\hat{c}}\mathrm{d}\xi \\ &= \left(\frac{\sqrt{2\kappa}}{\sigma\sqrt{e^{2\kappa(\tau-\hat{\tau})}-1}}\right)e^{(r-\kappa)(\hat{\tau}-\tau)} \\ &\quad \times \exp\left(\frac{-\left(\ln\widetilde{\gamma}(\tau) - e^{\kappa(\tau-\hat{\tau})}\ln\widetilde{\gamma}(\hat{\tau}) - \frac{\sigma^2}{2\kappa}\left(e^{\kappa(\tau-\hat{\tau})}-1\right) + \varphi(\tau,\hat{\tau}) - x\right)^2}{2\left(\frac{\sigma\sqrt{e^{2\kappa(\tau-\hat{\tau})}-1}}{\sqrt{2\kappa}}\right)^2}\right), \end{aligned} \tag{3.20}$$

when

$$\hat{a} = \frac{\sigma^2}{4\kappa}\left(e^{2\kappa(\tau-\hat{\tau})} - 1\right),$$
$$\hat{b} = i\left(\ln\widetilde{\gamma}(\tau) - e^{\kappa(\tau-\hat{\tau})}\ln\widetilde{\gamma}(\hat{\tau}) - \frac{\sigma^2}{2\kappa}\left(e^{\kappa(\tau-\hat{\tau})} - 1\right) + \varphi(\tau,\hat{\tau}) - x\right) \text{ and}$$
$$\hat{c} = (r-\kappa)(\hat{\tau}-\tau).$$

From (3.6), we can write

$$g(x,\tau) = (\phi K - \phi\widetilde{\gamma}(\tau)e^{-x})\mathbb{1}_{\left\{\phi x \geq \phi \ln\frac{\widetilde{\gamma}(\tau)}{K}\right\}}(x,\tau) - \widetilde{E}(x,\tau), \tag{3.21}$$

where $\mathbb{1}_{\{\cdot\}}(\cdot)$ is the indicator function. Thus,

$$\frac{\partial g(x,\tau)}{\partial \tau} = -\phi\widetilde{\gamma}'(\tau)e^{-x}\mathbb{1}_{\left\{\phi x \geq \phi \ln\frac{\widetilde{\gamma}(\tau)}{K}\right\}}(x,\tau) - \frac{\partial \widetilde{E}(x,\tau)}{\partial \tau}, \tag{3.22}$$
$$\frac{\partial g(x,\tau)}{\partial x} = \phi\widetilde{\gamma}(\tau)e^{-x}\mathbb{1}_{\left\{\phi x \geq \phi \ln\frac{\widetilde{\gamma}(\tau)}{K}\right\}}(x,\tau) - \frac{\partial \widetilde{E}(x,\tau)}{\partial x} \quad \text{and} \tag{3.23}$$
$$\frac{\partial^2 g(x,\tau)}{\partial x^2} = K\delta\left(x - \ln\frac{\widetilde{\gamma}(\tau)}{K}\right) - \phi\widetilde{\gamma}(\tau)e^{-x}\mathbb{1}_{\left\{\phi x \geq \phi \ln\frac{\widetilde{\gamma}(\tau)}{K}\right\}}(x,\tau) - \frac{\partial^2 \widetilde{E}(x,\tau)}{\partial x^2}, \tag{3.24}$$

where $\delta(\cdot)$ is the Dirac delta function. By substituting (3.21)–(3.24) into (3.7) and from (3.4)–(3.5), we get that, for $\phi x \in \mathcal{U}$ and $0 < \tau \leq T$,

$$\begin{aligned}
f(x,\tau) =& -\frac{\partial g(x,\tau)}{\partial \tau} + \frac{\sigma^2}{2}\frac{\partial^2 g(x,\tau)}{\partial x^2} \\
&- \left(-\frac{\sigma^2}{2} + \kappa\widetilde{\mu}(\tau) - \kappa\ln\widetilde{\gamma}(\tau) + \frac{\widetilde{\gamma}'(\tau)}{\widetilde{\gamma}(\tau)} + \kappa x\right)\frac{\partial g(x,\tau)}{\partial x} - rg(x,\tau) \\
=& \widehat{\mathcal{L}}\widetilde{E}(x,\tau) + \frac{\sigma^2 K}{2}\delta\left(x - \ln\frac{\widetilde{\gamma}(\tau)}{K}\right) + \left(\phi e^{-x}\left(-\frac{\sigma^2}{2}\widetilde{\gamma}(\tau) - \widetilde{\gamma}'(\tau)\right)\right. \\
&+ \left.\left(\phi\widetilde{\gamma}(\tau)e^{-x}\left(\frac{\sigma^2}{2} - \kappa\widetilde{\mu}(\tau) + \kappa\ln\widetilde{\gamma}(\tau) - \frac{\widetilde{\gamma}'(\tau)}{\widetilde{\gamma}(\tau)} - \kappa x + r\right) - \phi rK\right)\right) \\
&\times \mathbb{1}_{\left\{\phi x \geq \phi \ln\frac{\widetilde{\gamma}(\tau)}{K}\right\}}(x,\tau) \\
=& \frac{\sigma^2 K}{2}\delta\left(x - \ln\frac{\widetilde{\gamma}(\tau)}{K}\right) \\
&+ \left(\phi\widetilde{\gamma}(\tau)e^{-x}\left(-\kappa\widetilde{\mu}(\tau) + \kappa\ln\widetilde{\gamma}(\tau) - \kappa x + r\right) - \phi rK\right)\mathbb{1}_{\left\{\phi x \geq \phi \ln\frac{\widetilde{\gamma}(\tau)}{K}\right\}}(x,\tau).
\end{aligned} \tag{3.25}$$

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Observe that in the case where $\phi = -1$ and $\mathcal{U} = \mathbb{R}$, for the reason that the second term of the right hand side of (3.25) has the Fourier transform, we can first show that (3.19) holds on $\mathcal{U} = (C, \infty)$ for all $C < 0$. Precisely, we get that for each $C < 0$,

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_0^\tau e^{\kappa(\hat{\tau}-\tau)} \int_{-\infty}^{\infty} f_1(x-z,\tau,\hat{\tau}) f(ze^{\kappa(\hat{\tau}-\tau)},\hat{\tau}) \mathbb{1}_{\left\{ze^{\kappa(\hat{\tau}-\tau)}\in[C,\infty)\right\}}(z)\,\mathrm{d}z\,\mathrm{d}\hat{\tau} \tag{3.26}$$

satisfies PDE (3.8) on $\mathcal{U} = (C, \infty)$. Since the integrand in (3.26) in z-variable is bounded and dominated by the Guassian function f_1, applying the dominated convergence theorem [11] and taking $C \to \infty$, (3.26) then converges to (3.19). Thus, the result on $\mathcal{U} = \mathbb{R}$ is obtained.

Using (3.20) and (3.25), we rewrite (3.19) in the form

$$u(x,\tau) = \int_0^\tau \int_{-\infty}^{\infty} f_3(x-z,\tau,\hat{\tau}) f_2(z,\tau,\hat{\tau})\,\mathrm{d}z\,\mathrm{d}\hat{\tau},$$

where

$$\begin{aligned} f_2(z,\tau,\hat{\tau}) &= e^{r(\hat{\tau}-\tau)} f(ze^{\kappa(\hat{\tau}-\tau)},\hat{\tau}) \\ &= \frac{\sigma^2 K e^{(r-\kappa)(\hat{\tau}-\tau)}}{2} \delta\Big(z - e^{\kappa(\tau-\hat{\tau})} \ln \frac{\widetilde{\gamma}(\hat{\tau})}{K}\Big) \\ &\quad + \phi e^{r(\hat{\tau}-\tau)} \Big(\widetilde{\gamma}(\hat{\tau}) e^{-ze^{\kappa(\hat{\tau}-\tau)}} \big(r - \kappa\widetilde{\mu}(\hat{\tau}) + \kappa \ln \widetilde{\gamma}(\hat{\tau}) - \kappa z e^{\kappa(\hat{\tau}-\tau)}\big) \\ &\quad - rK\Big) \times \mathbb{1}_{\left\{\phi z \geq \phi e^{\kappa(\tau-\hat{\tau})} \ln \frac{\widetilde{\gamma}(\hat{\tau})}{K},\ \phi z \in \mathcal{U}\right\}}(z,\hat{\tau}), \end{aligned}$$

and

$$\begin{aligned} &f_3(x,\tau,\hat{\tau}) \\ &= \frac{e^{(\kappa-r)(\hat{\tau}-\tau)}}{\sqrt{2\pi}} f_1(x,\tau,\hat{\tau}) \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{\sqrt{2\kappa}}{\sigma\sqrt{e^{2\kappa(\tau-\hat{\tau})}-1}}\right) \\ &\quad \times \exp\left(\frac{-\Big(x - \ln\widetilde{\gamma}(\tau) + e^{\kappa(\tau-\hat{\tau})} \ln \widetilde{\gamma}(\hat{\tau}) + \frac{\sigma^2}{2\kappa}\big(e^{\kappa(\tau-\hat{\tau})}-1\big) - \varphi(\tau,\hat{\tau})\Big)^2}{2\left(\frac{\sigma\sqrt{e^{2\kappa(\tau-\hat{\tau})}-1}}{\sqrt{2\kappa}}\right)^2}\right). \end{aligned}$$

□

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3.2 European Commodity Option Pricing Based on PDE

In this section, a PDE for European option value on the underlying asset prices satisfy the model (3.1) is proposed and the solution of the PDE is derived by applying the result from Lemma 3.1.

By applying the Feynman-Kac formula in Theorem 2.6 with model (3.1), the European option value (3.2) satisfies the PDE

$$\frac{\partial v(S,t;\phi)}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 v(S,t;\phi)}{\partial S^2} + \kappa(\mu(t) - \ln S)S\frac{\partial v(S,t;\phi)}{\partial S} - rv(S,t;\phi) = 0, \tag{3.27}$$

for $S > 0$ and $0 \le t < T$, subject to the terminal condition

$$v(S,T;\phi) = (\phi K - \phi S)^+, \quad S \ge 0. \tag{3.28}$$

In order to derive our solution, we need boundary conditions for $v(S,t;\phi)$ provided in the following lemma.

Lemma 3.2. *Assume that the underlying asset spot price* $(S_t)_{t\ge 0}$ *follows the model* (3.1) *with integrable function* $\mu : [0,T] \to \mathbb{R}$*. Then,*

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$$\lim_{S\to 0} v(S,t;-1) = \lim_{S\to\infty} v(S,t;1) = 0 \tag{3.29}$$

for $0 \le t < T$.

Proof. Let $X_t = \ln S_t$. By applying Ito's formula in Theorem 2.4 to model (3.1) with the function $f(t,x) = \ln x$, the model can be written as the extended Vasicek or Hull-White model [14]

$$\mathrm{d}X_t = \kappa(\alpha(t) - X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \tag{3.30}$$

where $\alpha(t) = \mu(t) - \frac{\sigma^2}{2\kappa}$ and the solution of (3.30) is ([28], p.265)

$$X_t = X_0 e^{-\kappa t} + \kappa e^{-\kappa t} \int_0^t \alpha(u) e^{\kappa u} \, \mathrm{d}u + \sigma e^{-\kappa t} \int_0^t e^{\kappa u} \, \mathrm{d}W_u. \tag{3.31}$$

Hence, the solution of (3.1) is

$$S_t = S_0^{e^{-\kappa t}} \exp\left(\frac{\sigma^2}{2\kappa} \left(e^{-\kappa t} - 1 \right) + \kappa e^{-\kappa t} \int_0^t \mu(u) e^{\kappa u} \, \mathrm{d}u + \sigma e^{-\kappa t} \int_0^t e^{\kappa u} \, \mathrm{d}W_u \right).$$

It is easy to see from the integrability of μ that the power of the exponential term

$$\frac{\sigma^2}{2\kappa} \left(e^{-\kappa t} - 1 \right) + \kappa e^{-\kappa t} \int_0^t \mu(u) e^{\kappa u} \, \mathrm{d}u + \sigma e^{-\kappa t} \int_0^t e^{\kappa u} \, \mathrm{d}W_u$$

is bounded for all $0 \le t < T$. Thus, the condition

$$S_t \to 0 \text{ for some } 0 \le t < T \quad \text{implies that} \quad S_T \to 0 \text{ as well.} \tag{3.32}$$

By (3.2), for $0 \le t < T$,

$$\begin{aligned} \lim_{S \to 0} v(S, t; -1) &= \lim_{S \to 0} e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}} \left[(S_T - K)^+ \mid S_t = S \right] \\ &= e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}} \left[(S_T - K)^+ \mid S_T \to 0 \right] \\ &= 0. \end{aligned} \tag{3.33}$$

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Similar to (3.32), the condition

$$S_t \to \infty \text{ for some } 0 \le t < T \text{ implies } S_T \to \infty \text{ as well.} \tag{3.34}$$

Hence, similar to (3.33),

$$\lim_{S \to \infty} v(S, t; 1) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}} \left[(K - S_T)^+ \mid S_T \to \infty \right] = 0. \qquad \square$$

The analytical formula for European option is now derived by solving the PDE (3.27) subject to the terminal condition (3.28) and the boundary condition (3.29) as described in the following theorem.

Theorem 3.3 (European option pricing formula)**.** *Assume that $\mu : [0,T] \to \mathbb{R}$ is integrable. Then, the value of a European option $v(S,t;\phi)$ on the asset spot price S at time $t \le T$ with a strike price K and an expiration date T is represented by*

$$v(S,t;\phi) = \widetilde{u}(S,T-t;\phi) + (\phi K - \phi S)^{+}, \tag{3.35}$$

where

$$\widetilde{u}(S,\tau;\phi) = \int_0^{\tau} \Big(K\left(H_1 + \phi H_2\, N\left[\phi d_1(\rho)\right]\right) + S^{e^{-\kappa\rho}} M\left(H_3 + \phi H_4\, N\left[\phi d_2(\rho)\right]\right)\Big)\, \mathrm{d}\rho \tag{3.36}$$

with

$$\begin{aligned}
H_1 &= \frac{\sigma\sqrt{\kappa}e^{(\kappa-r)\rho-\frac{1}{2}d_1^2(\rho)}}{2\sqrt{\pi}\sqrt{e^{2\kappa\rho}-1}},\\
H_2 &= -re^{-r\rho},\\
H_3 &= -\frac{\sigma\sqrt{\kappa}}{2\sqrt{\pi}}\left(\sqrt{e^{2\kappa\rho}-1}\right)e^{-\frac{1}{2}d_2^2(\rho)},\\
H_4 &= (r-\kappa\mu(T-\tau+\rho))\,e^{\kappa\rho} + \kappa\ln S + \frac{\sigma^2}{2}\left(1-e^{-\kappa\rho}\right) + \kappa\varphi(\rho),\\
M &= \exp\left(-(r+\kappa)\rho - \frac{\sigma^2}{4\kappa}(1-e^{-\kappa\rho})^2 + e^{-\kappa\rho}\varphi(\rho)\right),\\
d_1(\rho) &= \frac{\sqrt{2\kappa}}{\sigma\sqrt{e^{2\kappa\rho}-1}}\left(\ln\frac{K}{S} + \left(\frac{\sigma^2}{2\kappa}+\ln K\right)(e^{\kappa\rho}-1) - \varphi(\rho)\right),\\
d_2(\rho) &= d_1(\rho) - \frac{\sigma\sqrt{1-e^{-2\kappa\rho}}}{\sqrt{2\kappa}},\\
\varphi(\rho) &= \kappa e^{\kappa\tau}\int_{\tau-\rho}^{\tau}\mu(T-w)e^{-\kappa w}\,\mathrm{d}w
\end{aligned}$$

and $N[\cdot]$ is the CDF of the standard normal distribution given by

$$N[z] := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{x^2}{2}}\,dx. \tag{3.37}$$

Proof. For convenience, we denote $v(S,t)$ for $v(S,t;\phi)$ in the proof. This proof is divided into 2 parts: (i) conversion and (ii) solution.

(i) Conversion:

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Note that the domain of the European option value $v(S,t)$ is

$$\{(S,t) \mid 0 < S < \infty \text{ and } 0 \le t \le T\}.$$

First, we change the variables as follows:

$$\tau := T - t, \quad x := -\ln S, \quad \widetilde{\mu}(\tau) := \mu(t) \;\text{ and }\; E(x,\tau) := v(S,t). \tag{3.38}$$

Then, the new domain is $\{(x,\tau) \mid x \in \mathbb{R} \text{ and } 0 \le \tau \le T\}$. By the chain rule,

$$\frac{\partial v(S,t)}{\partial t} = -\frac{\partial E(x,\tau)}{\partial \tau}, \quad \frac{\partial v(S,t)}{\partial S} = -\frac{1}{S}\frac{\partial E(x,\tau)}{\partial x} \quad \text{and}$$
$$\frac{\partial^2 v(S,t)}{\partial S^2} = \frac{1}{S^2}\frac{\partial^2 E(x,\tau)}{\partial x^2} + \frac{1}{S^2}\frac{\partial E(x,\tau)}{\partial x}.$$

Substituting into (3.27), we obtain a new PDE

$$\widehat{\mathcal{L}} E(x,\tau) = 0, \quad x \in \mathbb{R} \;\text{ and }\; 0 < \tau \le T, \tag{3.39}$$

where

$$\widehat{\mathcal{L}} := \frac{\partial}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \left(-\frac{\sigma^2}{2} + \kappa\widetilde{\mu}(\tau) + \kappa x\right)\frac{\partial}{\partial x} + r.$$

By (3.38), the terminal condition (3.28) becomes the initial condition

$$E(x,0) = (\phi K - \phi e^{-x})^+ \tag{3.40}$$

and the boundary conditions (3.29) become

$$\lim_{x\to\infty} E(x,\tau) = 0 \quad \text{if } \phi = -1 \;\text{ and} \tag{3.41}$$

$$\lim_{x\to-\infty} E(x,\tau) = 0 \quad \text{if } \phi = 1. \tag{3.42}$$

To solve the PDE (3.39) subject to (3.40)–(3.42), we apply the ideas of Underwood and Wang [30] by setting

$$E(x,\tau) = u(x,\tau) + g(x), \quad x \in \mathbb{R} \;\text{ and }\; 0 \le \tau \le T, \tag{3.43}$$

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where

$$g(x) = (\phi K - \phi e^{-x})^+. \tag{3.44}$$

Substituting (3.43) into (3.39) and (3.40)–(3.42), we have a new PDE

$$\widehat{\mathcal{L}}u(x,\tau) = f(x,\tau), \quad x \in \mathbb{R} \text{ and } 0 < \tau \le T, \tag{3.45}$$

where

$$f(x,\tau) = -\widehat{\mathcal{L}}g(x),$$

with the initial condition

$$u(x,0) = 0 \tag{3.46}$$

and the boundary condition

$$\lim_{\phi x \to -\infty} u(x,\tau) = \lim_{\phi x \to -\infty} (E(x,\tau) - g(x)) = - \lim_{\phi x \to -\infty} (\phi K - \phi e^{-x})^+ = 0. \tag{3.47}$$

Applying Lemma 3.1 to solve (3.45) subject to (3.46)–(3.47), by using $\mathcal{U} = \mathbb{R}$, $\widetilde{\gamma}(\cdot) = 1$ and $\widetilde{E}(\cdot,\cdot) = 0$, yields

$$u(x,\tau) = \int_0^\tau \int_{-\infty}^\infty f_3(x-z,\tau,\hat{\tau}) f_2(z,\tau,\hat{\tau}) \,\mathrm{d}z\,\mathrm{d}\hat{\tau}, \tag{3.48}$$

where

$$\begin{aligned} f_2(z,\tau,\hat{\tau}) = {} & \frac{\sigma^2 K e^{(r-\kappa)(\hat{\tau}-\tau)}}{2} \delta(z + e^{\kappa(\tau-\hat{\tau})} \ln K) + \phi \Big((r - \kappa\widetilde{\mu}(\hat{\tau}))\, e^{r(\hat{\tau}-\tau) - z e^{\kappa(\hat{\tau}-\tau)}} \\ & -\kappa z e^{(r+\kappa)(\hat{\tau}-\tau) - z e^{\kappa(\hat{\tau}-\tau)}} - rKe^{r(\hat{\tau}-\tau)} \Big) \mathbb{1}_{\{\phi z \ge -\phi e^{\kappa(\tau-\hat{\tau})} \ln K\}}(z) \end{aligned} \tag{3.49}$$

and

$$\begin{aligned} f_3(x,\tau,\hat{\tau}) = {} & \frac{1}{\sqrt{2\pi}} \left(\frac{\sqrt{2\kappa}}{\sigma\sqrt{e^{2\kappa(\tau-\hat{\tau})} - 1}} \right) \\ & \times \exp\left(\frac{-\left(x + \frac{\sigma^2}{2\kappa}(e^{\kappa(\tau-\hat{\tau})} - 1) - \kappa e^{\kappa\tau} \int_{\hat{\tau}}^{\tau} \widetilde{\mu}(w) e^{-\kappa w}\,\mathrm{d}w \right)^2}{2\left(\frac{\sigma\sqrt{e^{2\kappa(\tau-\hat{\tau})}-1}}{\sqrt{2\kappa}} \right)^2} \right). \end{aligned} \tag{3.50}$$

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(ii) Solution:

Note that

$$\int_{-\infty}^{\infty} h(z)\mathbb{1}_{\{\phi z\geq \phi m\}}(z)\,\mathrm{d}z = \lim_{\phi n\to\infty}\phi\int_{m}^{n} h(z)\,\mathrm{d}z \tag{3.51}$$

for any integrable function h and $m\in\mathbb{R}$. By applying (3.51) to (3.49), we can write the solution (3.48) in the form

$$u(x,\tau) = \int_0^{\tau} (I_1 + I_2 + I_3 + I_4)\,\mathrm{d}\hat{\tau}, \tag{3.52}$$

where

$$I_1 := \int_{-\infty}^{\infty} \frac{\sigma^2 K e^{(r-\kappa)(\hat{\tau}-\tau)}}{2}\,\delta(z + e^{\kappa(\tau-\hat{\tau})}\ln K)\, f_3(x-z,\tau,\hat{\tau})\,\mathrm{d}z, \tag{3.53}$$

$$I_2 := -\lim_{\phi n\to\infty}\int_{-e^{\kappa(\tau-\hat{\tau})}\ln K}^{n} rKe^{r(\hat{\tau}-\tau)} f_3(x-z,\tau,\hat{\tau})\,\mathrm{d}z, \tag{3.54}$$

$$I_3 := \lim_{\phi n\to\infty}\int_{-e^{\kappa(\tau-\hat{\tau})}\ln K}^{n} (r-\kappa\widetilde{\mu}(\hat{\tau}))\, e^{r(\hat{\tau}-\tau)-ze^{\kappa(\hat{\tau}-\tau)}} f_3(x-z,\tau,\hat{\tau})\,\mathrm{d}z \text{ and} \tag{3.55}$$

$$I_4 := -\lim_{\phi n\to\infty}\int_{-e^{\kappa(\tau-\hat{\tau})}\ln K}^{n} \kappa z e^{(r+\kappa)(\hat{\tau}-\tau)-ze^{\kappa(\hat{\tau}-\tau)}} f_3(x-z,\tau,\hat{\tau})\,\mathrm{d}z. \tag{3.56}$$

For convenience, we denote

$$\begin{cases} a := e^{\kappa(\tau-\hat{\tau})}, & b := \frac{\sigma^2}{2\kappa}(e^{\kappa(\tau-\hat{\tau})}-1) - \kappa e^{\kappa\tau}\int_{\hat{\tau}}^{\tau}\widetilde{\mu}(w)e^{-\kappa w}\,\mathrm{d}w, \\ c := \frac{\sigma\sqrt{e^{2\kappa(\tau-\hat{\tau})}-1}}{\sqrt{2\kappa}} & \text{and} \quad m := -e^{\kappa(\tau-\hat{\tau})}\ln K. \end{cases} \tag{3.57}$$

Then, (3.50) can be written as

$$f_3(x-z,\tau,\hat{\tau}) = \frac{1}{c\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{x-z+b}{c}\right)^2\right). \tag{3.58}$$

From (3.53), (3.57) and (3.58),

$$\begin{aligned} I_1 &= \frac{\sigma^2 K e^{(r-\kappa)(\hat{\tau}-\tau)}}{2} f_3\left(x + e^{\kappa(\tau-\hat{\tau})}\ln K,\tau,\hat{\tau}\right) = \frac{\sigma^2 K e^{(r-\kappa)(\hat{\tau}-\tau)}}{2} f_3\left(x-m,\tau,\hat{\tau}\right) \\ &= \frac{\sigma^2 K}{2c\sqrt{2\pi}}\exp\left((r-\kappa)(\hat{\tau}-\tau) - \frac{1}{2}\left(\frac{x-m+b}{c}\right)^2\right). \end{aligned} \tag{3.59}$$

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Applying (2.12), (3.37) and (3.58) to (3.54), we obtain

$$
\begin{aligned}
I_2 &= -rKe^{r(\hat{\tau}-\tau)} \lim_{\phi n \to \infty} \int_m^n f_3(x-z,\tau,\hat{\tau})\,\mathrm{d}z \\
&= -rKe^{r(\hat{\tau}-\tau)} \lim_{\phi n \to \infty} \left(\frac{1}{\sqrt{2\pi}} \int_{\frac{m-x-b}{c}}^{\frac{n-x-b}{c}} e^{-\frac{y^2}{2}}\,\mathrm{d}y \right) \\
&= -\phi rKe^{r(\hat{\tau}-\tau)} N\left[-\phi\left(\frac{m-x-b}{c}\right)\right]. \qquad (3.60)
\end{aligned}
$$

By using (3.55), (3.57) and (3.58), we get that

$$
I_3 = (r-\kappa\widetilde{\mu}(\hat{\tau}))\, e^{r(\hat{\tau}-\tau)} J_3, \qquad (3.61)
$$

where

$$
\begin{aligned}
J_3 &= \lim_{\phi n \to \infty} \int_m^n e^{-z/a} f_3(x-z,\tau,\hat{\tau})\,\mathrm{d}z \\
&= \lim_{\phi n \to \infty} \left(\frac{1}{c\sqrt{2\pi}} \int_m^n \exp\left(-\frac{z}{a} - \frac{1}{2}\left(\frac{x-z+b}{c}\right)^2 \right) \mathrm{d}z \right) \\
&= \exp\left(\frac{-x-b+c^2/2a}{a}\right) \\
&\quad \times \lim_{\phi n \to \infty} \left(\frac{1}{c\sqrt{2\pi}} \int_m^n \exp\left(-\frac{1}{2}\left(\frac{z-x-b+c^2/a}{c}\right)^2 \right) \mathrm{d}z \right) \qquad (3.62) \\
&= \exp\left(\frac{-x-b+c^2/2a}{a}\right) \lim_{\phi n \to \infty} \left(\frac{1}{\sqrt{2\pi}} \int_{\frac{m-x-b+c^2/a}{c}}^{\frac{n-x-b+c^2/a}{c}} e^{-\frac{y^2}{2}}\,\mathrm{d}y \right). \qquad (3.63)
\end{aligned}
$$

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Similar to (3.60), applying (2.12) and (3.37) to (3.63) yields

$$
J_3 = \phi \exp\left(\frac{-x-b+c^2/2a}{a}\right) N\left[-\phi\left(\frac{m-x-b+c^2/a}{c}\right)\right]. \qquad (3.64)
$$

Thus,

$$
\begin{aligned}
I_3 = {}& \phi\,(r-\kappa\widetilde{\mu}(\hat{\tau})) \exp\left(r(\hat{\tau}-\tau) + \frac{-x-b+c^2/2a}{a} \right) \\
& \times N\left[-\phi\left(\frac{m-x-b+c^2/a}{c}\right)\right]. \qquad (3.65)
\end{aligned}
$$

Similar to the way we obtain (3.61) and (3.62), we set

$$
\begin{aligned}
I_4 &= -\frac{\kappa e^{r(\hat{\tau}-\tau)}}{a} \lim_{\phi n \to \infty} \int_m^n z e^{-z/a} f_3(x-z,\tau,\hat{\tau})\,\mathrm{d}z \\
&= -\frac{\kappa}{a} \exp\left(r(\hat{\tau}-\tau) + \frac{-x-b+c^2/2a}{a} \right) J_4, \qquad (3.66)
\end{aligned}
$$

where

$$
J_4 = \lim_{\phi n \to \infty} \left(\frac{1}{c\sqrt{2\pi}} \int_m^n z \exp\left(-\frac{1}{2} \left(\frac{z-x-b+c^2/a}{c} \right)^2 \right) \mathrm{d}z \right). \qquad (3.67)
$$

Similar to (3.63), setting $h = -x-b+c^2/a$ in (3.67), we have

$$
\begin{aligned}
J_4 &= \lim_{\phi n \to \infty} \left(\frac{1}{c\sqrt{2\pi}} \int_m^n (z+h) \exp\left(-\frac{1}{2} \left(\frac{z+h}{c} \right)^2 \right) \mathrm{d}z \right. \\
&\qquad \left. - \frac{h}{c\sqrt{2\pi}} \int_m^n \exp\left(-\frac{1}{2} \left(\frac{z+h}{c} \right)^2 \right) \mathrm{d}z \right) \\
&= \frac{c}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} \left(\frac{m+h}{c} \right)^2 \right) - \phi h N\left[-\phi \left(\frac{m+h}{c} \right) \right]. \qquad (3.68)
\end{aligned}
$$

Substituing (3.68) into (3.66) yields

$$
\begin{aligned}
I_4 = &- \frac{\kappa c}{a\sqrt{2\pi}} \exp\left(r(\hat{\tau}-\tau) + \frac{-x-b+c^2/2a}{a} - \frac{1}{2} \left(\frac{m-x-b+c^2/a}{c} \right)^2 \right) \\
&- \phi\kappa \left(\frac{x+b-c^2/a}{a} \right) \exp\left(r(\hat{\tau}-\tau) + \frac{-x-b+c^2/2a}{a} \right) \\
&\times N\left[-\phi \left(\frac{m-x-b+c^2/a}{c} \right) \right]. \qquad (3.69)
\end{aligned}
$$

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Collecting (3.52), (3.59), (3.60), (3.65) and (3.69) after combining I_3 and I_4, we obtain

$$
\begin{aligned}
u(x,\tau) = \int_0^\tau \Bigg(& \frac{\sigma^2 K}{2c\sqrt{2\pi}} \exp\left((r-\kappa)(\hat{\tau}-\tau) - \frac{1}{2}\left(\frac{x-m+b}{c}\right)^2 \right) \\
& - \phi r K e^{r(\hat{\tau}-\tau)} N\left[\phi\left(\frac{x-m+b}{c}\right)\right] \\
& - \frac{\kappa c}{a\sqrt{2\pi}} \exp\left(r(\hat{\tau}-\tau) + \frac{-x-b+c^2/2a}{a} - \frac{1}{2}\left(\frac{x-m+b-c^2/a}{c}\right)^2 \right) \\
& + \phi\left(r - \kappa\widetilde{\mu}(\hat{\tau}) + \kappa\left(\frac{-x-b+c^2/a}{a}\right)\right) \\
& \times \exp\left(r(\hat{\tau}-\tau) + \frac{-x-b+c^2/2a}{a}\right) N\left[\phi\left(\frac{x-m+b-c^2/a}{c}\right)\right] \Bigg)\, \mathrm{d}\hat{\tau}.
\end{aligned}
\tag{3.70}
$$

Substituting $x = -\ln S$ and (3.57) back to (3.70) yields

$$
\begin{aligned}
& \widetilde{u}(S,\tau) = u(x,\tau) \\
& = \int_0^\tau \left(K\left(\widetilde{H}_1 + \phi\widetilde{H}_2 N\left[\phi\widetilde{d}_1(\hat{\tau})\right]\right) + S^{e^{\kappa(\hat{\tau}-\tau)}}\widetilde{M}\left(\widetilde{H}_3 + \phi\widetilde{H}_4 N\left[\phi\widetilde{d}_2(\hat{\tau})\right]\right)\right) \mathrm{d}\hat{\tau},
\end{aligned}
\tag{3.71}
$$

where

$$
\begin{aligned}
\widetilde{H}_1 &= \frac{\sigma\sqrt{\kappa}e^{(r-\kappa)(\hat{\tau}-\tau)-\frac{1}{2}\widetilde{d}_1^2(\hat{\tau})}}{2\sqrt{\pi}\sqrt{e^{2\kappa(\tau-\hat{\tau})}-1}}, \\
\widetilde{H}_2 &= -re^{r(\hat{\tau}-\tau)}, \\
\widetilde{H}_3 &= -\frac{\sigma\sqrt{\kappa}}{2\sqrt{\pi}}\left(\sqrt{e^{2\kappa(\tau-\hat{\tau})}-1}\right)e^{-\frac{1}{2}\widetilde{d}_2^2(\hat{\tau})}, \\
\widetilde{H}_4 &= \left(r-\kappa\widetilde{\mu}(\hat{\tau})\right)e^{\kappa(\tau-\hat{\tau})} + \kappa\ln S + \frac{\sigma^2}{2}\left(1-e^{\kappa(\hat{\tau}-\tau)}\right) + \kappa\widetilde{\varphi}(\hat{\tau}), \\
\widetilde{M} &= \exp\left((r+\kappa)(\hat{\tau}-\tau) - \frac{\sigma^2}{4\kappa}\left(1-e^{\kappa(\hat{\tau}-\tau)}\right)^2 + e^{\kappa(\hat{\tau}-\tau)}\widetilde{\varphi}(\hat{\tau})\right), \\
\widetilde{d}_1(\hat{\tau}) &= \frac{\sqrt{2\kappa}}{\sigma\sqrt{e^{2\kappa(\tau-\hat{\tau})}-1}}\left(\ln\frac{K}{S} + \left(\frac{\sigma^2}{2\kappa}+\ln K\right)\left(e^{\kappa(\tau-\hat{\tau})}-1\right) - \widetilde{\varphi}(\hat{\tau})\right), \\
\widetilde{d}_2(\hat{\tau}) &= \widetilde{d}_1(\hat{\tau}) - \frac{\sigma\sqrt{1-e^{2\kappa(\hat{\tau}-\tau)}}}{\sqrt{2\kappa}} \quad \text{and} \\
\widetilde{\varphi}(\hat{\tau}) &= \kappa e^{\kappa\tau}\int_{\hat{\tau}}^{\tau}\widetilde{\mu}(w)e^{-\kappa w}\,\mathrm{d}w.
\end{aligned}
$$

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From (3.38), (3.43), (3.44), (3.71), $\widetilde{\mu}(w) = \mu(T-w)$ and setting $\rho = \tau - \hat{\tau}$,

$$v(S,t) = u(x,\tau) + (\phi K - \phi e^{-x})^{+} = \widetilde{u}(S, T-t) + (\phi K - \phi S)^{+},$$

where $\widetilde{u}(S,\tau)$ is defined as $\widetilde{u}(S,\tau;\phi)$ in (3.36). □

Remark 3.4. The decomposition of the formula (3.35) as the sum of the integral and the known initial payoff in the second term can provide the bound for the option prices if one can estimate the integral term.

The following corollary describes the put-call parity for European option based on the result of Theorem 3.3.

Corollary 3.5. *Set $p(S,t) = v(S,t;1)$ and $c(S,t) = v(S,t;-1)$ denote the put and call option functions, respectively. Then,*

$$p(S,t) + S = c(S,t) + Ke^{-r(T-t)} + u_{pc}(S, T-t), \tag{3.72}$$

where

$$u_{pc}(S,\tau) = \int_0^{\tau} S^{e^{-\kappa\rho}} M H_4 \,\mathrm{d}\rho,$$

with M and H_4 are defined in Theorem 3.3.

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Proof. The result is obtained straight forward from Theorem 3.3. □

Remark 3.6. Note that the put-call parity formula (3.72) for European option on underlying asset following (3.1) is different from that of BS formula (for stock) with the addition of the last term $u_{pc}(S, T-t)$.

From Theorem 3.3, we note that (3.35) can be applied with any integrable long-run mean function. In Chapter 4, we compare the results computed from the obtained formula with those from MC simulations and BS-type formula in various kinds of long-run mean functions. Moreover, the behaviors of European option prices have been demonstrated and discussed.

3.3 American Commodity Option Pricing Based on PDE

In this section, we provide systems of PDE for American option and EEP and derive an analytical formula of American option value from the solution of the system of PDE for EEP in a similar way as presented in Theorem 3.3.

In the same way as the process following a geometric Brownian motion, the underlying commodity spot price satisfying the model (3.1) is a diffusion process. Then, under the condition that the risk-free interest rate is constant, for each time $t \in [0, T]$, there exists the critical commodity spot price $\gamma(t; \phi)$ below (above) which the American put (call) option should be exercised early ([7], eqs. 1.2 and 1.3), where $\phi = -1$ for a call option and $\phi = 1$ for a put option. The critical price function $\gamma(t; \phi)$ is called the optimal exercise boundary function, we denote for convenience by $\gamma(t) = \gamma(t; \phi)$. Denote the first passage time of the underlying commodity spot price to its boundary by

$$\tau_e^{(\phi)} = \inf\{t < u \leq T \mid S_u = \gamma(u; \phi)\} \tag{3.73}$$

and the American option value by $V(S, t; \phi, \gamma(t)) = V(S, t; \phi)$.

Note that the domain of $V(S, t; \phi, \gamma(t))$ is

$$\{(S, t) \mid S > 0,\ \phi S > \phi\gamma(t),\ 0 \leq t \leq T\}.$$

Since the underlying commodity spot price process following (3.1) is a Markovian diffusion process and the risk-free interest rate r is constant, by applying the result from [23] (Proposition 2 and the same argument as in the proof of Proposition 3), we have that the American option value (3.3) is the solution of a system of PDE as demonstrated in the following proposition.

Proposition 3.7. *Assume that the underlying commodity spot price S_t follows the mean-reverting process* (3.1) *and the risk-free interest rate r is constant. Then, the*

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value of an American option $V(S,t;\phi,\gamma(t))$ *on the commodity spot price* S *at time* t*, with strike price* K *and expiration date* T *satisfies the PDE*

$$\mathcal{L}V(S,t;\phi,\gamma(t)) = 0, \tag{3.74}$$

for $S > 0,\ \phi S > \phi\gamma(t)$ *and* $0 \le t < T$*, where*

$$\mathcal{L} := \frac{\partial}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2}{\partial S^2} + \kappa(\mu(t) - \ln S)S\frac{\partial}{\partial S} - r, \tag{3.75}$$

subject to the terminal condition

$$V(S,T;\phi,\gamma(t)) = (\phi K - \phi S)^+, \tag{3.76}$$

for $S > 0,\ \phi S > \phi\gamma(t)$ *and the boundary conditions*

$$\lim_{S\to 0} V(S,t;-1,\gamma(t)) = 0, \tag{3.77}$$

$$\lim_{S\to\infty} V(S,t;1,\gamma(t)) = 0 \ \ \textit{and} \tag{3.78}$$

$$\lim_{S\to\gamma(t)} V(S,t;\phi,\gamma(t)) = \phi K - \phi\gamma(t), \tag{3.79}$$

for $0 \le t < T$.

From Proposition 3.7, we may solve the system of PDE (3.74) subject to (3.76)–(3.79) directly in order to derive the formula for pricing the American option value.

Alternatively, since the underlying commodity spot price S_t has the Markov property, Nunes ([23], Proposition 1) showed that the value of an American option $V(S,t;\phi)$ in (3.3) can be decomposed into the sum of the value of the European counterpart $v(S,t;\phi)$ and the EEP $W(S,t;\phi,\gamma(t))$, i.e.,

$$V(S,t;\phi,\gamma(t)) = v(S,t;\phi) + W(S,t;\phi,\gamma(t)), \tag{3.80}$$

with $v(S,t;\phi)$ is defined as (3.2) and

$$W(S,t;\phi,\gamma(t)) = \int_t^T e^{-r(u-t)}\left(\phi K - \phi\gamma(u) - v(\gamma(u),u)\right)\mathbb{Q}(\tau_e^{(\phi)} \in \mathrm{d}u \mid \mathcal{F}_t), \tag{3.81}$$

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where $\mathbb{Q}(\tau_e^{(\phi)} \in \mathrm{d}u \mid \mathcal{F}_t)$ represents the probability density function of the first passage time $\tau_e^{(\phi)}$ defined by (3.73).

From (3.80), we instead separate the system of PDE for the American option into a PDE for the European option and a system of PDE for EEP. The result is stated in the following proposition.

Proposition 3.8. *Let $\phi \in \{-1, 1\}$ represents the type of functions, $K, T > 0$ and $\gamma(\cdot)$ be a function on $[0, T]$. We further let $v(S, t; \phi)$ and $W(S, t; \phi, \gamma(t))$ be functions. Suppose that $v(S, t; \phi)$ and $W(S, t; \phi, \gamma(t))$ satisfy the PDEs subject to the terminal conditions and boundary conditions written as follows.*

1.

$$\mathcal{L}v(S, t; \phi) = 0 \tag{3.82}$$

for $S > 0$ and $0 \le t < T$, where $\mathcal{L}$ is defined in (3.75), subject to the terminal condition

$$v(S, T; \phi) = (\phi K - \phi S)^+ \tag{3.83}$$

for $S > 0$ and the boundary conditions

$$\lim_{S \to 0} v(S, t, -1) = 0 \ \ \textit{and} \tag{3.84}$$

$$\lim_{S \to \infty} v(S, t, 1) = 0 \tag{3.85}$$

for $0 \le t < T$.

2.

$$\mathcal{L}W(S, t; \phi, \gamma(t)) = 0 \tag{3.86}$$

for $S > 0$, $\phi S > \phi\gamma(t)$ and $0 \le t < T$, where $\mathcal{L}$ is defined in (3.75), subject to the terminal condition

$$W(S, T; \phi, \gamma(t)) = 0 \tag{3.87}$$

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for $S > 0,\ \phi S > \phi\gamma(t)$ *and the boundary conditions*

$$\lim_{S\to 0} W(S,t;-1,\gamma(t)) = 0 \tag{3.88}$$

$$\lim_{S\to\infty} W(S,t;1,\gamma(t)) = 0 \ \ and \tag{3.89}$$

$$\lim_{S\to\gamma(t)} W(S,t;\phi,\gamma(t)) = \phi K - \phi\gamma(t) - v(\gamma(t),t) \tag{3.90}$$

for $0 \le t < T$, *where* $v(\cdot,\cdot,\phi)$ *satisfies* (3.82)–(3.85) .

Then, $v(S,t;\phi)$ *and* $W(S,t;\phi;\gamma(t))$ *satisfies* (3.2) *and* (3.81)*, respectively.*

Furthermore, if we set $V(S,t;\phi,\gamma(t)) = v(S,t;\phi) + W(S,t;\phi,\gamma(t))$ *for* $S > 0,\ \phi S > \phi\gamma(t)$ *and* $0 \le t \le T$*, then* $V(S,t;\phi,\gamma(t))$ *solves the system of PDE* (3.74) *subject to the terminal condition* (3.76) *and the boundary conditions* (3.77)–(3.79).

Proof. By applying the Feynman-Kac formula with the mean-reverting process (3.1) and the terminal condition (3.83), the solution of the PDE (3.82) can be written as (3.2).

By the same argument as in the proof of Proposition 2 and 3 of [23] and the facts (3.32) and (3.34), (3.81) is the solution of the system of the PDE (3.86) subject to the terminal condition (3.87) and the boundary conditions (3.88)–(3.90).

The last statement is obtained directly from (3.80) with (3.82)–(3.90). □

Next, the derivation of an analytical representation formulas for EEP is provide in Theorem 3.9 by solving the system of PDE (3.86) subject to (3.87)–(3.90).

Theorem 3.9 (EEP pricing formula). *Assume that* $\mu : [0,T] \to \mathbb{R}$ *is integrable and* $\gamma(\cdot) : [0,T] \to \mathbb{R}^+$ *is a differentiable function such that* $\phi\gamma(\cdot) \le \phi K$*. Then, the EEP* $W(S,t;\phi,\gamma(t))$ *on the underlying asset spot price* S *at time* $t \le T$ *with a strike price* K *and an expiration date* T *is represented by*

$$W(S,t;\phi,\gamma(t)) = \widetilde{U}(S,T-t;\phi,\gamma(T-t)) - \widetilde{u}(S,T-t;\phi), \tag{3.91}$$

where

$$\widetilde{U}(S,\tau;\phi,\gamma(\tau)) = \int_0^\tau \Big(H_1 + H_2\big(N\left[d_1(K)\right] - N\left[d_1(\gamma(T-\tau+\rho))\right]\big) + M\big(H_3 + H_4\left(N\left[d_2(K)\right] - N\left[d_2(\gamma(T-\tau+\rho))\right]\right)\big)\Big)\,\mathrm{d}\rho \tag{3.92}$$

with

$$\begin{aligned}
H_1 &= \frac{\sigma K\sqrt{\kappa}e^{(\kappa-r)\rho-\frac{1}{2}d_1^2(K)}}{2\sqrt{\pi}\sqrt{e^{2\kappa\rho}-1}},\\
H_2 &= rKe^{-r\rho},\\
H_3 &= \frac{\sigma\sqrt{\kappa}\sqrt{e^{2\kappa\rho}-1}}{2\sqrt{\pi}}\left(e^{-\frac{1}{2}d_2^2(\gamma(T-\tau+\rho))} - e^{-\frac{1}{2}d_2^2(K)}\right),\\
H_4 &= \left(\kappa\mu(T-\tau+\rho)-r\right)e^{\kappa\rho} - \kappa\ln S + \frac{\sigma^2}{2}\left(e^{-\kappa\rho}-1\right) - \kappa\varphi(\tau-\rho),\\
M &= S^{e^{-\kappa\rho}}\exp\left(-(r+\kappa)\rho - \frac{\sigma^2}{4\kappa}\left(1-e^{-\kappa\rho}\right)^2 + e^{-\kappa\rho}\varphi(\tau-\rho)\right),\\
d_1(P) &= \frac{\sqrt{2\kappa}}{\sigma\sqrt{e^{2\kappa\rho}-1}}\left(\ln\frac{S}{P} + (1-e^{\kappa\rho})\left(\ln P + \frac{\sigma^2}{2\kappa}\right) + \varphi(\tau-\rho)\right),\\
d_2(P) &= d_1(P) + \frac{\sigma\sqrt{1-e^{-2\kappa\rho}}}{\sqrt{2\kappa}},\\
\varphi(\rho) &= \kappa\int_\rho^\tau \mu(T-w)e^{\kappa(\tau-w)}\,\mathrm{d}w
\end{aligned}$$

and $\widetilde{u}(S,\tau;\phi)$ *and* $N[\cdot]$ *is defined as* (3.36) *and* (3.37) *in Theorem 3.3, respectively.*

Proof. For convenience in writing, we denote $W(\cdot,\cdot)$, $v(\cdot,\cdot)$ and $u(\cdot,\cdot)$ for $W(\cdot,\cdot;\phi,\gamma(t))$, $v(\cdot,\cdot;\phi)$ and $u(\cdot,\cdot;\phi,\gamma(t))$, respectively, and leave the representation ϕ for the type of options and the function $\gamma(\cdot)$ in this proof.

This proof is divided into 2 parts: (i) conversion and (ii) solution.

(i) Conversion:

From Proposition 3.8, the value of the EEP $W(S,t)$ is the solution of the system of the PDE (3.86) subject to the terminal condition (3.87) and boundary conditions (3.88)–(3.90).

Note that the domain of EEP $W(S,t)$ is

$$\{(S,t) \mid 0 < S,\ \phi\gamma(t) < \phi S \text{ and } 0 \le t \le T\}.$$

First, we change the variables as follows:

$$\begin{cases} \tau := T - t, \\ \widetilde{\gamma}(\tau) := \gamma(T-\tau), \\ x := \ln(\widetilde{\gamma}(\tau)/S), \\ \widetilde{\mu}(\tau) := \mu(T-\tau), \\ E(x,\tau) := \widetilde{v}(S,\tau) := v(S,t) \text{ and} \\ D(x,\tau) := \widetilde{W}(S,\tau) := W(S,t). \end{cases} \tag{3.93}$$

Then, the new domain is $\{(x,\tau) \mid \phi x \le 0 \text{ and } 0 \le \tau \le T\}$. By the chain rule,

$$\frac{\partial W(S,t)}{\partial t} = -\frac{\partial D(x,\tau)}{\partial \tau} - \frac{\widetilde{\gamma}'(\tau)}{\widetilde{\gamma}(\tau)}\frac{\partial D(x,\tau)}{\partial x},$$
$$\frac{\partial W(S,t)}{\partial S} = -\frac{1}{S}\frac{\partial D(x,\tau)}{\partial x} \text{ and}$$
$$\frac{\partial^2 W(S,t)}{\partial S^2} = \frac{1}{S^2}\frac{\partial^2 D(x,\tau)}{\partial x^2} + \frac{1}{S^2}\frac{\partial D(x,\tau)}{\partial x}.$$

Substituting into (3.86) and using (3.93) with (3.87)–(3.90), we obtain the new system of PDE

$$\widehat{\mathcal{L}}D(x,\tau) = 0, \quad \phi x < 0 \text{ and } 0 < \tau \le T, \tag{3.94}$$

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where

$$\widehat{\mathcal{L}} := \frac{\partial}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \left(\kappa\widetilde{\mu}(\tau) - \frac{\sigma^2}{2} - \kappa\ln\widetilde{\gamma}(\tau) + \frac{\widetilde{\gamma}'(\tau)}{\widetilde{\gamma}(\tau)} + \kappa x\right)\frac{\partial}{\partial x} + r.$$

By (3.93) the terminal condition becomes the initial condition

$$D(x,0) = 0 \tag{3.95}$$

and the boundary conditions (3.88)–(3.90) become

$$\lim_{\phi x \to -\infty} D(x,\tau) = 0 \text{ and} \tag{3.96}$$

$$D(0,\tau) = \phi K - \phi\widetilde{\gamma}(\tau) - E(0,\tau). \tag{3.97}$$

To solve the system of the PDE (3.94) subject to (3.95)–(3.97), we applies ideas from Underwood and Wang [30] again. We set

$$D(x,\tau) := u(x,\tau) + g(x,\tau), \quad \phi x \le 0 \text{ and } 0 \le \tau \le T, \tag{3.98}$$

where

$$g(x,\tau) := (\phi K - \phi\widetilde{\gamma}(\tau)e^{-x})^{+} - E(x,\tau) \tag{3.99}$$

and

$$u(0,\tau) = 0. \tag{3.100}$$

By using the assumption that $\phi\widetilde{\gamma}(\tau) \le \phi K$ for all $0 \le \tau \le T$, (3.98)–(3.100) satisfy the boundary condition (3.97). From (3.83) and (3.93),

$$E(x,0) = v(S,T) = (\phi K - \phi S)^{+} = (\phi K - \phi\widetilde{\gamma}(0)e^{-x})^{+}$$

for $\phi x \le 0$. Thus, (3.99) implies

$$g(x,0) = (\phi K - \phi\widetilde{\gamma}(0)e^{-x})^{+} - E(x,0) = 0, \quad \phi x \le 0. \tag{3.101}$$

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Substituting (3.98) into (3.94) and (3.95) and using (3.101), we have the new system of PDE

$$\widehat{\mathcal{L}}u(x,\tau) = f(x,\tau), \quad \phi x < 0 \text{ and } 0 < \tau \le T, \tag{3.102}$$

where

$$f(x,\tau) := -\widehat{\mathcal{L}}g(x,\tau), \tag{3.103}$$

with the initial condition

$$u(x,0) = 0, \quad \phi x \le 0. \tag{3.104}$$

Similarly, from (3.84), (3.85), (3.98) and (3.99), the boundary condition (3.96) becomes

$$\begin{aligned}
\lim_{\phi x \to -\infty} u(x,\tau) &= \lim_{\phi x \to -\infty} -g(x,\tau) \\
&= \lim_{\phi x \to -\infty} \big(-(\phi K - \phi\widetilde{\gamma}(\tau)e^{-x})^{+} + E(x,\tau)\big) \\
&= \lim_{\phi x \to -\infty} E(x,\tau) \\
&= \lim_{\phi \ln S \to \infty} v(S,t) \\
&= 0, \quad 0 < \tau \le T. \qquad (3.105)
\end{aligned}$$

To solve (3.102) subject to (3.104) and (3.105), we now apply Lemma 3.1. First, extend

$$u(x,\tau) = 0, \quad 0 \le \phi x \;\text{ and }\; 0 \le \tau \le T,$$

$$f(x,\tau) = 0, \quad 0 \le \phi x, 0 \le \tau \le T \text{ and } 0 \ge \phi x,\; \tau = 0.$$

Then, (3.102) and (3.104) becomes

$$\widehat{\mathcal{L}}u(x,\tau) = f(x,\tau), \quad x \in \mathbb{R} \;\text{ and }\; 0 \le \tau \le T, \qquad (3.106)$$

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with the initial condition

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$$u(x,0) = 0, \quad x \in \mathbb{R}. \qquad (3.107)$$

Note from (3.82) that

$$\widehat{\mathcal{L}}E(x,\tau) = \mathcal{L}v(S,t) = 0, \quad \phi x < 0 \;\text{ and }\; 0 \le \tau \le T. \qquad (3.108)$$

From (3.99), (3.103), (3.105) and (3.108), by using $\mathcal{U} = (-\infty, 0)$ and $\widetilde{E}(\cdot,\cdot) = E(\cdot,\cdot)$, applying Lemma 3.1 to solve (3.106) subject to (3.105) and (3.107) yields

$$u(x,\tau) = \int_0^{\tau}\int_{-\infty}^{\infty} f_3(x-z,\tau,\hat{\tau}) f_2(z,\tau,\hat{\tau})\,\mathrm{d}z\,\mathrm{d}\hat{\tau}, \qquad (3.109)$$

where

$$f_2(z,\tau,\hat{\tau}) = \frac{\sigma^2 K e^{(r-\kappa)(\hat{\tau}-\tau)}}{2}\delta\Big(z - e^{\kappa(\tau-\hat{\tau})}\ln\frac{\widetilde{\gamma}(\hat{\tau})}{K}\Big) - \phi r K e^{r(\hat{\tau}-\tau)}$$
$$+ \phi\widetilde{\gamma}(\hat{\tau})e^{r(\hat{\tau}-\tau)-ze^{\kappa(\hat{\tau}-\tau)}}\left(r - \kappa\widetilde{\mu}(\hat{\tau}) + \kappa\ln\widetilde{\gamma}(\hat{\tau}) - \kappa z e^{\kappa(\hat{\tau}-\tau)}\right)$$
$$\times \mathbb{1}_{\left\{\phi e^{\kappa(\tau-\hat{\tau})}\ln\frac{\widetilde{\gamma}(\hat{\tau})}{K}\le\phi z<0\right\}}(z,\hat{\tau}) \tag{3.110}$$

and

$$f_3(x,\tau,\hat{\tau})$$
$$= \frac{1}{\sqrt{2\pi}}\left(\frac{\sqrt{2\kappa}}{\sigma\sqrt{e^{2\kappa(\tau-\hat{\tau})}-1}}\right)$$
$$\times \exp\left(\frac{-\left(x - \ln\widetilde{\gamma}(\tau) + e^{\kappa(\tau-\hat{\tau})}\ln\widetilde{\gamma}(\hat{\tau}) + \frac{\sigma^2}{2\kappa}\left(e^{\kappa(\tau-\hat{\tau})}-1\right) - \varphi(\tau,\hat{\tau})\right)^2}{2\left(\frac{\sigma\sqrt{e^{2\kappa(\tau-\hat{\tau})}-1}}{\sqrt{2\kappa}}\right)^2}\right) \tag{3.111}$$

with $\varphi(\tau,\hat{\tau}) = \kappa e^{\kappa\tau}\int_{\hat{\tau}}^{\tau}\widetilde{\mu}(w)e^{-\kappa w}\mathrm{d}w.$

(ii) Solution:

Note from the assumption that $\widetilde{\gamma}(\hat{\tau}) \le K$ when $\phi = 1$. Thus, $\ln\frac{\widetilde{\gamma}(\hat{\tau})}{K} \le 0$. On a contrary, when $\phi = -1$, we have $\widetilde{\gamma}(\hat{\tau}) \ge K$. Thus, $\ln\left(\frac{\widetilde{\gamma}(\hat{\tau})}{K}\right) \ge 0$.

Similar to (3.51),

$$\int_{-\infty}^{\infty} h(z)\mathbb{1}_{\{\phi m\le\phi z<0\}}(z)\,\mathrm{d}z = \phi\int_m^0 h(z)\,\mathrm{d}z \tag{3.112}$$

for any integrable function h and $m \in \mathbb{R}$. By applying (3.112) to (3.110), we can write the solution (3.109) in the form

$$u(x,\tau) = \int_0^{\tau}(I_1 + I_2 + I_3 + I_4)\,\mathrm{d}\hat{\tau}, \tag{3.113}$$

where

$$I_1 := \int_{-\infty}^{\infty} \frac{\sigma^2 K e^{(r-\kappa)(\hat{\tau}-\tau)}}{2} \delta\Big(z - e^{\kappa(\tau-\hat{\tau})} \ln \frac{\widetilde{\gamma}(\hat{\tau})}{K}\Big) f_3(x-z,\tau,\hat{\tau}) \,\mathrm{d}z, \tag{3.114}$$

$$I_2 := -\int_{e^{\kappa(\tau-\hat{\tau})}\ln \frac{\widetilde{\gamma}(\hat{\tau})}{K}}^{0} rKe^{r(\hat{\tau}-\tau)} f_3(x-z,\tau,\hat{\tau}) \,\mathrm{d}z, \tag{3.115}$$

$$I_3 := \int_{e^{\kappa(\tau-\hat{\tau})}\ln \frac{\widetilde{\gamma}(\hat{\tau})}{K}}^{0} \widetilde{\gamma}(\hat{\tau}) e^{r(\hat{\tau}-\tau) - ze^{\kappa(\hat{\tau}-\tau)}} \left(r - \kappa\widetilde{\mu}(\hat{\tau}) + \kappa \ln \widetilde{\gamma}(\hat{\tau})\right) f_3(x-z,\tau,\hat{\tau}) \,\mathrm{d}z \ \text{ and} \tag{3.116}$$

$$I_4 := -\int_{e^{\kappa(\tau-\hat{\tau})}\ln \frac{\widetilde{\gamma}(\hat{\tau})}{K}}^{0} \kappa\widetilde{\gamma}(\hat{\tau}) z e^{(r+\kappa)(\hat{\tau}-\tau) - ze^{\kappa(\hat{\tau}-\tau)}} f_3(x-z,\tau,\hat{\tau}) \,\mathrm{d}z, \tag{3.117}$$

For convenience, we denote

$$\begin{cases} a := e^{\kappa(\tau-\hat{\tau})}, \quad b := -\ln \widetilde{\gamma}(\tau) + a \ln \widetilde{\gamma}(\hat{\tau}) + \frac{\sigma^2}{2\kappa}(a-1) - \varphi(\hat{\tau}), \\ c := \frac{\sigma\sqrt{e^{2\kappa(\tau-\hat{\tau})}-1}}{\sqrt{2\kappa}}, \quad m := e^{\kappa(\tau-\hat{\tau})} \ln \frac{\widetilde{\gamma}(\hat{\tau})}{K} \quad \text{and} \quad \varphi(\hat{\tau}) := \kappa \int_{\hat{\tau}}^{\tau} \widetilde{\mu}(w) e^{\kappa(\tau-w)} \mathrm{d}w. \end{cases} \tag{3.118}$$

Then, in the same way as (3.58) for the case of European option, (3.111) can be written as

$$f_3(x-z,\tau,\hat{\tau}) = \frac{1}{c\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-z+b}{c}\right)^2\right). \tag{3.119}$$

Similar to (3.59), from (3.114) and (3.119),

$$I_1 = \frac{\sigma^2 K}{2c\sqrt{2\pi}} \exp\left((r-\kappa)(\hat{\tau}-\tau) - \frac{1}{2}\left(\frac{x-m+b}{c}\right)^2\right). \tag{3.120}$$

In the same way as (3.60) by replacing $\lim\limits_{\phi n \to 0}$ instead of $\lim\limits_{\phi n \to \infty}$ and applying the fact that

$$N[b] - N[a] = \frac{1}{\sqrt{2\pi}} \int_b^a e^{-\frac{y^2}{2}}\, dy,$$

(3.115) becomes

$$\begin{aligned} I_2 &= -rKe^{r(\hat{\tau}-\tau)} \frac{1}{\sqrt{2\pi}} \int_{\frac{m-x-b}{c}}^{\frac{-x-b}{c}} e^{-\frac{y^2}{2}} \,\mathrm{d}y \\ &= rKe^{r(\hat{\tau}-\tau)} \left(N\left[\frac{m-x-b}{c}\right] - N\left[\frac{-x-b}{c}\right]\right). \end{aligned} \tag{3.121}$$

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Similar to (3.121) and the same argument as (3.61)–(3.63) and (3.64)–(3.65),

$$\begin{aligned}
I_3 =& \widetilde{\gamma}(\hat{\tau})e^{r(\hat{\tau}-\tau)}\left(r - \kappa\widetilde{\mu}(\hat{\tau}) + \kappa \ln \widetilde{\gamma}(\hat{\tau})\right) \int_m^0 e^{-z/a} f_3(x-z,\tau,\hat{\tau})\,\mathrm{d}z \\
=& \widetilde{\gamma}(\hat{\tau})e^{r(\hat{\tau}-\tau)}\left(r - \kappa\widetilde{\mu}(\hat{\tau}) + \kappa \ln \widetilde{\gamma}(\hat{\tau})\right) \\
& \times \exp\left(\frac{-x-b+c^2/(2a)}{a}\right)\left(\frac{1}{\sqrt{2\pi}}\int_{\frac{m-x-b+c^2/a}{c}}^{\frac{-x-b+c^2/a}{c}} e^{-\frac{y^2}{2}}\,\mathrm{d}y\right) \\
=& \widetilde{\gamma}(\hat{\tau})\left(r - \kappa\widetilde{\mu}(\hat{\tau}) + \kappa \ln \widetilde{\gamma}(\hat{\tau})\right)\exp\left(r(\hat{\tau}-\tau) + \frac{-x-b+c^2/(2a)}{a}\right) \\
& \times \left(N\left[\frac{-x-b+\frac{c^2}{a}}{c}\right] - N\left[\frac{m-x-b+\frac{c^2}{a}}{c}\right]\right).
\end{aligned} \tag{3.122}$$

Similar to (3.122) and the same argument as (3.66)–(3.69) by setting $h = -x-b+c^2/a$,

$$\begin{aligned}
I_4 =& -\frac{\kappa\widetilde{\gamma}(\hat{\tau})e^{r(\hat{\tau}-\tau)}}{a}\int_m^0 z e^{-ze^{\kappa(\hat{\tau}-\tau)}} f_3(x-z,\tau,\hat{\tau})\,\mathrm{d}z \\
=& -\frac{\kappa\widetilde{\gamma}(\hat{\tau})e^{r(\hat{\tau}-\tau)}}{a}\exp\left(\frac{-x-b+c^2/2a}{a}\right) \\
& \times \left(\frac{-c}{\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{h}{c}\right)^2} - e^{-\frac{1}{2}\left(\frac{m+h}{c}\right)^2}\right) - h\left(\frac{1}{\sqrt{2\pi}}\int_{\frac{m+h}{c}}^{\frac{h}{c}} e^{-\frac{y^2}{2}}\,\mathrm{d}z\right)\right) \\
=& \frac{\kappa\widetilde{\gamma}(\hat{\tau})}{a}\exp\left(r(\hat{\tau}-\tau) + \frac{-x-b+c^2/2a}{a}\right) \\
& \times \left(\frac{c}{\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{-x-b+\frac{c^2}{a}}{c}\right)^2} - e^{-\frac{1}{2}\left(\frac{m-x-b+\frac{c^2}{a}}{c}\right)^2}\right)\right. \\
& \left. + \left(-x-b+\frac{c^2}{a}\right)\left(N\left[\frac{-x-b+\frac{c^2}{a}}{c}\right] - N\left[\frac{m-x-b+\frac{c^2}{a}}{c}\right]\right)\right).
\end{aligned} \tag{3.123}$$

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Collecting (3.113) and (3.120)–(3.123), after combining I_3 and I_4 we obtain

$$
\begin{aligned}
u(x,\tau) = \int_0^\tau \Bigg(& \frac{\sigma^2 K}{2c\sqrt{2\pi}} \exp\left((r-\kappa)(\hat{\tau}-\tau) - \frac{1}{2}\left(\frac{m-x-b}{c}\right)^2 \right) \\
& + rKe^{r(\hat{\tau}-\tau)} \left(N\left[\frac{m-x-b}{c}\right] - N\left[\frac{-x-b}{c}\right] \right) \\
& + \widetilde{\gamma}(\hat{\tau}) \exp\left(r(\hat{\tau}-\tau) + \frac{-x-b+c^2/(2a)}{a} \right) \\
& \quad \times \Bigg(\frac{\kappa c}{a\sqrt{2\pi}} \left(e^{-\frac{1}{2}\left(\frac{-x-b+c^2/a}{c}\right)^2} - e^{-\frac{1}{2}\left(\frac{m-x-b+c^2/a}{c}\right)^2} \right) \\
& \quad\quad + \left(\kappa\widetilde{\mu}(\hat{\tau}) - r - \kappa \ln \widetilde{\gamma}(\hat{\tau}) - \frac{\kappa}{a}\left(-x-b+c^2/a\right) \right) \\
& \quad\quad\quad \times \left(N\left[\frac{m-x-b+c^2/a}{c}\right] - N\left[\frac{-x-b+c^2/a}{c}\right] \right) \Bigg) \Bigg) \mathrm{d}\hat{\tau} \\
= \int_0^\tau \Bigg(& \frac{\sigma^2 K}{2c\sqrt{2\pi}} \exp\left(r(\hat{\tau}-\tau) - \frac{1}{2}(d_{11})^2 \right) + rKe^{r(\hat{\tau}-\tau)} \left(N[d_{11}] - N[d_{12}] \right) \\
& + \widetilde{\gamma}(\hat{\tau}) \exp\left(r(\hat{\tau}-\tau) + d_3 \right) \Bigg(\frac{\kappa c}{a\sqrt{2\pi}} \left(e^{-\frac{1}{2}(d_{22})^2} - e^{-\frac{1}{2}(d_{21})^2} \right) \\
& \quad + \left(\kappa\widetilde{\mu}(\hat{\tau}) - r - \kappa \ln \widetilde{\gamma}(\hat{\tau}) - \frac{\kappa c d_{22}}{a} \right) \left(N[d_{21}] - N[d_{22}] \right) \Bigg) \Bigg) \mathrm{d}\hat{\tau},
\end{aligned}
\tag{3.124}
$$

where

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$$
\begin{aligned}
d_{11} &:= \frac{m-x-b}{c}, \\
d_{12} &:= \frac{-x-b}{c}, \\
d_{21} &:= \frac{m-x-b+c^2/a}{c}, \\
d_{22} &:= \frac{-x-b+c^2/a}{c} \quad \text{and} \\
d_3 &:= \frac{-x-b+c^2/(2a)}{a}.
\end{aligned}
$$

Defining

$$\widetilde{d}_1(P) := \frac{\sqrt{2\kappa}}{\sigma\sqrt{e^{2\kappa(\tau-\hat{\tau})}-1}}\left(\ln\frac{S}{P} + \left(1-e^{\kappa(\tau-\hat{\tau})}\right)\left(\ln P + \frac{\sigma^2}{2\kappa}\right) + \varphi(\hat{\tau})\right),$$
$$\widetilde{d}_2(P) := \widetilde{d}_1(P) + \frac{\sigma\sqrt{1-e^{2\kappa(\hat{\tau}-\tau)}}}{\sqrt{2\kappa}}$$

and substituting $x = \ln(\widetilde{\gamma}(\tau)/S)$ and (3.118), we rewrite

$$\begin{aligned}
d_{11} &= \frac{1}{c}\left(\ln S - a\ln K + \frac{\sigma^2}{2\kappa}(1-a) + \varphi(\hat{\tau})\right) \\
&= \frac{1}{c}\left(\ln(S/K) + \left(\ln K + \frac{\sigma^2}{2\kappa}\right)(1-a) + \varphi(\hat{\tau})\right) \\
&= \widetilde{d}_1(K), && (3.125) \\
d_{12} &= \frac{1}{c}\left(\ln S - a\ln\widetilde{\gamma}(\hat{\tau}) + \frac{\sigma^2}{2\kappa}(1-a) + \varphi(\hat{\tau})\right) \\
&= \frac{1}{c}\left(\ln(S/\widetilde{\gamma}(\hat{\tau})) + \left(\ln\widetilde{\gamma}(\hat{\tau}) + \frac{\sigma^2}{2\kappa}\right)(1-a) + \varphi(\hat{\tau})\right) \\
&= \widetilde{d}_1(\widetilde{\gamma}(\hat{\tau})), && (3.126) \\
d_{21} &= d_{11} + \frac{c^2/a}{c} = \widetilde{d}_1(K) + \frac{c^2/a}{c} \\
&= \frac{1}{c}\left(\ln(S/K) + \left(\ln K - \frac{\sigma^2}{2\kappa a}\right)(1-a) + \varphi(\hat{\tau})\right) \\
&= \widetilde{d}_2(K), && (3.127) \\
d_{22} &= d_{12} + \frac{c^2/a}{c} = \widetilde{d}_1(\widetilde{\gamma}(\hat{\tau})) + \frac{c^2/a}{c} \\
&= \frac{1}{c}\left(\ln(S/\widetilde{\gamma}(\hat{\tau})) + \left(\ln\widetilde{\gamma}(\hat{\tau}) - \frac{\sigma^2}{2\kappa a}\right)(1-a) + \varphi(\hat{\tau})\right) \\
&= \widetilde{d}_2(\widetilde{\gamma}(\hat{\tau})) \text{ and} && (3.128) \\
d_3 &= \frac{1}{a}\left(\ln S - a\ln\widetilde{\gamma}(\hat{\tau}) - \frac{\sigma^2}{4\kappa a}(a-1)^2 + \varphi(\hat{\tau})\right) \\
&= \ln S^{\frac{1}{a}} - \ln\widetilde{\gamma}(\hat{\tau}) - \frac{\sigma^2}{4\kappa}\left(1-\frac{1}{a}\right)^2 + \frac{1}{a}\varphi(\hat{\tau}). && (3.129)
\end{aligned}$$

Substituting (3.118) and (3.125)–(3.129) back to (3.124) yields

$$\begin{aligned}
u(x,\tau) = \int_0^{\tau} \Big(\widetilde{H}_1 + \widetilde{H}_2\left(N\left[\widetilde{d}_1(K)\right] - N\left[\widetilde{d}_1(\widetilde{\gamma}(\hat{\tau}))\right]\right) \\
+\widetilde{M}\left(\widetilde{H}_3 + \widetilde{H}_4\left(N\left[\widetilde{d}_2(K)\right] - N\left[\widetilde{d}_2(\widetilde{\gamma}(\hat{\tau}))\right]\right)\right)\Big)\,\mathrm{d}\hat{\tau}, && (3.130)
\end{aligned}$$

where

$$\widetilde{H}_1 = \frac{\sigma K\sqrt{\kappa}e^{(r-\kappa)(\hat{\tau}-\tau)-\frac{1}{2}\widetilde{d}_1^2(K)}}{2\sqrt{\pi}\sqrt{e^{2\kappa(\tau-\hat{\tau})}-1}},$$

$$\widetilde{H}_2 = rKe^{r(\hat{\tau}-\tau)},$$

$$\widetilde{H}_3 = \frac{\sigma\sqrt{\kappa}\sqrt{e^{2\kappa(\tau-\hat{\tau})}-1}}{2\sqrt{\pi}}\left(e^{-\frac{1}{2}\widetilde{d}_2^2(\widetilde{\gamma}(\hat{\tau}))} - e^{-\frac{1}{2}\widetilde{d}_2^2(K)}\right),$$

$$\widetilde{H}_4 = \left(\kappa\widetilde{\mu}(\hat{\tau}) - r\right)e^{\kappa(\tau-\hat{\tau})} - \kappa\ln S + \frac{\sigma^2}{2}\left(e^{\kappa(\hat{\tau}-\tau)}-1\right) - \kappa\varphi(\hat{\tau}),$$

$$\widetilde{M} = S^{e^{\kappa(\hat{\tau}-\tau)}}\exp\left((r+\kappa)(\hat{\tau}-\tau) - \frac{\sigma^2}{4\kappa}\left(1-e^{\kappa(\hat{\tau}-\tau)}\right)^2 + e^{\kappa(\hat{\tau}-\tau)}\varphi(\hat{\tau})\right) \quad \text{and}$$

$$\varphi(\hat{\tau}) = \kappa\int_{\hat{\tau}}^{\tau}\widetilde{\mu}(w)e^{\kappa(\tau-w)}\,\mathrm{d}w.$$

By changing the variable $\rho = \tau - \hat{\tau}$, substituting $\widetilde{\mu}(w) = \mu(T-w)$ and $\widetilde{\gamma}(w) = \gamma(T-w)$ in (3.130), we obtain

$$\begin{aligned}
\widetilde{U}(S,\tau) &:= u(x,\tau) \\
&= \int_0^{\tau}\Big(H_1 + H_2\left(N\left[d_1(K)\right] - N\left[d_1(\gamma(T-\tau+\rho))\right]\right) \\
&\qquad + M\left(H_3 + H_4\left(N\left[d_2(K)\right] - N\left[d_2(\gamma(T-\tau+\rho))\right]\right)\right)\Big)\,\mathrm{d}\rho \\
&= \int_0^{\tau}\Big(H_1 + H_2\left(N\left[d_1(K)\right] - N\left[d_1(\widetilde{\gamma}(\tau-\rho))\right]\right) \\
&\qquad + M\left(H_3 + H_4\left(N\left[d_2(K)\right] - N\left[d_2(\widetilde{\gamma}(\tau-\rho))\right]\right)\right)\Big)\,\mathrm{d}\rho, \qquad (3.131)
\end{aligned}$$

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where

$$
\begin{aligned}
H_1 &= \frac{\sigma K\sqrt{\kappa}e^{(\kappa-r)\rho-\frac{1}{2}d_1^2(K)}}{2\sqrt{\pi}\sqrt{e^{2\kappa\rho}-1}},\\
H_2 &= rKe^{-r\rho},\\
H_3 &= \frac{\sigma\sqrt{\kappa}\sqrt{e^{2\kappa\rho}-1}}{2\sqrt{\pi}}\left(e^{-\frac{1}{2}d_2^2(\gamma(T-\tau+\rho))}-e^{-\frac{1}{2}d_2^2(K)}\right),\\
H_4 &= (\kappa\mu(T-\tau+\rho)-r)\,e^{\kappa\rho}-\kappa\ln S+\frac{\sigma^2}{2}\left(e^{-\kappa\rho}-1\right)-\kappa\varphi(\tau-\rho),\\
M &= S^{e^{-\kappa\rho}}\exp\left(-(r+\kappa)\rho-\frac{\sigma^2}{4\kappa}\left(1-e^{-\kappa\rho}\right)^2+e^{-\kappa\rho}\varphi(\tau-\rho)\right),\\
d_1(P) &= \frac{\sqrt{2\kappa}}{\sigma\sqrt{e^{2\kappa\rho}-1}}\left(\ln\frac{S}{P}+(1-e^{\kappa\rho})\left(\ln P+\frac{\sigma^2}{2\kappa}\right)+\varphi(\tau-\rho)\right),\\
d_2(P) &= d_1(P)+\frac{\sigma\sqrt{1-e^{-2\kappa\rho}}}{\sqrt{2\kappa}} \text{ and}\\
\varphi(\rho) &= \kappa\int_\rho^\tau \mu(T-w)e^{\kappa(\tau-w)}\,\mathrm{d}w.
\end{aligned}
$$

Substituting (3.93) back to (3.98)–(3.99) yields

$$
\begin{aligned}
W(S,t) &= u(x,\tau)+(\phi K-\phi\widetilde{\gamma}(\tau)e^{-x})^+-E(x,\tau)\\
&= \widetilde{U}(S,\tau)+(\phi K-\phi S)^+-\widetilde{v}(S,\tau)\\
&= \widetilde{U}(S,T-t)+(\phi K-\phi S)^+-v(S,t), \qquad (3.132)
\end{aligned}
$$

where $\widetilde{U}(S,\tau)$ is defined as (3.131). Substituting (3.35) in Theorem 3.3 to (3.132), the proof is complete. □

From Theorem 3.9, an analytical representation formula for American option value is obtained as stated in the following theorem.

Theorem 3.10 (American option pricing formula)**.** *Assume that the conditions in Theorem 3.9 hold. Then, the American option $V(S,t;\phi,\gamma(t))$ on the underlying asset spot price S at time $t \le T$ with a strike price K and an expiration date T is represented by*

$$
V(S,t;\phi,\gamma(t)) = \widetilde{U}(S,T-t;\phi,\gamma(T-t))+(\phi K-\phi S)^+, \qquad (3.133)
$$

where $\widetilde{U}(S,\tau;\phi,\gamma(\tau))$ *is defined as* (3.92) *in Theorem 3.9.*

Proof.

The result is obtained directly from (3.35), (3.36), (3.80), (3.91) and (3.92). □

Remark 3.11. Since the optimal exercise boundary function $\gamma(\cdot)$ is not known explicitly, to compute the value of American option $V(S,t;\phi,\gamma)$ in Theorem 3.10, one need to use the formula (3.133) with the boundary condition (3.79) to derive a functional equation for solving for $\gamma(\cdot)$ and $V(S,t;\phi,\gamma)$, simultaneously. In particular, from (3.79) and (3.133), we get that

$$V(\gamma(t),t;\phi,\gamma) = \phi K - \phi\gamma(t) \tag{3.134}$$

and

$$\begin{aligned} V(\gamma(t),t;\phi,\gamma) &= \widetilde{U}(\gamma(t),T-t;\phi,\gamma) + (\phi K - \phi\gamma(t))^+ \\ &= \widetilde{U}(\gamma(t),T-t;\phi,\gamma) + \phi K - \phi\gamma(t), \end{aligned} \tag{3.135}$$

where $\widetilde{U}(S,\tau;\phi,\gamma)$ is defined as (3.92). Substituting (3.134) into to (3.135) yields

$$\widetilde{U}(\gamma(t),T-t;\phi,\gamma) = 0. \tag{3.136}$$

Consequently, numerical techniques such as the fixed-point iterative method or the implicit finite-difference method can be applied to solve the functional equation (3.136) for the optimal exercise boundary $\gamma(\cdot)$ in order to compute the value of $V(S,t;\phi,\gamma)$ from the formula (3.133). (see [5, 20] and [30] for examples and more details about these methods).

In this dissertation, although, we do not provide numerical results for American option computed from our formula (3.133), we investigate and verify our solutions in the case of European option computed from our formula (3.35) instead because of the same technique in derivation and similar PDO. The numerical results compared with other methods and discussions are provided in the next chapter.

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CHAPTER IV

NUMERICAL RESULTS AND DISCUSSIONS FOR EUROPEAN COMMODITY OPTIONS

In this chapter, we provide numerical results of European option prices computed from analytical formula (3.35) under some cases of long-run mean functions. This chapter is divided into two parts. In Section 4.1, the accuracy of the results computed from our analytical formula (3.35) are compared with MC simulations and BS-type formula. The examples of option price behaviors have been illustrated and discussed in Section 4.2.

In the following sections, we use parameters; strike price $K = 40$, expiration date $T = 1$, volatility $\sigma = 0.5$ and five cases of long-run mean functions; constant, linear, smooth periodic, piecewise differentiable and periodic piecewise continuous. The graphs and descriptions of these long-run mean functions $\mu : [0, 1] \to \mathbb{R}$ are displayed in Figure 4.1.

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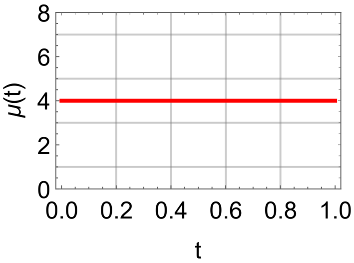
4.1 Comparisons with Other Solutions

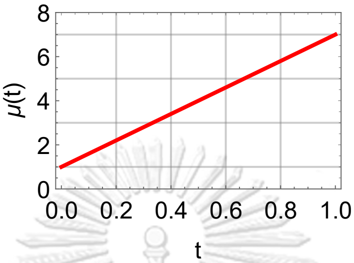
To verify the results from our analytical formula, we compare with the standard benchmark approaches such as MC simulations and BS-type formula, since they are quite accurate and simple to perform.

In our comparisons, we compute call and put option prices (3.2) on the under-



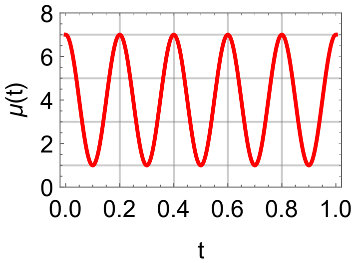
(a) constant

$\mu(t) = 4$



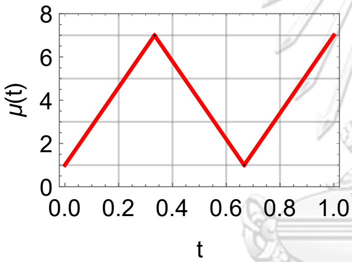
(b) linear

$\mu(t) = 1 + 6t$



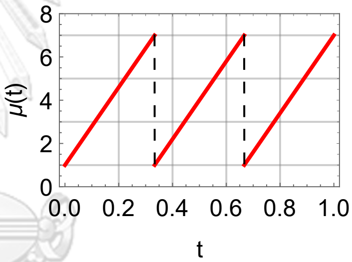
(c) smooth periodic

$\mu(t) = 4 + 3\sin(\frac{\pi}{2} + 10\pi t)$



(d) piecewise differentiable

$$\mu(t) = \begin{cases} 1 + 18t, & t \in [0, \frac{1}{3}] \\ 7 - 18(t - \frac{1}{3}), & t \in (\frac{1}{3}, \frac{2}{3}] \\ 1 + 18(t - \frac{2}{3}), & t \in (\frac{2}{3}, 1] \end{cases}$$



(e) periodic piecewise continuous

$$\mu(t) = \begin{cases} 1 + 18t, & t \in [0, \frac{1}{3}] \\ 1 + 18(t - \frac{1}{3}), & t \in (\frac{1}{3}, \frac{2}{3}] \\ 1 + 18(t - \frac{2}{3}), & t \in (\frac{2}{3}, 1] \end{cases}$$

Figure 4.1: Five cases of long-run mean functions.

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lying asset prices S at the initial time $t = 0$,

$$v(S, 0; \phi) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(\phi K - \phi S_T)^+ | S_0 = S\right], \tag{4.1}$$

by varying the asset spot price S with the fixed initial time $t = 0$, where $\phi = -1$ for call and $\phi = 1$ for put. The other parameters are risk-free interest rate $r = 0.05$ and speed of reversion $\kappa = 0.05$.

4.1.1 MC Simulation

Our MC simulation for computing (4.1) have employed the simple Euler-Maruyama discretization based on a simple simulation of the mean-reverting process following (3.1), namely,

$$S_{t_i} = S_{t_{i-1}} + \kappa(\mu(t_{i-1}) - \ln S_{t_{i-1}})S_{t_{i-1}}\,\Delta t + \sigma S_{t_{i-1}}\sqrt{\Delta t}\,Z_{t_i},$$

where Z_t is the standard normal random variable. We generate sample paths of S_t on $[0, T]$, using the time-step $\Delta t = 0.01$ with 100,000 sample paths.

4.1.2 BS-type Formula

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Given that at the initial time t, the initial asset price $S_t = S$. To compute the option value (3.2), one can directly use the definition of expectation with the PDF of the asset log-price at time T, $X_T := \ln S_T$. Similar to (3.31), X_T can be represented by

$$X_T = (\ln S)e^{-\kappa(T-t)} + \frac{\sigma^2}{2\kappa}\left(e^{-\kappa(T-t)} - 1\right) + \kappa e^{-\kappa T}\int_t^T \mu(u)e^{\kappa u}\,\mathrm{d}u$$
$$+ \sigma e^{-\kappa T}\int_t^T e^{\kappa u}\,\mathrm{d}W_u$$

which is normally distributed with mean

$$m(T) := (\ln S)e^{-\kappa(T-t)} + \frac{\sigma^2}{2\kappa}\left(e^{-\kappa(T-t)} - 1\right) + \kappa e^{-\kappa T}\int_t^T \mu(u)e^{\kappa u}\,\mathrm{d}u, \tag{4.2}$$

variance

$$g(T) := \sigma^2 e^{-2\kappa T} \int_t^T e^{2\kappa u}\,\mathrm{d}u = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa(T-t)}) \tag{4.3}$$

and the PDF

$$f_X(x) := \frac{1}{\sqrt{2\pi g(T)}} \exp\left(-\frac{(x - m(T))^2}{2g(T)}\right). \tag{4.4}$$

By using (4.4), the option value (3.2) can be computed by

$$v(S,t;\phi) = e^{-r(T-t)} \int_{-\infty}^{\infty} (\phi K - \phi e^x)^+ f_X(x)\,\mathrm{d}x. \tag{4.5}$$

Note that, to use the formula (4.5) directly, one needs to evaluate the improper integral.

For further simplification, the improper integral (4.5) is derived to get a closed-form formula similar to the BS formula for stocks by using the PDF of the asset log-price and the property of normal distribution. The BS-type formula is stated as follow.

Theorem 4.1. *The value of European option* (3.2) *can be represented by*

$$v(S,t;\phi) = \phi K e^{-r(T-t)} N\left[\phi d_1\right] - \phi e^{-r(T-t)+m(T)+\frac{g(T)}{2}} N\left[\phi d_2\right], \tag{4.6}$$

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where $m(T)$ *and* $g(T)$ *are defined as* (4.2) *and* (4.3)*, respectively,* $d_1 = \frac{\ln K - m(T)}{\sqrt{g(T)}}$ *and* $d_2 = d_1 - \sqrt{g(T)}$.

Proof. From (4.5), note that

$$\begin{aligned} v(S,t;\phi) &= e^{-r(T-t)} \phi \lim_{n\to-\infty} \int_{\phi n}^{\ln K} (\phi K - \phi e^x) f_X(x)\,\mathrm{d}x \\ &= e^{-r(T-t)} \left(K \lim_{n\to-\infty} \int_{\phi n}^{\ln K} f_X(x)\,\mathrm{d}x - \lim_{n\to-\infty} \int_{\phi n}^{\ln K} e^x f_X(x)\,\mathrm{d}x\right). \end{aligned} \tag{4.7}$$

Since X_T is normally distributed with mean $m(T)$ and variance $g(T)$, we have that $Z := \frac{X_T - m(T)}{\sqrt{g(T)}}$ is the standard normal random variable. Since $f_X(x)$ is the PDF of

X_T, we have

$$\begin{aligned}\lim_{n\to-\infty}\int_{\phi n}^{\ln K} f_X(x)\,\mathrm{d}x &= \phi\,\mathbb{Q}\left(\phi X_T \le \phi \ln K\right)\\ &= \phi\,\mathbb{Q}\left(\phi Z \le \phi\frac{\ln K - m(T)}{\sqrt{g(T)}}\right)\\ &= \phi\,\mathbb{Q}\left(Z \le \phi\frac{\ln K - m(T)}{\sqrt{g(T)}}\right)\\ &= \phi N\left[\phi d_1\right], \end{aligned} \tag{4.8}$$

where the third equality in (4.8) is obtained from the property of normal distribution $\mathbb{Q}\left(Z \ge x\right) = \mathbb{Q}\left(Z \le -x\right)$. Let $f_Y(x) := \frac{1}{\sqrt{2\pi g(T)}}\exp\left(-\frac{(x-m(T)-g(T))^2}{2g(T)}\right)$ be the PDF of a normal random variable with mean $m(T)+g(T)$ and variance $g(T)$. By using (4.4), we obtain

$$\begin{aligned}\lim_{n\to-\infty}\int_{\phi n}^{\ln K} e^x f_X(x)\,\mathrm{d}x &= e^{m(T)+\frac{g(T)}{2}}\lim_{n\to-\infty}\int_{\phi n}^{\ln K}\frac{1}{\sqrt{2\pi g(T)}}e^{-\frac{(x-m(T)-g(T))^2}{2g(T)}}\,\mathrm{d}x\\ &= e^{m(T)+\frac{g(T)}{2}}\lim_{n\to-\infty}\int_{\phi n}^{\ln K} f_Y(x)\,\mathrm{d}x\\ &= \phi e^{m(T)+\frac{g(T)}{2}}N\left[\phi d_2\right], \end{aligned} \tag{4.9}$$

where the last equality of (4.9) is obtained by the similar argument as (4.8). Substituting (4.8) and (4.9) into (4.7), the result is obtained. □

4.1.3 Comparison Results

Let

$$C(S) := v(S,0;-1),\ \ P(S) := v(S,0;1)$$

denote call and put option prices computed from our analytical formula (3.35), respectively, and

$$C_{MC}(S),\ \ P_{MC}(S) \text{ and } C_{BS}(S),\ \ P_{BS}(S)$$

denote call and put option prices obtained by MC simulations and BS-type formula (4.6), respectively.

In our numerical tests, we compute the values of $C(S)$, $P(S)$, $C_{MC}(S)$, $P_{MC}(S)$, $C_{BS}(S)$ and $P_{BS}(S)$ by using various underlying asset spot prices $S \in D_S := \{30, 32, \ldots, 48\}$. The option prices obtained from our analytical formula, MC simulations and BS-type formula are compared based on the five cases of long-run mean functions to illustrate the accuracy of the formula (3.35). The displays of the five comparison results are shown in Figure 4.2.

4.1.4 Accuracy and Efficiency

According to the comparison results shown in Figure 4.2, the level of accuracy of the results from formula (3.35) under the five cases of long-run mean functions are demonstrated by the average absolute differences with BS-type and the average percentage errors with MC simulations.

Define the average absolute difference and the average percentage error by

$$\bar{d}_{BS} := \left(\sum_{S \in D_S} d(S) \right) / |D_S| \quad \text{and} \quad \bar{\epsilon}_{MC} := \left(\sum_{S \in D_S} \epsilon(S) \right) / |D_S|,$$

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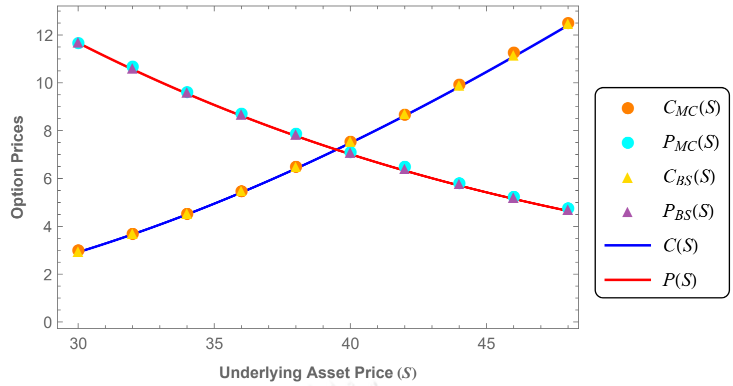
respectively, where

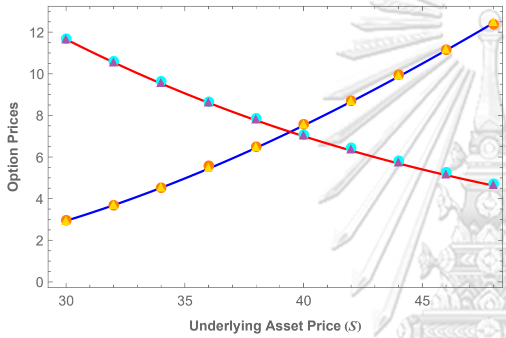
$$d(S) := |a(S) - x(S)| \quad \text{and} \quad \epsilon(S) := \left| \frac{a(S) - y(S)}{a(S)} \right| \times 100\%,$$

with $a(S)$ represents call/put option value from our formula and $x(S)$ and $y(S)$ represent call/put option values from BS-type formula and from MC simulations, respectively.

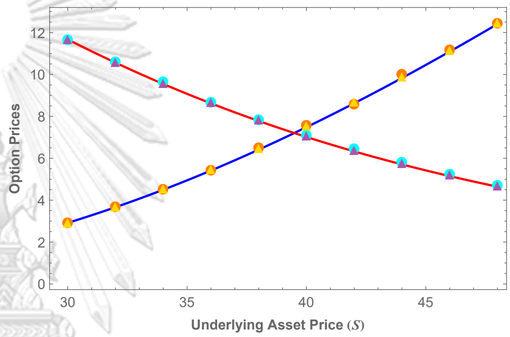
The average absolute differences, the average percentage errors and the average computational times for each approach corresponding to each long-run mean function are demonstrated in Table 4.1.



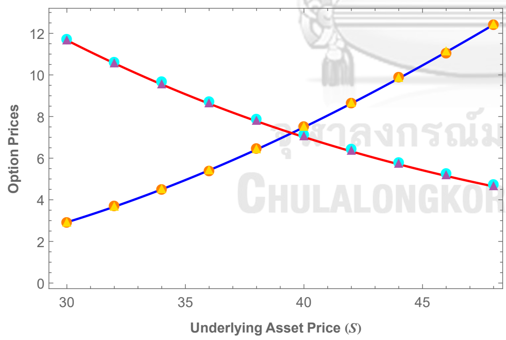
(a) constant



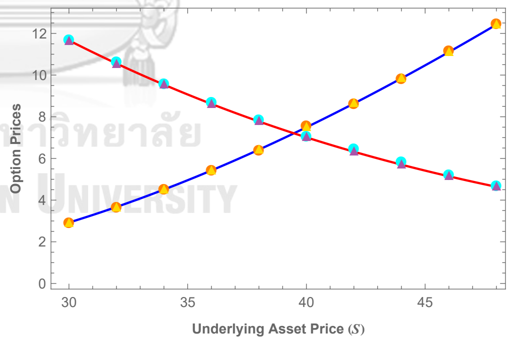
(b) linear



(c) smooth periodic



(d) piecewise differentiable



(e) periodic piecewise continuous

Figure 4.2: Five comparison results of call and put option prices obtained from our formula, MC simulations and BS-type formula corresponding to the five cases of long-run mean functions.

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Table 4.1: The average absolute differences ($\bar{d}_{BS}$) and the average percentage errors ($\bar{\epsilon}_{MC}$) when compare the results obtained from our formula with those from BS-type formula and MC simulations, respectively, and the average computational times for each approach

Long-run mean function	$\bar{d}_{BS}$ ($\times 10^{-8}$)		$\bar{\epsilon}_{MC}$ (%)		Average computational time (s)		
	call	put	call	put	Ours	BS-type	MC
(a) constant	3	3	0.28	0.37	0.02	0.0008	218
(b) linear	3	3	0.60	0.36	0.02	0.0008	227
(c) smooth periodic	2	2	0.79	0.25	0.02	0.0008	228
(d) piecewise differentiable	7	7	0.44	0.58	0.50	0.0008	292
(e) periodic piece-wise continuous	7	7	0.65	0.48	0.60	0.0008	282

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Table 4.1 shows that the results of call and put option values from our formula (3.35) are accurate as compared to the BS-type formula (4.6) with the average absolute differences less than 10^{-7} and to MC simulations with the average percentage errors less than 0.8% for all five cases of long-run mean functions. This fact verifies the validity of the formula (3.35).

Based on the average computational times, Table 4.1 confirms that the analytical formula (3.35) and the BS-type formula are much more efficient than MC simulations as expected, where the BS-type formula is clearly faster than our formula.

In the next section, we describe the behaviors of option values as functions of underlying asset price S and initial time t corresponding to various different long-run mean functions.

4.2 Examples of Option Price Behaviors on Different Long-run Mean Functions

In this section, we demonstrate the results of European option values based on the computation of our analytical formula (3.35).

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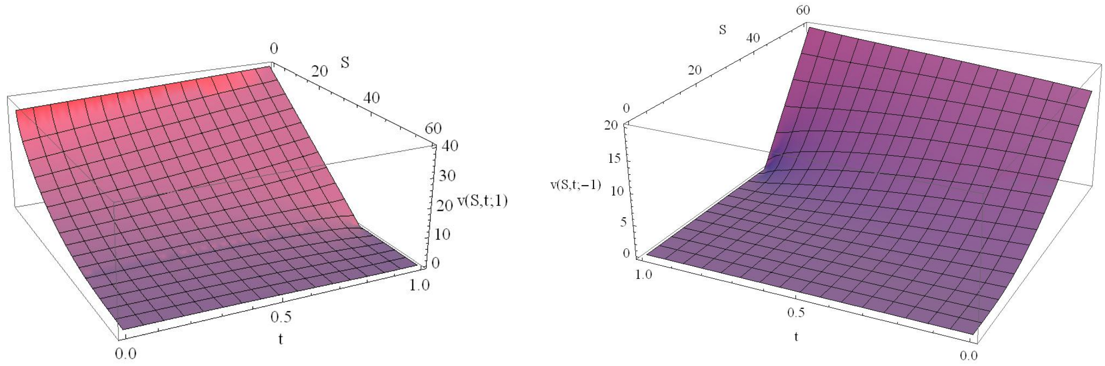
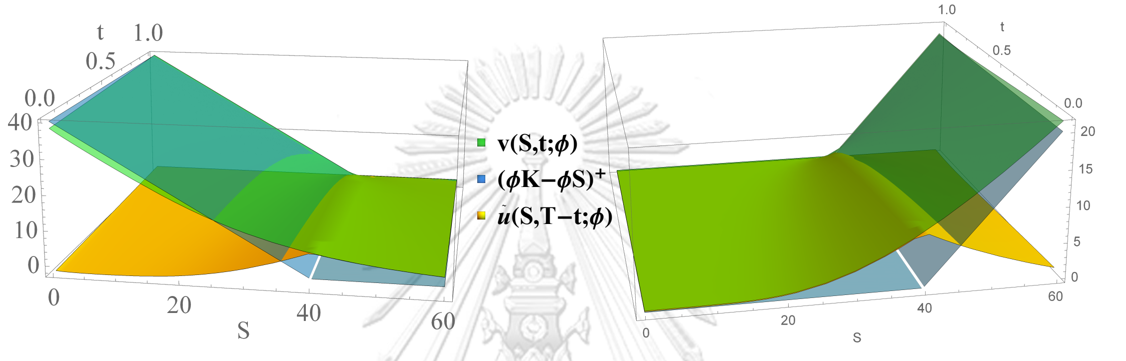
The following examples illustrate behaviors of both European put and call option values $v(S, t; \phi)$ on a mean-reverting asset spot price $S \in [0, 60]$ at time $t \in [0, T]$ corresponding to the five long-run mean functions with parameters $r = 0.1$ and $\kappa = 0.5$.

4.2.1 Constant

Example 4.2. We consider a constant long-run mean function shown in Figure 4.1a. In this case, the underlying asset spot prices do not exhibit seasonality,



(a) put option (b) call option



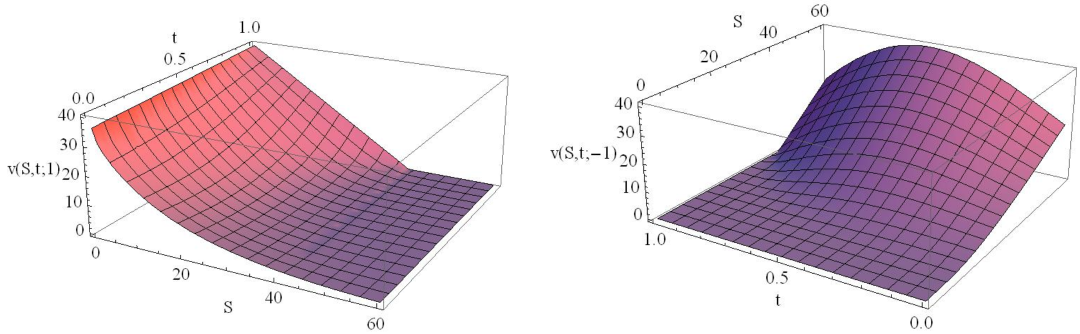
(c) decomposition of put option (d) decomposition of call option

Figure 4.3: Option values for constant $\mu(t) = 4$.

and follow the one-factor Schwartz model [27] used to represent oil and copper prices in commodity markets. The values of put and call options obtained from the analytical formula (3.35) are shown in Figures 4.3a and 4.3b, respectively.

Both figures show that as time t gets closer to the expiration date T, the option value gets closer to the terminal condition $v(S,T;\phi) = (\phi K - \phi S)^+$ as expected with continuous smooth curves in both directions except on the expiration date. Furthermore, when the time t changes and the asset spot price S is fixed, we see that most option values change linearly.

Figures 4.3c and 4.3d illustrate the combination of the option prices $v(S,t;\phi)$, the initial payoff $(\phi K - \phi S)^+$ and the integral term $\widetilde{u}(S,T-t;\phi)$ functions for put and call options in Theorem 3.3, respectively. The decomposition of the graph of



(a) put option (b) call option

Figure 4.4: Option values for linear $\mu(t) = 1 + 6t$.

option price into the others is clearly seen for both options.

4.2.2 Linear

Example 4.3. We consider the case of a linear long-run mean function shown in Figure 4.1b, where the spot prices of underlying asset normally have linear trend without seasonality. The option values obtained from the analytical formula are shown in Figure 4.4.

Similar to Example 4.2, as time t gets closer to the expiration date T, the option value gets closer to the terminal condition $v(S, T; \phi) = (\phi K - \phi S)^+$ as expected. However, the figures show quadratic trends when the time t changes and the asset spot price S is fixed. Surprisingly, the call option has the highest value not at the expiration date but around the midpoint of the lifetime of option.

4.2.3 Smooth Periodic

Example 4.4. We consider a smooth periodic long-run mean function shown in Figure 4.1c for the spot price of seasonal underlying asset. This behavior of assets is usually seen on most commodities such as agricultures, livestocks, energy and

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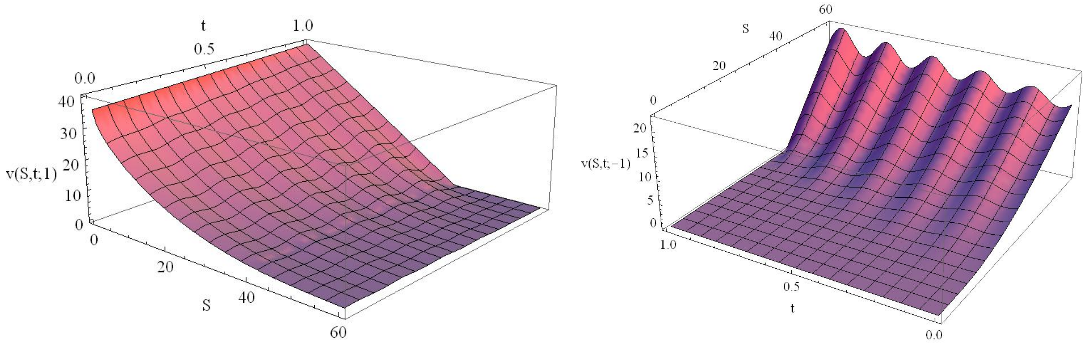

(a) put option (b) call option

Figure 4.5: Option values for smooth periodic $\mu(t) = 4 + 3\sin(\pi/2 + 10\pi t)$.

manufactured metal. The option values corresponding to this case are demonstrated in Figure 4.5.

The figures show that the terminal condition $v(S,T;\phi) = (\phi K - \phi S)^+$ holds as in the previous examples. In addition, there are many smooth oscillations on both option values as the time changes with fixed spot price, but the call has stronger oscillations than the put.

4.2.4 Piecewise Differentiable

Example 4.5. In this example, we consider an asset with long-run mean described by a piecewise differentiable function shown in Figure 4.1d. The option values obtained from the analytical formula are shown in Figure 4.6.

Both options show that the terminal condition $v(S,T;\phi) = (\phi K - \phi S)^+$ holds as in the previous examples, with smooth curves even though its long-run mean function is not. The behavior of the values is similar to those in Example 4.4, there are a few smooth oscillations as the time changes with fixed spot price and the call has stronger oscillations.

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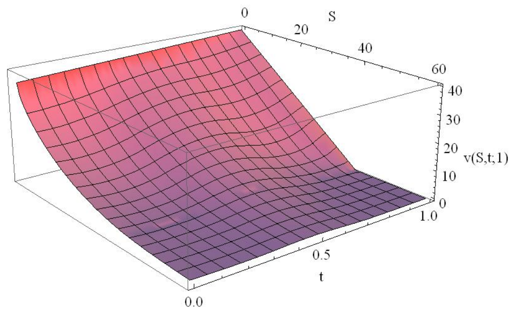
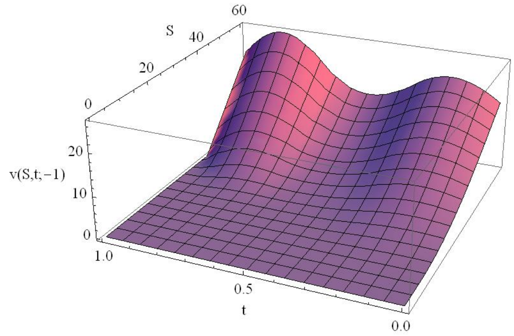


(a) put option



(b) call option

Figure 4.6: Option values for piecewise differentiable

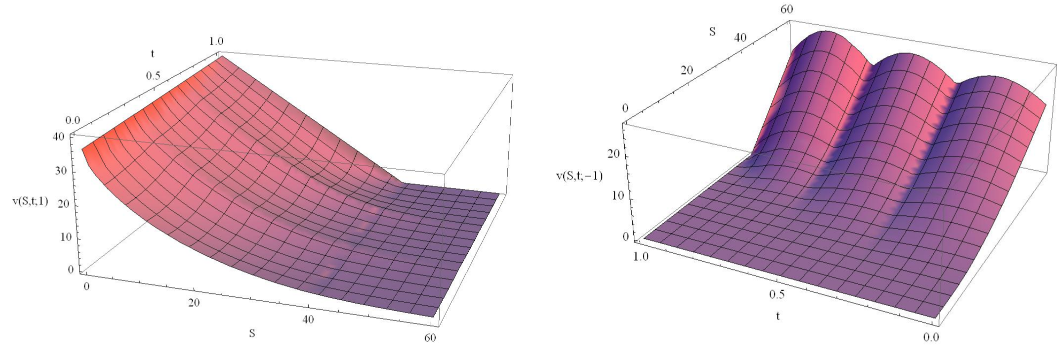
$$\mu(t) = \begin{cases} 1 + 18t, & t \in [0, \frac{1}{3}] \\ 7 - 18(t - \frac{1}{3}), & t \in (\frac{1}{3}, \frac{2}{3}] \\ 1 + 18(t - \frac{2}{3}), & t \in (\frac{2}{3}, 1]. \end{cases}$$

4.2.5 Periodic Piecewise Continuous

Example 4.6. In this case, we consider a model for assets with seasonality described by a periodic piecewise continuous long-run mean function shown in Figure 4.1e, where the spot price has rapid changed when it reaches the high peak. The option values corresponding to this case are demonstrated in Figure 4.7.

Similar to all previous examples, the results satisfy the terminal condition $v(S,T;\phi) = (\phi K - \phi S)^+$ as expected. Both option values are still continuous and smooth even though their long-run mean function is piecewise continuous. In addition, there are some large smooth waves when the time changes, which are shallower for the put option. There are also some lines between the waves that are changing level very fast corresponding to the behavior of jumps on the long-run mean function.

In this section, Examples 4.2–4.6 demonstrate that both European put and



(a) put option (b) call option

Figure 4.7: Option values for periodic piecewise continuous

$$\mu(t) = \begin{cases} 1 + 18t, & t \in [0, \frac{1}{3}] \\ 1 + 18(t - \frac{1}{3}), & t \in (\frac{1}{3}, \frac{2}{3}] \\ 1 + 18(t - \frac{2}{3}), & t \in (\frac{2}{3}, 1]. \end{cases}$$

call option values obtained from the analytical formula (3.35) satisfy the terminal condition (3.83) as expected with continuous smooth curves, even though the corresponded long-run mean functions are not continuous or smooth in the domain. In addition, the provided long-run mean functions show strong affection in both options such as linear, quadratic and oscillation behaviors as time changes with fixed spot price, but the calls are stronger impacted than the puts. This suggests that the analytical formula can be applied for any kinds of mean-reverting assets with integrable long-run mean functions describing both seasonal and nonseasonal behaviors. In addition, Example 4.2 illustrates the decompositions of option prices between the initial payoff term and the integral term from formula (3.35).

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CHAPTER V

CONCLUSION

In this dissertation, we derive integral representation formulas for pricing European option and American option on the underlying asset, especially, commodities, whose prices follow a mean-reverting model with time-dependent long-run mean function called an ESC model.

Both analytical formulas for both put and call options are derived based on a PDE approach together with the Fourier transformation in Chaper 3. The method can be applied and simply modified for both European and American options. Although another analytical method such as probability approach is simpler to obtain for European style but much more difficult to handle with American option. We use this advantage by preparing a useful lemma in Section 3.1. In Section 3.2, the European option pricing formula and the put-call parity are presented differently from the traditional BS formula. In Section 3.3, the American option pricing formula is alternatively derived via the solution of the EEP because of the decomposition of American option price. For both option styles, the formulas obtained from our techniques compose of two terms: the payoff at the initial time which is known in the beginning and the time integral over the lifetime driven by the long-run mean function which required only to be integrable. This implies that the formulas can be applied for commodities that may have seasonality in price described not only by continuous but also discontinuous long-run mean functions.

In Chaper 4, the accuracy and the computational time of the European option

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pricing formula under various kinds of long-run mean functions has been verified by comparing with MC simulation and the BS-type formula which is derived from a probability approach and can be used as an explicit formula in Section 4.1. The comparison results validate the accuracy of the formula (3.35) with much less computational time as compared with MC simulation and a little more than that of BS-type formula due to the computation of the integral term. In Section 4.2, the behaviors of European option prices based on our formula are illustrated and discussed in the examples of option prices corresponding to these long-run mean functions.

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