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STOCHASTIC CONTROL MODEL WITH CARRYING CAPACITY OF POPULATION MANAGEMENT POLICY FOR SQUIRRELS IN DURIAN ORCHARDS

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ในวิทยานิพนธ์ฉบับนี้ เราสนใจปัญหาที่กระรอกได้สร้างตำหนิให้แก่ทุเรียนซึ่งเป็นพืชเศรษฐกิจของประเทศไทย เราจึงต้องการหากลยุทธ์ในการกำจัดกระรอกให้แก่เจ้าของสวนทุเรียนภายใต้เงื่อนไขว่ากระรอกไม่ใช่เอเลี่ยนสปีชีส์ในประเทศไทย อีกทั้งยังทำประโยชน์ให้กับระบบนิเวศน์อีกด้วย ปัญหานี้ถูกแก้ไขด้วยการนำตัวแบบควบคุมสโตแคสติกมาประยุกต์ใช้ โดยใช้สมการเชิงอนุพันธ์สโตแคสติกแทนการเปลี่ยนแปลงของประชากรกระรอกและกำหนดให้จำนวนประชากรของกระรอกมีได้อย่างจำกัด เนื่องจากเราพิจารณาจำนวนกระรอกในพื้นที่สวนที่มีอย่างจำกัด ดัชนีควบคุมที่แสดงถึงประโยชน์ทั้งหมดที่ได้จากการใช้กลยุทธ์กำจัดกระรอกต่อเจ้าของสวนทุเรียนได้ถูกกำหนด ดัชนีควบคุมประกอบไปด้วยค่าใช้จ่ายในการกำจัดกระรอก ความเสียหายที่เกิดจากกระรอก และประโยชน์ของกระรอกต่อสิ่งแวดล้อม ดัชนีควบคุมที่เหมาะสมที่สุดสามารถคำนวณได้จากอสมการการแปรผันที่สามารถหาผลเฉลยได้โดยระเบียบวิธีผลต่างจำกัด เราได้ทำการวิเคราะห์และนำเสนอกลยุทธ์ที่เหมาะสมที่สุดในการกำจัดกระรอกเป็นผลเฉลยเชิงตัวเลข

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In this thesis, the problem that squirrels ruin durian, which is an economic important fruit in Thailand, is considered. We seek for a strategy on squirrel elimination under the consideration that squirrels are not alien species in Thailand and also orchard ecosystem. The problem is solved through a stochastic control model. The population dynamics of squirrels is constructed as a controlled stochastic differential equation with carrying capacity, since we consider the squirrel population in a confined orchard. A performance index indicating the total benefit of a given squirrel elimination strategy is provided. The index comprises the countermeasure cost, resources loss, and squirrel's benefit. The optimal performance index is numerically solved through the variational inequality using the finite difference method. The corresponding optimal strategy to control the squirrel population is also given numerically.

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CHAPTER I

INTRODUCTION

Squirrels, which are indigenous to the Americas, Europe, Asia and Africa, are members of the family *Sciuridae*, a family that includes small or medium-sized rodents. They are omnivores, meaning that they can consume meat, plants, and especially fruits [12]. They consume a variety of fruits, one of which is durian. It is the edible fruit belonging to the genus *Durio* [3]. Durian is known widely in Asia, and there are many desserts made with durian. Moreover, durian is one of the most popular exported Thai fruits [2], and the number of exports has been increasing since 2010. Nowadays, Thailand exports durian to other countries around 500-600 million kilograms annually. However, durian has a sweet taste with unique strong smell attracting squirrels to bite it. Defected durians are unable to be exported and have a lower price. Therefore, durian farmers have to eliminate squirrels from their orchards by gun shooting, using firecracker or drugs. On the other hand, squirrels are not alien species, at least in Thailand, and provide ecosystem service. Hence, we need to establish a strategy that allows durian farmers to kill squirrels with ecosystem service by squirrels in consideration.

The predator and prey model [6] is applied to approach the strategy following the mentioned issue above by considering the squirrel as predator and durian as prey. The predator and prey model can be represented by a system of ordinary differential equations (ODEs) [8], partial differential equations (PDEs) [1] or stochastic differential equations (SDEs) [13]. However, a stochastic optimal control model, which is normally studied in the field of finance and economics, can be employed to control the squirrel population [5]. A controlled SDE [7] and a performance index are the composition of a stochastic control model. The value of the performance index should be minimized or maximized by selecting an appropriate control.

In 2018, Yuta Yeagashi et al. [13] formulated a singular stochastic control model to manage the population of *Phalacrocorax carbo* (P. carbo; Great Cormorant). *P. carbo* eats *Plecoglossus altivelis* (P. altivelis; Ayu), which is one of the most economically and culturally important inland fishery resources in Japan. Therefore, local fishery cooperatives and governments need to seek for a sustainable policy, which can subjugate the severe predation of the bird to the riverine fishes, to manage *P. carbo* population. On the other hand, the bird should not be exterminated because it is not an alien species and provides nutrient-cycling system. Yuta Yeagashi et al. [13] adapted this concept to manage the bird population by using Ito's SDE to represent the bird population dynamics

$$dX_t = \mu X_t dt + \sigma X_t dW_t - d\eta_t, \quad X_0 = x \geq 0. \tag{1.1}$$

Here, X_t is the total number of the bird at time $t, \mu > 0$ is the deterministic growth rate of the population, $\sigma > 0$ is the magnitude of stochastic fluctuation involved in the population dynamics, W_t is a one-dimensional standard Brownian motion, and η_t is the right-continuous adapted (with respect to the natural filtration) process representing the decrease of the population by gun shooting.

The bird population dynamics is controlled by a performance index that should be maximized by the decision-maker, a local fishery cooperative or the government, through selecting an optimal control $\eta_t = \eta_t^*$. They denote the performance index for an admissible η_t as $v = v(x; \eta)$ which is set as

$$v(x;\eta) = E\left[\int_0^\infty e^{-\delta s}(RX_s^M - SX_s^m)\, ds - \int_0^\infty e^{-\delta s}\, d\eta_s\right]. \tag{1.2}$$

Here, $E[\cdot]$ is the expectation conditioned on $X_0 = x \geq 0$, $\delta > 0$ is the discount rate of the profit, and S, R, m, M are model parameters which satisfy $S, R \geq 0$ and $0 < M < 1 < m \leq 2$. The performance index v represents the expected net profit of the decision-maker. The discount rate δ represents the attitude of the decision-makers on the management of *P. carbo* population.

Since the performance index must be maximized [13], they defined the maximized performance index v to be the function

$$V(x) = \sup_{\eta} v(x;\eta) = v(x;\eta^*). \tag{1.3}$$

Then, they apply the dynamic programming principle of stochastic control model [9] and receive an exact solution by application of an analytical technique as

$$V(x) = \begin{cases} ax^k + Ax^m + Bx^M & (0 < x \leq \bar{x}) \\ b - x & (x > \bar{x}) \end{cases}, \tag{1.4}$$

where k, A, B are the values that can be computed from $\mu, \sigma, \delta, m, M, S$ and R, and $a, b, \bar{x}$ are the unknowns that can be received by solving a nonlinear equation of $V(x)$. Here, $\bar{x}$ represents the threshold for suppression.

In this work, we use a controlled SDE to quantify the dynamic of the squirrel population while a control threshold, which can vary in time, is obtained by finding the optimal strategy of performance index, which corresponds to the damage from squirrels on durian and the ecosystem service of squirrels. The maximized performance index is solved by the variational inequality. We solve this variational inequality numerically by using the finite difference method (FDM) and analyze the population-control threshold and the optimal control following the numerical solution.

The remaining of this thesis are organized as follows: basic knowledge in Chapter 2; methodology, which introduces the squirrel model, setting values of parameters, variational inequality and numerical method in Chapter 3; the reward function in Chapter 4; the result in Chapter 5; and conclusion and future work in Chapter 6.

CHAPTER II

BASIC KNOWLEDGE

This chapter provides the basic knowledge of SDE in Section 2.1. Euler-Maruyama method, which is an easy numerical method to approximate solutions of SDEs, is also presented in Section 2.2. Furthermore, we introduce the stochastic optimization problem in Section 2.3 and a numerical method to solve it in Section 2.4.

2.1 Introduction to SDE

Definition 2.1. (Stochastic Process)
Let I be a subset of $\mathbb{R}$. A family of random variables $\{X_t\}_{t\in I}$, indexed by I, is called **stochastic process**.

Definition 2.2. (Continuous Sample Paths)
Let $\{X_t\}_{t\in I}$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$. For a fixed $\omega \in \Omega$, a function $X_{.}(\omega) : I \to \mathbb{R}$ is called a **sample path**. The process $\{X_t\}_{t\in I}$ is said to have **continuous sample paths** when $X_{.}(\omega)$ is a continuous function on I for almost all $\omega \in \Omega$.

Definition 2.3. (Standard Brownian Motion)
A stochastic process $\{W_t\}_{t\in[0,T]}$ is a scalar **standard Brownian motion**, or **standard Wiener process**, if the following conditions hold.

1. $W_0 = 0$, with probability 1.
2. The increment $W_t - W_s$ has normal distribution with mean 0 and variance $t - s$ for any $0 \leq s < t$.
3. $\{W_t\}_{t\in[0,T]}$ has independent increments.
4. $\{W_t\}_{t\in[0,T]}$ has continuous sample paths.

SDEs can be written in differential form as

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \quad 0 \le t < T, \qquad X_0 = x, \tag{2.1}$$

where $\mu(t, X_t)$ and $\sigma(t, X_t)$ are real-valued functions, $\{W_t\}_{t\in[0,T]}$ is a standard Brownian motion and $x \in \mathbb{R}$. The equation 2 can also be written as stochastic integral equation

$$X_t = x_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s.$$

$\int_0^t \mu(s, X_s)ds$ is interpreted in the Riemann sense and $\int_0^t \sigma(s, X_s)dW_s$ is interpreted in the Itô sense.

2.2 Euler-Maruyama Method for SDEs

Euler-Maruyama method is a numerical solution of an SDE. For an SDE

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \qquad t \in [0, T], \tag{2.2}$$

where X_0 is constant, and W_t is a standard Brownian motion. The procedure of Euler-Maruyama method is as follows.

1. Discretize an interval time, $[0, T]$, into N equal length stages for some $N \in \mathbb{N}$ and let $\triangle t = \frac{T}{N}$.

2. Define $t_n = n\triangle t$ and denote X_{t_n} by x_n for $n = 0, 1, ..., N$.

3. The Euler-Maruyama scheme for (2.2) has the following form

$$x_0 = X_0,$$
$$x_n = x_{n-1} + \mu(t_{n-1}, x_{n-1})\triangle t + \sigma(t_{n-1}, x_{n-1})\triangle W_n, \quad \text{for } n = 1, 2, ..., N,$$

where $\triangle W_n = W_{t_n} - W_{t_{n-1}}$ has normal distribution with mean 0 and variant $\triangle t$.

2.3 Stochastic Control Model

In this section, we introduce the stochastic control model which we studied from Continuous-time stochastic control and optimization with financial applications [9].

2.3.1 Stochastic Optimization Problem

In general terms, a stochastic optimization problem includes the following characteristics.

- **State of the system**: A dynamic system is defined by its state at any time and evolves in an uncertain environment, which is formalized by a probability space $(\Omega, \mathcal{F}, P)$. The state of the system is the set of quantitative variables needed to characterize the problem. The state of the system is denoted at time t in a world scenario $\omega \in \Omega$ by $X_t(\omega)$. Here, we consider $X = \{X_t\}_{t\in\mathbb{T}}$ when $\mathbb{T} = [0, T]$ for some $T > 0$ or $\mathbb{T} = [0, \infty)$.
- **Control**: The dynamics $t \to X_t$ of the system is typically affected by a control which is described as a continuous-time process $\alpha = (\alpha_t)$, the value of which is determined at any time t in function of the available information. The control α, called admissible control, should satisfy some constraints.
- **Performance/cost criterion**: To maximize (or minimize) a functional $J(X, \alpha)$ over all admissible controls is the aim. The typically objective functionals is considered in the form

$$E\left[\int_0^T f(X_t, \omega, \alpha_t)\, dt + g(X_T, \omega)\right] \tag{2.3}$$

on a finite horizon $T < \infty$, and

$$E\left[\int_0^\infty e^{-\beta t} f(X_t, \omega, \alpha_t)\, dt\right], \tag{2.4}$$

on an infinite horizon. The function f represents a running profit function, g represents a terminal reward function, and $\beta > 0$ represents a discount factor. The controller α may directly determine the end time of the objective in other cases. The associated optimization problem is known as optimal stopping time. In a general formulation, the control can be combined, consisting of a pair (α, τ) of control and stopping time, and the objective functional has the form

$$J(X, \alpha, \tau) = E\left[\int_0^\tau f(X_t, \alpha_t)\, dt + g(X_\tau)\right]. \tag{2.5}$$

The maximum value is defined by

$$v = \sup_{\alpha,\tau} J(X, \alpha, \tau), \tag{2.6}$$

which is called the value function. The aim of a stochastic optimization problem is to discover the maximizing control process and/or stopping time that achieves the desired value function.

2.3.2 Dynamic Programming Principle

We consider an SDE on $\mathbb{R}^n$ to govern the state of the system of a control model:

$$dX_s = b(X_s, \alpha_s)ds + \delta(X_s, \alpha_s)dW_s, \tag{2.7}$$

where W is a d-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$ satisfying the usual condition, i.e., $\bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_t$ for every non-maximal $t \in \mathbb{T}$ and $\{A \subseteq \Omega \mid \exists B \in \mathcal{F}, A \subseteq B \text{ and } P(B) = 0\} \subseteq \mathcal{F}_0$. The control process $\alpha = (\alpha_s)$ is a progressively measurable (with respect to $\mathbb{F}$) process, valued in A, which is a subset of $\mathbb{R}^m$. The measurable function $b : \mathbb{R}^n \times A \to \mathbb{R}^n$ and $\delta : \mathbb{R}^n \times A \to \mathbb{R}^{n\times d}$ satisfy a uniform Lipchitz condition in A, i.e., there exists $C \geq 0$ such that $\forall x, y \in \mathbb{R}^n, \forall a \in A$,

$$|b(x, a) - b(y, a)| + |\delta(x, a) - \delta(y, a)| \leq C|x - y|. \tag{2.8}$$

For a finite horizon $\mathbb{T} = [0, T]$ for some $0 < T < \infty$, the set of control process

$\alpha := \{\alpha_t\}_{t\in[0,T]}$ such that

$$E\left[\int_0^T |b(0,\alpha_t)|^2 + |\delta(0,\alpha_t)|^2 dt\right] < \infty \tag{2.9}$$

is denoted by $\mathcal{A}$.

Remark 2.1. The conditions (2.8) and (2.9) guarantee the existence and uniqueness of a strong solution with almost surely continuous sample paths to the SDE (2.7) which start from x at $s = t$, denoted by $\{X_s^{t,x}, t \leq s \leq T\}$, for any initial condition $(t,x) \in [0,T] \times \mathbb{R}^n$ and for all $\alpha \in \mathcal{A}$. By these conditions on b, δ and α, we get

$$E\left[\sup_{t\leq s\leq T} \left|X_s^{t,x}\right|^2\right] < \infty, \tag{2.10}$$

$$\lim_{h\to 0^+} E\left[\sup_{s\in[t,t+h]} \left|X_s^{t,x} - x\right|^2\right] = 0. \tag{2.11}$$

Functional objective

Let $f : [0,T] \times \mathbb{R}^n \times A \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ be two measurable functions. Suppose that f is a continuous function and

(i) g is lower-bounded

or (ii) g satisfies a quadratic growth condition, i.e., there exists a constant C which is independent of x such that

$$|g(x)| \leq C(1 + |x|^2), \quad \forall x \in \mathbb{R}^n. \tag{2.12}$$

The subset of controls α in $\mathcal{A}$ such that

$$E\left[\int_t^T |f(s, X_s^{t,x}, \alpha_s)| ds\right] < \infty$$

for all $(t,x) \in [0,T] \times \mathbb{R}^n$ is denoted by $\mathcal{A}(t,x)$. Assume that $\mathcal{A}(t,x)$ is not an empty set for all $(t,x) \in [0,T] \times \mathbb{R}^n$. Then, the gain function under the two conditions of g for all

$(t,x) \in [0,T] \times \mathbb{R}^n$ and $\alpha \in \mathcal{A}(t,x)$ is defined by

$$J(t,x,\alpha) = E\left[\int_t^T f(s, X_s^{t,x}, \alpha_s) ds + g(X_T^{t,x})\right]. \tag{2.13}$$

The aim is to maximize the gain function J over control processes. Then, we define the corresponding value function:

$$v(t,x) = \sup_{\alpha \in \mathcal{A}(t,x)} J(t,x,\alpha). \tag{2.14}$$

Proposition 2.1. Assume that (2.10) holds. If f satisfies a quadratic growth condition in x, i.e., there exists a positive constant C and a positive function $\psi : A \to \mathbb{R}_+$ such that

$$|f(t,x,a)| \leq C(1 + |x^2|) + \psi(a) \tag{2.15}$$

for any $(t,x,a) \in [0,T] \times \mathbb{R}^n \times A$, then for all $(t,x) \in [0,T] \times \mathbb{R}^n$ and for any constant control $\alpha = a \in A$

$$E\left[\int_t^T |f(s, X_s^{t,x}, a)| ds\right] < \infty.$$

Recall that (2.8) and (2.9) imply (2.10). Therefore, if (2.8), (2.9) and (2.15) hold, then $\mathcal{A}(t,x)$ contains all constant controls in A. Furthermore, the condition (2.9) and (2.10) demonstrates that for all $(t,x) \in [0,T] \times \mathbb{R}^n$, for any control $\alpha \in \mathcal{A}$

$$E\left[\int_t^T |f(s, X_s^{t,x}, \alpha_s)| ds\right] < \infty,$$

if there exists a positive constant C such that

$$\psi(a) \leq C(1 + |b(0,a)|^2 + |\sigma(0,a)|^2),$$

for all $a \in A$. This means that $\mathcal{A}(t,x) = \mathcal{A}$.

Let $\mathcal{T}_{t,T}$ denote the set of stopping times valued in $[t,T]$. The dynamic programming principle (DPP), formulated below, is a fundamental principle in the theory of stochastic control.

Theorem 2.1. ***(Dynamic programming principle for the finite horizon)*** *Let $(t,x) \in [0,T] \times \mathbb{R}^n$. Then, we have that*

1. *For all $\alpha \in \mathcal{A}(t,x)$ and $\theta \in \mathcal{T}_{t,T}$*

$$v(t,x) \geq E\left[\int_t^{\theta} f(s, X_s^{t,x}, \alpha_s)ds + v(\theta, X_{\theta}^{t,x})\right]. \tag{2.16}$$

2. *For any $\varepsilon > 0$, there exists $\alpha \in \mathcal{A}(t,x)$, which depends on ε, such that for all $\theta \in \mathcal{T}_{t,T}$*

$$v(t,x) - \varepsilon \leq E\left[\int_t^{\theta} f(s, X_s^{t,x}, \alpha_s)ds + v(\theta, X_{\theta}^{t,x})\right].$$

From this theorem, we have that

$$v(t,x) = \sup_{\alpha \in \mathcal{A}(t,x)} E\left[\int_t^{\theta} f(s, X_s^{t,x}, \alpha_s)ds + v(\theta, X_{\theta}^{t,x})\right] \tag{2.17}$$

for any stopping time $\theta \in \mathcal{T}_{t,T}$.

2.3.3 Hamiltonian-Jacobi-Bellman (HJB) equation

We consider a constant control $\alpha_s = a$ for some arbitrary a in A and the time $\theta = t + h$, by the relation (2.16) in Theorem 2.1, we have that

$$v(t,x) \geq E\left[\int_t^{t+h} f(s, X_s^{t,x}, a)ds + v(t+h, X_{t+h}^{t,x})\right]. \tag{2.18}$$

We apply Itô's formula between t and $t+h$ to v which is smooth enough:

$$v(t+h, X_{t+h}^{t,x}) = v(t,x) + \int_t^{t+h} \left(\frac{\partial v}{\partial t} + \mathcal{L}^a v\right)(s, X_s^{t,x})ds + \int_t^{t+h} D_x v(s, X_s^{t,x}) \cdot \delta(x,a)dW_s,$$

where $\mathcal{L}^a$ is the operator corresponded to (2.7) for the constant control a, and defined by

$$\mathcal{L}^a = b(x,a) \cdot D_x v + \frac{1}{2}\mathrm{tr}\left(\delta(x,a)\delta'(x,a)D_x^2 v\right).$$

Note that $\int_t^{t+h} D_x v(s, X_s^{t,x}) \cdot \delta(x,a) dW_s$ is a martingale. By substituting $v(t+h, X_{t+h}^{t,x})$ into (2.18), we obtain that

$$0 \geq E\left[\int_t^{t+h} \left(\left(\frac{\partial v}{\partial t} + \mathcal{L}^a v\right)(s, X_s^{t,x}) + f(s, X_s^{t,x}, a)\right) ds\right].$$

Note that $\dfrac{\partial v}{\partial t}, \mathcal{L}^a v$ and f are continuous functions. Dividing by h and letting h to 0, we have by the mean-value theorem for definite integral that

$$0 \geq \frac{\partial v}{\partial t}(t,x) + \mathcal{L}^a v(t,x) + f(t,x,a).$$

Since the above inequality is true for any $a \in A$, we get the inequality

$$-\frac{\partial v}{\partial t}(t,x) - \sup_{a \in A}\left[\mathcal{L}^a v(t,x) + f(t,x,a)\right] \geq 0. \tag{2.19}$$

Assume that α^* is an optimal control. Then, in (2.17), we get

$$v(t,x) = E\left[\int_t^{t+h} f(s, X_s^*, \alpha_s^*) ds + v(t+h, X_{t+h}^*)\right],$$

where X^* is the state system solution to (2.7) starting from x at t, with the control α^*. By the related reason as above, we obtain that

$$-\frac{\partial v}{\partial t}(t,x) - \mathcal{L}^{\alpha_t^*} v(t,x) - f(t,x,\alpha_t^*) = 0. \tag{2.20}$$

We combine (2.20) with (2.19), and it suggests that v should satisfy

$$-\frac{\partial v}{\partial t}(t,x) - \sup_{a \in A}\left[\mathcal{L}^a v(t,x) + f(t,x,a)\right] = 0,$$

for all $(t,x) \in [0,T) \times \mathbb{R}^n$, when the supremum in a is finite. This PDE is rewritten in the form

$$-\frac{\partial v}{\partial t}(t,x) - H(t,x,D_x v(t,x), D_x^2 v(t,x)) = 0, \quad \forall (t,x) \in [0,T) \times \mathbb{R}^n, \tag{2.21}$$

for $(t,x,p,M) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}_n$ where

$$H(t,x,p,M) = \sup_{a \in A}\left[b(x,a)p + \frac{1}{2}\mathrm{tr}(\delta\delta'(x,a)M) + f(t,x,a)\right]$$

and $\mathcal{S}_n$ is the set of symmetric $n \times n$ matrices. Equation (2.21) is called the dynamic programming equation or HJB equation and the function H is called the Hamiltonian of associated control problem. The regular terminal condition corresponding to this PDE is

$$v(T,x) = g(x), \quad \forall x \in \mathbb{R}^n, \tag{2.22}$$

which is a direct consequence from (2.13) and (2.14) of the value function v at the horizon time T.

The Hamiltonian may take the value ∞ in some domain of (t,x,p,M), if the control space A is unbounded. Assume that there exists a continuous function $G(t,x,p,M)$ on $[0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}_n$ such that

$$H(t,x,p,M) < \infty \Longleftrightarrow G(t,x,p,M) \geq 0.$$

Then, by the arguments leading to the HJB equation (2.21), the inequality (2.19) is true for any $a \in A$. Hence, we obtain that

$$G(t,x,D_x v(t,x), D_x^2 v(t,x)) \geq 0, \tag{2.23}$$

$$\text{and } -\frac{\partial v}{\partial t}(t,x) - H(t,x,D_x v(t,x), D_x^2 v(t,x)) \geq 0. \tag{2.24}$$

Moreover, if the inequality (2.23) is strict at some $(t,x) \in [0,T) \times \mathbb{R}^n$, then there exists a neighborhood of $(t,x,D_x v(t,x), D_x^2 v(t,x))$ on which H is finite. Then, the neighborhood of (t,x) should contain the optimal control. By considering inequality (2.24), we have that

$$\begin{aligned} &(i) && -\frac{\partial v}{\partial t}(t,x) - H(t,x,D_x v(t,x), D_x^2 v(t,x)) = 0 \\ \text{or } &(ii) && -\frac{\partial v}{\partial t}(t,x) - H(t,x,D_x v(t,x), D_x^2 v(t,x)) > 0. \end{aligned}$$

If case (i) happens, it is obviously the HJB equation which provides the optimal control. Then, the solution of $v(t,x)$ can be directly obtained from this PDE. Otherwise, the optimal control will be provided by the equation $G(t,x,D_x v(t,x), D_x^2 v(t,x)) = 0$. Hence, the **variational inequality** for the dynamic programming equation is as follow:

$$\min\Big[-\frac{\partial v}{\partial t}(t,x) - H(t,x,D_x v(t,x), D_x^2 v(t,x)), \\ G(t,x,D_x v(t,x), D_x^2 v(t,x))\Big] = 0. \quad (2.25)$$

If we solve for

$$-\frac{\partial v}{\partial t}(t,x) - H(t,x,D_x v(t,x), D_x^2 v(t,x)) = 0$$

and

$$G(t,x,D_x v(t,x), D_x^2 v(t,x)) = 0$$

separately, we will have two surfaces and we would like to choose the higher one as the solution of the variational inequality so that the solution is not differentiable on the intersection between the two surfaces. Thus, this control problem is called a **singular** control problem, which differs from the regular case of (2.21). A typical case of a singular problem happens when the control contributes linearly the gain function and the dynamics of the system.

For example of the singular control problem, in the one-dimensional case $n = 1, A = \mathbb{R}_+$, and

$$b(x,a) = \hat{b}(x) - a, \quad \delta(x,a) = \hat{\delta}(x), \quad f(t,x,a) = \hat{f}(t,x) - a,$$

for some functions $\hat{b}, \hat{\delta}$ and $\hat{f}$, we have that

$$\begin{aligned} H(t,x,p,M) &= \sup_{a\in A}\left[(\hat{b}(x)-a)p+\frac{1}{2}\hat{\delta}(x)^2M+\hat{f}(t,x)-a\right] \\ &= \sup_{a\in A}\left[\hat{b}(x)p+\frac{1}{2}\hat{\delta}(x)^2M-\hat{f}(t,x)-a(p+1)\right]. \end{aligned}$$

Therefore,

$$H(t,x,p,M) = \begin{cases} \hat{b}(x)p+\frac{1}{2}\hat{\delta}(x)^2M+\hat{f}(t,x) & \text{if } p+1\geq 0 \\ \infty & \text{if } p+1<0. \end{cases}$$

Then, the variational inequality is written as

$$\min\left[-\frac{\partial v}{\partial t}(t,x)-\hat{b}(x)\frac{\partial v}{\partial x}(t,x)-\frac{1}{2}\hat{\delta}(x)^2\frac{\partial^2 v}{\partial x^2}(t,x)-\hat{f}(t,x), \frac{\partial v}{\partial x}(t,x)+1\right]=0.$$

2.4 Finite Difference Method (FDM)

In this work, we investigate in second order linear PDEs. Let $u(x,y)\in\mathbb{R}^2\to\mathbb{R}$. For the general second order linear PDE

$$Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=G, \quad (x,y)\in\mathbb{R}^2, \tag{2.26}$$

where A,B,C,D,E,F,G are continuous functions on $\mathbb{R}^2$. The procedure for FDM is as follows.

1. Discretize the interval of x, $[0,X]$ and y, $[0,Y]$, into m and n stages, respectively, for some $m,n\in\mathbb{N}$. Let $\Delta x=\frac{X}{m}$ and $\Delta y=\frac{Y}{n}$.

2. Define $x_i=i\Delta x$, $y_j=j\Delta y$ and denote $u(x_i,y_j)$ by u_{ij} for $i=0,1,2,...,m$ and $j=0,1,2,...,n$.

3. Approximate derivatives by difference quotients of FDM scheme:

- Forward in x: $u_x(x_i, y_j) = \dfrac{u_{i+1,j} - u_{ij}}{\Delta x}$
- Backward in x: $u_x(x_i, y_j) = \dfrac{u_{ij} - u_{i-1,j}}{\Delta x}$
- Forward in y: $u_y(x_i, y_j) = \dfrac{u_{i,j+1} - u_{ij}}{\Delta y}$
- Backward in y: $u_y(x_i, y_j) = \dfrac{u_{ij} - u_{i,j-1}}{\Delta y}$
- Central in space x: $u_{xx}(x_i, y_j) = \dfrac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{(\Delta x)^2}$
- Central in space y: $u_{yy}(x_i, y_j) = \dfrac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{(\Delta y)^2}$

4. Set the boundary $u(0, y), u(X, y), u(x, 0), u(x, Y)$ and substitute the quotients in (2.26) to find $u(x_i, y_j)$ for all i, j.

This numerical method is used in Section 3.4 to find the solution of the variational inequality which gives the control threshold in our work.

Here, we provide all necessary knowledge related to our work. Next, the explanation of the methodology is described in Chapter 3.

CHAPTER III

METHODOLOGY

This chapter describes (1) the squirrel model, including the squirrel' dynamics and performance index, which is the criterion to seek for the optimal strategy to control the squirrels' population, in Section 3.1, (2) parameters setting in Section 3.2, (3) the variational inequality to solve the maximized performance index in Section 3.3 and (4) the numerical scheme to solve variational inequality in Section 3.4.

3.1 Squirrel Model

We employ a stochastic control model to control the squirrel population which includes an SDE representing squirrel dynamics and the performance index.

3.1.1 Squirrel dynamics

We consider the situation in the harvest season of durian; therefore, we set the index set of time to be $[0, T]$ where T is the length of the harvest season. The population dynamics of squirrels [4] is defined by

$$dX_t = \mu X_t \left(1 - \frac{X_t}{K}\right) dt + \sigma X_t dW_t - d\eta_t, \quad X_0 = x. \tag{3.1}$$

where X_t is the total number of the squirrel at time t, $x > 0$ is the initial amount of squirrels, $\mu > 0$ is the deterministic growth rate of the population, $\sigma > 0$ is the magnitude of stochastic fluctuation involved in the population dynamics, K is the carrying capacity of squirrels in any orchards, W_t is a one-dimensional standard Brownian motion, and η_t is the right-continuous adapted process representing the decrease of the population by countermeasure such as gun shooting, firecrackers and poisoning. η_t is in the form of

$\alpha_t \in [0, \infty)$, which is a measurable process, as

$$\eta_t = \int_0^t \alpha_s ds, \tag{3.2}$$

where α_t represents the eliminated squirrel population by the durian orchard owner per unit of time. We assume that $\eta_0 = 0$ therefore η_t represents the total number of squirrels eliminated by the farmer during the time interval $(0, t)$. We can rewrite the SDE (3.1) as

$$dX_t = \left[\mu X_t \left(1 - \frac{X_t}{K}\right) - \alpha_t\right] dt + \sigma X_t dW_t, \quad X_0 = x.$$

Thus, the functions $b(x, a) = \mu x \left(1 - \frac{x}{K}\right) - a$ and $\delta(x, a) = \sigma x$ satisfy the conditions (2.8) and (2.9).

Figures 3.1, 3.2 and 3.3 illustrate examples of a sample path of X_t with and without the corresponding process η_t. For the upper graphs in each figure, the blue graph is a sample path of X_t without the controlled process η_t, i.e., the SDE

$$dX_t = \mu X_t \left(1 - \frac{X_t}{K}\right) dt + \sigma X_t dW_t, \quad X_0 = x; \tag{3.3}$$

the red graph is a sample path of X_t controlled by a population control threshold, i.e., the SDE (3.1), where the control threshold is represented by the dashed line; the green graph is the addition of the red process and the control process η_t. The controlled process η_t is obtained by accumulating the excessive of the blue process (3.3) over the control threshold. When the population size of squirrels at t, X_t, is more than the control threshold, we will eliminate the squirrels to have the population size equal to the threshold value and collect the amount of eliminated squirrels at t as α_t. Then, we can simulate $\eta_t = \int_0^t \alpha_s ds$ as shown in the lower graphs of Figures 3.1, 3.2 and 3.3. The control threshold may vary in time or be a constant value. Figure 3.1 shows the case that the control threshold is a constant, while Figures 3.2 and 3.3 are cases that the control threshold is a function of the variable t. This work aims to seek the optimal value of the population control threshold through solving the performance index defined in Section 3.1.2.

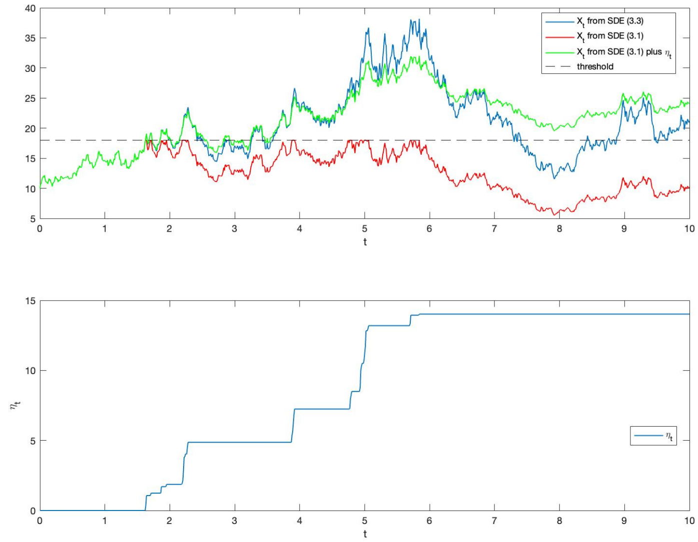


Figure 3.1: A sample path of X_t with and without the control threshold (the upper graph) and the corresponding η_t (the lower graph) by using parameters $\mu = 0.09, K = 30, \sigma = 0.3, x = 10$ and the control threshold is a constant 18.

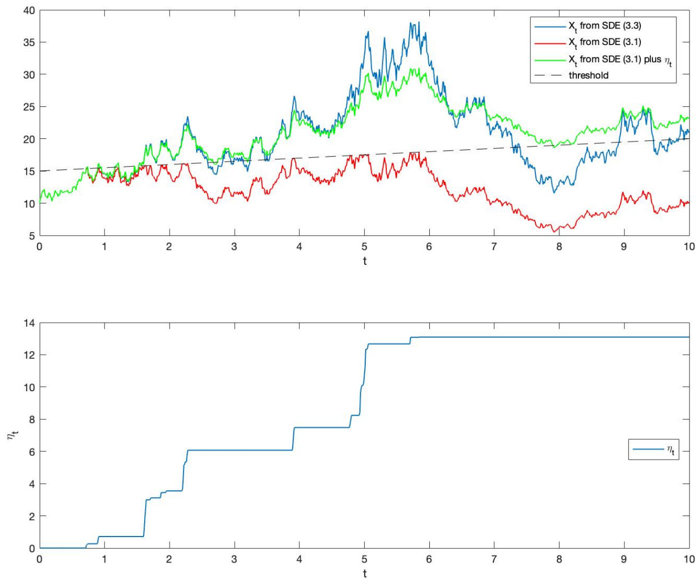


Figure 3.2: A sample path of X_t with and without the control threshold (the upper graph) and the corresponding η_t (the lower graph) by using parameters $\mu = 0.09, K = 30, \sigma = 0.3, x = 10$ and the control threshold is a function $0.5t + 15$.

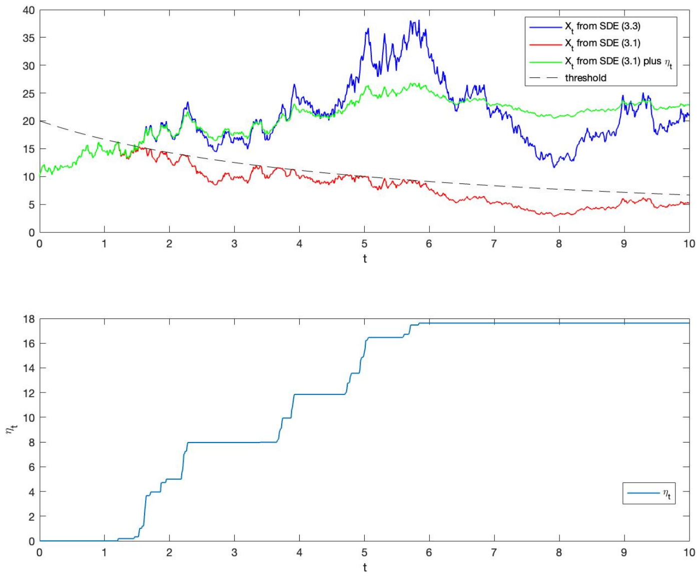


Figure 3.3: A sample path of X_t with and without the control threshold (the upper graph) and the corresponding η_t (the lower graph) by using parameters $\mu = 0.09, K = 30, \sigma = 0.3, x = 10$ and the control threshold is a function $100\left(\frac{1}{t+5}\right)$.

3.1.2 Performance index

The squirrel population dynamics is controlled by a performance index which should be maximized by the decision-maker (an orchard owner) through selecting an optimal control $\eta_t = \eta_t^*$ in order to achieve the highest performance index. The form of performance index defined in our work is modulated from Yuta Yesgashi et al. [13], which comprises the terms of predator's benefit, resource loss and the cost of countermeasure. Here, we consider the squirrels as the predator and define the performance index $\tilde{v} = \tilde{v}(x;\eta)$ by

$$\tilde{v}(x;\eta) = E\left[\int_0^T (\tilde{R}X_s^m - \tilde{S}X_s^M)ds - \gamma\eta_T + \tilde{g}(X_T) \,\middle|\, X_0 = x\right], \tag{3.4}$$

where $x \geq 0$ is the initial population of squirrels, η is the squirrel elimination strategy, $\gamma > 0$ is the constant value representing the countermeasure cost for a squirrel. Hence, we can divide γ out of (3.4) to eliminate one more variable from our model and get $v(x;\eta)$

as

$$v(x;\eta) = E\left[\int_0^T (RX_s^m - SX_s^M)\, ds - \eta_T + g(X_T) \;\middle|\; X_0 = x\right], \tag{3.5}$$

where the performance index $v = \frac{\tilde{v}}{\gamma}$ represents the expected net profit of the decision-makers, $T > 0$ is the end time, $g = \frac{\tilde{g}}{\gamma}$ is the reward function contributed by the remaining squirrels left at time T such as eating durian pests, removing small plants that compete with durians for nutrients, etc., and S, R, m, M are model parameters which satisfy $S = \frac{\tilde{S}}{\gamma} \geq 0, R = \frac{\tilde{R}}{\gamma} \geq 0$ and $0 < m < 1 < M \leq 2$. The condition of parameters m, M also obtain by the performance index of Yuta Yeagashi et al. [13], which represent the squirrels' benefit and durian loss, respectively. The values of $\tilde{R}$ and $\tilde{S}$ are weighted by the decision of the durian farmer in the sense of the positive and negative effect of squirrels on the durian orchard. The parametrs R, S, m, M are explained in details in Section 3.2. We can rewrite $v(x;\eta)$ in (3.5) as

$$v(x;\eta) = E\left[\int_0^T (RX_s^m - SX_s^M - \alpha_s)\, ds + g(X_T) \;\middle|\; X_0 = x\right].$$

Since $0 < m, M < 2$, the function $f(t, x, a) = Rx^m - Sx^M - a$ satisfies the condition (2.15).

$RX_s^m - SX_s^M$ represents the total value of squirrels, including the benefit and the damage to the orchard. The term RX_s^m represents the benefit of squirrels to the orchard ecosystem. The squirrels are known as the seed dispersers [11]. This behaviour will increase the amount of green space in the environment, which helps to reduce seasonal swings. However, squirrels are rodent. They will turn to ruin the ecosystem as they gnaw the xylem of the plants too much when their population size is too large; therefore, their contribution to the environment will be less beneficial [10]. This is the reason why m has to be between 0 and 1. An example of the graph RX_s^m is shown in Figure 3.4.

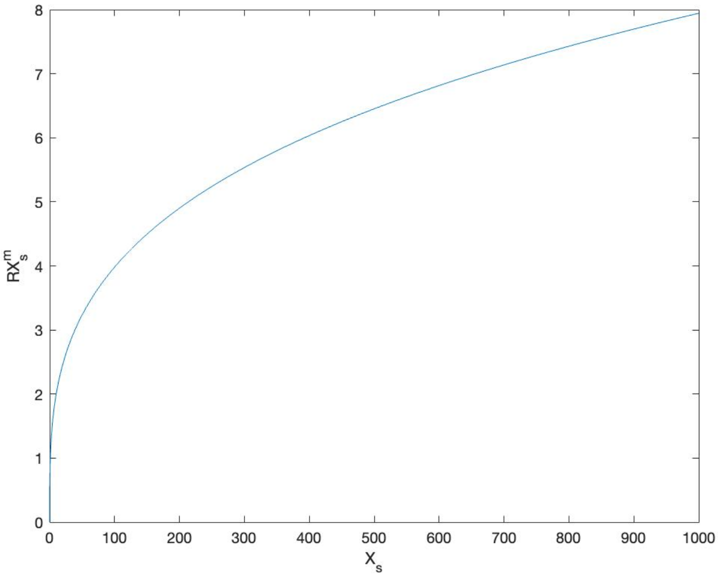


Figure 3.4: An example of the term RX_s^m when we set $R = 1, m = 0.3$ and $x \in [0, 1000]$.

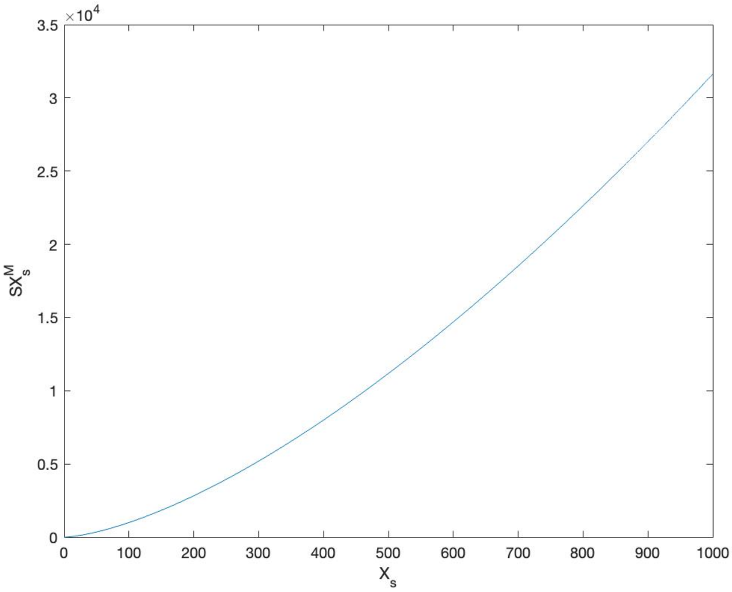


Figure 3.5: An example of the term SX_s^M when we set $S = 1, M = 1.5$ and $x \in [0, 1000]$.

The term $-SX_s^M$ represents the negative impact of squirrels on durian orchard. The higher the squirrel population, the more harmful the effect on durian. Hence, the value of M should be more than 1. However, the negative effect should not be too strong. Here, we assume that M should be at most 2. An example of the graph SX_s^M is shown in Figure 3.5.

3.2 How to set the parameters

The values of all parameters should be computed by the actual data. Unfortunately, we do not have any actual data to compute. Hence, we attempt to set the meaningful values of all parameters to be close to actual data as much as possible. The values of parameters are set as in Table 3.1.

Parameter	Value
μ	0.05
σ	0.03
K	200
T	4
R	0.00003
S	0.3
m	0.3
M	1.2

Table 3.1: The prototype of our parameters

The values of each parameter in Table 3.1 are obtained from the situation in a 12,800 square meters durian orchard in Chanthaburi, Thailand, during the fruiting season. The period of this season is around four months due to the climate changes and inconstancy of natural. Hence, we take time $T = 4$. The durian fruiting season starts around February as same as the beginning of the squirrel mating season. The female squirrel has been pregnant for a month. Young squirrels begin to come out of their burrows at about three months of age, and during that time, they feed on their mother's milk. Therefore, the number of squirrels that could be the enemy of farmer's products will remain relatively low in the period we consider. Then, we take $\mu = 0.05$. Squirrels are also solitude-loving animals. Even in the mating season, males and females will stay together for about two weeks. Therefore, we assume that the population of squirrels will not fluctuate highly during the period under consideration. However, it could be that the smell of durian will attract some squirrels to come in the orchard. Hence, we let $\sigma = 0.03$. There are many scales for planting the durian trees depending on individual durian farmers. In this case, we assume that a farmer plants one durian tree per 8×8 square meters. This means that there will be 25 durian trees in 1,600 square meters. Moreover, we suppose that each durian tree can accommodate only one squirrel due to its solitude-loving behaviour. Therefore, on the space of 12,800 square meters, there will be at most 200 squirrels; we set $K = 200$. We assume that a durian orchard owner prioritizes the harm the squirrels do to durian fruit over the benefit they provide to the ecosystem. We consider R and S in the sense of the price that a squirrel contributes to and harms the orchard, respectively. If we suppose that a squirrel can contribute only 1 baht to the orchard, which means 1 baht for 15,000 kilograms of durian, we assume that there are 30 durian fruits per a tree and a fruit weights around 2.5 kilograms. Also, for S, we suppose that a squirrel can harm the orchard for 9,000 baht. Moreover, we assume that the cost of countermeasure for a squirrel is 2 baht; $\gamma = 2$. Hence, we set $R = \dfrac{1}{15,000 \times 2} \approx 0.00003$ and $S = \dfrac{9,000}{15,000 \times 2} = 0.3$. By the condition that $0 < m < 1 < M \leq 2$ [13], the value of m and M are set to be 0.3 and 1.2, respectively, since we take the assumption that the orchard owner focuses on the damage by squirrels and do not appreciate the benefit to environment.

3.3 Variational inequality

The performance index (3.5) is maximized over all strategies η as the value function

$$V(x) := \sup_{\eta} v(x;\eta) = v(x;\eta^*). \tag{3.6}$$

Here, η^* represents the optimal plan for controlling the squirrel population for durian farmers.

According to the dynamic programming principle for stochastic control problems [9], our work associates with the one-dimensional case of singular control problem. Let $a \in \mathcal{A} = \mathbb{R}_+$ be a constant control. In our case,

$$b(x,a) = \mu x\left(1 - \frac{x}{K}\right) - a, \delta(x,a) = \sigma x, \text{ and } f(t,x,a) = Rx^m - Sx^M - a.$$

The function b, δ and f satisfy the conditions (2.8), (2.9) and (2.15) . Hence, the Hamiltonian is

$$\begin{aligned} H(t,x,v_x(t,x),v_{xx}(t,x)) &= \sup_{a\in\mathcal{A}}\Bigg[\left(\mu x\left(1-\frac{x}{K}\right)-a\right)v_x(t,x) + \frac{1}{2}\sigma^2x^2v_{xx}(t,x) \\ &\qquad + (Rx^m - Sx^M - a)\Bigg] \\ &= \sup_{a\in\mathcal{A}}\Bigg[\mu x\left(1-\frac{x}{K}\right)v_x(t,x) + \frac{1}{2}\sigma^2x^2v_{xx}(t,x) + (Rx^m - Sx^M) \\ &\qquad - a\left(v_x(t,x)+1\right)\Bigg]. \end{aligned}$$

Since $a \in \mathcal{A} = \mathbb{R}_+$, the Hamiltonian may take the value ∞ when $v_x(t,x) + 1 < 0$ which means that

$$H(t,x,p,Q) = \begin{cases} \mu x\left(1-\frac{x}{K}\right)p + \frac{1}{2}\sigma^2x^2Q + (Rx^m - Sx^M), & \text{if } p+1 \geq 0, \\ \infty, & \text{if } p+1 < 0, \end{cases}$$

where $p = v_x(t,x)$ and $Q = v_{xx}(t,x)$. Therefore,

$$H(t,x,p,Q) < \infty \Longleftrightarrow v_x(t,x) + 1 \geq 0.$$

Then, the value function $V(x)$ is a solution $u(0,x)$ to the variational inequality

$$\min\left[-u_t - \mu x\left(1 - \frac{x}{K}\right)u_x - \frac{1}{2}\sigma^2 x^2 u_{xx} + Sx^M - Rx^m, u_x + 1\right] = 0 \qquad (3.7)$$

for $(t,x) \in [0,T) \times (0,\infty)$ with the conditions that $u(T,x) = g(x)$ for all $x \in (0,\infty)$ and $u(t,0) = 0$ for all $t \in [0,T)$. In (3.7), the left term in the min operator quantifies the situation where we should not eliminate squirrels, whereas the other term has the opposite meaning.

3.4 Applying a numerical scheme

From variational inequality (3.7), the intersection of those two terms in the min operator represents the optimal strategy to control the squirrels population. Then, we adapt FDM to the following two equations:

$$-u_t - \mu x\left(1 - \frac{x}{K}\right)u_x - \frac{1}{2}\sigma^2 x^2 u_{xx} + Sx^M - Rx^m = 0, \qquad (3.8)$$

$$u_x + 1 = 0. \qquad (3.9)$$

For $(t,x) \in [0,T] \times [0,\infty)$, we take $u(T,x) = 0$ and $u(t,0) = 0$ to be the initial conditions. We apply FDM, which is backward in time, forward in x for u_x and central in space x for u_{xx} to (3.8). Due to the smoothness of the equation (3.8), applying the different order of x will not affect the calculation of the numerical solution. An interval time t is discretized to be $L+1$ stages for $L \in \mathbb{N}$ with the time step $l := \frac{T}{L}$. We discretize an interval of $x \in [0,K]$ to be $N+1$ stages for $N \in \mathbb{N}$ with the step $h := \frac{K}{N}$. Let $t_i = il$ where $i \in \{0,1,2,...,L\}$, and $x_j = jh$ where $j \in \{0,1,...,N\}$. Let $u_{i,j} := u(t_i, x_j)$. Then, we have that

$$\begin{aligned}
u_t(t_i,x_j) &\approx \frac{u_{i,j}-u_{i-1,j}}{l} && \text{for } i=1,...,L, j=0,1,...,N,\\
u_x(t_i,x_j) &\approx \frac{u_{i,j+1}-u_{i,j}}{h} && \text{for } i=0,1,...,L, j=0,1,...,N-1,\\
u_{xx}(t_i,x_j) &\approx \frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{h^2} && \text{for } i=0,1,...,L, j=1,2,...,N-1.
\end{aligned}$$

We apply all above equations to (3.8),

$$\begin{aligned}
-\frac{u_{i,j}-u_{i-1,j}}{l} - \mu x_j\left(1-\frac{x_j}{K}\right)\left(\frac{u_{i,j+1}-u_{i,j}}{h}\right)\\
-\frac{1}{2}\sigma^2 x_j^2\left(\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{h^2}\right) + Sx_j^M - Rx_j^m = 0.
\end{aligned}$$

Then, we obtain that

$$\begin{aligned}
u_{i-1,j} = l\Bigg[\frac{u_{i,j}}{l} + \mu x_j\left(1-\frac{x_j}{K}\right)\left(\frac{u_{i,j+1}-u_{i,j}}{h}\right)\\
+\frac{1}{2}\sigma^2 x_j^2\left(\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{h^2}\right) - Sx_j^M + Rx_j^m\Bigg], \quad (3.10)
\end{aligned}$$

where i start from L to 1 and $j = 1, 2, ..., N-1$.

The computation of (3.9) is FDM which forward in x.

$$\frac{u_{i,j+1}-u_{i,j}}{h} + 1 = 0.$$

Then, we obtain

$$u_{i,j+1} = u_{i,j} - h, \quad (3.11)$$

where $i = 0, 1, ..., L$ and $j = 0, 1, ..., N-1$.

By the constraints, we have no boundary at $u(t, K)$. Hence, we have to extend the boundary $u_{L,j} = 0$ for $j = 0, 1, .., N$ to $j = 0, 1, ..., N+L$ as the mapping nodes shown in Figure 3.6.

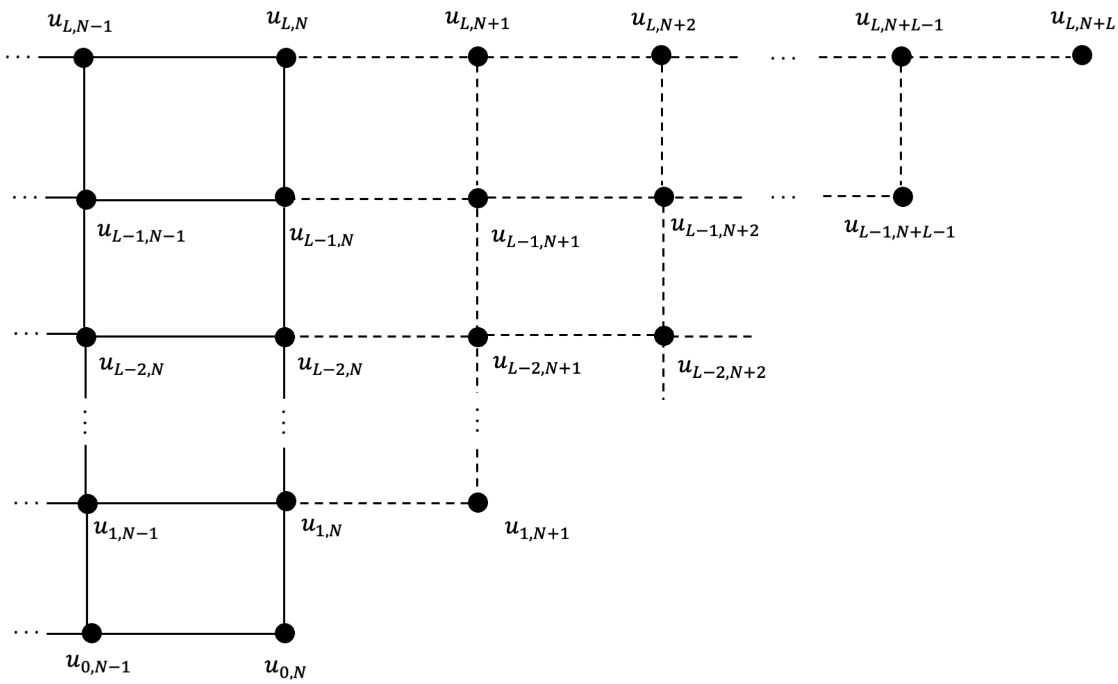


Figure 3.6: The mapping node of $u_{i,j}$.

More precisely, we simulate the solution of the variation inequality with the boundaries $u_{L,j} = g(x_j)$ and $u_{i,0}$ follow the algorithm:

Given h, l, t_i, x_j and $g(x_j)$:

1. Let u^1 be a $(L+1)\times(N+L+1)$ matrix and u^2 be a $(L+1)\times(N+1)$ matrix.
2. Let $u^1_{L,j} = g(x_j)$ for $j = 0$ to $L+N$,

   $u^1_{i,0}, u^2_{i,0} = 0$ for $i = 0$ to L,

   $u^2_{L,j} = g(x_j)$ for $j = 0$ to N.
3. Compute u^1:

   For each $i = 0$ to $L-1$:

   For each $j \in 1$ to $L+N-i$:

$$u^1_{i-1,j} = l\left[\frac{u^1_{i,j}}{l} + \mu x_j\left(1-\frac{x_j}{K}\right)\left(\frac{u^1_{i,j+1}-u^1_{i,j}}{h}\right) + \frac{1}{2}\sigma^2 x_j^2\left(\frac{u^1_{i,j+1}-2u^1_{i,j}+u^1_{i,j-1}}{h^2}\right) - Sx_j^M + Rx_j^m\right].$$

4. Compute u^2:

For each $i = 0$ to L:

For each $j = 0$ to $N - 1$:

$$u^2_{i,j+1} = u^2_{i,j} - h$$

5. Plot the surface of u^1 and u^2.

The simulations of (3.8) and (3.9) applied FDM are provided with different data sets and discussed in Chapter 5.

CHAPTER IV

REWARD FUNCTION

The reward function demonstrates the benefit of the squirrels left at the end time. In this work, we consider 2 cases: (1) farmers do not receive any reward from the remaining squirrels at time T, and (2) the farmers obtain some benefit contributed by the squirrels left at time T.

4.1 Non-benefit of remaining squirrels

We let $g(X_T)$ be 0, since the durian orchard owner does not get any reward from the remaining squirrels. Obviously, the function g satisfies the condition (2.12). By substituting $g(X_T)$ in (3.5), we get the performance index

$$v(x;\eta) = E\left[\int_0^T (RX_s^m - SX_s^M)\, ds - \eta_T \;\middle|\; X_0 = x\right], \tag{4.1}$$

which can be solved by the variational inequality

$$\min\left[-u_t - \mu x\left(1 - \frac{x}{K}\right)u_x - \frac{1}{2}\sigma^2 x^2 u_{xx} + Sx^M - Rx^m, u_x + 1\right] = 0 \tag{4.2}$$

for $(t,x) \in [0,T) \times (0,\infty)$ with the conditions that $u(T,x) = 0$ for all $x \in (0,\infty)$ and $u(t,0) = 0$ for all $t \in [0,T)$.

4.2 Benefit contributed by remaining squirrels

After the fruiting season of durian, the alive squirrels in the orchard may turn to eat bugs or worms, which are durian pests, or bite the small plants that compete with durian for nutrients. The durian trees will be more healthy and produce more fruits in the next season which means that there will be more profit for the durian orchard owner. However, squirrels may destroy the the orchard in the situation that they gnaw small branches of durian trees so that the durian may halt their growth. We assume $g(X_T) := \kappa X_T$ where

$\kappa \in \mathbb{R}$ is the parameter representing the benefit or damage that squirrels can provide after the end time. Hence, κ can also be positive and negative depending on the decision of the durian orchard owner. Since function g satisfies the condition (2.12), and by substituting $g(X_T)$ in (3.5), we then obtain the performance index

$$v(x;\eta) = E\left[\int_0^T (RX_s^m - SX_s^M)\, ds - \eta_T + \kappa X_T \;\middle|\; X_0 = x\right],$$

Here, $\kappa := \dfrac{\tilde{\kappa}}{\gamma}$ and we assume that a squirrel can contribute 2 baht for the environment after the end time. Hence, $\tilde{\kappa} = 2$. The performance index, in this case, is

$$v(x;\eta) = E\left[\int_0^T (RX_s^m - SX_s^M)\, ds - \eta_T + X_T \;\middle|\; X_0 = x\right], \tag{4.3}$$

which can be solved by the variational inequality

$$\min\left[-u_t - \mu x\left(1 - \frac{x}{K}\right) u_x - \frac{1}{2}\sigma^2 x^2 u_{xx} + Sx^M - Rx^m, u_x + 1\right] = 0 \tag{4.4}$$

for $(t, x) \in [0, T) \times (0, \infty)$ with the conditions that $u(T, x) = x$ for all $x \in (0, \infty)$ and $u(t, 0) = 0$ for all $t \in [0, T)$.

In this study, we compare the simulation of both 2 cases when using the set of parameters designated in Table 3.1, which is discussed in Chapter 5.

CHAPTER V

RESULT

In this chapter, the simulations of the control threshold applied with 2 cases of the reward functions and adjusted by different values of 4 parameters, R, S, m, M, are presented by using the starting parameters as in Table 3.1. The FDM is employed to simulate the model with the squirrel population increment $\Delta x = 1$ and the time increment $\Delta t = 0.01$.

Firstly, we simulate the performance index with the reward function $g(X_T) = 0$ as shown in Figure 5.1. It shows the numerical solution $u(t, x)$ of (3.8) and (3.9). There is an intersection curve between the 2 graphs representing the squirrel population's control threshold. In other cases when we change some parameters in the model, the simulations of $u(t, x)$ will be similar to Figure 5.1 and also have the intersection curve.

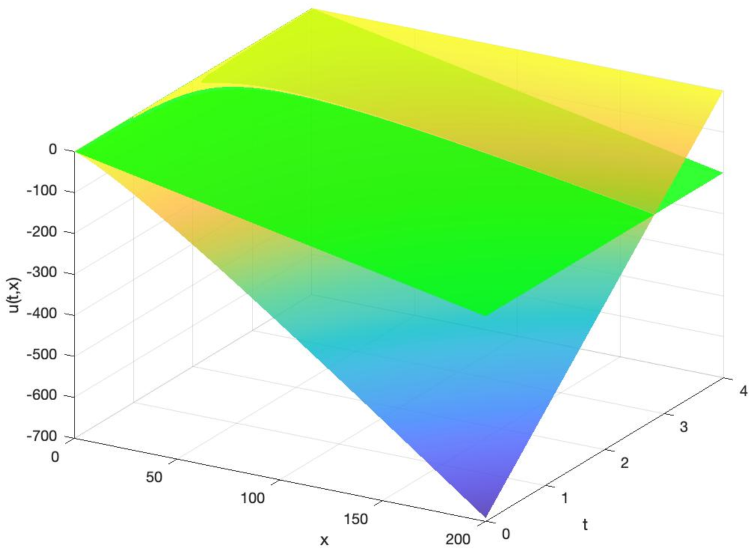
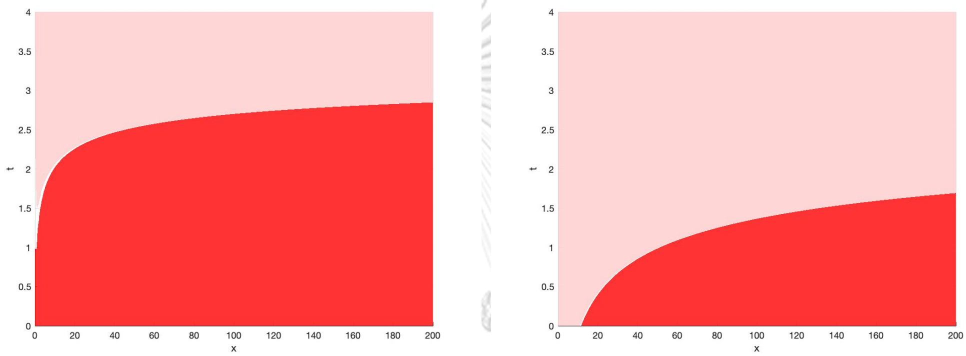


Figure 5.1: Surfaces of $u(t, x)$ corresponding to (3.9) (the green surface) and (3.8) (the other one) when $g(X_T) = 0$.

The control thresholds for 2 cases of reward functions are shown in Figure 5.2. Evidently, the control threshold varies on time, which tells us that the farmers should control squirrels according to the control threshold line to maximize the performance index. The strategy in Figure 5.2(a) tells us that the farmer should eliminate squirrels all in the first month, which is quite severe and inappropriate for the environment. On the other hand, the strategy obtained from Figure 5.2(b) shows that the farmer should allow 12 squirrels to stay in the orchard at the beginning and should not kill the squirrels after 2 months. Here, we show the result with the maximum number of squirrels equal to K due to the carrying capacity of squirrels in the orchard.



(a) **(b)**

Figure 5.2: Simulation of population control threshold when (a) $g(X_T) = 0$ and (b) $g(X_T) = X_T$.

5.1 Sensitivity analysis

Due to the lacking of actual data, we simulate the control threshold from the different values of 4 parameters, R, S, m, M, in the case that $g(X_T) = X_T$ to analyze how these parameters affect the strategy and to know which parameter should be collected carefully in the future.

We first consider the parameter S, which tells how much the squirrels can destroy the orchard following the decision of the farmer. We simulate the strategies by setting S to be $0.00003, 0.3$ and 0.6. Figure 5.3 shows the control threshold obtained from 3 different values of S. When the value of S is small, $S = 0.00003$, the result tells that the farmer should not eliminate squirrels at any time, as shown in Figure 5.3(a). Moreover, the strategy gets stricter when the value of S is higher. Hence, it seems that parameter S significantly affects the strategy.

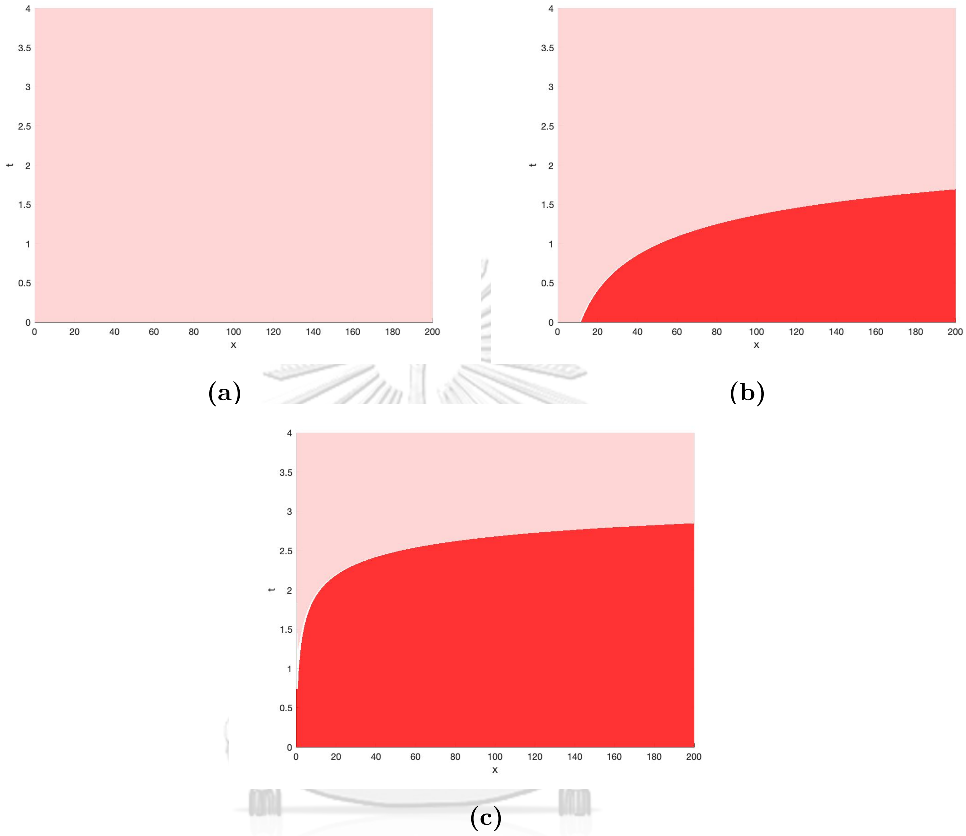
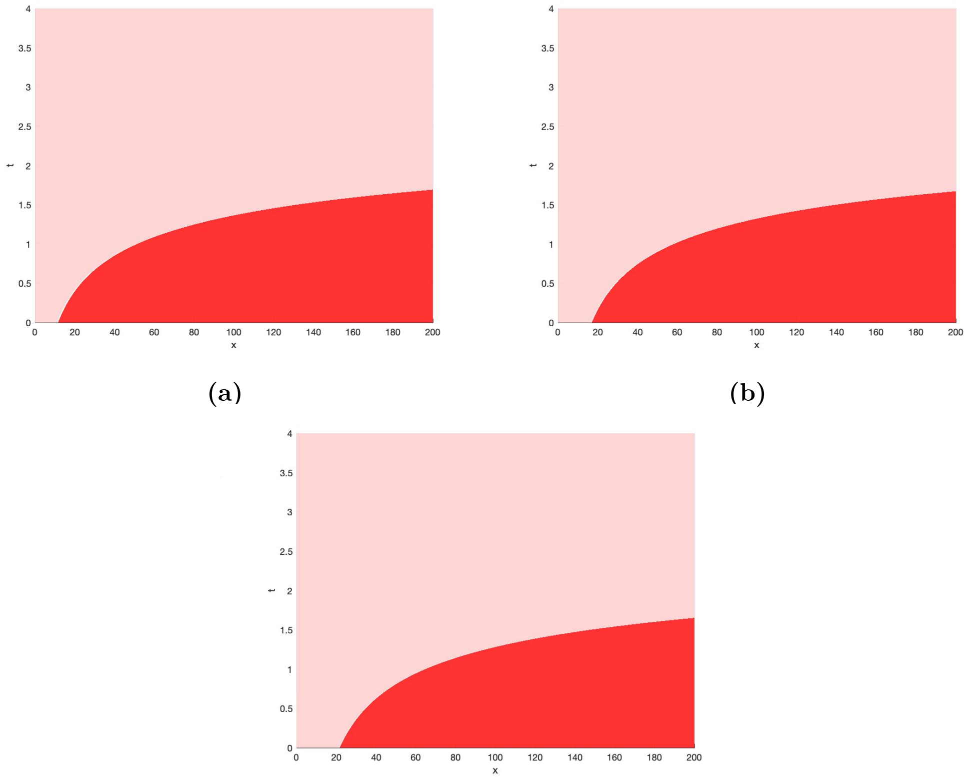


Figure 5.3: Simulation of population control threshold when $g(X_T) = X_T$ with 3 different values of S: (a) $S = 0.00003$, (b) $S = 0.3$ and (c) $S = 0.6$.

Next, the parameter R is considered. The value of this parameter represents how much the squirrels can contribute to the orchard, which depends on the farmer's decision. Figure 5.4 shows the control threshold from 3 different values of R: $R = 0.00003 << 0.3 = S$, $R = 0.3 = S$ and $R = 0.6 = 2S$. We can see that 3 strategies shown in Figure 5.4 are not quite different which contrasts with the big change in the value of R. This may cause by the small value of m considered from the term Rx^m in (3.5). We then prove by simulating the control threshold when $m = 0.7$, which means that the benefit of squirrels to the ecosystem is greater, as shown in Figure 5.5.

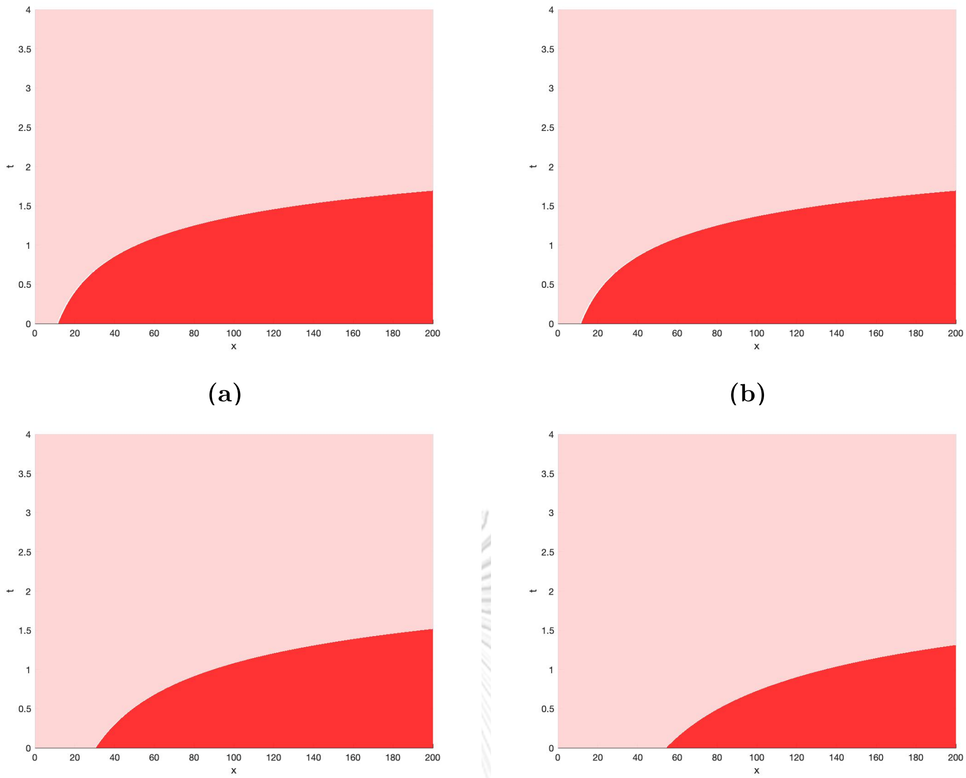


(a) **(b)**

(c)

Figure 5.4: Simulation of population control threshold when $g(X_T) = X_T$ with 3 different values of R: (a) $R = 0.00003$, (b) $R = 0.3$ and (c) $R = 0.6$.

The control threshold in Figure 5.5(a), using $m = 0.3$, and 5.5(b), using $m = 0.7$, are not quite different, although the value of m is different. Then, we simulate the control threshold when $m = 0.7$ with other 2 greater values of R, 0.3 and 0.6, in Figures 5.5(c) and 5.5(d), respectively. The strategy in Figure 5.5 becomes lighter when the value of R and m get bigger, which is what we expect.



(a) (b) (c) (d)

Figure 5.5: Simulation of population control threshold when $g(X_T) = X_T$ with (a) $m = 0.3$ and $R = 0.00003$ (b) $m = 0.7$ and $R = 0.00003$, (c) $m = 0.7$ and $R = 0.3$ and (d) $m = 0.7$ and $R = 0.6$.

Figure 5.6 is the simulation of the control threshold when we adjust the value of M to be 1.8, which means that the squirrels severely destroy the orchard. The strategy shown in Figure 5.6 is pretty severe. It tells that the farmer should eliminate squirrels when their population size is more than 2 in the first 3 months. It seems that the parameter M significantly affects the strategy.

The last case is the simulation in Figure 5.7, which shows the control threshold when we set $m = 0.9$ and $M = 1.1$. It represents the situation that squirrel is really important and beneficial to the environment, also lightly destroy the orchard. The control threshold shown in Figure 5.7 is high, around 160 to 200, and control only in the first 3 days which related to the meaning of the parameters.

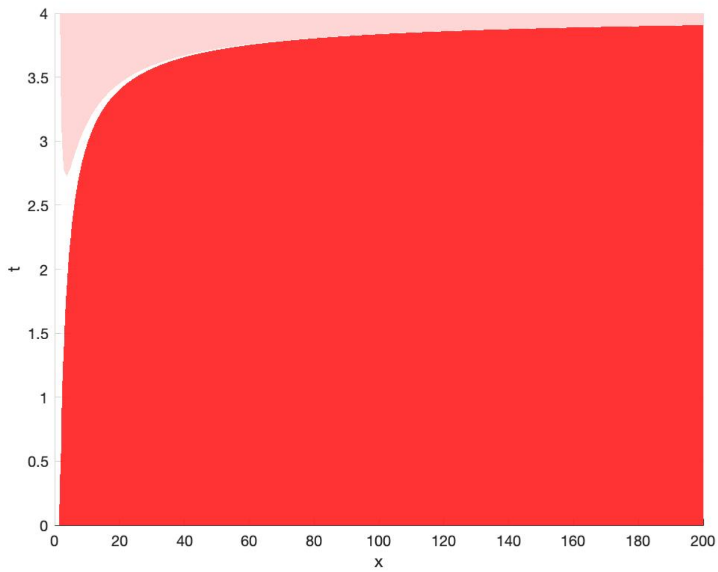


Figure 5.6: Simulation of population control threshold when $g(X_T) = X_T$ with $M = 1.8$.

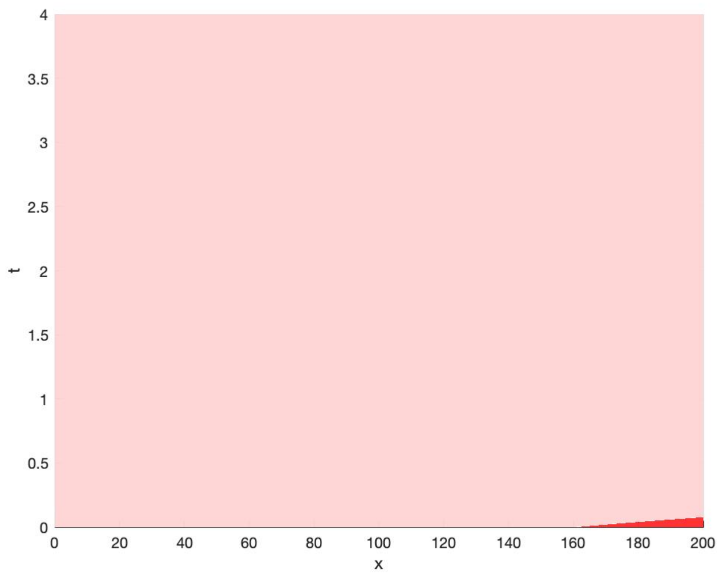


Figure 5.7: Simulation of population control threshold when $g(X_T) = X_T$ with $M = 1.1$ and $m = 0.9$.

5.2 Simulations of SDE

In this section, we simulate a sample path of X_t when $x_0 = 100$ for 2 cases: (1) the values of parameters follow Table 3.1, shown in Figure 5.8, and (2) the values of parameters are as in Table 3.1 but $m = 0.9$ and $M = 1.1$, shown in Figure 5.9.

Figures 5.8 and 5.9 show that the control threshold at $t = 0$ greatly effects the population size after that. This may cause by the small values of μ and σ. Therefore, we simulate Figure 5.10, which uses the value of $\mu = 0.5$ and $\sigma = 0.1$. Figure 5.10 shows the simulation of X_t controlled by the control threshold for 5 random sets. Although we use the greater values of μ and σ, the population size of squirrels is still effected mainly by the control threshold at the beginning, as shown in Figure 5.10. The control threshold associated with each scenario at time $t = 0$, $\bar{x}_0$, is presented in Table 5.1. The values of $\bar{x}_0$ are computed from the case that we set the reward function $g(X_T) = X_T$.

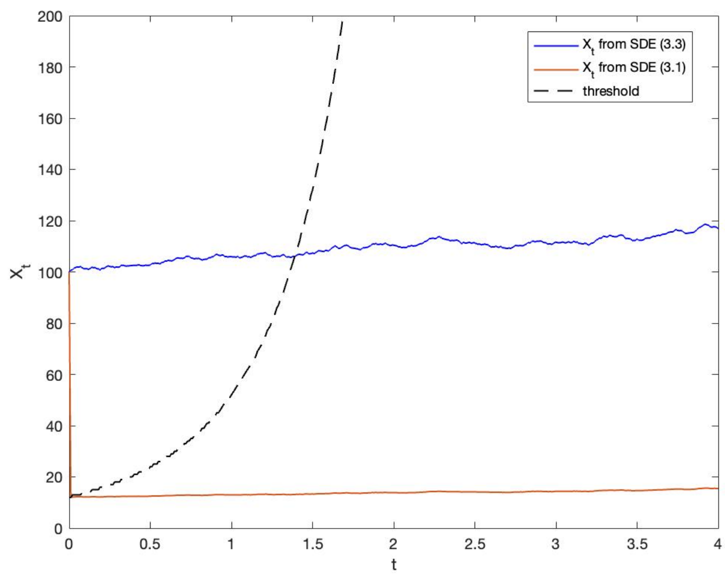


Figure 5.8: Simulation of population size X_t without controlling (the blue line), the population size X_t controlled by the control threshold (the red line) and the control threshold (the dash line) when the values of the parameters are as in Table 3.1.

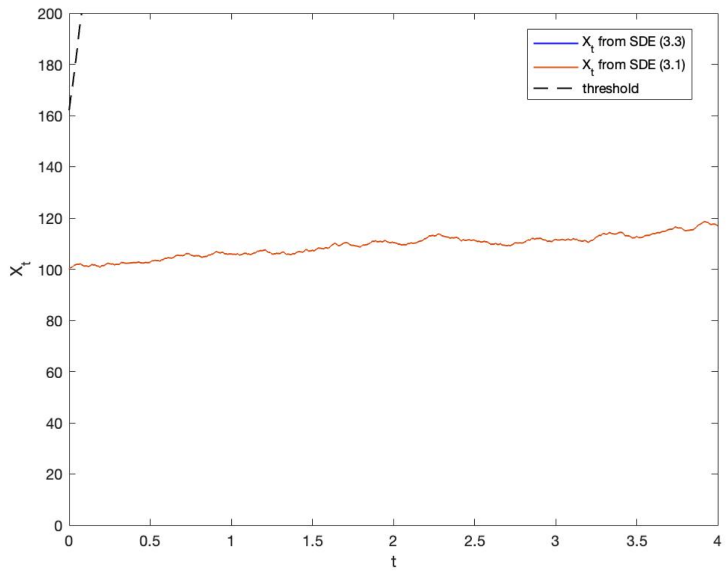


Figure 5.9: Simulation of population size X_t without controlling (the blue line), the population size X_t controlled by the control threshold (the red line) and the control threshold (the dash line) when the values of the parameters are as in Table 3.1 but $m = 0.9$ and $M = 1.1$.

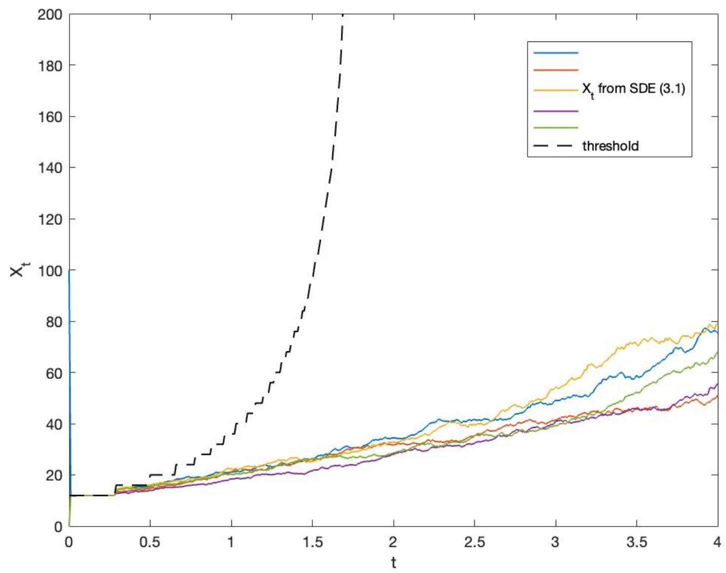


Figure 5.10: Simulation of population size X_t controlled by the control threshold (the 5 colors line) and the control threshold (the dash line) when the values of the parameters are as in Table 3.1 but $\mu = 0.5$ and $\sigma = 0.1$.

Parameter	μ	σ	K	T	R	S	m	M	$\bar{x}_0$
Scenario 1	0.05	0.03	200	4	0.00003	0.3	0.3	1.2	12
Scenario 2	0.05	0.03	200	4	0.00003	0.00003	0.3	1.2	-
Scenario 3	0.05	0.03	200	4	0.00003	0.6	0.3	1.2	1
Scenario 4	0.05	0.03	200	4	0.3	0.3	0.3	1.2	18
Scenario 5	0.05	0.03	200	4	0.6	0.3	0.3	1.2	22
Scenario 7	0.05	0.03	200	4	0.00003	0.3	0.7	1.2	12
Scenario 8	0.05	0.03	200	4	0.3	0.3	0.7	1.2	31
Scenario 9	0.05	0.03	200	4	0.6	0.3	0.7	1.2	55
Scenario 10	0.05	0.03	200	4	0.00003	0.3	0.3	1.8	2
Scenario 11	0.05	0.03	200	4	0.00003	0.3	0.9	1.1	162
Scenario 12	0.5	0.1	200	4	0.00003	0.3	0.9	1.1	12

Table 5.1: The set of parameters with corresponding $\bar{x}_0$.

From Table 5.1 and all simulated results, the parameters S and M seem to affect significantly the control threshold. Also, parameters R and m have a more significant effect on the control threshold when their values increase together. Therefore, we should carefully collect the data to estimate parameters R, S, m, M.

CHAPTER VI

CONCLUSION AND FUTURE WORK

6.1 Conclusion

In this thesis, we proposed a stochastic control model

$$dX_t = \mu X_t \left(1 - \frac{X_t}{K}\right) dt + \sigma X_t dW_t - d\eta_t, \quad X_0 = x, \tag{3.1}$$

based on the logistic SDE with carrying capacity for the squirrel population due to the confined space of the durian orchard. This model seeks for population management policy for squirrels by maximizing the performance index

$$v(x;\eta) = E\left[\int_0^T (RX_s^m - SX_s^M)\, ds - \eta_T + g(X_T) \;\Big|\; X_0 = x\right], \tag{3.5}$$

defined by the primary consideration of the ecosystem service and the damage to the orchards by squirrels. The cost of squirrel elimination is also included in the performance index. The reward function can also be added to the performance index; we consider 2 situations where we exclude a reward function and include the reward function $g(X_T) = X_T$. The HJB PDE is formulated by DPP for the finite horizon problem. Consequently, the variational inequality

$$\min\left[-u_t - \mu x\left(1 - \frac{x}{K}\right) u_x - \frac{1}{2}\sigma^2 x^2 u_{xx} + Sx^M - Rx^m, u_x + 1\right] = 0 \tag{3.7}$$

is formulated. The maximized performance index

$$V(x) := \sup_{\eta} v(x;\eta) = v(x;\eta^*). \tag{3.6}$$

is the solution to the variational inequality at time zero. Moreover, the numerical solutions of the variational inequality are presented by FDM, and the regions for performing the countermeasure are discussed. In addition, the optimal strategy to control the squirrel population is given numerically, and the sensitivity discussions of parameters S, R, m, M in the case $g(X_T) = X_T$ are also presented. The simulation shows that parameters S, M are greatly sensitive and parameters R, m are sensitive when both values are higher. Furthermore, sample paths of X_t are simulated and we can see that the control threshold around time $t = 0$ greatly affects the population size despite the high values of μ and σ. Therfore, finding the control threshold at time $t = 0$, $\bar{x}_0$, is very important. The values of $\bar{x}_0$ corresponding to each scenario are presented in Table 5.1.

6.2 Future work

Notice that with our performance index (3.5), the optimal strategies to eliminate squirrels at $t = 0$ significantly impact the population size at later time, as shown in all simulation results and Table 5.1 in Chapter 5. Moreover, it seems that parameters S and M significantly affect the strategy. These suggest that we should modify the performance index and define more conditions on the relation between the ecosystem service provided by squirrels and the damage from squirrels to the durian orchard. Furthermore, we should collect actual data and perform parameter estimation to have realistic values for all parameters in the model. In addition, the stochastic control model (3.1) can also be modified with the dynamics of durians in the orchard to obtain a system of SDEs; consequently, we can add a term regarding durians into the performance index.

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