จำนวนแรมซีย์ขนาดเชื่อมโยงสำหรับการจับคู่และกราฟบางกราฟ

นายจิรวัฒน์ มุ่งตุ้มกลาง

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
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CONNECTED SIZE RAMSEY NUMBERS OF MATCHING AND SOME GRAPHS

Mr. Jirawat Mungtumklang

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By	Mr. Jirawat Mungtumklang
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Thesis Advisor	Assistant Professor Teeradej Kittipassorn, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

Polkit Sangvanich
.... Dean of the Faculty of Science

(Professor Polkit Sangvanich, Ph.D.)

THESIS COMMITTEE

.... Chairman

(Associate Professor Chariya Uiyyasathian, Ph.D.)

.... Thesis Advisor

(Assistant Professor Teeradej Kittipassorn, Ph.D.)

Nitthi Rung
.... Examiner

(Nithi Rungtanapirom, Dr.rer.nat.)

.... External Examiner

(Associate Professor Varanoot Khemmani, Ph.D.)

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จำนวนแรมซีย์ขนาดเชื่อมโยงสำหรับกราฟ G และ H ซึ่งเขียนแทนด้วยสัญลักษณ์โดย $\hat{r}_c(G,H)$ คือจำนวนนับที่น้อยที่สุดที่จะมีกราฟเชื่อมโยง F ที่มีเส้นเชื่อม k เส้นและการระบายสีบนเส้นเชื่อมของกราฟ F ด้วยสีแดงหรือน้ำเงินต้องประกอบด้วยสำเนาของกราฟ G ที่เป็นสีแดงหรือสำเนาของกราฟ F ที่เป็นสีน้ำเงิน Assiyatun, Baskoro และ Rahadjeng แสดงไว้ว่า $\hat{r}_c(nK_2,P_3) \leq \lfloor\frac{5n-1}{2}\rfloor$ เมื่อไม่นานมานี้ Wang, Song, Zhang และ Zhang ได้พิสูจน์ว่า $\hat{r}_c(nK_2,P_3) = \lfloor\frac{5n-1}{2}\rfloor$ ในวิทยานิพนธ์นี้เราได้ทำการแสดงบทพิสูจน์ $\hat{r}_c(nK_2,P_3) = \lfloor\frac{5n-1}{2}\rfloor$ โดยใช้วิธีพิสูจน์ที่แตกต่างกัน

ภาควิชา	คณิตศาสตร์และวิทยาการคอมพิวเตอร์	ลายมือชื่อนิสิต	จิรวัฒน์
สาขาวิชา	คณิตศาสตร์	ลายมือชื่อ อ.ที่ปรึกษาหลัก	
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JIRAWAT MUNGTUMKLANG : CONNECTED SIZE RAMSEY NUMBERS OF MATCHING AND SOME GRAPHS. ADVISOR : ASST. PROF. TEERADEJ KITTIPASSORN, Ph.D., 21 pp.

The connected size Ramsey number of graphs G and H, denoted by $\hat{r}_c(G, H)$, is the smallest natural number k such that there exists a connected graph F with k edges and any red-blue edge-coloring of F contains a red copy of G or a blue copy of H. Assiyatun, Baskoro and Rahadjeng showed that $\hat{r}_c(nK_2, P_3) \leq \left\lfloor \frac{5n-1}{2} \right\rfloor$. Recently, Wang, Song, Zhang, and Zhang proved that $\hat{r}_c(nK_2, P_3) = \left\lfloor \frac{5n-1}{2} \right\rfloor$. In this thesis, we give an alternative proof of this result.

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CHAPTER I

INTRODUCTION

1.1 Background on graph theory

Throughout this thesis, a graph G consists of a set of vertices, denoted by $V(G)$, and a set of edges that connect pairs of vertices, denoted by $E(G)$. Moreover, we write $e(G)$ for the number of edges of graph G. We only consider simple graphs in this thesis.

For example, let G be a graph in Figure 1.1. Then $V(G) = \{v_1, v_2, v_3, v_4\}$ and $E(G) = \{v_1v_2, v_1v_3\}$.

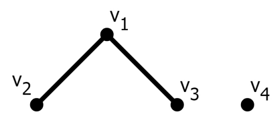

Figure 1.1: The graph G.

There are many types of graphs, however, in this thesis, we will focus on the following basic graphs.

A *path* of order n, denoted by P_n, is a graph with $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and $E(P_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n\}$. A *cycle* of order n, denoted by C_n, is a graph with $V(C_n) = \{v_1, v_2, 4 \dots, v_n\}$ and $E(C_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\}$.

For example, the path P_4 and the cycle C_5 are shown in Figures 1.2 and 1.3.

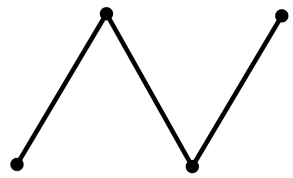

Figure 1.2: The path P_4.

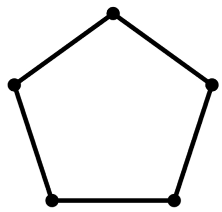

Figure 1.3: The cycle C_5.

A *complete graph*, denoted by K_n, is a graph consisting of n vertices such that any two vertices are adjacent. A graph G is said to be *bipartite* if its vertices can be partitioned into two parts so that all edges are between the two parts. A *complete bipartite graph*, denoted by $K_{m,n}$, is a bipartite graph with two parts $\{u_1, u_2, \ldots, u_m\}$ and $\{v_1, v_2, \ldots, v_n\}$ where u_i and v_j are adjacent for all $i \in [m]$ and $j \in [n]$.

For example, the complete graph K_4, the graph G, and the complete bipartite graph $K_{3,2}$ are shown in Figures 1.4, 1.5, and 1.6, respectively.

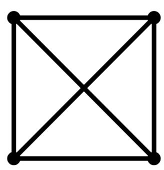

Figure 1.4: The complete graph K_4.

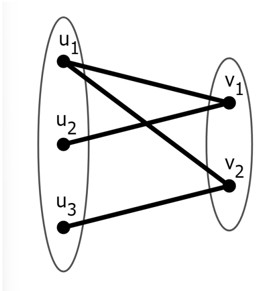

Figure 1.5: The bipartite graph G.

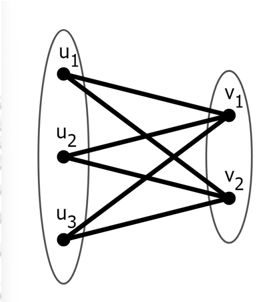

Figure 1.6: The complete bipartite graph $K_{3,2}$.

A *matching* of size n, denoted by nK_2, is a graph consisting of $2n$ vertices and n edges such that no two edges share common vertices. For example, the matching $4K_2$ is shown in Figure 1.7. In general, we write nG for a graph consisting of n disjoint copies of a graph G.

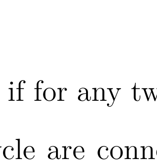

Figure 1.7: The matching $4K_2$.

A graph is said to be *connected* if for any two vertices, there exists a path between them. Observe that a path and a cycle are connected, while the graphs in Figures 1.1 and 1.7 are disconnected.

Interested readers can further investigate basic definitions and results in graph theory in West [1].

1.2 Ramsey number

Ramsey theory is a branch in graph theory concerning colorings on the edges of complete graphs.

We define $R(m,n)$ to be the minimum positive integer r such that for any red-blue edge-coloring of K_r, there exists a red copy of K_m or a blue copy of K_n. This number is called *a Ramsey number*. Ramsey [2] showed that the Ramsey numbers $R(m,n)$ are well defined.

First, we start with some trivial facts about the Ramsey numbers. Observe that, for any positive integers m and n, we have $R(m,n) = R(n,m)$ and $R(2,n) = n$.

The easiest non-trivial value of the Ramsey numbers is $R(3,3) = 6$. Indeed, we can color the edges of the K_5 as shown in Figure 1.8. The edge-coloring contains neither red copies of K_3 nor blue copies of K_3, thus $R(3,3) > 5$. Conversely, it can be shown by the pigeonhole principle that for any red-blue edge-coloring on K_6, there exists a red copy of K_3 or a blue copy of K_3.

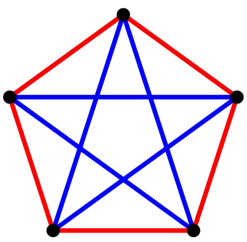

Figure 1.8: The edge-coloring of K_5 containing neither red copies of K_3 nor blue copies of K_3.

Currently, we only know the exact values of the Ramsey numbers for some small m and n. For example, $R(3,4) = 9$, $R(3,5) = 14$, $R(4,4) = 18$ and $R(4,5) = 25$ (see [3–7]).

It is extremely difficult to compute the Ramsey numbers exactly for the remaining cases, so we only have lower and upper bounds.

1.3 Size Ramsey number

Ramsey number has been studied intensively by many mathematicians (see, for example, [8–10]) who decided to generalize this concept in various ways, one of which is replacing complete graphs with arbitrary graphs. We start by introducing the following notation.

Given graphs F, G and H, the notation $F \to (G,H)$ means for any red-blue edge-coloring of F, there exists a red copy of G or a blue copy of H. It can be easily seen that $R(m,n)$ in the previous section can also be defined as the smallest positive integer r such that $K_r \to (K_m, K_n)$.

Burr, Erdős, Faudree and Rousseau [11] defined *the size Ramsey numbers of graphs G and H*, denoted by $\hat{r}(G,H)$, to be the smallest positive integer k such that there exists a graph F with k edges and $F \to (G,H)$. Observe that, for any graphs G and H, $\hat{r}(G,H) \leq \binom{R(|V(G)|,|V(H)|)}{2}$. Then $\hat{r}(G,H)$ is well-defined.

We give an example of the size Ramsey numbers. We will determine the value of $\hat{r}(2K_2, K_3)$. First, the upper bound can be found by constructing a graph G with $G \to (2K_2, K_3)$. A graph G in Figure 1.9 has such a property and G has 6 edges. This implies that $\hat{r}(2K_2, K_3) \leq 6$.

Conversely, we want to show that for any graph H with 5 edges, there exists a red-blue edge-coloring containing neither red copies of $2K_2$ nor blue copies of K_3. This can be shown by checking all cases as follows. We separate the cases by considering whether H contains a copy of K_3. If H contains no copies of K_3, we color all edges of H by blue. If H contains a copy of K_3, we color the edges of K_3 by red, and the remaining edges are colored by blue. This implies that $\hat{r}(2K_2, K_3) \geq 6$. Hence $\hat{r}(2K_2, K_3) = 6$.

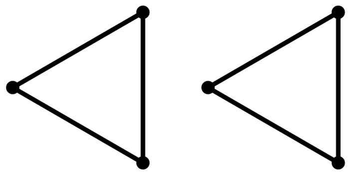

Figure 1.9: The graph G with $G \to (2K_2, K_3)$.

The size Ramsey numbers of many pairs of graphs are already determined, e.g, K_m versus K_n and K_m versus $K_{1,n}$. The size Ramsey numbers involving a matching was first considered by Burr, Erdős, Faudree, Rousseau and Schelp [12] in 1978. They showed that $\hat{r}(mK_{1,s}, nK_{1,t}) = (m+n-1)(s+t-1)$ for all natural numbers m, n, s and t. In 1981, Erdős and Faudree [13] determined the size Ramsey numbers of a matching versus several classic graphs including a path, a cycle, a complete graph and a complete bipartite graph. For instance, they proved that $\hat{r}(nK_2, P_4) = \lceil \frac{5n}{2} \rceil$ and $\hat{r}(nK_2, P_5) = 3n + l_n$ where $l_n = 0$ if n is even and $l_n = 1$ if n is odd.

1.4 Connected size Ramsey number

The size Ramsey numbers was modified by adding connectedness to F by Assiyatun, Baskoro and Rahadjeng [14]. They defined the *connected size Ramsey number* of G and H, denoted by $\hat{r}_c(G, H)$, to be the smallest natural number k such that there exists a connected graph F with k edges and $F \to (G, H)$. Using the same bound as $\hat{r}(G, H)$, we have that $\hat{r}_c(G, H)$ is well-defined.

We illustrate concept for the value of $\hat{r}_c(2K_2, K_3)$. We cannot apply the graph in Figure 1.9 because that graph is not connected. Therefore, we use the connected graph in Figure 1.10. This implies that $\hat{r}_c(2K_2, K_3) \leq 7$.

Conversely, let H be a connected graph with 6 edges. We separate the cases by considering whether H contains a copy of K_3. If H contains no copies of K_3, we color all edges of H by blue. Suppose H contains a copy of K_3 and we let $H' = H - E(K_3)$. If H' does not contain copies of K_3, we color the edges of K_3 by red and the edges of H'

by blue. If H' contains a copy of K_3, then H must be the graph in Figure 1.11 and we can color as shown. This implies that $\hat{r}_c(2K_2, K_3) \geq 7$. Hence, $\hat{r}_c(2K_2, K_3) = 7$.

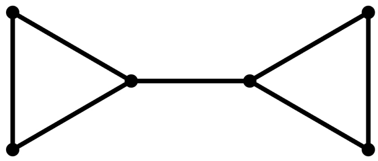

Figure 1.10: The connected graph G with $G \to (2K_2, K_3)$.

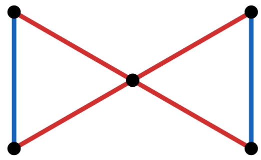

Figure 1.11: The edge-coloring of H containing neither red copies of $2K_2$ nor blue copies of K_3.

Assiyatun, Baskoro and Rahadjeng [14–17] determined upper bounds for $\hat{r}_c(nK_2, P_3)$, $\hat{r}_c(nK_2, P_4)$, $\hat{r}_c(nK_2, P_5)$, $\hat{r}_c(nK_2, 2P_3)$, and proved that they are sharp for small n. We are interested in the upper bound $\hat{r}_c(nK_2, P_3) \leq \left\lfloor \frac{5n-1}{2} \right\rfloor$. Recently, Wang, Song, Zhang, and Zhang [18] proved that this upper bound matches the exact value. In this thesis, we give an alternative proof of this result.

Theorem 1.1. $\hat{r}_c(nK_2, P_3) = \left\lfloor \frac{5n-1}{2} \right\rfloor$ *for all* $n \in \mathbb{N}$.

The rest of this thesis is organized as follows. In Chapter 2, we prove the main theorem. We discuss some questions regarding generalizations of the main theorem in Chapter 3.

CHAPTER II

PROOF OF THE MAIN THEOREM

It suffices to show that $\hat{r}_c(nK_2, P_3) \geq \left\lfloor \frac{5n-1}{2} \right\rfloor$, i.e., for any connected graph F with at most $\left\lfloor \frac{5n-1}{2} \right\rfloor - 1$ edges, we have $F \nrightarrow (nK_2, P_3)$. In the other words, there exists a red-blue edge-coloring of F such that F contains neither red copies of nK_2 nor blue copies of P_3.

Recall that *the matching number* of G, denoted by $\nu(G)$, is the size of a maximum matching of a graph G. In this thesis, given a red-blue coloring c on G, we define *the red matching number* of G, denoted by $\nu_r^c(G)$, to be the size of a maximum red matching of a graph G. We use the notation $\Delta(G)$ for the maximum degree of G.

If H is a subgraph of G, $G - H$ denotes the graph with $V(G - H) = V(G)$, $E(G - H) = E(G) - E(H)$. If A is a subset of $V(G)$, $G - A$ denotes the graph resulted by removing all vertices in A and all edges involving vertices in A. This notations will be used in our proof.

Before we begin to prove the main theorem, we give a required coloring for a path or a cycle.

Lemma 2.1. *If F is a path or a cycle with k edges, then there exists a coloring c on F containing no blue copies of P_3 with $\nu_r^c(F) \leq \left\lceil \frac{k}{3} \right\rceil$, i.e., $F \nrightarrow \big((\left\lceil \frac{k}{3} \right\rceil + 1)K_2, P_3\big)$.*

For example, the path P_6 and the cycle C_6 can be colored c as in Figures 2.1 and 2.2, respectively, containing no blue copies of P_3 with $\nu_r^c(H) \leq \left\lceil \frac{5}{3} \right\rceil = 2$ for $H = P_6, C_6$.

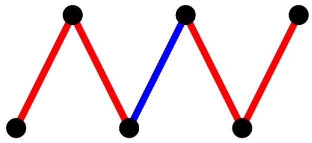

Figure 2.1: The edge-coloring of P_6 in Lemma 2.1.

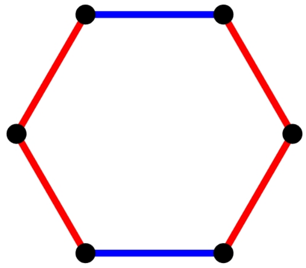

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Figure 2.2: The edge-coloring of C_6 in Lemma 2.1.

Proof of Lemma 2.1. Suppose F is a path or a cycle with k edges. Starting from an edge incident to a leaf if F is a path, we color all edges as we walk along F by red, red, blue, red, red, blue and so on along all edges of F. Observe that F contains no blue copies of P_3 and the size of a maximum red matching of F is at most $\lceil \frac{k}{3} \rceil$. Then $F \nrightarrow ((\lceil \frac{k}{3} \rceil + 1)K_2, P_3)$. $\square$

We will introduce three lemmas that simplify the main proof as follows.

Lemma 2.2. *Let G be a connected graph. Then G contains a copy of P_3 such that $G - P_3$ has one non-trivial component.*

Proof of Lemma 2.2. By considering a longest path $v_1v_2 \ldots v_t$ in G, we let G_1 be the graph consisting of the edges v_1v_2 and v_2v_3 if $N(v_2) = \{v_1, v_3\}$ and let G_1 be the graph consisting of the edges v_1v_2 and v_2u for some $u \in N(v_2) \setminus \{v_1, v_3\}$ if $N(v_2) \neq \{v_1, v_3\}$. Note that G_1 is a copy of P_3. We will show that G_2, the graph obtained from G by deleting the edges of G_1 and isolated vertices, is connected.

Claim. Let H be a connected graph with a longest path $u_1u_2\ldots u_m$. If $d(u_1) \geq 2$, then $H - u_1u_2$ is connected.

Proof. Suppose $d(u_1) \geq 2$. Since $u_1u_2\ldots u_m$ is a longest path, there exists an edge between u_1 and u_i for some $i \in \{3, 4, \ldots, t\}$ and so we can walk from u_1 to any vertex in $H - u_1u_2$ through u_i. Hence $H - u_1u_2$ is connected. □

Case 1. $N(v_2) = \{v_1, v_3\}$.

If $d(v_1) = 1$, then $G_2 = G - \{v_1, v_2\}$ is connected. We may assume that $d(v_1) \geq 2$. We may assume that $d(v_1) \geq 2$. Since $v_1v_2\ldots v_t$ is a longest path, there exists an edge between v_1 and v_i for some $i \in \{3, 4, \ldots, t\}$ and so we can walk from v_1 to any vertex in $G_2 = G - v_2$ through v_i. Hence G_2 is connected.

Case 2. $N(v_2) \neq \{v_1, v_3\}$.

Case 2.1. $d(v_1) = 1 = d(u)$.

Then $G_2 = G - \{v_1, u\}$ is connected.

Case 2.2. $d(v_1) \geq 2$ and $d(u) = 1$.

By applying the claim with $G - u$ and a longest path $v_1v_2\ldots v_t$ with $d(v_1) \geq 2$, we have that $G_2 = G - u - v_1v_2$ is connected.

Case 2.3. $d(v_1) = 1$ and $d(u) \geq 2$.

Similar to the Case 2.2.

Case 2.4. $d(v_1) \geq 2$ and $d(u) \geq 2$.

By applying the claim with G and a longest path $v_1v_2\ldots v_t$ with $d(v_1) \geq 2$, we have that $G - v_1v_2$ is connected. By applying the claim with $G - v_1v_2$ and the longest path $uv_2\ldots v_t$ with $d(u) \geq 2$, we have that $G_2 = G - v_1v_2 - uv_2$ is connected. □

Lemma 2.3. *Let $n \in \mathbb{N}$. Let G be a connected graph with $e(G) = 5m + l \leq \lfloor \frac{5n-1}{2} \rfloor - 4$ where $0 \leq l \leq 4$. Suppose that for all $k < n$ and any connected graph H with at most $\lfloor \frac{5k-1}{2} \rfloor - 1$ edges, $H \nrightarrow (kK_2, P_3)$. Then $G \nrightarrow \big((2m + \lfloor \frac{l}{2} \rfloor + 1)K_2, P_3\big)$.*

Proof of Lemma 2.3. The statement that for all $k < n$ and any connected graph H with at most $\lfloor \frac{5k-1}{2} \rfloor - 1$ edges, $H \nrightarrow (kK_2, P_3)$ is equivalent to $\hat{r}_c(kK_2, P_3) > \lfloor \frac{5k-1}{2} \rfloor - 1$.

Since

$$2m + \left\lfloor \frac{l}{2} \right\rfloor + 1 \leq \frac{2}{5}\left(\frac{5n-9}{2} - l\right) + \frac{l}{2} + 1 = n + \frac{l}{10} - \frac{4}{5} < n,$$

we have

$$\begin{aligned}
\hat{r}_c\left(\left(2m + 1 + \left\lfloor \frac{l}{2} \right\rfloor\right) K_2, P_3\right) &\geq \left\lfloor \frac{5(2m + 1 + \lfloor \frac{l}{2} \rfloor) - 1}{2} \right\rfloor \\
&= \left\lfloor 5m + \frac{4}{2} + \frac{5}{2}\left\lfloor \frac{l}{2} \right\rfloor \right\rfloor \\
&= 5m + 2 + \frac{3}{4}l \\
&> 5m + l
\end{aligned}$$

for all $l = 0, 1, 2, 3, 4$. Therefore, there exists a coloring c on G containing no blue copies of P_3 with $\nu_r^c(G) \leq 2m + \lfloor \frac{l}{2} \rfloor$. $\square$

Lemma 2.4. *Let $n \in \mathbb{N}$. Let G be a connected graph with an edge uv where v is a leaf and $e(G) = 5m + 5 \leq \lfloor \frac{5n-1}{2} \rfloor - 2$. Suppose that for all $k < n$ and any connected graph H with at most $\lfloor \frac{5k-1}{2} \rfloor - 1$ edges, $H \nrightarrow (kK_2, P_3)$. Then $G \nrightarrow ((2m+3)K_2, P_3)$ and uv is colored by red.*

Proof of Lemma 2.4. We will induct on m.

For $m = 0$, let G be a connected graph with an edge uv where v is a leaf and $e(G) = 5$. Clearly, $\nu(G) \leq 3$. If $\nu(G) \leq 2$, all edges are colored by red and so the size of a maximum red matching of G is at most 2. We may assume that $\nu(G) = 3$ with a matching $\{e_1, e_2, e_3\}$. By connectedness, the two remaining edges connect between e_1, e_2, and e_3. Then $uv \in \{e_1, e_2, e_3\}$. Thus we color $e \in \{e_1, e_2, e_3\} \setminus \{uv\}$ by blue and the remaining edges by red. Hence there exists a coloring c containing no blue copies of P_3 with $\nu_r^c(G) \leq 2$.

For $m \geq 1$, let G be a connected graph with an edge uv where v is a leaf and $e(G) = 5m + 5$. Let $u' \in N(u) \setminus \{v\}$. We will consider the graph $G' = G - \{uu', uv\}$ ignoring any isolated vertices. Then G' contains at most two components. Note that G' has $5m + 3$ edges.

Case A. G' is connected.

Since $e(G') = e(G) - 2 \leq \lfloor \frac{5n-1}{2} \rfloor - 4$, by Lemma 2.3, there exists a coloring c' on G' containing no blue copies of P_3 with $\nu_r^{c'}(G') \leq 2m + 1$. Coloring uu' and uv by red gives the coloring c containing no blue copies of P_3 with $\nu_r^c(G) \leq 2m + 2$.

Case B. G' contains two components.

Let C_1, C_2 be the components in G' with $e(C_i) = 5m_i + l_i$ for all $i = 1, 2$ where $4 \geq l_1 \geq l_2 \geq 0$. Then $5m + 3 = e(G') = e(C_1) + e(C_2) = 5(m_1 + m_2) + (l_1 + l_2)$. Thus, there are two possibilities for $l_1 + l_2$ as follows.

Case B1. $l_1 + l_2 = 3$.

In this case, we have $m_1 + m_2 = m$. There are only two subcases, $l_1 = 3$, $l_2 = 0$ and $l_1 = 2$, $l_2 = 1$. Since $e(C_i) = 5m_i + l_i \leq e(G) - 3 \leq \lfloor \frac{5n-1}{2} \rfloor - 4$ for $i = 1, 2$, by applying Lemma 2.3 to C_1 and C_2, there exist colorings c_1 and c_2 on C_1 and C_2 containing no blue copies of P_3 with $\nu_r^{c_1}(C_1) \leq 2m_1 + 1$ and $\nu_r^{c_2}(C_2) \leq 2m_2$, respectively. Therefore, there exists a coloring c containing no blue copies of P_3 with $\nu_r^c(G') \leq 2m_1 + 1 + 2m_2 = 2m + 1$ and so we are done as in Case A.

Case B2. $l_1 + l_2 = 8$.

We have $l_1 = l_2 = 4$ since $4 \geq l_1 \geq l_2$, and so $m_1 + m_2 = m - 1$. Without loss of generality, let $u' \in V(C_1)$ and $u \in V(C_2)$. By applying the induction hypothesis to $C_1 \cup \{uu'\}$ and $C_2 \cup \{uv\}$, there exist colorings c_1 and c_2 on $C_1 \cup \{uu'\}$ and $C_2 \cup \{uv\}$, respectively, containing no blue copies of P_3 with $\nu_r^{c_1}(C_1) \leq 2m_1 + 2$ and $\nu_r^{c_2}(C_2) \leq 2m_2 + 2$, and uu', uv are colored by red. Hence there exists a coloring c containing no blue copies of P_3 with $\nu_r^c(G) \leq 2m_1 + 2 + 2m_2 + 2 = 2m + 2$ and uv is colored by red. $\square$

We are now ready to prove the main theorem.

Proof of Theorem 1.1. We will use induction on n. For $n = 1$, let F be a connected graph with one edge. Then we color the edge by blue and so we have $F \nrightarrow (K_2, P_3)$.

Let $n \geq 2$ be such that, for all $k < n$ and any connected graph G with at most $\lfloor \frac{5k-1}{2} \rfloor - 1$ edges, we have $G \nrightarrow (kK_2, P_3)$. First, we suppose that n is even. Let F be a connected graph with at most $\frac{5n}{2} - 2$ edges. We need to show that $F \nrightarrow (nK_2, P_3)$.

By Lemma 2.2, F contains a copy of P_3 such that $F - P_3$ has one non-trivial component, say F' with

$$e(F') \leq \frac{5n}{2} - 4 = \left\lfloor \frac{5(n-1) - 1}{2} \right\rfloor - 1.$$

Thus, by applying the induction hypothesis to F', we have $F' \nrightarrow ((n-1)K_2, P_3)$, that is, there exists a coloring c' on F' containing no blue copies of P_3 with $\nu_r^{c'}(F') \leq n - 2$. We color F' by such a coloring and color all edges of the copy of P_3 by red. Then there exists a coloring c on F containing no blue copies of P_3 with $\nu_r^c(F') \leq n - 1$. Hence $F \nrightarrow (nK_2, P_3)$.

Now we may assume that n is odd. Let F be a connected graph with at most $\frac{5n-3}{2}$ edges. We need to show that $F \nrightarrow (nK_2, P_3)$. Without loss of generality, $e(F) = \frac{5n-3}{2}$ since any subgraph F' of F will also satisfy $F' \nrightarrow (nK_2, P_3)$.

We will delete a vertex of F whose degree is at least three, and use the induction hypothesis for the remaining edges. If $\Delta(F) \le 2$, then F is a path or a cycle and so we are done by Lemma 2.1.

So we may assume that $\Delta(F) \ge 3$. Let $v \in V(F)$ with $d(v) \ge 3$. Suppose that $F - v$ contains p components, say $C_1, C_2, \ldots, C_p$ with $e(C_i) = 5m_i + l_i$ where $0 \le l_i \le 4$ for all $i \in \{1, 2, \ldots, p\}$.

Since $e(C_i) = 5m_i + l_i \le e(F) - 3 = \left\lfloor \frac{5n-1}{2} \right\rfloor - 4$, by applying Lemma 2.3 to C_i, there exists a coloring c_i on C_i containing no blue copies of P_3 with $\nu_r^{c_i}(C_i) \le 2m_i + \left\lfloor \frac{l_i}{2} \right\rfloor$ for all $i \in \{1, 2, \ldots, p\}$. Then these give a coloring c on F containing no blue copies of P_3 with

$$\begin{aligned} \nu_r^c(F) \le \sum_{i=1}^{p} \left(2m_i + \left\lfloor \frac{l_i}{2} \right\rfloor\right) + 1 &\le \frac{2}{5}\left(e(F) - d(v) - \sum_{i=1}^{p} l_i\right) + \sum_{i=1}^{p} \left\lfloor \frac{l_i}{2} \right\rfloor + 1, \\ &= n + \frac{2}{5}\left(1 - d(v) - \sum_{i=1}^{p} l_i\right) + \sum_{i=1}^{p} \left\lfloor \frac{l_i}{2} \right\rfloor, \end{aligned}$$

since $\sum_{i=1}^{p}(5m_i + l_i) + d(v) = e(F) = \frac{5n-3}{2}$.

We have $\nu_r^c(F) < n$, if $\frac{2}{5}\left(1 - d(v) - \sum_{i=1}^{p} l_i\right) + \sum_{i=1}^{p} \left\lfloor \frac{l_i}{2} \right\rfloor < 0$. Equivalently, we are done if $d(v) > 1 + \sum_{i=1}^{p} \left(\frac{5}{2}\left\lfloor \frac{l_i}{2} \right\rfloor - l_i\right)$.

We may assume that $d(v) \le 1 + \sum_{i=1}^{p} \left(\frac{5}{2}\left\lfloor \frac{l_i}{2} \right\rfloor - l_i\right)$. Observe that

$$\frac{5}{2}\left\lfloor \frac{l}{2} \right\rfloor - l = \begin{cases} 0 & \text{if } l = 0, \\ -1 & \text{if } l = 1, \\ \frac{1}{2} & \text{if } l = 2, \\ -\frac{1}{2} & \text{if } l = 3, \\ 1 & \text{if } l = 4. \end{cases}$$

Then

$$p \le d(v) \le 1 + \sum_{i=1}^{p} \left(\frac{5}{2}\left\lfloor \frac{l_i}{2} \right\rfloor - l_i\right) \le 1 + p.$$

Case. $d(v) = p + 1$.

This is an equality for the above inequality on the right. This implies that

$$\frac{5}{2}\left\lfloor \frac{l_i}{2} \right\rfloor - l_i = 1.$$

Indeed, $l_i = 4$ for all $i \in \{1, 2, \ldots, p\}$. Since $p = d(v) - 1 \geq 2$, there exists $i \in \{1, 2, \ldots, p\}$ with exactly one edge between v and C_i. Then we are done by the following claim.

Claim. If there exists $i \in \{1, 2, \ldots, p\}$ with $l_i = 4$ such that there is exactly one edge e between v and C_i, then there exists a coloring c on F containing no blue copies of P_3 with $\nu_r^c(F) \leq n - 1$.

Proof of Claim. Since $e(C_i \cup \{e\}) = 5m_i + 5 \leq e(F) - 2 = \left\lfloor \frac{5n-1}{2} \right\rfloor - 2$, by applying Lemma 2.4 to $C_i \cup \{e\}$, then there exists a coloring c_i on $C_i \cup \{e\}$ containing no blue copies of P_3 with $\nu_r^{c_i}(C_i \cup \{e\}) \leq 2m_i + 2$ and e is colored by red. Since

$$e(F - C_i) = e(F) - (5m_i + 5) = 5\left(\frac{n-3}{2} - m_i\right) - 5 < \left\lfloor \frac{5n-1}{2} \right\rfloor - 4,$$

by applying Lemma 2.3 to $F - C_i$, there exists a coloring c' on $F - C_i$ containing no blue copies of P_3 with $\nu_r^{c'}(F - C_i) \leq 2\left(\frac{n-3}{2} - m_i\right)$. Thus there exists a coloring c containing no blue copies of P_3 with $\nu_r^c(F) \leq 2\left(\frac{n-3}{2} - m_i\right) + (2m_i + 2) = n - 1$. $\square$

Case. $d(v) = p$.

This case means C_i has exactly one edge incident to v for each $i \in \{1, 2, \ldots, p\}$. If there exists $i \in \{1, 2, \ldots, p\}$ with $l_i = 4$, we are done by the Claim. We may assume that $l_i \leq 3$ for all $i \in \{1, 2, \ldots, p\}$. Then $d(v) \leq 1 + \sum_{i=1}^{p}\left(\frac{5}{2}\left\lfloor \frac{l_i}{2} \right\rfloor - l_i\right) \leq 1 + \frac{p}{2} \leq 1 + \frac{d(v)}{2}$ and so $d(v) \leq 2$, we get a contradiction. $\square$

CHAPTER III

CONCLUSIONS

We give an alternative proof of the result that $\hat{r}_c(nK_2, P_3) = \left\lfloor \frac{5n-1}{2} \right\rfloor$ for all $n \geq 1$, which was first proved by Wang, Song, Zhang, and Zhang. In contrast to their proof which used the concept of *block* along with some techniques in graph theory, our proof does not require any prerequisite. The problem that we considered can be generalized in various ways as follows.

First, we may consider $\hat{r}_c(nK_2, P_m)$. Assiyatun, Baskoro and Rahadjeng [17] proved that $\hat{r}_c(nK_2, P_4) \leq 3n + l_n$ where $l_n = -1$ if n is even and $l_n = 0$ if n is odd. They proved that this upper bound is sharp for some small n. Then they conjecture that the upper bound matches the exact value as follows.

Conjecture 3.1. For $n \geq 1$, $\hat{r}_c(nK_2, P_4) = 3n + l_n$ where $l_n = -1$ if n is even, and $l_n = 0$ if n is odd.

Vito, Nabila, Safitri, and Silaban [19] proved that $\hat{r}_c(nK_2, P_m) \leq \left\lfloor \frac{(m+2)n-1}{2} \right\rfloor + l_n$ where $l_n = 0$ if n is even, and $l_n = 1$ if n is odd. They proved that the upper bound is sharp for $n = 2$. This thesis shows that the upper bound is sharp for $m = 3$. Thus, they asked whether all values above match the exact value generalizing Conjecture 3.1 as follows.

Problem 3.2. For $n \geq 1$ and $m \geq 3$, is $\hat{r}_c(nK_2, P_m) = \left\lfloor \frac{(m+2)n-1}{2} \right\rfloor + l_n$ where $l_n = 0$ if n is even, and $l_n = 1$ if n is odd?

On the other hand, if we view P_3 as $K_{1,2}$, we may consider $\hat{r}_c(nK_2, K_{1,m})$. Assiyatun, Baskoro and Rahadjeng [14] proved that $\hat{r}_c(nK_2, K_{1,m}) \leq nm + n - 1$. We conjecture that the upper bound matches the exact value as follows.

Conjecture 3.3. For $n \geq 1$ and $m \geq 3$, $\hat{r}_c(nK_2, K_{1,m}) = nm + n - 1$.

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BIOGRAPHY

Name	Mr. Jirawat Mungtumklang
Date of Birth	19 November 1996
Place of Birth	Bangkok, Thailand
Educations	B.Sc. (Mathematics), Chulalongkorn University, 2015–2018
	M.Sc. (Mathematics), Chulalongkorn University, 2019–Present
Scholarships	ทุนการศึกษาระดับบัณฑิตศึกษา “ทุนวิทยบัณฑิต” ประจำปีการศึกษา 2562–2565

Publications

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