การประมาณค่าเชิงตัวเลขของความน่าจะเป็นการล้มละลายของการวางนัยทั่วไปของตัวแบบความเสี่ยงแบบฉบับ

นายกิตติวัฒน์ วรเกตุ

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NUMERICAL APPROXIMATION TO RUIN PROBABILITY OF GENERALIZATION OF CLASSICAL RISK MODEL

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แบบจำลองความเสี่ยงเป็นเครื่องมือสำคัญในการประเมินความเสี่ยงของธุรกิจในบริษัทประกัน แบบจำลองความเสี่ยงได้ถูกนำไปใช้กับกระบวนการรายได้เบี้ยประกันของบริษัทประกันที่เป็นกระบวนการปัวซงเชิงประกอบ ในขณะเดียวกัน กระบวนการนับของจำนวนการเอาประกันและการเวนคืนเป็นกระบวนการการทำให้บาง พี และกระบวนการการทำให้บาง คิว ของกระบวนการรายได้เบี้ยประกัน โดยที่ความเสี่ยงของบริษัทมักจะประเมินผ่านค่าความน่าจะเป็นของการล้มละลาย อย่างไรก็ตาม การคำนวณค่าความน่าจะเป็นของการล้มละลายโดยทั่วไปมีความยาก ดังนั้น จึงมีการใช้การหาค่าขอบเขตบนในบทประยุกต์ต่างๆ ในวิทยานิพนธ์ฉบับนี้ เราจะศึกษาการประมาณค่าเชิงตัวตัวเลขที่เหมาะสมในการประมาณค่าความน่าจะเป็นของการล้มละลายของการวางนัยทั่วไปของตัวแบบความเสี่ยงแบบฉบับ แทนการใช้ค่าประมาณค่าขอบบน นอกจากนี้ เรายังทำการศึกษาตัวอย่างเชิงตัวเลขและประสิทธิภาพของการประมาณค่าเชิงตัวตัวเลขที่ได้รับ

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The risk model is an important tool for insurance companies to evaluate the risk of business. Risk models have been introduced into the premium income process of insurance companies to follow the compound Poisson process. The counting processes of claim and surrender are the p-thinning process and the q-thinning process of the premium income process. In general, the risk is usually evaluated via the ruin probability. However, it is quite difficult to calculate in general. Therefore, the upper bound is commonly used in many applications. In this thesis, instead of using an upper bound, we investigate some suitable numerical approximations to the ruin probability of a generalization of the classical risk model. Moreover, we perform a numerical study to investigate the performance of the obtained numerical approximation.

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CHAPTER I

INTRODUCTION

The risk model is an important tool for insurance company to evaluate the risk of business. Several risk models have been introduced for different types of insurance contracts. One type of popular insurance contracts is the classical risk model, defined by

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i \qquad \text{for } t \geq 0, \tag{1.1}$$

where $u \geq 0$ is the initial reserve, $c > 0$ is the premium rate, and $\{N(t),\ t \geq 0\}$ is a Poisson process with intensity $\lambda > 0$, representing the number of claims up to time t. The individual claim sizes $X_1, X_2, \ldots,$ independent of $\{N(t),\ t \geq 0\}$, are independent and identically distributed (i.i.d.) non-negative random variables with the common distribution function F_X. The classical risk model is a basic model of the total cost of insurance which is often used for insurance risk management.

The most important quantity in the insurance risk model is the ruin probability which is the probability that the surplus eventually becomes negative. In particular, the ruin probability is defined as

$$\psi(u) = Pr[\ U(t) < 0 \ \text{ for some } \ t \geq 0 \mid U(0) = u]. \tag{1.2}$$

However, the ruin probability is usually difficult to calculate in general. Therefore, it is commonly estimated by an upper bound called as the Lundberg upper bound of the ruin probability defined as

$$\psi(u) \leq e^{-Ru}, \tag{1.3}$$

where R is the unique positive solution of the adjustment equation $\lambda M_X(r) - \lambda - cr = 0$ called as the adjustment coefficient and M_X is the moment generating function of the claim size.

Beside of using the Lundberg upper bound, there is an interest in obtaining a numerical approximation of the ruin probability among researchers. For example, in 1991, Grandell [4] proposed several numerical approximations of the ruin probability for the classical risk model. Those numerical approximations are the Laplace transform method, the De Vylder approximation, and the Cramér-Lundberg approximation. Since then, the concept of numerical approximations of the ruin probability has been extended to more general risk models. For example, in 2003, Boikov [2] proposed a numerical approximation for risk models with stochastic premium process. In 2010, Seixas and Reis [14] proposed a numerical approximation for risk models with interference. In 2015, Mishura, Ragulina and Stroyev [9] proposed a numerical approximation for risk models with additional funds. In 2020, Ragulina [11] proposed a numerical approximation for risk models with stochastic premiums and a constant dividend strategy.

In this thesis, we investigate suitable numerical approximations to the ruin probability of a more generalized risk model. Moreover, we perform numerical studies to investigate the performance of the obtained numerical approximation comparing to Monte Carlo approximation and the Lundberg upper bound.

The rest of this thesis is organized as follows. Chapter 2 introduces the content, definitions, and theories that will be encountered in this thesis. Chapter 3 studies numerical approximation of the ruin probability for the risk model with constant premiums and surrenders subject to dependence thinning involving the Cramér approximation, the Laplace transform method, the De-Vylder approximation, and the Lundberg inequality. Chapter 4 studies numerical approximation of the ruin probability for the risk model with stochastic premiums and surrenders subject to dependence thinning involving the Cramér approximation, the Laplace transform method, the De-Vylder approximation, and the Lundberg inequality. Chapter 5 studies numerical approximation of the ruin probability for the renewal risk model with constant premiums and surrenders involving the Cramér approximation and the Laplace transform method. Chapter 6 gives discusses and conclusions of our study.

CHAPTER II

PRELIMINARY

In this chapter, we will introduce the basics, definitions, and theories of probability theory to be used in this project.

2.1 Basic real analysis and applied analysis

In this section, we introduce important, definitions and theorems in real analysis and applied analysis used in this project.

Definition 2.1. A set $A \subseteq \mathbb{R}$ is **bounded above**, if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. The number b is called an **upper bound** for A. Similarly, the set A is **bounded below**, if there exists a number $l \in \mathbb{R}$ satisfying $l \leq a$ for every $a \in A$. The number l is called a **lower bound** for A. If A have both upper bound and lower bound, we say that A is a bounded set.

Definition 2.2. Assume $g(x) \neq 0$ for all $x \neq a$ in some interval containing a.
The **Little-oh notation** is a notation representing the behavior of a limit of a function at a given value. The statement

$$f(x) = o(g(x)) \text{ as } x \to a$$

can be intuitively interpreted as saying that $g(x)$ grows much faster than $f(x)$ at a or mean that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0.$$

The symbol $f(x) = o(g(x))$ is read "$f(x)$ is **little-oh** of $g(x)$" or "$f(x)$ is of smaller order than $g(x)$" as $x \to a$.

Theorem 2.1. *[10] Properties of the Little-oh notation are as follows.*

1. *If c is a nonzero constant and $f = o(g)$, then $c \cdot f = o(g)$.*
2. *If $f = o(F)$ and $g = o(G)$, then $f \cdot g = o(F \cdot G)$.*
3. *If $f = o(g)$ and $g = o(h)$, then $f = o(h)$.*

Theorem 2.2. *[18]* ***Vieta's theorem***

Let r_1 and r_2 be the roots of the quadratic equation $ax^2 + bx + c = 0$. Then, the two identities

$$r_1 + r_2 = -\frac{b}{a} \quad \text{and} \quad r_1 r_2 = \frac{c}{a}$$

both hold.

In the same way, let r_1, r_2 and r_3 be the roots of the cubic equation $ax^3+bx^2+cx+d = 0$. Then, we have

$$r_1 + r_2 + r_3 = -\frac{b}{a}\ , \ r_1r_2 + r_2r_3 + r_1r_3 = \frac{c}{a}, \quad \text{and} \quad r_1r_2r_3 = -\frac{d}{a}.$$

Definition 2.3. [1] **The Fundamental Theorem of Calculus**

If f is continuous on $[0, b]$, then the function F defined by $F(x) = \int_0^x f(t)\, dt$, for all $x \in [0, b]$, is continuous on $[0, b]$ and differentiable on $(0, b)$ and

$$F'(x) = f(x) \quad \text{for all } x \in (0, b).$$

Definition 2.4. The Leibniz integral rule

Let $f(x, t)$ be a function such that both $f(x, t)$ and its partial derivative $f_x(x, t)$ are continuous in t and x in some region of the xt-plane including $a(x) \leq t \leq b(x)$, for $x_0 \leq x \leq x_1$. Also, suppose that the functions $a(x)$ and $b(x)$ are both continuous and both have continuous derivatives for $x_0 \leq x \leq x_1$. Then, for $x_0 \leq x \leq x_1$,

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\, dt\right) = f\big(x, b(x)\big) \cdot \frac{d}{dx}b(x) - f\big(x, a(x)\big) \cdot \frac{d}{dx}a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\, dt.$$

Definition 2.5. Let f be a complex-valued function of a real variable so that it can be decomposed as

$$f(x) = g(x) + ih(x),$$

where g and h are real-valued functions. The **complex conjugate** of f, denoted by $\overline{f}$, is defined by

$$\overline{f}(x) = \overline{f(x)} = g(x) - ih(x).$$

Definition 2.6. The **cross-correlation** of two complex functions $f(x)$ and $g(x)$ of a real variable x on $[0, \infty)$, denoted by $f \star g$, is defined by

$$[f \star g](x) = \int_0^\infty f(x+y)\overline{g(y)}\, dy.$$

Definition 2.7. Let $f : [0, \infty) \longrightarrow \mathbb{R}$ be a continuous function such that $e^{-\xi_0 x} f(x)$ is in $L^1([0, \infty))$ for some $\xi_0 \in \mathbb{R}$. Its **Laplace transform** is the function defined by

$$\mathscr{L}[f(x)](s) = f^*(s) := \int_0^\infty e^{-sx} f(x)\, dx, \quad \text{for all } s \in \mathbb{C} \text{ s.t. } \mathrm{Re}(s) > \xi_0.$$

Here, for the Laplace transform of f', we assume that f is continuously differentiable on $(0, \infty)$ and $\lim_{x \to 0^+} f'(x)$ is finite.

Theorem 2.3. *[3] Properties of the Laplace transform are as follows:*

$$\begin{aligned}
\mathscr{L}[af(x) + bg(x)](s) &= af^*(s) + bg^*(s), \\
\mathscr{L}[1](s) &= 1/s, \\
\mathscr{L}\left[e^{-ax}\right](s) &= 1/(s+a), \\
\mathscr{L}\left[f'(x)\right](s) &= sf^*(s) - f(0), \\
\mathscr{L}\left[\int_0^x f(t)\, dt\right](s) &= f^*(s)/s, \\
\mathscr{L}\left[\int_0^x f(t)g(x-t)\, dt\right](s) &= f^*(s) \cdot g^*(s), \\
\mathscr{L}\left[\int_0^\infty f(x+y)\overline{g(y)}\, dy\right](s) &= f^*(s)\overline{g^*(-\overline{s})},
\end{aligned}$$

where a and b are constant.

2.2 Basic probability theory

In this section, we will use some techniques to find the probabilistic properties of random variables.

Definition 2.8. A **random experiment** is any activity or process whose outcome is subject to uncertainty.

Definition 2.9. The set of all possible outcomes of a random experiment is called a **sample space** denoted by Ω. Each outcome in a sample space is called a **sample point**.

Definition 2.10. A collection $\mathscr{F} \subseteq 2^{\Omega}$ of subsets of Ω is called a **$\boldsymbol{\sigma}$-field** (also σ-algebra or event space) on Ω, if it has the following three properties.

1. $\Omega \in \mathscr{F}$.
2. If $A \in \mathscr{F}$, then $A^c \in \mathscr{F}$. (closed under complement)
3. If $A_i \in \mathscr{F}$ for all $i \in \mathbb{N}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathscr{F}$. (closed under countable union)

Each element in a σ-field is referred to as an **event**.

For a sample space Ω, let $\mathcal{A}$ be a collection of events, and let $\sigma(\mathcal{A})$ represent the smallest σ-field containing $\mathcal{A}$. Thus, $\sigma(\mathcal{A})$ is called the σ-field "generated" by $\mathcal{A}$.

Definition 2.11. Let $\mathscr{F}$ be a σ-field on a sample space Ω. A set function $P : \mathscr{F} \to [0, 1]$ is called a **probability measure**, if it has the following two properties.

1. $P(\Omega) = 1$
2. If $A_i \in \mathscr{F}$ for all $i \in \mathbb{N}$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$, then we have that

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i). \qquad \text{(countably additive, } \sigma\text{-additive)}$$

The triple $(\Omega, \mathscr{F}, P)$ is called a **probability space**.

Remark 2.1. We can call $(\Omega, \mathscr{F})$ a measurable space and $(\Omega, \mathscr{F}, P)$ a measure space.

For a topological space Ω, let $\mathscr{B}(\Omega)$ represent the σ-field generated by all open sets in Ω. Thus, $\mathscr{B}(\Omega)$ is called the **Borel $\boldsymbol{\sigma}$-field**. Each set in this σ-field is called a **Borel set**.

Definition 2.12. If $\mathscr{F}$ is a σ-field on Ω, then a function $X : \Omega \to \mathbb{R}$ is said to be $\mathscr{F}$**-measurable** or $(\mathscr{F}, \mathscr{B}(\mathbb{R}))$-measurable, if

$$X^{-1}(B) \in \mathscr{F}$$

for every Borel set $B \in \mathscr{B}(\mathbb{R})$. If $(\Omega, \mathscr{F}, P)$ is a probability space, then such a function is called a **random variable**.

Definition 2.13. Let X be a random variable on $(\Omega, \mathscr{F}, P)$. Define a probability measure P_X on $\mathbb{R}$ by

$$P_X(A) = P\left(X^{-1}(A)\right) = P(X \in A) \quad \text{for all } A \in \mathscr{B}(\mathbb{R})$$

and call it the **probability distribution** of X. The function $F_X : \mathbb{R} \to [0, 1]$ defined by

$$F_X(x) = P_X((-\infty, x]) = P(X \leq x) \quad \text{for all } x \in \mathbb{R}$$

is called the **distribution function** or **cumulative distribution function (CDF)** of X.

Definition 2.14. The random variable X is called **discrete**, if it takes values in some countable subset of $\mathbb{R}$. The discrete random variable X has **probability mass function (PMF)** $f : \mathbb{R} \to [0, 1]$ given by

$$f(x) = P(X = x) \text{ for all } x \in \mathbb{R}.$$

Definition 2.15. The random variable X is called **continuous**, if its distribution can be expressed as

$$F_X(x) = \int_{-\infty}^{x} f(u)\, du \quad \text{for all } x \in \mathbb{R}$$

for some integrable function $f : \mathbb{R} \to [0, \infty)$ called the **probability density function (PDF)** of X.

Remark 2.2. From definition of probability density function and the fundamental theorem of calculus in theorem 2.3 , we get

$$F'_X(x) = f(x) \quad \text{for } f \text{ is continuous at } x.$$

Definition 2.16. Law of total probability for a discrete random variable
Let $(\Omega, \mathscr{F}, P)$ be a probability space. Suppose X is a discrete random variable with distribution function F_X, and A an event on $(\Omega, \mathscr{F}, P)$. Then

$$P(A) = \sum_x P(A \mid X = x) P(X = x).$$

Definition 2.17. Law of total probability for a continuous random variable
Let $(\Omega, \mathscr{F}, P)$ be a probability space. Suppose X is a continuous random variable with distribution function F_X, and A an event on $(\Omega, \mathscr{F}, P)$. Then,

$$P(A) = \int_{-\infty}^{\infty} P(A \mid X = x)\, dF_X(x).$$

Definition 2.18. If $P(B) > 0$, then the **conditional probability** that A occurs given that B occurs is defined to be

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

Definition 2.19. Events A and B are **independent**, if

$$P(A \cap B) = P(A)P(B).$$

More generally, a family of events $\{A_i \mid i \in I\}$ is called **independent** or **mutually independent**, if

$$P\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} P(A_i)$$

for any finite subset J of I.

Definition 2.20. We say that random variables $X_1, X_2, \ldots, X_n$ are **independent**, if the σ-field $\sigma(X_1), \sigma(X_2), \ldots, \sigma(X_n)$ are independent, i.e., for any $A_1, A_2, \ldots, A_n \in \mathscr{B}(\mathbb{R})$, we have that

$$P(X_1 \in A_1, X_2 \in A_2, \ldots, X_n \in A_n) = P(X_1 \in A_1)P(X_2 \in A_2) \cdots P(X_n \in A_n).$$

More generally, a family $\{X_i\}_{i \in I}$ of random variables is said to be independent, if every finite subfamily is.

Definition 2.21. Let X be a random variable on a probability space $(\Omega, \mathscr{F}, P)$. The **expected value** or **expectation** of X, denoted by $E(X)$, is defined by

$$E(X) = \int_{\Omega} X \, dP.$$

If X is a discrete or continuous random variable. Then,

$$E(X) = \begin{cases} \displaystyle\sum_{x \in ImX} xf(x), & \text{if } X \text{ is discrete with PMF } f, \\ \displaystyle\int_{-\infty}^{\infty} xf(x)\, dx, & \text{if } X \text{ is continuous with PDF } f. \end{cases}$$

Theorem 2.4. *[8] Let X be a random variable with finite expected value. Then, for any constant a and b,*

$$E(aX + b) = aE(X) + b.$$

Remark 2.3. Let X and Y be random variables with finite expected values. Then,

1. $E(X \pm Y) = E(X) \pm E(Y)$;
2. if $X \geq 0$, then $E(X) \geq 0$;
3. if $X \geq Y$, so $X - Y \geq 0$, then $E(X) - E(Y) = E(X - Y) \geq 0$, i.e.,

$$E(X) \geq E(Y).$$

Theorem 2.5. *[8] If $X_1, X_2, \ldots, X_n$ are independent random variables and $E|X_i| < \infty$ for all i, then*

$$E\left[\prod_{i=1}^{n} X_i\right] = \prod_{i=1}^{n} E[X_i],$$

i.e., the expectation on the left exists and has the value given on the right.

Theorem 2.6. *[8] Random variables X and Y are independent if and only if*

$$E(f(X)g(Y)) = E(f(X))E(g(Y)).$$

for all bounded Borel measurable functions f and g.

Definition 2.22. Let X be a random variable with finite expected value μ, the **variance** of X, denoted by $Var(X)$, is defined by

$$Var(X) := E[(X-\mu)^2] = E(X^2) - \mu^2.$$

The quantity $\sqrt{Var(X)}$ is called the **standard deviation** of X, denoted by $SD(X)$.

Remark 2.4. Let $X_1, \ldots, X_n$ be independent random variables. Then,

$$Var\Big(\sum_{i=1}^{n} a_i X_i\Big) = \sum_{i=1}^{n} a_i^2 Var(X_i) \qquad \text{for all constant } a_i.$$

Definition 2.23. Let X be a random variable. Then, the **generating function (GF** of X, denoted by G_X, is defined as

$$G_X(t) = E[t^X] \text{ for all } t \in \mathbb{R} \text{ for which the expected value exists in } \mathbb{R}.$$

Remark 2.5. Let X be a discrete or continuous random variable. Then,

$$G_X(t) = \begin{cases} \displaystyle\sum_{x \in ImX} t^x f(x), & \text{if } X \text{ is discrete with PMF } f, \\ \displaystyle\int_{-\infty}^{\infty} t^x f(x)dx, & \text{if } X \text{ is continuous with PDF } f. \end{cases}$$

Theorem 2.7. *[8] If the generating function G_X of a random variable X exists, then $G_X^{(n)}(1) = E\big(X(X-1)\cdots(X-(n-1))\big)$ for all $n \in \mathbb{N}$.*

Theorem 2.8. *[8] Let X and Y be random variables, and a and b be real numbers. Then,*

1. $G_{X+a}(t) = t^a G_X(t)$,
2. $G_{bX}(t) = G_X(t^b)$,
3. $G_{bX+a}(t) = t^a G_X(t^b)$,
4. $G_{X+Y}(t) = G_X(t)G_Y(t)$,*if X and Y are independent.*

Theorem 2.9. *[8] Let X and Y be random variables. Then, $G_X(t) = G_Y(t)$ for all $t \in \mathbb{R}$ if and only if X and Y have the same distribution.*

Definition 2.24. Let X be a random variable. Then, the **moment generating function (MGF)** of X, denoted by M_X, is defined by

$$M_X(t) = E[e^{tX}].$$

We say that the moment generating function of X **exists**, if there exists $\delta > 0$ such that $M_X(t)$ is finite for all $t \in (-\delta, \delta)$. The domain of M_X is the set $\{t \in \mathbb{R} \mid M_X(t) < \infty\}$.

Remark 2.6. Let X be a discrete or continuous random variable. Then,

$$M_X(t) = \begin{cases} \displaystyle\sum_{x \in ImX} e^{tx} f(x), & \text{if } X \text{ is discrete with PMF } f, \\ \displaystyle\int_{-\infty}^{\infty} e^{tx} f(x)dx, & \text{if } X \text{ is continuous with PDF } f. \end{cases}$$

Theorem 2.10. *[8] If the moment generating function M_X of a random variable X exists, then $M_X^{(n)}(0) = E(X^n)$ for all $n \in \mathbb{N}$.*

Theorem 2.11. *[8] Let X and Y be random variables, and a and b be real numbers. Then,*

1. $M_{X+a}(t) = e^{at} M_X(t)$,
2. $M_{bX}(t) = M_X(bt)$,
3. $M_{bX+a}(t) = e^{at} M_X(bt)$,
4. $M_{X+Y}(t) = M_X(t)M_Y(t)$, *if X and Y are independent.*

Theorem 2.12. *[8] Let X and Y be random variables. Then, $M_X(t) = M_Y(t)$ for all $t \in \mathbb{R}$ if and only if X and Y have the same distribution.*

Remark 2.7. $G_X(t) = E[t^X] = E[e^{ln(t)X}] = M_X(ln(t))$.

Definition 2.25. Let X be a random variable. Then, the **cumulant generating function (CGF)** of X, denoted by K_X, is defined as

$$K_X(t) = ln(E[e^{tX}]) = ln(M_X(t)) \text{ for all } t \text{ in the domain of } M_X.$$

Theorem 2.13. *[8] Let X and Y be random variables. Then,*

1. $K_{X+Y}(t) = K_X(t) + K_Y(t)$, *if X and Y are independent,*
2. $K'_X(0) = E(X)$,
3. $K''_X(0) = E([X - E(X)]^2) = Var(X)$,
4. $K'''_X(0) = E([X - E(X)]^3)$.

Definition 2.26. Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $X : \Omega \to \mathbb{R}$ be a random variable such that $E(|X|) < \infty$. If $\mathcal{G} \subseteq \mathscr{F}$ is a σ-field on the probability space $(\Omega, \mathscr{F}, P)$, then the **conditional expectation** of X given $\mathcal{G}$, denoted by $E(X \mid \mathcal{G})$ is a $\mathcal{G}$-measurable function such that $\int_A E(X \mid \mathcal{G})\, dP = \int_A X\, dP$ for any $A \in \mathcal{G}$.

Definition 2.27. A random variable X is said to have a **Poisson distribution** with parameter λ (for some $\lambda > 0$), denoted as $X \sim Poi(\lambda)$, if

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad \text{for } x = 0, 1, 2, \ldots$$

Theorem 2.14. *[8] Let $X \sim Poi(\lambda)$. Then,*

1. $E(X) = \lambda$,
2. $Var(X) = \lambda$,
3. $G_X(t) = e^{\lambda(t-1)}$ *for $t \in \mathbb{R}$,*
4. $M_X(t) = e^{\lambda(e^t - 1)}$ *for $t \in \mathbb{R}$.*

Definition 2.28. A random variable X is said to have a **generalized exponential distribution** with shape parameter $\alpha > 0$ and scale parameter $\eta > 0$, denoted as $X \sim GExp(\alpha, \eta)$, if its probability density function is defined as

$$f(x) = \begin{cases} \alpha\eta(1-e^{-\eta x})^{\alpha-1}e^{-\eta x}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0. \end{cases}$$

Theorem 2.15. *[5] Let $X \sim GExp(\alpha, \eta)$. Then,*

1. $E(X) = \frac{1}{\eta}[\psi(\alpha+1) - \psi(1)]$,

2. $Var(X) = \frac{1}{\eta^2}[\psi'(1) - \psi'(\alpha+1)]$,

3. $M_X(t) = \frac{\Gamma(\alpha+1)\Gamma(1-\frac{t}{\eta})}{\Gamma(\alpha-\frac{t}{\eta}+1)}$, *for* $t < \eta$,

4. $F(x) = (1-e^{-\eta x})^{\alpha}$, *for* $x \geq 0$,

where Γ, ψ, *and* ψ' *are gamma, digamma, and trigamma functions, respectively.*

Definition 2.29. A random variable X is said to have an **exponential distribution** with parameter η (for some $\eta > 0$), denoted as $X \sim Exp(\eta)$, if its probability density function is defined as

$$f(x) = \begin{cases} \eta e^{-\eta x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Theorem 2.16. *[8] Let $X \sim Exp(\eta)$. Then,*

1. $E(X) = \frac{1}{\eta}$,

2. $Var(X) = \frac{1}{\eta^2}$,

3. $M_X(t) = \frac{\eta}{\eta - t}$, *for* $t < \eta$,

4. $E[X^n] = \frac{n!}{\lambda^n}$, *for* $n \in \mathbb{N}$.

Definition 2.30. A random variable X is said to have a **gamma distribution** with parameters β ($\beta > 0$) and α ($\alpha > 0$), denoted as $X \sim Gamma(\alpha, \beta)$, if its probability density function is defined as

$$f(x) = \begin{cases} \dfrac{\beta e^{-\beta x}(\beta x)^{\alpha-1}}{\Gamma(\alpha)} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0, \end{cases}$$

where the gamma function $\Gamma : (0, \infty) \to \mathbb{R}$ is defined as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

Theorem 2.17. *[8] Let $X \sim Gamma(\alpha, \beta)$. Then,*

1. $E(X) = \dfrac{\alpha}{\beta}$,
2. $Var(X) = \dfrac{\alpha}{\beta^2}$,
3. $M_X(t) = \left(\dfrac{\beta}{\beta - t}\right)^\alpha$ *for* $t < \beta$,
4. $E[X^n] = \dfrac{\Gamma(n+\alpha)}{\beta^n \Gamma(\alpha)}$ *for* $n \in \mathbb{N}$.

Theorem 2.18. *[8] For a probability space $(\Omega, \mathscr{F}, P)$, let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be random variables with finite first moment and $\mathcal{G}$ is a σ-field on the probability space $(\Omega, \mathscr{F}, P)$ such that $\mathcal{G} \subseteq \mathscr{F}$, then the properties of conditional expectation are as follows.*

1. $E(aX + bY \mid \mathcal{G}) = aE(X \mid \mathcal{G}) + bE(Y \mid \mathcal{G})$ *for all* $a, b \in \mathbb{R}$;
2. *if* $X \geq 0$, *then* $E(X \mid \mathcal{G}) \geq 0$;
3. *if* $X \leq Y$, *then* $E(X \mid \mathcal{G}) \leq E(Y \mid \mathcal{G})$;
4. *if* $\mathscr{F}_1$ *is a* σ*-field such that* $\mathscr{F}_1 \subseteq \mathscr{F}$, $\mathscr{F}_2$ *is a* σ*-field such that* $\mathscr{F}_2 \subseteq \mathscr{F}$ *and* $\mathscr{F}_1 \subseteq \mathscr{F}_2$, *then* $E(E(X \mid \mathscr{F}_2) \mid \mathscr{F}_1) = E(X \mid \mathscr{F}_1)$;
5. $E(E(X \mid \mathcal{G})) = E(X)$;
6. *if* X *is independent of* $\mathcal{G}$, *then* $E(X \mid \mathcal{G}) = E(X)$;
7. *if* Y *is* $\mathcal{G}$*-measurable, then* $E(XY \mid \mathcal{G}) = Y(X \mid \mathcal{G})$.

Definition 2.31. Let $X : \Omega \to \mathbb{R}$ be a random variable. The **conditional variance** of X given σ-field $\mathcal{G} \subseteq \mathscr{F}$, denoted by $Var(X \mid \mathcal{G})$, is defined by

$$Var(X \mid \mathcal{G}) \coloneqq E\big((X - E(X|\mathcal{G}))^2 \mid \mathcal{G}\big) = E(X^2 \mid \mathcal{G}) - (E(X \mid \mathcal{G}))^2.$$

Theorem 2.19. *[8] Assume that $X : \Omega \to \mathbb{R}$, $\mathcal{G} \subseteq \mathscr{F}$ is a σ-field on the probability space $(\Omega, \mathscr{F}, P)$, $\mathscr{F}$ is σ-field on Ω, $E(|X|) < \infty$, and $Var(|X|) < \infty$. Then,*

$$Var(X) = E(Var(X \mid \mathcal{G})) + Var(E(X \mid \mathcal{G})).$$

Theorem 2.20. *[15]* ***Markov's Inequality***

Let X be a non-negative random variable and a a positive real number. Then,

$$P(X \geq a) \leq \frac{E(X)}{a}.$$

Theorem 2.21. *[6]* ***Hoeffding's Inequality***

Suppose that $X_1, \ldots, X_n$ are independent random variables such that $a_i \leq X_i \leq b_i$ and $E[X_i] = \mu$. Then, for any $t > 0$

$$P(|\bar{X} - \mu| \geq t) \leq 2\ \exp\left(-\frac{2n^2t^2}{\sum_{i=1}^{N}(b_i - a_i)^2}\right),$$

where $\bar{X}_n = n^{-1}\sum_i X_i$. When $a \leq X_i \leq b$, this becomes

$$P(|\bar{X} - \mu| \geq t) \leq 2\ \exp\left(-\frac{2n^2t^2}{(b-a)^2}\right).$$

2.3 Basic stochastic processes

In this section, we will introduce the definitions, properties and theories of stochastic processes which consists of stochastic processes, counting processes, poisson processes to be used in this project.

Definition 2.32. A **stochastic process** is a collection of random variables $\underline{X} = \{X_t \mid t \in T\}$ defined on a common probability space $(\Omega, \mathcal{F}, P)$, i.e., X_t is a random variable for all $t \in T$. The index set T is called the **parameter space**, the set S containing all possible values of X_t for $t \in T$ is called the **state space**, and each member in S is called a **state**. If T is a countable set, such as $\mathbb{N}$ and $\mathbb{N} \cup \{0\}$, then $\underline{X}$ is called a **discrete-time stochastic process**. If T is an interval in $\mathbb{R}$, then $\underline{X}$ is called a **continuous-time stochastic process**.

Definition 2.33. Let $X_1, X_2, \ldots$ be independent and identically distributed (i.i.d.) random variables and let N be a random variable taking values in $\{0, 1, 2, 3, \ldots\}$ which is independent of $\{X_i\}_{i=1}^{\infty}$. Let

$$S_N = X_1 + X_2 + \cdots + X_N = \sum_{i=1}^{N} X_i,$$

with $S_N = 0$ if $N = 0$. The random variable S_N is called a **random sum**. The distribution of a random sum is said to be a **compound distribution**.

Definition 2.34. For a stochastic process $X(t)$ and time $s < t$, the random variable $X(t) - X(s)$ is called an **increment** of the process, since it gives the increase (or decrease) in the value over the period running from time s to t.
We say that the process has **independent increments**, if the increments over disjoint time intervals are independent.

Definition 2.35. We say that the process has **stationary increments**, if the distribution of any increment depends only on the length of the time interval and not the particular starting point, i.e., given any $h > 0$ and time s and t, we require that

$$X(s + h) - X(s) \sim X(t + h) - X(t).$$

Definition 2.36. A stochastic process $\{N(t) : t \geq 0\}$ is said to be a **counting process**, if $N(t)$ represents the total number of "events" that have occurred up to time t.
A counting process $N(t)$ must satisfy:

1. $N(t) \geq 0 \ \forall t \geq 0$;
2. $N(t)$ is integer-valued;
3. If $0 \leq s < t$, then $N(s) \leq N(t)$.

For a counting process $\{N(t) : t \geq 0\}$ and $s < t$, $N(t) - N(s)$ is the number of events occurring in the time interval $(s, t]$.

Definition 2.37. A counting process $N(t)$ is called a **Poisson process** with rate λ, if it has stationary and independent increments and if, for all $t > 0$,

$$N(t) \sim Poi(\lambda t) \ .$$

From the definition, a stochastic process $N(t)$ is a Poisson process with rate $\lambda > 0$ if]]

1. $N(0) = 0$;
2. the process has independent increments;
3. for $t \geq 0$ and $s > 0$, $N(t + s) - N(t)$ has a Poisson distribution with mean λs.

Lemma 2.1. *[12] If we consider a very short interval of length Δt, then the number of arrivals in this interval has the same distribution as $N(\Delta t)$. We can write*

$$\begin{aligned} P(\textit{no event occurs in the interval}) = P(N(\Delta t) = 0) = &\ 1 - \alpha \Delta t + o(\Delta t), \\ P(\textit{one event occurs in the interval}) = P(N(\Delta t) = 1) = &\ \alpha \Delta t + o(\Delta t), \\ P(\textit{more than one event occur in the interval}) = P(N(\Delta t) \geq 2) = &\ o(\Delta t). \end{aligned}$$

Definition 2.38. Let $\{N(t)\}_{t \geq 0}$ be a Poisson process with parameter λ, and let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of independent and identically distributed random variables, each with distribution function F, independent of $N(t)$ for all $t > 0$. We define a process $\{S(t)\}_{t \geq 0}$ by

$$S(t) = \sum_{i=1}^{N(t)} Y_i,$$

with $S(t) = 0$ when $N(t) = 0$. The process $\{S(t)\}_{t \geq 0}$ is said to be a **compound Poisson process** with Poisson parameter λ.

Theorem 2.22. *[12] A compound Poisson process $S(t)$ has the following properties.*

1. *Expectation :* $E(S(t)) = \lambda t E(Y)$.
2. *Variance :* $Var(S(t)) = \lambda t E(Y^2)$.
3. *Moment generating function :* $M_{S(t)}(z) = e^{\lambda t(M_Y(z)-1)}$.
4. *For $t > 0$, the random variable $S(t)$ has a compound Poisson distribution with Poisson parameter λt.*
5. *The compound Poisson Process have stationary and independent increments.*

Definition 2.39. Let M be any non-negative integer-value random variable and $X_1, X_2, \ldots$ be i.i.d. Bernoulli random variables with parameter α $(0 \leq \alpha \leq 1)$. Then,

$$\alpha \circ M = \sum_{i=1}^{M} X_i$$

is called the **binomial thinning operator** of M.

Theorem 2.23. *[7] The binomial thinning operator in definition 2.39 has the following property.*

*If M is a Poisson random variable with parameter λ, then $\alpha \circ M$ is a Poisson random variable with parameter $\alpha\lambda$ and $\alpha \circ M$ is called α-**thinning**.*

Definition 2.40. A continuous-time stochastic process $X(t)$ is a **martingale**, if

1. $E(|X(t)|) < \infty$ for all $t > 0$,
2. $E(X(t) \mid X(u),\ 0 \leq u \leq s) = X(s)$ for all $t \geq s$.

Definition 2.41. A random variable T is a **stopping time** with respect to the filtration $\{\mathcal{F}_t\}$, if $\{T \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$.

Theorem 2.24. *[13] The Martingale Stopping Time Theorem*

Let $\{Z_t\}$ be a martingale and T a stopping time. If any one of the following conditions holds:

1. *T is bounded;*
2. *$E[T] < \infty$, and there is an $M < \infty$ such that*

$$E[|Z_{n+1} - Z_n| \mid Z_0, Z_1, Z_2, \ldots, Z_n] < M.$$

Then,

$$E[Z_T] = E[Z_0].$$

2.4 Basic risk theory

In this section, we will introduce the definitions, properties and theories of risk theory which consists of compound sum, risk model to be used in this project.

Definition 2.42. The **classical risk model** is a model of total capital values, defined by

$$U(t) = u + ct - S(t),$$

when

$U(t)$	is the total capital values at time t,
u	is the amount of initial reserves,
c	is a constant rate of premium per unit of time,
$S(t)$	is the aggregated claims up to time t,

such that

$$S(t) = \sum_{i=1}^{N(t)} Y_i,$$

when

$N(t)$	is the number of claims up to time t, which is a counting process,
$\{Y_i\}_{i\geq 1}$	is a sequence of the amount of the i^{th} claims which are independent and identically distributed (i.i.d.) random variables.

Definition 2.43. Ruin Time or Time of Ruin

Time of ruin T is the first time at which the surplus process become negative, defined by

$$T = \inf\{t \geq 0 \mid U(t) < 0\}.$$

Definition 2.44. Ruin Probability or Probability of Ruin

Let $\psi(u)$ be a ruin probability when the initial reserves $u > 0$. It is defined by

$$\begin{aligned}\psi(u) &= P[\ T < \infty \mid U(0) = u\ ] \\ &\equiv P[\ U(s) < 0 \text{ for some } s \geq 0 \mid U(0) = u\ ].\end{aligned}$$

where T is the time of ruin.

Definition 2.45. Let $\psi(u, t)$ be the probability of ruin at some point in the time interval $(0, t]$, given initial reserves $u > 0$. It is defined by

$$\begin{aligned}\psi(u,t) &= P[\ T\ < t \mid U(0) = u\ ] \\ &\equiv P[\ U(s) < 0 \text{ for some } s \in [0,t] \mid U(0) = u\ ].\end{aligned}$$

Definition 2.46. Non-Ruin Probability or Probability of Survival

Let $\phi(u)$ be a survival probability when the initial reserves $u > 0$. It is defined by

$$\phi(u) = P[\ U(s) \geq 0 \text{ for all } s \geq 0 \mid U(0) = u\ ].$$

Definition 2.47. Let $\phi(u, t)$ be the probability of survival at some point in the time interval $(0, t]$, given initial reserves $u > 0$. It is defined by

$$\phi(u,t) = P[\ U(s) \geq 0 \text{ for all } s \in [0,t] \mid U(0) = u\ ].$$

Remark 2.8. The ruin probability and the non-ruin probability have the following properties.

1. $\psi(u) = 1 - \phi(u)$,
2. $\psi(u, t) = 1 - \phi(u, t)$.

CHAPTER III

RISK MODEL WITH CONSTANT PREMIUMS AND SURRENDERS SUBJECT TO DEPENDENCE THINNING

In this chapter, we study numerical approximations of a risk model with constant premiums and surrenders subject to dependence thinning. In our study, we first introduce the risk model and ruin probability and evaluate its properties. Then, we obtain formula for numerical approximation of the ruin probability by using the Cramér approximation, the Laplace transforms method, and the De Vylder Approximations. Moreover, perform numerical studies to see performance of the three methods and compare them with the Lundberg upper bound and the Monte Carlo approximation.

The organization of this chapter is as follows. Section 3.1 introduces the classical risk model. Section 3.2 studies some properties of the risk model with constant premiums and surrenders subject to dependence thinning. Section 3.3 derives the analytical approximation of the ruin probability. Section 3.4 derives the Lundberg's upper bound of the ruin probability. Section 3.5 performs experimental simulations.

3.1 Introduction to the classical risk model

In this section, we will introduce the classical risk model and the ruin probability.

Definition 3.1. The classical risk model is the model of total capital values, defined by

$$U(t) = u + ct - \sum_{i=1}^{N(t)} Y_i, \tag{3.1}$$

when

$U(t)$	is the total capital values at time t,
u	is the amount of initial reserves,
c	is a constant rate of premium per unit of time,
$N(t)$	is the number of claims up to time t, which is a Poisson process,
$\{Y_i\}_{i\geq 1}$	is a sequence of the amount of the i^{th} claims which are independent and identically distributed (i.i.d.) random variables.

The time to ruin, denoted by T, is defined as

$$T = \inf\{t \geq 0 \mid U(t) < 0\}. \tag{3.2}$$

The ruin probability with an initial surplus $u > 0$, $\psi(u)$, is

$$\begin{aligned} \psi(u) &= P[\ T < \infty \mid U(0) = u\ ], \\ &\equiv P[\ U(s) < 0 \text{ for some } s \geq 0 \mid U(0) = u\ ]. \end{aligned} \tag{3.3}$$

The non-ruin probability with an initial surplus $u > 0$, $\phi(u)$, is

$$\phi(u) = P[\ U(s) \geq 0 \text{ for all } s \geq 0 \mid U(0) = u\ ]. \tag{3.4}$$

From (3.3) and (3.4) we see that $\psi(u) + \phi(u) = 1$.

3.2 The risk model with constant premiums and surrenders subject to dependence thinning (CPST)

In this section, we introduce the risk model with constant premiums and surrenders subject to dependence thinning, denoted as CPST. The concept of dependence thinning arises from the fact that, in reality, the variance of the claim number following a Poisson distribution exceeds the mean of the claim number. This occurs due to certain events where the policyholder may not to claim for compensation in the event of an accident, leading to a situation where the number of claims is lower than the actual number of accidents. Similarly, the number of surrenders is lower than the actual number of contract cancellations. Therefore, we are interested in scenarios where the expectation number of claims and surrenders is lower than the expectation number of premiums, as both values depend on the number of premiums. This allows us to apply the thinning process.

The risk model consists of the initial capital, premiums, claims, and surrenders, where premiums are assumed to be equal for all customers and surrenders represent the amounts lost due to cancellation of the contract. In particular, the model is presented as

$$U(t) = u + cN(t) - \sum_{i=1}^{N(t,p)} Y_i - \sum_{i=1}^{N(t,q)} Z_i, \tag{3.5}$$

where u represents the initial capital, c is the constant rate of premium, $N(t)$ is the Poisson process with intensity $\lambda > 0$, denoting the number of premiums up to time t. Particularly, $N(t) \sim \text{Poisson}(\lambda t)$. $N(t,p)$, where $0 < p < 1$, is the p-thinning process of $N(t)$ denoting the number of claims up to time t. In particular, it is defined as $\sum_{i=1}^{M(t)} Q_i$ where Q_i are i.i.d. Bernoulli random variables with parameter p and $M(t)$ is independent and identically distributed with $N(t)$. The individual claim size $\{Y_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. non-negative random variables with a cumulative distribution function G. $N(t,q)$, where $0 < q < 1$, is the q-thinning process of $N(t)$ denoting the number of surrenders up to time t. The sequence of i.i.d. non-negative random variables $\{Z_i\}_{i=1}^{\infty}$ represents the amount of the i-th payment of insurance policy with a cumulative distribution function H. In addition, we suppose that $\{N(t)\}_{t\geq 0}, \{N(t,p)\}_{t\geq 0}, \{N(t,q)\}_{t\geq 0}, \{Y_i\}_{i=1}^{\infty}$, and $\{Z_i\}_{i=1}^{\infty}$ are mutually independent.

In order to ensure the insurance company's stable business, we assume that

$$E\left[cN(t) - \sum_{i=1}^{N(t,p)} Y_i - \sum_{i=1}^{N(t,q)} Z_i \right] > 0. \tag{3.6}$$

Since

$$\begin{aligned} E\left[cN(t) - \sum_{i=1}^{N(t,p)} Y_i - \sum_{i=1}^{N(t,q)} Z_i \right] &= E\left[cN(t)\right] - E\left[\sum_{i=1}^{N(t,p)} Y_i\right] - E\left[\sum_{i=1}^{N(t,q)} Z_i\right] \\ &= c\lambda t \ - \ \lambda pt\mu_Y - \ \lambda qt\mu_Z, \end{aligned}$$

the assumption becomes

$$c - p\mu_Y - q\mu_Z > 0, \tag{3.7}$$

which is called as the "**net profit condition**".

Lemma 3.1. *Define the profits process by* $\{S(t); t \geq 0\}$ *as*

$$S(t) = cN(t) - \sum_{i=1}^{N(t,p)} Y_i - \sum_{i=1}^{N(t,q)} Z_i.$$

Then, the profits process $S(t)$ *has the following properties:*

1. $S(0) = 0$,

2. $E[S(t)] = [\ c\lambda - \lambda p\mu_Y - \lambda q\mu_Z\ ]t$,

3. $Var[S(t)] = \big(\ c\lambda + \lambda pE[Y^2] + \lambda qE[Z^2]\ \big)\, t$,

4. $M_{S(t)}(s) = \exp\left\{t\left[\lambda\left(e^{sc} - 1\right) + \lambda p\left(M_Y(-s) - 1\right) + \lambda q\left(M_Z(-s) - 1\right)\right]\right\}$,

5. $\{S(t); t \geq 0\}$ *has stationary and independent increments.*

Proof.

(1) Since $N(t)$, $N(t,p)$, $N(t,q)$ are Poisson processes, $N(0) = 0$, $N(0,p) = 0$, and $N(0,q) = 0$. Then,

$$\begin{aligned} S(0) &= cN(0) - \sum_{i=1}^{N(0,p)} Y_i - \sum_{i=1}^{N(0,q)} Z_i, \\ &= 0 - \sum_{i=1}^{0} Y_i - \sum_{i=1}^{0} Z_i \\ &= 0. \end{aligned}$$

(2) By the property of expectation

$$E[S(t)] = E\left[cN(t)\right] - E\left[\sum_{i=1}^{N(t,p)} Y_i\right] - E\left[\sum_{i=1}^{N(t,q)} Z_i\right].$$

From Theorem 2.22,

$$E[S(t)] = [\ c\lambda - \lambda p\mu_Y - \lambda q\mu_Z\ ]t.$$

(3) By the property of variance and the independence of $Y_i, Z_i, N(t), N(t,p)$, and $N(t,q)$,

$$Var[S(t)] = Var\left[cN(t)\right] + Var\left[\sum_{i=1}^{N(t,p)} Y_i\right] + Var\left[\sum_{i=1}^{N(t,q)} Z_i\right].$$

From Theorem 2.22,

$$Var[S(t)] = \left(\ c\lambda + \lambda pE[Y^2] + \lambda qE[Z^2]\ \right)t.$$

(4) We know that

$$M_{S(t)}(s) = E[e^{sS(t)}].$$

By the independence property of the three terms of $S(t)$,

$$M_{S(t)}(s) = E\left[e^{scN(t)}\right] E\left[e^{-s\sum_{i=1}^{N(t,p)} Y_i}\right] E\left[e^{-s\sum_{i=1}^{N(t,q)} Z_i}\right].$$

The three terms are computed as follows

1) $$E\left[e^{scN(t)}\right] = M_{N(t)}(sc) = e^{\lambda t(e^{sc}-1)}.$$

2) $$E\left[e^{-s\sum_{i=1}^{N(t,p)} Y_i}\right] = M_{\sum_{i=1}^{N(t,p)} Y_i}(-s) = G_{N(t,p)}\left[M_Y(-s)\right] = e^{\lambda pt[M_Y(-s)-1]}.$$

3) $$E\left[e^{-s\sum_{i=1}^{N(t,q)} Z_i}\right] = M_{\sum_{i=1}^{N(t,q)} Z_i}(-s)$$
$$= G_{N(t,q)}\left[M_Z(-s)\right]$$
$$= e^{\lambda qt[M_Z(-s)-1]}.$$

Therefore,

$$M_{S(t)}(s) = \exp\left\{t\left[\lambda\left(e^{sc}-1\right)+\lambda p\left(M_Y(-s)-1\right)+\lambda q\left(M_Z(-s)-1\right)\right]\right\}.$$

(5) Since $N(t,p)$ has stationary increments and $\{Y_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. non-negative random variables, we get

$$\sum_{i=1}^{N(t+h,p)} Y_i - \sum_{i=1}^{N(t,p)} Y_i \quad \text{is identically distributed as} \quad \sum_{i=1}^{N(t+h,p)-N(t,p)} Y_i$$

and

$$\sum_{i=1}^{N(t+h,p)-N(t,p)} Y_i \quad \text{is identically distributed as} \quad \sum_{i=1}^{N(s+h,p)-N(s,p)} Y_i.$$

Therefore, $\sum_{i=1}^{N(t,p)} Y_i$ has stationary increments.

To prove that the process has independent increments, let $s_1 < s_2 \leq s_3 < s_4$. Since $N(t,p)$ has independent increments and $\{Y_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. non-negative random variables, we get

$$\sum_{i=N(s_1,p)+1}^{N(s_2,p)} Y_i \quad \text{is independent with} \sum_{i=N(s_3,p)+1}^{N(s_4,p)} Y_i.$$

Therefore, $\sum_{i=1}^{N(t,p)} Y_i$ has independent increments.

By the same technique, we can show that $\sum_{i=1}^{N(t,q)} Z_i$ has stationary and independent increments. Thus, $\{S(t); t \geq 0\}$ has stationary and independent increments. $\square$

3.3 Approximation to the ruin probability of the risk model

In this section, we will study analytical approximation of the ruin probability for the CPST model (3.5). We will start by obtaining the integro-differential equation for the ruin probability. Then we obtain an approximation of the ruin probability using the Cramér approximation, the Laplace transforms method, and the De Vylder Approximations. To obtain the three approximations, we first obtain the integro-differential equations stated in Theorem 3.1 below.

Theorem 3.1. *The ruin probability* $\psi(u)$ *for risk model* (3.5) *satisfies the integro-differential equation*

$$\begin{aligned}\psi'(u) \quad = \quad & \left[\frac{p}{c}+\frac{q}{c}\right]\psi(u)-\frac{q}{c}[1-H(u)]-\frac{p}{c}[1-G(u)] \\ & -\frac{p}{c}\int_0^u \psi(u-y)\,dG(y)-\frac{q}{c}\int_0^u \psi(u-z)\,dH(z), \quad u\geq 0,\end{aligned} \tag{3.8}$$

where G *and* H *are cumulative distribution functions of the individual claims sizes and the amount of surrenders with probability density functions* g *and* h*, respectively.*

Proof. To compute the non-ruin probability $\phi(u)$, we consider five different possible disjoint events of the number of premiums, the number of claims, and the number of surrenders during an infinitesimal period $[0, \Delta t]$ as follows.

Case 1:

There is no premiums, no claims, and no surrenders in the interval when $\Delta t \to 0$. The event occurs with the probability

$$\begin{aligned}& P\left(N(\Delta t)=0, N(\Delta t,p)=0, N(\Delta t,q)=0\right) \\ & \quad = P\left(N(\Delta t)=0\right)P\left(N(\Delta t,p)=0\right)P\left(N(\Delta t,q)=0\right) \\ & \quad = (1-\lambda\Delta t+o(\Delta t))(1-\lambda p\Delta t+o(\Delta t))(1-\lambda q\Delta t+o(\Delta t)) \\ & \quad = 1-\lambda\Delta t-\lambda p\Delta t-\lambda q\Delta t+o(\Delta t).\end{aligned}$$

Case 2:

There is no premiums, no claims, and one surrender in the interval when $\Delta t \to 0$. The event occurs with the probability

$$
\begin{aligned}
&P\left(N(\Delta t) = 0, N(\Delta t, p) = 0, N(\Delta t, q) = 1\right) \\
&\quad = P\left(N(\Delta t) = 0\right) P\left(N(\Delta t, p) = 0\right) P\left(N(\Delta t, q) = 1\right) \\
&\quad = (1 - \lambda \Delta t + o(\Delta t))(1 - \lambda p \Delta t + o(\Delta t))(\lambda q \Delta t + o(\Delta t)) \\
&\quad = \lambda q \Delta t + o(\Delta t).
\end{aligned}
$$

Case 3:

There is no premiums, one claims, and no surrenders in the interval when $\Delta t \to 0$. The event occurs with the probability

$$
\begin{aligned}
&P\left(N(\Delta t) = 0, N(\Delta t, p) = 1, N(\Delta t, q) = 0\right) \\
&\quad = P\left(N(\Delta t) = 0\right) P\left(N(\Delta t, p) = 1\right) P\left(N(\Delta t, q) = 0\right) \\
&\quad = (1 - \lambda \Delta t + o(\Delta t))(\lambda p \Delta t + o(\Delta t))(1 - \lambda q \Delta t + o(\Delta t)) \\
&\quad = \lambda p \Delta t + o(\Delta t).
\end{aligned}
$$

Case 4:

There is one premium, no claims, and no surrenders in the interval when $\Delta t \to 0$. The event occurs with the probability

$$
\begin{aligned}
&P\left(N(\Delta t) = 1, N(\Delta t, p) = 0, N(\Delta t, q) = 0\right) \\
&\quad = P\left(N(\Delta t) = 1\right) P\left(N(\Delta t, p) = 0\right) P\left(N(\Delta t, q) = 0\right) \\
&\quad = (\lambda \Delta t + o(\Delta t))(1 - \lambda p \Delta t + o(\Delta t))(1 - \lambda q \Delta t + o(\Delta t)) \\
&\quad = \lambda \Delta t + o(\Delta t).
\end{aligned}
$$

Case 5:

There are more than one event of premiums, claims, and surrenders combined in the interval when $\Delta t \to 0$. The event occurs with the probability

$$P\left(N(\Delta t) + N(\Delta t, p) + N(\Delta t, q) > 1\right) = o(\Delta t).$$

From the law of total probability for discrete random variable in Definition 2.16, it follows that

$$\begin{aligned}
\phi(u) = \quad & P[N(\Delta t) = 0]P[N(\Delta t, p) = 0]P[N(\Delta t, q) = 0] \\
& \quad \cdot P[U(t) \geq 0, \forall t > 0 | N(\Delta t) = 0, N(\Delta t, p) = 0, N(\Delta t, q) = 0] \\
& + P[N(\Delta t) = 0]P[N(\Delta t, p) = 0]P[N(\Delta t, q) = 1] \\
& \quad \cdot P[U(t) \geq 0, \forall t > 0 | N(\Delta t) = 0, N(\Delta t, p) = 0, N(\Delta t, q) = 1] \\
& + P[N(\Delta t) = 1]P[N(\Delta t, p) = 0]P[N(\Delta t, q) = 0] \\
& \quad \cdot P[U(t) \geq 0, \forall t > 0 | N(\Delta t) = 1, N(\Delta t, p) = 0, N(\Delta t, q) = 0] \\
& + P[N(\Delta t) = 0]P[N(\Delta t, p) = 1]P[N(\Delta t, q) = 0] \\
& \quad \cdot P[U(t) \geq 0, \forall t > 0 | N(\Delta t) = 0, N(\Delta t, p) = 1, N(\Delta t, q) = 0] \\
& + P[N(\Delta t) + N(\Delta t, p) + N(\Delta t, q) > 1] \\
& \quad \cdot P[U(t) \geq 0, \forall t > 0 | N(\Delta t) + N(\Delta t, p) + N(\Delta t, q) > 1].
\end{aligned}$$

Then

$$\begin{aligned}
\phi(u) = \quad & [1 - \lambda \Delta t + o(\Delta t)][1 - \lambda p \Delta t + o(\Delta t)][1 - \lambda q \Delta t + o(\Delta t)] \\
& \quad \cdot P[U(t) \geq 0, \forall t > 0 | N(\Delta t) = 0, N(\Delta t, p) = 0, N(\Delta t, q) = 0] \\
& + [1 - \lambda \Delta t + o(\Delta t)][1 - \lambda p \Delta t + o(\Delta t)][\lambda q \Delta t + o(\Delta t)]] \\
& \quad \cdot P[U(t) \geq 0, \forall t > 0 | N(\Delta t) = 0, N(\Delta t, p) = 0, N(\Delta t, q) = 1] \\
& + [\lambda \Delta t + o(\Delta t)][1 - \lambda p \Delta t + o(\Delta t)][1 - \lambda q \Delta t + o(\Delta t)] \\
& \quad \cdot P[U(t) \geq 0, \forall t > 0 | N(\Delta t) = 1, N(\Delta t, p) = 0, N(\Delta t, q) = 0] \\
& + [1 - \lambda \Delta t + o(\Delta t)][\lambda p \Delta t + o(\Delta t)][1 - \lambda q \Delta t + o(\Delta t)] \\
& \quad \cdot P[U(t) \geq 0, \forall t > 0 | N(\Delta t) = 0, N(\Delta t, p) = 1, N(\Delta t, q) = 0] \\
& + o(\Delta t) \cdot \; P[U(t) \geq 0, \forall t > 0 | N(\Delta t) + N(\Delta t, p) + N(\Delta t, q) > 1].
\end{aligned}$$

By the properties of little-oh in Theorem 2.1 for $\Delta t \to 0$ and the law of total probability for continuous random variable Y_i and Z_i in Definition 2.17.

$$\begin{aligned}\phi(u) = \; & (1-\lambda\Delta t-\lambda p\Delta t-\lambda q\Delta t)\ P[U(t)\geq 0,\forall t>0 \mid U(\Delta t)=u] \\ & +\lambda q\Delta t\int_0^u P[U(t)\geq 0,\forall t>0 \mid U(\Delta t)=u-z]\,dH(z) \\ & +\lambda p\Delta t\int_0^u P[U(t)\geq 0,\forall t>0 \mid U(\Delta t)=u-y]\,dG(y) \\ & +\lambda\Delta t\ P[U(t)\geq 0,\forall t>0 \mid U(\Delta t)=u+c]+o(\Delta t).\end{aligned}$$

According to the concept of stationary, we can treat Δt as a new start time. Therefore, we can express $U(\Delta t)$ as $U(0)$. This implies that we are starting a new at Δt and can use $U(0)$ as the starting point,

$$\begin{aligned}\phi(u) = \; & (1-\lambda\Delta t-\lambda p\Delta t-\lambda q\Delta t)\ P[U(t)\geq 0,\forall t>0 \mid U(0)=u] \\ & +\lambda q\Delta t\int_0^u P[U(t)\geq 0,\forall t>0 \mid U(0)=u-z]\,dH(z) \\ & +\lambda p\Delta t\int_0^u P[U(t)\geq 0,\forall t>0 \mid U(0)=u-y]\,dG(y) \\ & +\lambda\Delta t\ P[U(t)\geq 0,\forall t>0 \mid U(0)=u+c]+o(\Delta t).\end{aligned}$$

Then, we get

$$\begin{aligned}\phi(u) \;&=\; (1-\lambda\Delta t-\lambda p\Delta t-\lambda q\Delta t)\ \phi(u)+\lambda q\Delta t\int_0^u \phi(u-z)\,dH(z) \\ &\quad +\lambda p\Delta t\int_0^u \phi(u-y)\,dG(y)+\lambda\Delta t\phi(u+c)+o(\Delta t).\end{aligned}$$

By the Taylor series expansion in $\phi(u+c)$ around $x_0 = u$, particularly, $\phi(u+c) = \phi(u)+c\phi'(u)+o(\Delta t)$ for $\Delta t \to 0$, we get

$$\begin{aligned}-\lambda\Delta t\phi'(u)c \;&=\; -\lambda p\Delta t\ \phi(u)-\lambda q\Delta t\ \phi(u)+\lambda q\Delta t\int_0^u \phi(u-z)\,dH(z) \\ &\quad +\lambda p\Delta t\int_0^u \phi(u-y)\,dG(y)+o(\Delta t).\end{aligned}$$

Dividing both sides by Δt and letting Δt approach to 0, we have

$$\phi'(u)=\left[\frac{p}{c}+\frac{q}{c}\right]\phi(u)-\frac{p}{c}\int_0^u \phi(u-y)\,dG(y)-\frac{q}{c}\int_0^u \phi(u-z)\,dH(z), \quad u\geq 0. \quad (3.9)$$

Using the property that $\phi(u) = 1 - \psi(u)$, we get

$$\begin{aligned} -\psi'(u) &= \left[\frac{p}{c} + \frac{q}{c}\right] - \left[\frac{p}{c} + \frac{q}{c}\right]\psi(u) - \frac{p}{c}\int_0^u 1\, dG(y) - \frac{q}{c}\int_0^u 1\, dH(z) \\ &\quad + \frac{p}{c}\int_0^u \psi(u-y)\, dG(y) + \frac{q}{c}\int_0^u \psi(u-z)\, dH(z), \quad u \geq 0. \end{aligned}$$

Therefore,

$$\begin{aligned} -\psi'(u) &= \left[\frac{p}{c} + \frac{q}{c}\right] - \left[\frac{p}{c} + \frac{q}{c}\right]\psi(u) - \frac{p}{c}G(u) - \frac{q}{c}H(u) \\ &\quad + \frac{p}{c}\int_0^u \psi(u-y)\, dG(y) + \frac{q}{c}\int_0^u \psi(u-z)\, dH(z), \quad u \geq 0. \end{aligned}$$

Thus,

$$\begin{aligned} \psi'(u) &= \left[\frac{p}{c} + \frac{q}{c}\right]\psi(u) - \frac{q}{c}[1 - H(u)] - \frac{p}{c}[1 - G(u)] \\ &\quad - \frac{p}{c}\int_0^u \psi(u-y)\, dG(y) - \frac{q}{c}\int_0^u \psi(u-z)\, dH(z), \quad u \geq 0. \end{aligned}$$

□

Corollary 3.1. *For risk model* (3.5)*,*

$$\psi(0) = \frac{p}{c}E[Y] + \frac{q}{c}E[Z].$$

Proof. Integrate the integro-differential equations (3.9) over the interval $(0, t)$ on u yields

$$\begin{aligned} \int_0^t \phi'(u)\, du = \left[\frac{p}{c} + \frac{q}{c}\right]\int_0^t \phi(u)\, du - \frac{p}{c}\int_0^t\int_0^u \phi(u-y)\, dG(y)\, du \\ - \frac{q}{c}\int_0^t\int_0^u \phi(u-z)\, dH(z)\, du. \end{aligned} \tag{3.10}$$

Consider $-\frac{q}{c}\int_0^t\int_0^u \phi(u-z)\, dH(z)\, du$ and the property of CDF H, we can show that

$$\begin{aligned} -\frac{q}{c}\int_0^t\int_0^u \phi(u-z)\, dH(z)\, du &= \frac{q}{c}\int_0^t\int_0^u \phi(u-z)\, d[1 - H(z)]\, du, \\ &= \frac{q}{c}\int_0^t \Big(\phi(0)(1 - H(u)) - \phi(u) \\ &\quad + \int_0^u (1 - H(z))\phi'(u-z)\, dz\Big) du. \end{aligned}$$

Then,

$$-\frac{q}{c}\int_0^t\int_0^u \phi(u-z)\,dH(z)\,du = \frac{q}{c}\int_0^t \phi(0)(1-H(u))\,du - \frac{q}{c}\int_0^t \phi(u)\,du + \frac{q}{c}\int_0^t\int_0^u (1-H(z))\phi'(u-z)\,dz\,du. \tag{3.11}$$

By the same technique, we can show that

$$-\frac{p}{c}\int_0^t\int_0^u \phi(u-y)\,dG(y)\,du = \frac{p}{c}\int_0^t \phi(0)(1-G(u))\,du - \frac{p}{c}\int_0^t \phi(u)\,du + \frac{p}{c}\int_0^t\int_0^u (1-G(y))\phi'(u-y)\,dy\,du. \tag{3.12}$$

Substituting (3.11) and (3.12) into (3.10), we get

$$\begin{aligned}\int_0^t \phi'(u)\,du = {} & \frac{q}{c}\int_0^t \phi(0)(1-H(u))\,du + \frac{p}{c}\int_0^t \phi(0)(1-G(u))\,du \\ & + \frac{q}{c}\int_0^t\int_0^u (1-H(z))\phi'(u-z)\,dz\,du \\ & + \frac{p}{c}\int_0^t\int_0^u (1-G(y))\phi'(u-y)\,dy\,du.\end{aligned} \tag{3.13}$$

Consider $\frac{q}{c}\int_0^t\int_0^u (1-H(z))\phi'(u-z)\,dz\,du$, we can show that

$$\begin{aligned}\frac{q}{c}\int_0^t\int_0^u (1-H(z))\phi'(u-z)\,dz\,du & = \frac{q}{c}\int_0^t\int_z^t (1-H(z))\phi'(u-z)\,du\,dz, \\ & = \frac{q}{c}\int_0^t (1-H(z))\int_z^t \phi'(u-z)\,d(u-z)\,dz.\end{aligned}$$

Then,

$$\frac{q}{c}\int_0^t\int_0^u (1-H(z))\phi'(u-z)\,dz\,du = \frac{q}{c}\int_0^t (1-H(z))\phi(t-z)\,dz - \frac{q}{c}\int_0^t (1-H(z))\phi(0)\,dz. \tag{3.14}$$

By the same technique, we can show that

$$\frac{p}{c}\int_0^t\int_0^u(1-G(y))\phi'(u-y)\,dy\,du=\frac{p}{c}\int_0^t(1-G(y))\phi(t-y)\,dy-\frac{p}{c}\int_0^t(1-G(y))\phi(0)\,dy. \tag{3.15}$$

Substituting (3.14) and (3.15) into (3.13), we get

$$\phi(t)-\phi(0)=\frac{q}{c}\int_0^t(1-H(z))\phi(t-z)\,dz+\frac{p}{c}\int_0^t(1-G(y))\phi(t-y)\,dy.$$

Letting t approach to ∞ and using the property that $\lim_{u\to\infty}\phi(u)=1$, we get,

$$1-\phi(0)=\frac{q}{c}\int_0^\infty(1-H(z))\,dz+\frac{p}{c}\int_0^\infty(1-G(y))\,dy.$$

Since $\int_0^\infty(1-H(z))\,dz=E[Z]$ and $\int_0^\infty(1-G(y))\,dy=E[Y]$, therefore,

$$1-\phi(0)=\frac{q}{c}E[Z]+\frac{p}{c}E[Y].$$

Using the property that $\phi(u)=1-\psi(u)$, we get

$$\psi(0)=\frac{q}{c}E[Z]+\frac{p}{c}E[Y].$$

□

3.3.1 The Cramér approximation

In this section, we obtain the Cramér approximation to the ruin probability when amounts of claims and surrenders follow exponential distributions. In particular, the probability density functions of the claim sizes and premiums are

$$g(y)=ae^{-ay}\quad\text{and}\quad h(z)=be^{-bz},y,z\geq 0, \tag{3.16}$$

corresponding to CDF's are G and H, respectively, in Theorem 3.1.

Theorem 3.2. *For the risk model* (3.5) *where the amounts of claims size and surrenders follow exponential distributions with parameters a and b, respectively, if the net profit condition* (3.7) *is satisfied, then the Cramér approximation of the ruin probability $\psi_C(u)$ is*

$$\psi_C(u) = C_1 e^{r_1 u} + C_2 e^{r_2 u} \quad \textit{for all } u \geq 0, \tag{3.17}$$

where $C_1, C_2, r_1,$ and r_2 are as follows

$$C_1 = \frac{C_{11}}{C_D}\ ,\ C_2 = \frac{C_{21}}{C_D}\ ,$$

$$r_1 = \frac{-\left[\frac{1}{a} + \frac{1}{b} - \frac{(p+q)}{acb}\right] - \sqrt{D}}{\frac{2}{ab}},$$

$$r_2 = \frac{-\left[\frac{1}{a} + \frac{1}{b} - \frac{(p+q)}{acb}\right] + \sqrt{D}}{\frac{2}{ab}},$$

which

$$D = \left[\frac{1}{a} + \frac{1}{b} - \frac{(p+q)}{acb}\right]^2 - \frac{4}{ab}\left[1 - \frac{p}{ca} - \frac{q}{bc}\right],$$

$$C_{11} = -abcp + bp^2 - abcq + apq + bpq + aq^2 - c(bp+aq)r_2,$$

$$C_{21} = abcp - bp^2 + abcq - apq - bpq - aq^2 + c(bp+aq)r_1,$$

and

$$C_D = abc^2(r_1 - r_2).$$

Proof.

Observe that CDF G and PDF g, satisfy $dG(u) = g(u)du$, as mentioned in Remark 2.2, including CDF H and PDF h.

Substituting the density functions of Y_i and Z_i with CDF's are G and H, respectively, into (3.8), we have

$$\begin{aligned}\psi'(u) =& \left[\frac{p}{c}+\frac{q}{c}\right]\psi(u) - \frac{q}{c}[e^{-bu}] - \frac{p}{c}[e^{-au}] - \frac{p}{c}\int_0^u \psi(u-y)ae^{-ay}\,dy \\ & -\frac{q}{c}\int_0^u \psi(u-z)be^{-bz}\,dz. \end{aligned} \tag{3.18}$$

Differentiating the equation with respect to u, we have

$$\begin{aligned}\psi''(u) =& \left[\frac{p}{c}+\frac{q}{c}\right]\psi'(u) + \frac{pa}{c}[e^{-au}] + \frac{qb}{c}[e^{-bu}] + \left[-\frac{qb}{c}-\frac{pa}{c}\right]\ \psi(u) \\ & +\frac{pa}{c}\int_0^u \psi(u-y)ae^{-ay}\,dy + \frac{qb}{c}\int_0^u \psi(u-z)be^{-bz}\,dz. \end{aligned}$$

Multiplying the equation by $\frac{1}{a}$, we have

$$\begin{aligned}\frac{\psi''(u)}{a} =& \left[\frac{p}{ca}+\frac{q}{ca}\right]\psi'(u) + \frac{p}{c}[e^{-au}] + \frac{qb}{ca}[e^{-bu}] + \left[-\frac{qb}{ca}-\frac{p}{c}\right]\ \psi(u) \\ & +\frac{p}{c}\int_0^u \psi(u-y)ae^{-ay}\,dy + \frac{qb}{ca}\int_0^u \psi(u-z)be^{-bz}\,dz. \end{aligned} \tag{3.19}$$

Adding the terms of each side of (3.19) and (3.18), we have

$$\begin{aligned}&\frac{\psi''(u)}{a} + \left(1-\frac{p}{ca}-\frac{q}{ca}\right)\psi'(u) \\ &= \left[\frac{q}{c}-\frac{qb}{ca}\right]\ \psi(u) + \left[\frac{qb}{ca}-\frac{q}{c}\right]e^{-bu} + \left[\frac{qb}{ca}-\frac{q}{c}\right]\int_0^u \psi(u-z)be^{-bz}\,dz. \end{aligned} \tag{3.20}$$

Differentiating the equation with respect to u, we have

$$\begin{aligned}&\frac{\psi'''(u)}{a} + \left(1-\frac{p}{ca}-\frac{q}{ca}\right)\psi''(u) \\ &= \left[\frac{q}{c}-\frac{qb}{ca}\right]\ \psi'(u) - \left[\frac{qb^2}{ca}-\frac{qb}{c}\right]e^{-bu} + \left[\frac{qb^2}{ca}-\frac{qb}{c}\right]\psi(u) \\ &\quad - \left[\frac{qb^2}{ca}-\frac{qb}{c}\right]\int_0^u \psi(u-z)be^{-bz}\,dz. \end{aligned}$$

Multiplying the equation by $\dfrac{1}{b}$, we have

$$\begin{aligned} &\frac{\psi'''(u)}{ab} + \left(\frac{1}{b} - \frac{p}{acb} - \frac{q}{acb}\right)\psi''(u) \\ &\quad = \left[\frac{q}{bc} - \frac{q}{ca}\right]\ \psi'(u) - \left[\frac{qb}{ca} - \frac{q}{c}\right] e^{-bu} + \left[\frac{qb}{ca} - \frac{q}{c}\right]\psi(u) \\ &\quad\quad - \left[\frac{qb}{ca} - \frac{q}{c}\right]\int_0^u \psi(u-z) b e^{-bz}\, dz. \end{aligned} \tag{3.21}$$

Adding the terms of each side of (3.21) and (3.20), we have

$$\frac{\psi'''(u)}{ab} + \left(\frac{1}{a} - \frac{p}{acb} - \frac{q}{acb} + \frac{1}{b}\right)\psi''(u) + \left[1 - \frac{p}{ca} - \frac{q}{bc}\right]\psi'(u) = 0. \tag{3.22}$$

The equivalent characteristic equation is

$$\frac{r^3}{ab} + \left(\frac{1}{a} - \frac{p}{acb} - \frac{q}{acb} + \frac{1}{b}\right) r^2 + \left[1 - \frac{p}{ca} - \frac{q}{bc}\right] r = 0. \tag{3.23}$$

Solving the equation, we obtain the three roots as

$$r_1 = \frac{-\left[\dfrac{1}{a} + \dfrac{1}{b} - \dfrac{(p+q)}{acb}\right] - \sqrt{D}}{\dfrac{2}{ab}},$$

$$r_2 = \frac{-\left[\dfrac{1}{a} + \dfrac{1}{b} - \dfrac{(p+q)}{acb}\right] + \sqrt{D}}{\dfrac{2}{ab}},$$

$$r_3 = 0,$$

where

$$D = \left[\frac{1}{a} + \frac{1}{b} - \frac{(p+q)}{acb}\right]^2 - \frac{4}{ab}\left[1 - \frac{p}{ca} - \frac{q}{bc}\right].$$

Therefore, the general solution of $\psi(u)$ is

$$\psi(u) = C_1 e^{r_1 u} + C_2 e^{r_2 u} + C_3. \tag{3.24}$$

Since

$$\begin{aligned} D &= \left[\frac{1}{a}+\frac{1}{b}-\frac{(p+q)}{acb}\right]^2-\frac{4}{ab}\left[1-\frac{p}{ca}-\frac{q}{bc}\right] \\ &= \left[\frac{1}{b}-\frac{1}{a}-\frac{(p-q)}{acb}\right]^2+\frac{4pq}{a^2b^2c^2} \ > \ 0. \end{aligned}$$

Then, r_1 and r_2 are distinct real roots.

Since

$$\frac{1}{a}+\frac{1}{b}-\frac{(p+q)}{acb}=\left[\frac{1}{cb}+\frac{1}{ca}\right]\left[c-\frac{p}{a}-\frac{q}{b}\right]+\frac{p}{a^2c}+\frac{q}{b^2c} \ > \ 0, \tag{3.25}$$

by the Vieta's theorem in Theorem 2.2 and (3.23), we get

$$r_1r_2=\frac{1-\frac{p}{ca}-\frac{q}{bc}}{ab}>0 \tag{3.26}$$

and

$$r_1+r_2=-\frac{\left[\frac{1}{a}+\frac{1}{b}-\frac{(p+q)}{acb}\right]}{ab}<0. \tag{3.27}$$

From (3.26) and the net profit condition (3.7), we can see that r_1 and r_2 have the same sign. From (3.27) and (3.25), we get

$$r_1<0 \text{ and } r_2<0.$$

Next, once we know the values of r_1 and r_1, we will then determine the values of C_1, C_2 and C_3 for (3.24) by using the initial conditions follow as,

1. $\lim_{u\to\infty} \psi(u)=0$, since $r_1, r_2<0$ which yields $C_3=0$.
2. Substituting, $\psi(0)=\frac{q}{cb}+\frac{p}{ca}$ in (3.24), we get

$$C_1+C_2=\frac{q}{cb}+\frac{p}{ca}. \tag{3.28}$$

3. Letting $u=0$ in (3.18) and using $\psi(u)$ from (3.24), we get

$$C_1r_1+C_2r_2=\left[\frac{p}{c}+\frac{q}{c}\right]\left[\frac{q}{cb}+\frac{p}{ca}\right]-\frac{p}{c}-\frac{q}{c}. \tag{3.29}$$

Solving system of (3.28) and (3.29), we get

$$C_1 = \frac{C_{11}}{C_D} \text{ and } C_2 = \frac{C_{21}}{C_D} \ ,$$

where

$$\begin{aligned} C_{11} &= -abcp + bp^2 - abcq + apq + bpq + aq^2 - c(bp + aq)r_2, \\ C_{21} &= abcp - bp^2 + abcq - apq - bpq - aq^2 + c(bp + aq)r_1, \\ C_D &= abc^2(r_1 - r_2). \end{aligned}$$

□

To calculate the approximated ruin probability using the Cramér approximation described in (3.17), we can use the R programming for computation.

3.3.2 The Laplace transform

In this section, we obtain an approximation of ruin probability using the Laplace transforms in conjunction with integral equation of ruin probability for the CPST model (3.5).

Theorem 3.3. *The Laplace transform of ruin probability* $\psi(u)$ *for risk model* (3.5) *is*

$$\psi^*(s) = \frac{-p[1 - g^*(s)] - q[1 - h^*(s)] + cs\psi(0)}{s\left[cs - p[1 - g^*(s)] - q[1 - h^*(s)] - q[1 - h^*(s)]\right]}, \tag{3.30}$$

where $\psi(0) = \frac{q}{c}E[Z] + \frac{p}{c}E[Y]$ *and* g^*, h^* *are the Laplace transforms of probability density functions for the amount of claims size* g *and surrender* h*, respectively.*

Proof. Taking the Laplace transform of (3.8) and formula in Theorem 2.3, we get

$$\begin{aligned} s\psi^*(s) - \psi(0) &= \left[\frac{p}{c} + \frac{q}{c}\right]\psi^*(s) - \frac{p}{c}\left[\frac{1}{s} - \frac{g^*(s)}{s}\right] - \frac{q}{c}\left[\frac{1}{s} - \frac{h^*(s)}{s}\right] \\ &\quad -\frac{p}{c}\psi^*(s)g^*(s) - \frac{q}{c}\psi^*(s)h^*(s). \end{aligned}$$

Multiplying both sides by $-cs$, we have

$$\begin{aligned}-cs^2\psi^*(s)+cs\psi(0) &= -s\left[p+q\right]\psi^*(s)+p\left[1-g^*(s)\right]+q\left[1-h^*(s)\right]\\ &\quad +ps\ \psi^*(s)g^*(s)+qs\ \psi^*(s)h^*(s).\end{aligned}$$

Therefore,

$$-p\left[1-g^*(s)\right]-q\left[1-h^*(s)\right]+cs\psi(0) = \left[cs^2-s\left[p+q\right]+qs\ h^*(s)+ps\ g^*(s)\right]\psi^*(s).$$

Thus,

$$\psi^*(s)=\frac{-p\left[1-g^*(s)\right]-q\left[1-h^*(s)\right]+cs\psi(0)}{s\left[cs-p\left[1-g^*(s)\right]-q\left[1-h^*(s)\right]\right]}.$$

□

Corollary 3.2. *Assume the risk model described in* (3.5) *where the amount of claims size and surrender follow exponential distributions according to* (3.16)*, probability density functions denoted as* g *and* h*, respectively, and with parameters* a *and* b*. If the net profit condition given by* (3.7) *holds, then the Laplace transform of the ruin probability* $\psi(u)$ *is*

$$\psi_{\mathscr{L}}(u)=\frac{b^2p+a^2q+(bp+aq)s_1}{abc(s_1-s_2)}e^{s_1u}+\frac{-b^2p-a^2q-(bp+aq)s_2}{abc(s_1-s_2)}e^{s_2u}, \qquad (3.31)$$

where

$$s_1=\frac{-ac-bc+p+q-\sqrt{S}}{2c},$$

$$s_2=\frac{-ac-bc+p+q+\sqrt{S}}{2c},$$

and

$$S=(ac+bc-p-q)^2-4c(abc-bp-aq).$$

Proof.

Substituting the Laplace transforms of the density functions of Y_i and Z_i with CDF's are G and H, respectively, into (3.30), we have

$$\psi^*(s) = \frac{-p\left[1 - \dfrac{a}{s+a}\right] - q\left[1 - \dfrac{b}{s+b}\right] + cs\left[\dfrac{p}{ca} + \dfrac{q}{cb}\right]}{s\left(cs - p\left[1 - \dfrac{a}{s+a}\right] - q\left[1 - \dfrac{b}{s+b}\right]\right)}.$$

Let $R(s) = abc - bp - aq + (ac + bc - p - q)s + cs^2$ and rearrange the equation for $\psi^*(s)$, we get

$$\psi^*(s) = \frac{b^2p + a^2q + (bp + aq)s}{abR(s)}. \tag{3.32}$$

Let $S = (ac + bc - p - q)^2 - 4c(abc - bp - aq)$. Then, $S > 0$.

Factoring $R(s)$, we will obtain that

$$\psi^*(s) = \frac{b^2p + a^2q + (bp + aq)s}{abc(s - s_1)(s - s_2)}, \tag{3.33}$$

where

$$s_1 = \frac{-ac - bc + p + q - \sqrt{S}}{2c},$$
$$s_2 = \frac{-ac - bc + p + q + \sqrt{S}}{2c}.$$

Since $S > 0$, then s_1 and s_2 are distinct real roots.

Since

$$-ac - bc + p + q = -\left[\frac{1}{b} + \frac{1}{a}\right](abc - bp - aq) - \frac{bp}{a} - \frac{aq}{b} < 0, \tag{3.34}$$

by the Vieta's theorem in Theorem 2.2 and equation $R(s)$, we get

$$s_1 s_2 = \frac{abc - bp - aq}{c} > 0 \tag{3.35}$$

and

$$s_1 + s_2 = \frac{-(ac + bc - p - q)}{c} < 0. \tag{3.36}$$

From (3.35) and the net profit condition (3.7), we can see that s_1 and s_2 have the same sign. From (3.36) and (3.34), we get

$$s_1 < 0 \text{ and } s_2 < 0.$$

Applying partial fraction decomposition to (3.33) with respect to s, we obtain

$$\psi^*(s) = \frac{b^2p + a^2q + (bp + aq)s_1}{abc(s_1 - s_2)(s - s_1)} + \frac{-b^2p - a^2q - (bp + aq)s_2}{abc(s_1 - s_2)(s - s_2)}. \tag{3.37}$$

Taking the inverse Laplace transform (3.37) with respect to s, we obtain

$$\psi_{\mathscr{L}}(u) = \frac{b^2p + a^2q + (bp + aq)s_1}{abc(s_1 - s_2)} e^{s_1 u} + \frac{-b^2p - a^2q - (bp + aq)s_2}{abc(s_1 - s_2)(s - s_2)} e^{s_2 u}.$$

□

It can be observed that the ruin probability of the Cramér approximation in (3.17) Theorem 3.2 and the Laplace transforms in (3.31) Theorem 3.2 are equal. This can be proven by showing that the formulas of both approximations yield the same value, as mentioned in Remark 3.1.

Remark 3.1. For the amount of claims size and surrender follow exponential distributions according to (3.16), probability density functions denoted as g and h, respectively, and with parameters a and b. The ruin probability of the Cramér approximation $\psi_C(u)$ (3.17) and the Laplace transforms $\psi_{\mathscr{L}}(u)$ (3.31) yield the same value, for all $u \geq 0$

$$\psi_C(u) = C_1 e^{r_1 u} + C_2 e^{r_2 u},$$
$$\psi_{\mathscr{L}}(u) = \frac{b^2p + a^2q + (bp + aq)s_1}{abc(s_1 - s_2)} e^{s_1 u} + \frac{-b^2p - a^2q - (bp + aq)s_2}{abc(s_1 - s_2)} e^{s_2 u},$$

where C_1, C_2, r_1, r_2, s_1 and s_2 are as follows

$$C_1 = \frac{-abcp + bp^2 - abcq + apq + bpq + aq^2 - c(bp + aq)r_2}{abc^2(r_1 - r_2)} ,$$

$$C_2 = \frac{abcp - bp^2 + abcq - apq - bpq - aq^2 + c(bp+aq)r_1}{abc^2(r_1 - r_2)} \ ,$$

$$r_1 = \frac{-\left[\frac{1}{a} + \frac{1}{b} - \frac{(p+q)}{acb}\right] - \sqrt{\left[\frac{1}{a} + \frac{1}{b} - \frac{(p+q)}{acb}\right]^2 - \frac{4}{ab}\left[1 - \frac{p}{ca} - \frac{q}{bc}\right]}}{\frac{2}{ab}},$$

$$r_2 = \frac{-\left[\frac{1}{a} + \frac{1}{b} - \frac{(p+q)}{acb}\right] + \sqrt{\left[\frac{1}{a} + \frac{1}{b} - \frac{(p+q)}{acb}\right]^2 - \frac{4}{ab}\left[1 - \frac{p}{ca} - \frac{q}{bc}\right]}}{\frac{2}{ab}},$$

$$s_1 = \frac{-ac - bc + p + q - \sqrt{(ac+bc-p-q)^2 - 4c(abc - bp - aq)}}{2c},$$

$$s_2 = \frac{-ac - bc + p + q + \sqrt{(ac+bc-p-q)^2 - 4c(abc - bp - aq)}}{2c}.$$

Proof. We want to show that the various coefficients and constants have the same value demonstrated as follows.

$$\begin{aligned} r_1 &= \frac{-\left[\frac{1}{a} + \frac{1}{b} - \frac{(p+q)}{acb}\right] - \sqrt{\left[\frac{1}{a} + \frac{1}{b} - \frac{(p+q)}{acb}\right]^2 - \frac{4}{ab}\left[1 - \frac{p}{ca} - \frac{q}{bc}\right]}}{\frac{2}{ab}} \\ &= \frac{-ac - bc + p + q - \sqrt{(ac+bc-p-q)^2 - 4c(abc - bp - aq)}}{2c} \\ &= s_1, \end{aligned}$$

$$\begin{aligned} r_2 &= \frac{-\left[\frac{1}{a} + \frac{1}{b} - \frac{(p+q)}{acb}\right] + \sqrt{\left[\frac{1}{a} + \frac{1}{b} - \frac{(p+q)}{acb}\right]^2 - \frac{4}{ab}\left[1 - \frac{p}{ca} - \frac{q}{bc}\right]}}{\frac{2}{ab}} \\ &= \frac{-ac - bc + p + q + \sqrt{(ac+bc-p-q)^2 - 4c(abc - bp - aq)}}{2c} \\ &= s_2, \end{aligned}$$

and

$$\begin{aligned} C_1 &= \frac{-abcp + bp^2 - abcq + apq + bpq + aq^2 - c(bp+aq)r_2}{abc^2(r_1 - r_2)} \\ &= \frac{-abp + \frac{bp^2}{c} - abq + \frac{apq}{c} + \frac{bpq}{c} + \frac{aq^2}{c} - (bp+aq)s_2}{abc(s_1 - s_2)} \\ &= \frac{b^2p + a^2q + (bp+aq)s_1}{abc(s_1 - s_2)}, \end{aligned}$$

$$\begin{aligned} C_2 &= \frac{abcp - bp^2 + abcq - apq - bpq - aq^2 + c(bp+aq)r_1}{abc^2(r_1 - r_2)} \\ &= \frac{abp - \frac{bp^2}{c} + abq - \frac{apq}{c} - \frac{bpq}{c} - \frac{aq^2}{c} + (bp+aq)s_1}{abc(s_1 - s_2)} \\ &= \frac{-b^2p - a^2q - (bp+aq)s_2}{abc(s_1 - s_2)}. \end{aligned}$$

Therefore,

$$\psi_C(u) = \psi_{\mathscr{L}}(u).$$

□

To calculate the approximated ruin probability using the Laplace transform for money amounts which follow exponential distributions described in (3.31), we can use the MATLAB commands “`partfrac`” and “`ilaplace`” for computation.

In the case that the money amounts follow gamma distributions, we also use MATLAB to calculate the approximated ruin probability. We use the general Laplace transforms for the ruin probability (3.30) with gamma distributions instead of exponential distributions.

3.3.3 The De-Vylder approximation

In this section, we consider the CPST (3.5) where claim sizes and surrenders follow other distributions rather than exponential distributions. The method used in this topic is the De-Vylder approximation which is to approximate the risk process by the classical risk model where the numbers of premiums, claims, and surrenders are exponentially distributed. In particular, the model (3.5) is approximated by the risk model:

$$\widetilde{U}(t) = u + \widetilde{c}\widetilde{N}(t) - \sum_{i=1}^{\widetilde{N}(t,\widetilde{p})} \widetilde{Y}_i - \sum_{i=1}^{\widetilde{N}(t,\widetilde{q})} \widetilde{Z}_i, \tag{3.38}$$

where $\widetilde{Y}_i$ and $\widetilde{Z}_i$ have exponential distributions with parameters $\widetilde{a}$ and $\widetilde{b}$, respectively. Also, $\widetilde{N}(t), \widetilde{N}(t,\widetilde{p})$, and $\widetilde{N}(t,\widetilde{q})$ are Poisson processes with intensities $\widetilde{\lambda}$, $\widetilde{\lambda}\widetilde{p}$, and $\widetilde{\lambda}\widetilde{q}$, respectively.

Since in this risk model the process $\{\widetilde{U}(t)\}_{t\geq 0}$ is determined by six parameters $\widetilde{\lambda}$, $\widetilde{p}$, $\widetilde{q}$, $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$, six equalities are required to determine these parameters. Therefore, we need to compute the first six moments of $\widetilde{U}(t)$ described in [17].

Theorem 3.4. *For the risk model* (3.5)*, let $M_Y(s)$ and $M_Z(s)$ be the moment generating functions of the random variables Y_i and Z_i, respectively. Then, for any s in the domain of $M_{U(t)}$, we have*

$$M_{U(t)}(s) = \exp\left\{su + t\lambda\big(M(s) - 1 - p - q\big)\right\},$$

$$M'_{U(t)}(s) = M_{U(t)}(s)\big(u + t\lambda M'(s)\big),$$

$$M''_{U(t)}(s) = M_{U(t)}(s)\Big(\big(u + t\lambda M'(s)\big)^2 + t\lambda M''(s)\Big),$$

$$M'''_{U(t)}(s) = M_{U(t)}(s)\Big(\big(u + t\lambda M'(s)\big)^3 + t\lambda M'''(s) + 3t\lambda\big(u + t\lambda M'(s)\big)M''(s)\Big),$$

$$\begin{aligned} M^{(4)}_{U(t)}(s) = M_{U(t)}(s)\Big(&\big(u + t\lambda M'(s)\big)^4 + 6t\lambda M''(s)\big(u + t\lambda M'(s)\big)^2 \\ &+4t\lambda M'''(s)\big(u + t\lambda M'(s)\big) + 3t^2\lambda^2\left[M''(s)\right]^2 + t\lambda M^{(4)}(s)\Big), \end{aligned}$$

$$\begin{aligned} M^{(5)}_{U(t)}(s) = M_{U(t)}(s)\Big(&\big(u + t\lambda M'(s)\big)^5 + 10t\lambda M''(s)\big(u + t\lambda M'(s)\big)^3 \\ &+10t\lambda M'''(s)\big(u + t\lambda M'(s)\big)^2 + 5t\lambda M^{(4)}(s)\big(u + t\lambda M'(s)\big) \\ &+15t^2\lambda^2\left[M''(s)\right]^2\big(u+t\lambda M'(s)]\big)+10t^2\lambda^2 M''(s)M'''(s)+t\lambda M^{(5)}(s)\Big), \end{aligned}$$

$$M^{(6)}_{U(t)}(s) = M_{U(t)}(s)\Big(\big(u + t\lambda M'(s)\big)^6 + 15t\lambda M''(s)\big(u + t\lambda M'(s)\big)^4$$

$$
\begin{aligned}
&+20t\lambda M'''(s)\big(u+t\lambda M'(s)\big)^3+15t\lambda M^{(4)}(s)\big(u+t\lambda M'(s)\big)^2\\
&+45t^2\lambda^2\left[M''(s)\right]^2\big(u+t\lambda M'(s)\big)^2+6t\lambda M^{(5)}(s)\big(u+t\lambda M'(s)\big)\\
&+60t^2\lambda^2\big(u+t\lambda M'(s)\big)M''(s)M'''(s)+t\lambda M^{(6)}(s)\\
&+15t^2\lambda^2M''(s)M^{(4)}(s)+10t^2\lambda^2\left[M'''(s)\right]^2+15t^3\lambda^3\left[M''(s)\right]^3\Bigg),
\end{aligned}
$$

where $M(s)=e^{sc}+pM_Y(-s)+qM_Z(-s)$.

Proof. By the formula for the moment generating function of $S(t)$ in Lemma 3.1, we have

$$
\begin{aligned}
M_{U(t)}(s)&=E[e^{s(u+S(t))}]\\
&=\exp\left\{su+t\lambda\left[(e^{sc}-1)+p\left(M_Y(-s)-1\right)+q\left(M_Z(-s)-1\right)\right]\right\}\\
&=\exp\left\{su+t\lambda\big(M(s)-1-p-q\big)\right\}.
\end{aligned}
$$

Differentiating with respect to s on both sides of the equation, we have that

$$
\begin{aligned}
M'_{U(t)}(s)&=\exp\left\{su+t\lambda\big(M(s)-1-p-q\big)\right\}\cdot\big(u+t\lambda M'(s)\big)\\
&=M_{U(t)}(s)\big(u+t\lambda M'(s)\big).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
M''_{U(t)}(s)&=M_{U(t)}(s)t\lambda M''(s)+M'_{U(t)}(s)\big(u+t\lambda M'(s)\big)\\
&=M_{U(t)}(s)\Big(\big(u+t\lambda M'(s)\big)^2+t\lambda M''(s)\Big).
\end{aligned}
$$

Straightforwardly, we can calculate $M'''_{U(t)}(s), M^{(4)}_{U(t)}(s), M^{(5)}_{U(t)}(s)$ and $M^{(6)}_{U(t)}(s)$ and obtain the following results.

$$
\begin{aligned}
M'''_{U(t)}(s)&=M_{U(t)}(s)\Big(\big(u+t\lambda M'(s)\big)^3+t\lambda M'''(s)+3t\lambda\big(u+t\lambda M'(s)\big)M''(s)\Big),\\
M^{(4)}_{U(t)}(s)&=M_{U(t)}(s)\Big(\big(u+t\lambda M'(s)\big)^4+6t\lambda M''(s)\big(u+t\lambda M'(s)\big)^2\\
&\qquad+4t\lambda M'''(s)\big(u+t\lambda M'(s)\big)+3t^2\lambda^2\left[M''(s)\right]^2+t\lambda M^{(4)}(s)\Big),\\
M^{(5)}_{U(t)}(s)&=M_{U(t)}(s)\Big(\big(u+t\lambda M'(s)\big)^5+10t\lambda M''(s)\big(u+t\lambda M'(s)\big)^3\\
&\qquad+10t\lambda M'''(s)\big(u+t\lambda M'(s)\big)^2+5t\lambda M^{(4)}(s)\big(u+t\lambda M'(s)\big)
\end{aligned}
$$

$$+15t^2\lambda^2\left[M''(s)\right]^2\left(u+t\lambda M'(s)]\right)+10t^2\lambda^2 M''(s)M'''(s)+t\lambda M^{(5)}(s)\Big),$$

$$\begin{aligned}M^{(6)}_{U(t)}(s) = M_{U(t)}(s)\Big(&\big(u+t\lambda M'(s)\big)^6 + 15t\lambda M''(s)\big(u+t\lambda M'(s)\big)^4\\ &+20t\lambda M'''(s)\big(u+t\lambda M'(s)\big)^3 + 15t\lambda M^{(4)}(s)\big(u+t\lambda M'(s)\big)^2\\ &+45t^2\lambda^2\left[M''(s)\right]^2\big(u+t\lambda M'(s)\big)^2 + 6t\lambda M^{(5)}(s)\big(u+t\lambda M'(s)\big)\\ &+60t^2\lambda^2\big(u+t\lambda M'(s)\big)M''(s)M'''(s) + t\lambda M^{(6)}(s)\\ &+15t^2\lambda^2 M''(s)M^{(4)}(s)+10t^2\lambda^2\left[M'''(s)\right]^2+15t^3\lambda^3\left[M''(s)\right]^3\Big). \quad \square\end{aligned}$$

For $k \in \{1, 2, \ldots, 6\}$, since $M^k_{U(t)}(s)$ is in the form of $M^k(s)$, we can find the equation for $M^k(s)$ for $k \in \{1, 2, \ldots, 6\}$ from the Remark 3.2.

Remark 3.2. For $n \in \mathbb{N}$, the n^{th} derivative of the function $M(s) = e^{sc} + pM_Y(-s) + qM_Z(-s)$ is given by

$$M^{(n)}(s) = c^n e^{sc} + (-1)^n pM_Y^{(n)}(-s) + (-1)^n qM_Z^{(n)}(-s)$$

Corollary 3.3. *For the risk model (3.5), we assume that Y_i and Z_i have finite first six moments. Then, for all $t \geq 0$, we have*

$$\begin{aligned}E[U(t)] &= u + t\lambda\big(c - pE[Y] - qE[Z]\big),\\ E[U^2(t)] &= (E[U(t)])^2 + t\lambda\left(c^2 + pE[Y^2] + qE[Z^2]\right),\\ E[U^3(t)] &= (E[U(t)])^3 + t\lambda\left(c^3 - pE[Y^3] - qE[Z^3]\right)\\ &\quad + 3t\lambda E[U(t)]\left(c^2 + pE[Y^2] + qE[Z^2]\right),\\ E[U^4(t)] &= (E[U(t)])^4 + 6t\lambda\left(c^2 + pE[Y^2] + qE[Z^2]\right)(E[U(t)])^2\\ &\quad + 4t\lambda\left(c^3 - pE[Y^3] - qE[Z^3]\right)E[U(t)]\\ &\quad + 3t^2\lambda^2\left(c^2 + pE[Y^2] + qE[Z^2]\right)^2\\ &\quad + t\lambda\left(c^4 + pE[Y^4] + qE[Z^4]\right),\\ E[U^5(t)] &= (E[U(t)])^5 + 10t\lambda\left(c^2 + pE[Y^2] + qE[Z^2]\right)(E[U(t)])^3\\ &\quad + 10t\lambda\left(c^3 - pE[Y^3] - qE[Z^3]\right)(E[U(t)])^2\\ &\quad + 5t\lambda\left(c^4 + pE[Y^4] + qE[Z^4]\right)E[U(t)]\\ &\quad + 15t^2\lambda^2\left(c^2 + pE[Y^2] + qE[Z^2]\right)^2 E[U(t)]\end{aligned}$$

$$
\begin{aligned}
&+ 10t^2\lambda^2 \left(c^2 + pE[Y^2] + qE[Z^2]\right)\left(c^3 - pE[Y^3] - qE[Z^3]\right) \\
&+ t\lambda \left(c^5 - pE[Y^5] - qE[Z^5]\right),
\end{aligned}
$$

$$
\begin{aligned}
E[U^6(t)] = &\,(E[U(t)])^6 + 15t\lambda \left(c^2 + pE[Y^2] + qE[Z^2]\right)(E[U(t)])^4 \\
&+ 20t\lambda \left(c^3 - pE[Y^3] - qE[Z^3]\right)(E[U(t)])^3 \\
&+ 15t\lambda \left(c^4 + pE[Y^4] + qE[Z^4]\right)(E[U(t)])^2 \\
&+ 45t^2\lambda^2 \left(c^2 + pE[Y^2] + qE[Z^2]\right)^2 (E[U(t)])^2 \\
&+ 6t\lambda \left(c^5 - pE[Y^5] - qE[Z^5]\right) E[U(t)] \\
&+ 60t^2\lambda^2 E[U(t)] \left(c^2 + pE[Y^2] + qE[Z^2]\right)\left(c^3 - pE[Y^3] - qE[Z^3]\right) \\
&+ t\lambda \left(c^6 + pE[Y^6] + qE[Z^6]\right) \\
&+ 15t^2\lambda^2 \left(c^2 + pE[Y^2] + qE[Z^2]\right)\left[c^4 + pE[Y^4] + qE[Z^4]\right] \\
&+ 10t^2\lambda^2 \left(c^3 - pE[Y^3] - qE[Z^3]\right)^2 \\
&+ 15t^3\lambda^3 \left(c^2 + pE[Y^2] + qE[Z^2]\right)^3.
\end{aligned}
$$

Proof. Since $E[U^n(t)] = M^{(n)}_{U(t)}(0)$ and $M^{(n)}(0) = c^n + (-1)^n pE[Y^n] + (-1)^n qE[Z^n]$ for all $n \in \mathbb{N}$, substituting $s = 0$ into the formulas in Theorem (3.4) yields the desired results. □

For the risk model (3.38) where $\widetilde{Y}_i$ and $\widetilde{Z}_i$ have exponential distributions with parameters $\widetilde{a}$ and $\widetilde{b}$, respectively, let $\widetilde{A} = \dfrac{1}{\widetilde{a}}$ and $\widetilde{B} = \dfrac{1}{\widetilde{b}}$ so that the mean of $\widetilde{Y}_i$ and $\widetilde{Z}_i$ are $\widetilde{A}$ and $\widetilde{B}$, respectively. We will deal with parameters $\widetilde{A}$ and $\widetilde{B}$ instead of $\widetilde{a}$ and $\widetilde{b}$ for the sake of simplicity of the final formula.

Theorem 3.5. *We can approximate the process $\{U(t)\}_{t\geq 0}$ in the risk model* (3.5) *by a process $\{\widetilde{U}(t)\}_{t\geq 0}$ in the risk model* (3.38) *with parameters $\widetilde{\lambda}$, $\widetilde{p}$, $\widetilde{q}$, $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{c}$ by matching the first six moments, i.e., $E[U(t)^k] = E[\widetilde{U}(t)^k]$ for $k = 1, 2, \ldots, 6$. The desired parameters $\widetilde{\lambda}$, $\widetilde{p}$, $\widetilde{q}$, $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{c}$ can be solved from the system of equations:*

$$
\widetilde{\lambda} = \frac{\gamma_3 + 6\gamma_1\widetilde{A}\widetilde{B} + 3\gamma_2\big(\widetilde{A} + \widetilde{B}\big)}{\widetilde{c}\Big(\widetilde{c}^2 + 6\widetilde{A}\widetilde{B} + 3\widetilde{c}\big(\widetilde{A} + \widetilde{B}\big)\Big)},
$$

$$\widetilde{q} = \frac{\gamma_2 + 2\gamma_1\widetilde{A} - \widetilde{c}(\widetilde{c} + 2\widetilde{A})\widetilde{\lambda}}{2\widetilde{B}(\widetilde{B} - \widetilde{A})\widetilde{\lambda}},$$

$$\widetilde{p} = \frac{\widetilde{\lambda}\widetilde{c} - \gamma_1 - \widetilde{\lambda}\widetilde{q}\widetilde{B}}{\widetilde{\lambda}\widetilde{A}},$$

and

$$\begin{aligned} F1 \cdot \gamma_4 &= E11 \cdot \gamma_1 + E12 \cdot \gamma_2 + E13 \cdot \gamma_3, \\ F2 \cdot \gamma_5 &= E21 \cdot \gamma_2 + E22 \cdot \gamma_3 + E23 \cdot \gamma_4, \\ F3 \cdot \gamma_6 &= E31 \cdot \gamma_3 + E32 \cdot \gamma_4 + E33 \cdot \gamma_5, \end{aligned}$$

where

$$\begin{aligned} F1 &= \widetilde{c}^2 + 6\widetilde{A}\widetilde{B} + 3\widetilde{c}(\widetilde{A} + \widetilde{B}), \\ E11 &= 6\widetilde{c}\widetilde{A}\widetilde{B}\left(\widetilde{c}^2 + 12\widetilde{A}\widetilde{B} + 4\widetilde{c}(\widetilde{A} + \widetilde{B})\right), \\ E12 &= 3\left(-24\widetilde{A}^2\widetilde{B}^2 + \widetilde{c}^3(\widetilde{A} + \widetilde{B}) + 4\widetilde{c}^2(\widetilde{A}^2 + \widetilde{A}\widetilde{B} + \widetilde{B}^2)\right), \\ E13 &= \widetilde{c}^3 - 24\widetilde{A}\widetilde{B}(\widetilde{A} + \widetilde{B}) - 12\widetilde{c}(\widetilde{A}^2 + \widetilde{A}\widetilde{B} + \widetilde{B}^2), \end{aligned}$$

$$\begin{aligned} F2 &= \widetilde{c}^2 + 12\widetilde{A}\widetilde{B} + 4\widetilde{c}(\widetilde{A} + \widetilde{B}), \\ E21 &= 12\widetilde{c}\widetilde{A}\widetilde{B}\left(\widetilde{c}^2 + 20\widetilde{A}\widetilde{B} + 5\widetilde{c}(\widetilde{A} + \widetilde{B})\right), \\ E22 &= 4\left(-60\widetilde{A}^2\widetilde{B}^2 + \widetilde{c}^3(\widetilde{A} + \widetilde{B}) + 5\widetilde{c}^2(\widetilde{A}^2 + \widetilde{A}\widetilde{B} + \widetilde{B}^2)\right), \\ E23 &= \widetilde{c}^3 - 60\widetilde{A}\widetilde{B}(\widetilde{A} + \widetilde{B}) - 20\widetilde{c}(\widetilde{A}^2 + \widetilde{A}\widetilde{B} + \widetilde{B}^2), \end{aligned}$$

$$\begin{aligned} F3 &= \widetilde{c}^2 + 20\widetilde{A}\widetilde{B} + 5\widetilde{c}(\widetilde{A} + \widetilde{B}), \\ E31 &= 20\widetilde{c}\widetilde{A}\widetilde{B}\left(\widetilde{c}^2 + 30\widetilde{A}\widetilde{B} + 6\widetilde{c}(\widetilde{A} + \widetilde{B})\right), \\ E32 &= 5\left(-120\widetilde{A}^2\widetilde{B}^2 + \widetilde{c}^3(\widetilde{A} + \widetilde{B}) + 6\widetilde{c}^2(\widetilde{A}^2 + \widetilde{A}\widetilde{B} + \widetilde{B}^2)\right), \\ E33 &= \widetilde{c}^3 - 120\widetilde{A}\widetilde{B}(\widetilde{A} + \widetilde{B}) - 30\widetilde{c}(\widetilde{A}^2 + \widetilde{A}\widetilde{B} + \widetilde{B}^2), \end{aligned}$$

and

$$\begin{aligned} \gamma_1 &= \lambda\left(c - pE[Y] - qE[Z]\right), \\ \gamma_2 &= \lambda\left(c^2 + pE[Y^2] + qE[Z^2]\right), \\ \gamma_3 &= \lambda\left(c^3 - pE[Y^3] - qE[Z^3]\right), \\ \gamma_4 &= \lambda\left(c^4 + pE[Y^4] + qE[Z^4]\right), \\ \gamma_5 &= \lambda\left(c^5 - pE[Y^5] - qE[Z^5]\right), \\ \gamma_6 &= \lambda\left(c^6 + pE[Y^6] + qE[Z^6]\right). \end{aligned}$$

Proof. Taking the k-th moment of the random variable that are exponentially distributed into the equation $E[U(t)^k] = E[\widetilde{U}(t)^k]$, we have the system of equations

$$\gamma_1 = \widetilde{\lambda}\widetilde{c} - \widetilde{\lambda}\widetilde{p}\widetilde{A} - \widetilde{\lambda}\widetilde{q}\widetilde{B}, \tag{3.39}$$

$$\gamma_2 = \widetilde{\lambda}\widetilde{c}^2 + 2\widetilde{\lambda}\widetilde{p}\widetilde{A}^2 + 2\widetilde{\lambda}\widetilde{q}\widetilde{B}^2, \tag{3.40}$$

$$\gamma_3 = \widetilde{\lambda}\widetilde{c}^3 - 6\widetilde{\lambda}\widetilde{p}\widetilde{A}^3 - 6\widetilde{\lambda}\widetilde{q}\widetilde{B}^3, \tag{3.41}$$

$$\gamma_4 = \widetilde{\lambda}\widetilde{c}^4 + 24\widetilde{\lambda}\widetilde{p}\widetilde{A}^4 + 24\widetilde{\lambda}\widetilde{q}\widetilde{B}^4, \tag{3.42}$$

$$\gamma_5 = \widetilde{\lambda}\widetilde{c}^5 - 120\widetilde{\lambda}\widetilde{p}\widetilde{A}^5 - 120\widetilde{\lambda}\widetilde{q}\widetilde{B}^5, \tag{3.43}$$

$$\gamma_6 = \widetilde{\lambda}\widetilde{c}^6 + 720\widetilde{\lambda}\widetilde{p}\widetilde{A}^6 + 720\widetilde{\lambda}\widetilde{q}\widetilde{B}^6. \tag{3.44}$$

Now, our aim is to find the constants $\widetilde{\lambda}$, $\widetilde{p}$, $\widetilde{q}$, $\widetilde{B}$, $\widetilde{A}$ and $\widetilde{c}$ from this system. From (3.39), we have $\widetilde{\lambda}\widetilde{p}\widetilde{A} = \widetilde{\lambda}\widetilde{c} - \gamma_1 - \widetilde{\lambda}\widetilde{q}\widetilde{B}$. Substituting this into (3.40)–(3.44), we get

$$\gamma_2 = -2\gamma_1\widetilde{A} + \widetilde{c}\left(\widetilde{c} + 2\widetilde{A}\right)\widetilde{\lambda} + 2\widetilde{B}\left(\widetilde{B} - \widetilde{A}\right)\widetilde{\lambda}\widetilde{q}, \tag{3.45}$$

$$\gamma_3 = 6\gamma_1\widetilde{A}^2 + \widetilde{c}\left(\widetilde{c}^2 - 6\widetilde{A}^2\right)\widetilde{\lambda} + 6\widetilde{B}\left(\widetilde{A}^2 - \widetilde{B}^2\right)\widetilde{\lambda}\widetilde{q}, \tag{3.46}$$

$$\gamma_4 = -24\gamma_1\widetilde{A}^3 + \widetilde{c}\left(\widetilde{c}^3 + 24\widetilde{A}^3\right)\widetilde{\lambda} + 24\widetilde{B}\left(\widetilde{B}^3 - \widetilde{A}^3\right)\widetilde{\lambda}\widetilde{q}, \tag{3.47}$$

$$\gamma_5 = 120\gamma_1\widetilde{A}^4 + \widetilde{c}\left(\widetilde{c}^4 - 120\widetilde{A}^4\right)\widetilde{\lambda} + 120\widetilde{B}\left(\widetilde{A}^4 - \widetilde{B}^4\right)\widetilde{\lambda}\widetilde{q}, \tag{3.48}$$

$$\gamma_6 = -720\gamma_1\widetilde{A}^5 + \widetilde{c}\left(\widetilde{c}^5 + 720\widetilde{A}^5\right)\widetilde{\lambda} + 720\widetilde{B}\left(\widetilde{B}^5 - \widetilde{A}^5\right)\widetilde{\lambda}\widetilde{q}. \tag{3.49}$$

Next, from (3.45) we have $2\widetilde{B}\left(\widetilde{B} - \widetilde{A}\right)\widetilde{\lambda}\widetilde{q} = \gamma_2 + 2\gamma_1\widetilde{A} - \widetilde{c}\left(\widetilde{c} + 2\widetilde{A}\right)\widetilde{\lambda}$. Substituting this into (3.46)–(3.49), we obtain

$$\gamma_3 = -6\gamma_1\widetilde{A}\widetilde{B} - 3\gamma_2\left(\widetilde{A} + \widetilde{B}\right) + \widetilde{c}\left(\widetilde{c}^2 + 6\widetilde{A}\widetilde{B} + 3\widetilde{c}\left(\widetilde{A} + \widetilde{B}\right)\right)\widetilde{\lambda}, \tag{3.50}$$

$$\begin{aligned}\gamma_4 = {} & 24\gamma_1\widetilde{A}\widetilde{B}\left(\widetilde{A} + \widetilde{B}\right) + 12\gamma_2\left(\widetilde{A}^2 + \widetilde{A}\widetilde{B} + \widetilde{B}^2\right) \\ & + \widetilde{c}\left(\widetilde{c}^3 - 24\widetilde{A}\widetilde{B}\left(\widetilde{A} + \widetilde{B}\right) - 12\widetilde{c}\left(\widetilde{A}^2 + \widetilde{A}\widetilde{B} + \widetilde{B}^2\right)\right)\widetilde{\lambda},\end{aligned} \tag{3.51}$$

$$\begin{aligned}\gamma_5 = \ & -120\gamma_1\widetilde{A}\widetilde{B}\left(\widetilde{A}^2+\widetilde{A}\widetilde{B}+\widetilde{B}^2\right)-60\gamma_2\left(\widetilde{A}+\widetilde{B}\right)\left(\widetilde{A}^2+\widetilde{B}^2\right)\\ & +\widetilde{c}\left(\widetilde{c}^4+120\widetilde{A}\widetilde{B}\left(\widetilde{A}^2+\widetilde{A}\widetilde{B}+\widetilde{B}^2\right)+60\widetilde{c}\left(\widetilde{A}^3+\widetilde{A}^2\widetilde{B}+\widetilde{A}\widetilde{B}^2+\widetilde{B}^3\right)\right)\widetilde{\lambda},\end{aligned} \tag{3.52}$$

$$\begin{aligned}\gamma_6 = \ & 720\gamma_1\widetilde{A}\widetilde{B}\left(\widetilde{A}+\widetilde{B}\right)\left(\widetilde{A}^2+\widetilde{B}^2\right)+360\gamma_2\left(\widetilde{A}^4+\widetilde{A}^3\widetilde{B}+\widetilde{A}^2\widetilde{B}^2+\widetilde{A}\widetilde{B}^3+\widetilde{B}^4\right)\\ & +\widetilde{c}\left[\widetilde{c}^5-720\widetilde{A}\widetilde{B}\left(\widetilde{A}^3+\widetilde{A}^2\widetilde{B}+\widetilde{A}\widetilde{B}^2+\widetilde{B}^3\right)-360\widetilde{c}\left(\widetilde{A}^4+\widetilde{A}^3\widetilde{B}+\widetilde{A}^2\widetilde{B}^2+\widetilde{A}\widetilde{B}^3+\widetilde{B}^4\right)\right].\end{aligned} \tag{3.53}$$

Next, from (3.50) we have $\widetilde{c}\left(\widetilde{c}^2+6\widetilde{A}\widetilde{B}+3\widetilde{c}\left(\widetilde{A}+\widetilde{B}\right)\right)\widetilde{\lambda}=\gamma_3+6\gamma_1\widetilde{A}\widetilde{B}+3\gamma_2\left(\widetilde{A}+\widetilde{B}\right)$. Substituting this into (3.51)–(3.53), we obtain

$$\begin{aligned}F1\gamma_4 = \ & 6\widetilde{c}\widetilde{A}\widetilde{B}\left(\widetilde{c}^2+12\widetilde{A}\widetilde{B}+4\widetilde{c}\left(\widetilde{A}+\widetilde{B}\right)\right)\gamma_1\\ & +3\left(-24\widetilde{A}^2\widetilde{B}^2+\widetilde{c}^3\left(\widetilde{A}+\widetilde{B}\right)+4\widetilde{c}^2\left(\widetilde{A}^2+\widetilde{A}\widetilde{B}+\widetilde{B}^2\right)\right)\gamma_2\\ & +\left(-24\widetilde{A}\widetilde{B}\left(\widetilde{A}+\widetilde{B}\right)+\widetilde{c}^3-12\widetilde{c}\left(\widetilde{A}^2+\widetilde{A}\widetilde{B}+\widetilde{B}^2\right)\right)\gamma_3,\end{aligned} \tag{3.54}$$

$$\begin{aligned}F1\gamma_5 = \ & 6\widetilde{c}\widetilde{A}\widetilde{B}\left(\widetilde{c}^3-60\widetilde{A}\widetilde{B}\left(\widetilde{A}+\widetilde{B}\right)-20\widetilde{c}\left(\widetilde{B}^2+\widetilde{A}\widetilde{B}+\widetilde{B}^2\right)\right)\gamma_1\\ & +3\left(\widetilde{A}+\widetilde{B}\right)\left(\widetilde{c}^4-20\widetilde{c}^2\widetilde{A}^2-20\widetilde{c}^2\widetilde{B}^2+120\widetilde{A}^2\widetilde{B}^2\right)\gamma_2\\ & +\left(\widetilde{c}^4+120\widetilde{A}\widetilde{B}\left(\widetilde{A}^2+\widetilde{A}\widetilde{B}+\widetilde{B}^2\right)+60\widetilde{c}\left(\widetilde{A}^3+\widetilde{A}^2\widetilde{B}+\widetilde{A}\widetilde{B}^2+\widetilde{B}^3\right)\right)\gamma_3,\end{aligned} \tag{3.55}$$

$$\begin{aligned}F1\gamma_6 = \ & 6\widetilde{c}\widetilde{A}\widetilde{B}\left(\widetilde{c}^4+360\widetilde{A}\widetilde{B}\left(\widetilde{B}^2+\widetilde{A}\widetilde{B}+\widetilde{B}^2\right)+120\widetilde{c}\left(\widetilde{A}^3+\widetilde{A}^2\widetilde{B}+\widetilde{A}\widetilde{B}^2+\widetilde{B}^3\right)\right)\gamma_1\\ & +3\left(\widetilde{c}^5\left(\widetilde{A}+\widetilde{B}\right)-720\widetilde{A}^2\widetilde{B}^2\left(\widetilde{A}^2+\widetilde{A}\widetilde{B}+\widetilde{B}^2\right)\right.\\ & \qquad \left.+120\widetilde{c}^2\left(\widetilde{A}^4+\widetilde{A}^3\widetilde{B}+\widetilde{A}^2\widetilde{B}^2+\widetilde{A}\widetilde{B}^3+\widetilde{B}^4\right)\right)\gamma_2\\ & +\left(\widetilde{c}^5+720\widetilde{A}\widetilde{B}\left(\widetilde{A}^3+\widetilde{A}^2\widetilde{B}+\widetilde{A}\widetilde{B}^2+\widetilde{B}^3\right)\right.\\ & \qquad \left.-360\widetilde{c}\left(\widetilde{A}^4+\widetilde{A}^3\widetilde{B}+\widetilde{A}^2\widetilde{B}^2+\widetilde{A}\widetilde{B}^3+\widetilde{B}^4\right)\right)\gamma_3.\end{aligned} \tag{3.56}$$

Multiplying (3.54) by $\dfrac{-\left(\widetilde{c}^3-60\widetilde{A}\widetilde{B}\left(\widetilde{A}+\widetilde{B}\right)-20\widetilde{c}\left(\widetilde{A}^2+\widetilde{A}\widetilde{B}+\widetilde{B}^2\right)\right)}{F2}$ and adding (3.55), we get

$$\begin{aligned} F2\gamma_5 = \quad & 12\widetilde{c}\widetilde{A}\widetilde{B}\left(\widetilde{c}^2 + 20\widetilde{A}\widetilde{B} + 5\widetilde{c}\left(\widetilde{A}+\widetilde{B}\right)\right)\gamma_2 \\ & +4\left(-60\widetilde{A}^2\widetilde{B}^2 + \widetilde{c}^3\left(\widetilde{A}+\widetilde{B}\right) + 5\widetilde{c}^2\left(\widetilde{A}^2+\widetilde{A}\widetilde{B}+\widetilde{B}^2\right)\right)\gamma_3 \\ & +\left(\widetilde{c}^3 - 60\widetilde{A}\widetilde{B}\left(\widetilde{A}+\widetilde{B}\right) - 20\widetilde{c}\left(\widetilde{A}^2+\widetilde{A}\widetilde{B}+\widetilde{B}^2\right)\right)\gamma_4. \end{aligned} \tag{3.57}$$

Multiplying (3.54) by $\dfrac{-\left(\widetilde{c}^4 + 360\widetilde{A}\widetilde{B}\left(\widetilde{A}^2+\widetilde{A}\widetilde{B}+\widetilde{B}^2\right) + 120\widetilde{c}\left(\widetilde{A}^3+\widetilde{A}^2\widetilde{B}+\widetilde{A}\widetilde{B}^2+\widetilde{B}^3\right)\right)}{F2}$ and adding (3.56), we get

$$\begin{aligned} F2\gamma_6 = \quad & 12\widetilde{c}\widetilde{A}\widetilde{B}\left(\widetilde{c}^3 - 120\widetilde{A}\widetilde{B}\left(\widetilde{A}+\widetilde{B}\right) - 30\widetilde{c}\left(\widetilde{A}^2+\widetilde{A}\widetilde{B}+\widetilde{B}^2\right)\right)\gamma_2 \\ & +4\left(\widetilde{c}^4\left(\widetilde{A}+\widetilde{B}\right) + 360\widetilde{A}^2\widetilde{B}^2\left(\widetilde{A}+\widetilde{B}\right) - 30\widetilde{c}^2\left(\widetilde{A}^3+\widetilde{A}^2\widetilde{B}+\widetilde{A}\widetilde{B}^2+\widetilde{B}^3\right)\right)\gamma_3 \\ & +\left(\widetilde{c}^4 + 360\widetilde{A}\widetilde{B}\left(\widetilde{A}^2+\widetilde{A}\widetilde{B}+\widetilde{B}^2\right) + 120\widetilde{c}\left(\widetilde{A}^3+\widetilde{A}^2\widetilde{B}+\widetilde{A}\widetilde{B}^2+\widetilde{B}^3\right)\right)\gamma_4. \end{aligned} \tag{3.58}$$

Multiplying (3.57) by $\dfrac{-\left(\widetilde{c}^3 - 120\widetilde{A}\widetilde{B}\left(\widetilde{A}+\widetilde{B}\right) - 30\widetilde{c}\left(\widetilde{A}^2+\widetilde{A}\widetilde{B}+\widetilde{B}^2\right)\right)}{F3}$ and adding (3.58), we get

$$\begin{aligned} F3\gamma_6 = \quad & 20\widetilde{c}\widetilde{A}\widetilde{B}\left(\widetilde{c}^2 + 30\widetilde{A}\widetilde{B} + 6\widetilde{c}\left(\widetilde{A}+\widetilde{B}\right)\right)\gamma_3 \\ & +5\left(-120\widetilde{A}^2\widetilde{B}^2 + \widetilde{c}^3\left(\widetilde{A}+\widetilde{B}\right) + 6\widetilde{c}^2\left(\widetilde{A}^2+\widetilde{A}\widetilde{B}+\widetilde{B}^2\right)\right)\gamma_4 \\ & +\left(\widetilde{c}^3 - 120\widetilde{A}\widetilde{B}\left(\widetilde{A}+\widetilde{B}\right) - 30\widetilde{c}\left(\widetilde{A}^2+\widetilde{A}\widetilde{B}+\widetilde{B}^2\right)\right)\gamma_5. \end{aligned} \tag{3.59}$$

Hence, we get the desired system of equations. □

Theorem 3.6. ***The De-Vylder approximation***

For the risk model (3.5) *under assumptions that* Y_i *and* Z_i *have finite sixth moments and that the net profit condition* (3.7) *holds, the De-Vylder approximation of ruin probability* $\psi_{De}(u)$ *is given by*

$$\psi_{De}(u) = C_1 e^{r_1 u} + C_2 e^{r_2 u} \quad \text{for all } u \geq 0, \tag{3.60}$$

where

$$r_2 = \frac{A+\sqrt{D}}{B} \quad , \quad r_1 = \frac{A-\sqrt{D}}{B},$$

$$A = -\left[\widetilde{A} + \widetilde{B} - \frac{\widetilde{A}\widetilde{B}(\widetilde{p}+\widetilde{q})}{\widetilde{c}}\right],$$

$$B = 2\widetilde{A}\widetilde{B},$$

$$D = \left[\widetilde{A} + \widetilde{B} - \frac{\widetilde{A}\widetilde{B}(\widetilde{p}+\widetilde{q})}{\widetilde{c}}\right]^2 - 4\widetilde{A}\widetilde{B}\left[1 - \frac{\widetilde{p}\widetilde{A}}{\widetilde{c}} - \frac{\widetilde{q}\widetilde{B}}{\widetilde{c}}\right],$$

$$C_1 = \frac{C_{11}}{C_D} \ , \ C_2 = \frac{C_{21}}{C_D},$$

which

$$\begin{aligned}
C_{11} &= -\frac{\widetilde{c}\widetilde{p}}{\widetilde{A}\widetilde{B}} + \frac{\widetilde{p}^2}{\widetilde{B}} - \frac{c\widetilde{q}}{\widetilde{A}\widetilde{B}} + \frac{\widetilde{p}\widetilde{q}}{\widetilde{A}} + \frac{\widetilde{p}\widetilde{q}}{\widetilde{B}} + \frac{\widetilde{q}^2}{\widetilde{A}} - \widetilde{c}\left(\frac{\widetilde{p}}{\widetilde{B}} + \frac{\widetilde{q}}{\widetilde{A}}\right) r_2, \\
C_{21} &= \frac{\widetilde{c}\widetilde{p}}{\widetilde{A}\widetilde{B}} - \frac{\widetilde{p}^2}{\widetilde{B}} + \frac{c\widetilde{q}}{\widetilde{A}\widetilde{B}} - \frac{\widetilde{p}\widetilde{q}}{\widetilde{A}} - \frac{\widetilde{p}\widetilde{q}}{\widetilde{B}} - \frac{\widetilde{q}^2}{\widetilde{A}} + \widetilde{c}\left(\frac{\widetilde{p}}{\widetilde{B}} - \frac{\widetilde{q}}{\widetilde{A}}\right) r_1, \\
C_D &= \frac{\widetilde{c}^2(r_1 - r_2)}{\widetilde{A}\widetilde{B}},
\end{aligned}$$

and the constants $\widetilde{\lambda}$, $\widetilde{p}$, $\widetilde{q}$, $\widetilde{A}$, $\widetilde{B}$ *and* $\widetilde{c}$ *are obtained from solving the system of equations stated in Theorem 3.5 which have the following values:*

$$\widetilde{\lambda} = \frac{\gamma_3 + 6\gamma_1\widetilde{A}\widetilde{B} + 3\gamma_2(\widetilde{A}+\widetilde{B})}{\widetilde{c}\left(\widetilde{c}^2 + 6\widetilde{A}\widetilde{B} + 3\widetilde{c}(\widetilde{A}+\widetilde{B})\right)},$$

$$\widetilde{q} = \frac{\gamma_2 + 2\gamma_1\widetilde{A} - \widetilde{c}(\widetilde{c}+2\widetilde{A})\widetilde{\lambda}}{2\widetilde{B}(\widetilde{B}-\widetilde{A})\widetilde{\lambda}},$$

$$\widetilde{p} = \frac{\widetilde{\lambda}\widetilde{c} - \gamma_1 - \widetilde{\lambda}\widetilde{q}\widetilde{B}}{\widetilde{\lambda}\widetilde{A}},$$

and $\widetilde{A}$, $\widetilde{B}$ *and* $\widetilde{c}$ *are obtained from solving the system of equations*

$$\begin{aligned}
F1 \cdot \gamma_4 &= E11 \cdot \gamma_1 + E12 \cdot \gamma_2 + E13 \cdot \gamma_3, \\
F2 \cdot \gamma_5 &= E21 \cdot \gamma_2 + E22 \cdot \gamma_3 + E23 \cdot \gamma_4, \\
F3 \cdot \gamma_6 &= E31 \cdot \gamma_3 + E32 \cdot \gamma_4 + E33 \cdot \gamma_5,
\end{aligned}$$

where

$$
\begin{aligned}
F1 &= \widetilde{c}^2 + 6\widetilde{A}\widetilde{B} + 3\widetilde{c}\big(\widetilde{A}+\widetilde{B}\big),\\
E11 &= 6\widetilde{c}\widetilde{A}\widetilde{B}\left(\widetilde{c}^2 + 12\widetilde{A}\widetilde{B} + 4\widetilde{c}\big(\widetilde{A}+\widetilde{B}\big)\right),\\
E12 &= 3\left(-24\widetilde{A}^2\widetilde{B}^2 + \widetilde{c}^3\big(\widetilde{A}+\widetilde{B}\big) + 4\widetilde{c}^2\big(\widetilde{A}^2+\widetilde{A}\widetilde{B}+\widetilde{B}^2\big)\right),\\
E13 &= \widetilde{c}^3 - 24\widetilde{A}\widetilde{B}\big(\widetilde{A}+\widetilde{B}\big) - 12\widetilde{c}\big(\widetilde{A}^2+\widetilde{A}\widetilde{B}+\widetilde{B}^2\big),
\end{aligned}
$$

$$
\begin{aligned}
F2 &= \widetilde{c}^2 + 12\widetilde{A}\widetilde{B} + 4\widetilde{c}\big(\widetilde{A}+\widetilde{B}\big),\\
E21 &= 12\widetilde{c}\widetilde{A}\widetilde{B}\left(\widetilde{c}^2 + 20\widetilde{A}\widetilde{B} + 5\widetilde{c}\big(\widetilde{A}+\widetilde{B}\big)\right),\\
E22 &= 4\left(-60\widetilde{A}^2\widetilde{B}^2 + \widetilde{c}^3\big(\widetilde{A}+\widetilde{B}\big) + 5\widetilde{c}^2\big(\widetilde{A}^2+\widetilde{A}\widetilde{B}+\widetilde{B}^2\big)\right),\\
E23 &= \widetilde{c}^3 - 60\widetilde{A}\widetilde{B}\big(\widetilde{A}+\widetilde{B}\big) - 20\widetilde{c}\big(\widetilde{A}^2+\widetilde{A}\widetilde{B}+\widetilde{B}^2\big),
\end{aligned}
$$

$$
\begin{aligned}
F3 &= \widetilde{c}^2 + 20\widetilde{A}\widetilde{B} + 5\widetilde{c}\big(\widetilde{A}+\widetilde{B}\big),\\
E31 &= 20\widetilde{c}\widetilde{A}\widetilde{B}\left(\widetilde{c}^2 + 30\widetilde{A}\widetilde{B} + 6\widetilde{c}\big(\widetilde{A}+\widetilde{B}\big)\right),\\
E32 &= 5\left(-120\widetilde{A}^2\widetilde{B}^2 + \widetilde{c}^3\big(\widetilde{A}+\widetilde{B}\big) + 6\widetilde{c}^2\big(\widetilde{A}^2+\widetilde{A}\widetilde{B}+\widetilde{B}^2\big)\right),\\
E33 &= \widetilde{c}^3 - 120\widetilde{A}\widetilde{B}\big(\widetilde{A}+\widetilde{B}\big) - 30\widetilde{c}\big(\widetilde{A}^2+\widetilde{A}\widetilde{B}+\widetilde{B}^2\big),
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma_1 &= \lambda\left(c - pE[Y] - qE[Z]\right),\\
\gamma_2 &= \lambda\left(c^2 + pE[Y^2] + qE[Z^2]\right),\\
\gamma_3 &= \lambda\left(c^3 - pE[Y^3] - qE[Z^3]\right),\\
\gamma_4 &= \lambda\left(c^4 + pE[Y^4] + qE[Z^4]\right),\\
\gamma_5 &= \lambda\left(c^5 - pE[Y^5] - qE[Z^5]\right),\\
\gamma_6 &= \lambda\left(c^6 + pE[Y^6] + qE[Z^6]\right).
\end{aligned}
$$

To calculate the approximated ruin probability using the De-Vylder approximation described in (3.60), we can use the MATLAB commands "`solve`" for computation.

3.4 Lundberg's inequality

In this section, we will study the martingale and stopping time. This will allow us to find the adjustment coefficient equation, Lundberg's inequality for the ruin probability, which it can be used to create as Lemma, Theorem and Corollary.

Theorem 3.7. *For the profits process* $\{S(t); t \geq 0\}$,

$$E[e^{-rS(t)}] = e^{tg(r)}, \tag{3.61}$$

where

$$g(r) = -\lambda\left[1 - e^{-rc}\right] - \lambda p\left[1 - M_Y(r)\right] - \lambda q\left[1 - M_Z(r)\right]. \tag{3.62}$$

Proof. Since $Y_i, Z_i, N(t), N(t,p)$ and $N(t,q)$ are mutually independent,

$$\begin{aligned} E\left[e^{-rS(t)}\right] &= E\left[\exp\{-rcN(t)\} \cdot \exp\{r\sum_{i=1}^{N(t,p)} Y_i\} \cdot \exp\{r\sum_{i=1}^{N(t,q)} Z_i\}\right] \\ &= E\left[\exp\{-rcN(t)\}\right] \cdot E\left[\exp\{r\sum_{i=1}^{N(t,p)} Y_i\}\right] \cdot E\left[\exp\{r\sum_{i=1}^{N(t,q)} Z_i\}\right]. \end{aligned}$$

By definition of MGF in Definition 2.24 and Theorem 2.22, we get

$$\begin{aligned} E\left[e^{-rS(t)}\right] &= M_{N(t)}(-rc) \cdot M_{\sum_{i=1}^{N(t,p)} Y_i}(r) \cdot M_{\sum_{i=1}^{N(t,q)} Z_i}(r) \\ &= e^{-\lambda t[1-e^{-rc}]} e^{-\lambda pt[1-M_Y(r)]} e^{-\lambda qt[1-M_Z(r)]} \\ &= \exp\left\{t\left(-\lambda\left[1-e^{-rc}\right] - \lambda p\left[1 - M_Y(r)\right] - \lambda q\left[1 - M_Z(r)\right]\right)\right\}. \end{aligned}$$

Therefore,

$$E[e^{-rS(t)}] = e^{tg(r)},$$

where

$$g(r) = -\lambda\left[1 - e^{-rc}\right] - \lambda p\left[1 - M_Y(r)\right] - \lambda q\left[1 - M_Z(r)\right].$$

Then, we obtain (3.61).

□

Theorem 3.8. *Equation*

$$g(r) = 0 \tag{3.63}$$

has a unique positive solution $r = R > 0$*, we can call* (3.63) *is said to be an adjustment coefficient equation of the risk model* (3.5)*, with* $R > 0$ *is said to be an adjustment coefficient.*

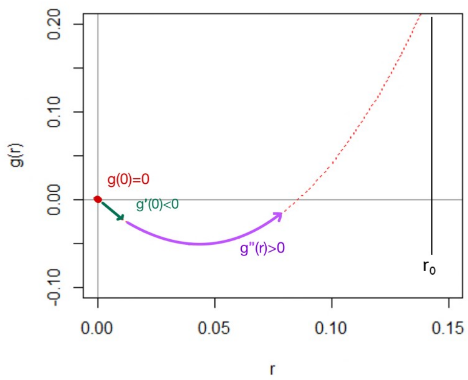
Proof. We will show that the adjustment coefficient equation has the unique positive solution, by proving the following properties of $g(r)$.

(1) $g(0) = 0$,

(2) $g'(0) < 0$,

(3) $g''(r) > 0$ for all $r > 0$,

(4) $\lim_{r \to +\infty} g(r) = \infty$.



From the definition of MGF in Definition 2.24, $M_Y(0) = 1$ and $M_Z(0) = 1$, we get

(1) From (3.62), then $g(0) = 0$.

(2) From (3.62), then

$$g'(r) = -\lambda c e^{-rc} + \lambda p E[Y_i e^{rY_i}] + \lambda q E[Z_i e^{rZ_i}]. \tag{3.64}$$

Therefore,

$$g'(0) = -\lambda c + \lambda p E[Y_i] + \lambda q E[Z_i].$$

From net profit condition (3.7), we get

$$\begin{aligned} g'(0) &= -\lambda c + \lambda p E[Y_i] + \lambda q E[Z_i], \\ &< -\lambda c + \lambda c = 0. \end{aligned}$$

Therefore, $g'(0) < 0$.

(3) Let $r > 0$. Due to the explanation of $g'(r)$ in (3.64), we have that

$$g''(r) = \lambda c^2 e^{-rc} + \lambda p E[Y_i^2 e^{rY_i}] + \lambda q E[Z_i^2 e^{rZ_i}].$$

Since Y_i, Z_i are non-negative random variables and $r > 0$,
$c^2 e^{-rc} > 0$, $E[Y_i^2 e^{rY_i}] \geq 0$ and $E[Z_i^2 e^{rZ_i}] \geq 0$.
Therefore,

$$g''(r) = \lambda c^2 e^{-rc} + \lambda p E[Y_i^2 e^{rY_i}] + \lambda q E[Z_i^2 e^{rZ_i}] > 0.$$

Thus,

$$g''(r) > 0 \quad \text{for all} \quad r > 0.$$

(4) Since (3.62) and the definition of MGF in Definition 2.24,

$$\begin{aligned} \lim_{r\to+\infty} g(r) &= -\lim_{r\to+\infty} \lambda \left[1 - e^{-rc}\right] - \lim_{r\to+\infty} \lambda p \left[1 - M_Y(r)\right] \\ &\quad - \lim_{r\to+\infty} \lambda q \left[1 - M_Z(r)\right]. \end{aligned}$$

Since $\lim_{r\to+\infty} e^{-rc} = 0$, $\lim_{r\to+\infty} M_Y(r) = \infty$, and $\lim_{r\to+\infty} M_Z(r) = \infty$,

$$\lim_{r\to+\infty} g(r) = \infty.$$

□

To determine the value of R based in the theory, it can be obtained as the unique positive solution of $g(r) = 0$, as indicated in Theorem 3.8. The function $g(r)$ is determined by (3.62). In practical applications, we will use the R command "`uniroot`" to compute the value of adjustment coefficient R.

For the profits process $\{S(t);\ t \geq 0\}$, let $F_t^S = \sigma\{S(v);\ v \leq t\}$ be a filtration.

Theorem 3.9. *The random process* $\{H_u(t);\ F_t^S;\ t \geq 0\}$ *is a martingale, where* $H_u(t) = \dfrac{e^{-r(u+S(t))}}{e^{tg(r)}}$.

Proof. Let $v < t$, we will show that $E[H_u(t) \mid F_v^S] = H_u(v)$.
Consider

$$\begin{aligned} E[H_u(t) \mid F_v^S] &= E\left[\frac{e^{-r(u+S(t))}}{e^{tg(r)}} \middle| F_v^S\right] \\ &= E\left[\frac{e^{-r(u+S(t))}}{e^{tg(r)}} \cdot \frac{e^{-rS(v)+rS(v)}}{e^{vg(r)-vg(r)}} \middle| F_v^S\right] \\ &= E\left[\frac{e^{-r(u+S(v))}}{e^{vg(r)}} \cdot \frac{e^{-r(S(t)-S(v))}}{e^{(t-v)g(r)}} \middle| F_v^S\right] \\ &= \frac{e^{-r(u+S(v))}}{e^{vg(r)}} \cdot \frac{1}{e^{(t-v)g(r)}} \cdot E[e^{-r(S(t)-S(v))} \mid F_v^S]. \end{aligned} \tag{3.65}$$

Consider

$$S(t) - S(v) = c[N(t) - N(v)] - \sum_{i=N(v,p)+1}^{N(t,p)} Y_i - \sum_{i=N(v,q)+1}^{N(t,q)} Z_i.$$

Since Y_i, Z_i are i.i.d. and $N(t)$ is stationary increment, we get

$$\begin{aligned} S(t) - S(v) &\overset{d}{\cong} c[N(t) - N(v)] - \sum_{i=1}^{N(t,p)-N(v,p)} Y_i - \sum_{i=1}^{N(t,q)-N(v,q)} Z_i \\ &\overset{d}{\cong} c[N(t-v)] - \sum_{i=1}^{N(t-v,p)} Y_i - \sum_{i=1}^{N(t-v,q)} Z_i \\ &= S(t-v). \end{aligned}$$

Therefore, (3.65) is become

$$E[H_u(t)|F_v^S] = \frac{e^{-r(u+S(v))}}{e^{vg(r)}} \cdot \frac{1}{e^{(t-v)g(r)}} \cdot E[e^{-r(S(t-v))}|F_v^S].$$

Since $S(t-v)$ and F_v^S are mutually independent and Theorem 2.18,

$$E[H_u(t)|F_v^S] \;=\; \frac{e^{-r(u+S(v))}}{e^{vg(r)}} \cdot \frac{1}{e^{(t-v)g(r)}} \cdot E[e^{-r(S(t-v))}].$$

From Theorem 3.7, we get

$$\begin{aligned} E[H_u(t)|F_v^S] &= \frac{e^{-r(u+S(v))}}{e^{vg(r)}} \cdot \frac{1}{e^{(t-v)g(r)}} \cdot e^{(t-v)g(r)} \\ &= H_u(v). \end{aligned}$$

Therefore,

$$E[H_u(t)|F_v^S] \;=\; H_u(v),$$

i.e., the random process $H_u(t)$ is a martingale. □

Lemma 3.2. *The ruin time T is the stopping time of F_t^S.*

Proof. Let T be the ruin time where $U(T) < 0$ and $F_t^S = \sigma\{S(v);\ v \leq t\}$.
From (3.5) and Lemma 3.1, then

$$U(t) \;=\; u + S(t).$$

Since F_t^S or $\sigma - algebra$ generated by random process $S(v)$ from time 0 to t occurs, it gives information $S(t)$ from time 0 to t.
Hence, event $\{T \leq t\}$ is a member of F_t^S.
Therefore,

$$T \text{ is the stopping time of } F_t^S.$$

□

Theorem 3.10. *For the surplus process $\{U(t); t \geq 0\}$, the ruin probability $\psi(u)$ satisfies Lundberg inequality:*

$$\psi(u) \leq e^{-Ru} \quad , u \geq 0, \tag{3.66}$$

where R is adjustment coefficient.

Proof. Let T be the ruin time, $t_0 > 0$ be a fixed time and $t_0 \wedge T = min(t_0, T)$, then $t_0 \wedge T$ is a stopping time. Hence, $t_0 \wedge T$ is a bounded stopping time.
From, $H_u(t) = \dfrac{e^{-r(u+S(t))}}{e^{tg(r)}}$ and Lemma 3.1, then

$$e^{-ru} = E[H_u(0)].$$

By Theorem 2.24 (The Martingale Stopping Time Theorem), we have that

$$E[H_u(0)] = E[H_u(T \wedge t_0)].$$

Therefore,

$$e^{-ru} = E[H_u(T \wedge t_0)].$$

Since

$$T \wedge t_0 = min(T, t_0) = \begin{cases} T, & \text{if } T \leq t_0, \\ t_0, & \text{if } T > t_0, \end{cases}$$

$$\begin{aligned} E[H_u(T \wedge t_0)] &= E[H_u(T \wedge t_0) \cdot [\mathbb{1}_{T \leq t_0} + \mathbb{1}_{T > t_0}]] \\ &= E[H_u(T \wedge t_0) \cdot \mathbb{1}_{T \leq t_0}] + E[H_u(T \wedge t_0) \cdot \mathbb{1}_{T > t_0}] \\ &= E[H_u(T) | T \leq t_0] \cdot P(T \leq t_0) + E[H_u(t_0) | T > t_0] \cdot P(T > t_0). \end{aligned}$$

Let $r = R$. Therefore,

$$e^{-Ru} = E[e^{-RU(T)} | T \leq t_0] \cdot P(T \leq t_0) + E[e^{-RU(t_0)} | T > t_0] \cdot P(T > t_0). \tag{3.67}$$

By the fact that $0 \leq E[e^{-RU(t_0)} | T > t_0] \leq 1$ and Theorem 2.20 (Markov's inequality), we obtain

$$\lim_{t_0 \to \infty} \Big[E[e^{-RU(t_0)} | T > t_0] \cdot P(T > t_0) \Big] = 0.$$

From (3.67), we get that

$$\begin{aligned}\lim_{t_0\to\infty} e^{-Ru} &= \lim_{t_0\to\infty}\Big[E[e^{-RU(T)}|T\le t_0]\cdot P(T\le t_0)+E[e^{-RU(t_0)}|T>t_0]\cdot P(T>t_0)\Big]\\ &= \lim_{t_0\to\infty}\Big[E[e^{-RU(T)}|T\le t_0]\cdot P(T\le t_0)\Big]+\lim_{t_0\to\infty}\Big[E[e^{-RU(t_0)}|T>t_0]\cdot P(T>t_0)\Big]\\ &= \lim_{t_0\to\infty}\Big[E[e^{-RU(T)}|T\le t_0]\cdot P(T\le t_0)\Big].\end{aligned}$$

Then,

$$\begin{aligned}e^{-Ru} &= E[e^{-RU(T)}|T\le\infty]\cdot P(T\le\infty)\\ &= E[e^{-RU(T)}|T\le\infty]\cdot\psi(u).\end{aligned}$$

Hence,

$$\psi(u)=\frac{e^{-Ru}}{E[e^{-RU(T)}|T\le\infty]}. \tag{3.68}$$

Since $U(T)<0$, we have that $\dfrac{1}{e^{-RU(T)}}<1$.

From (3.68), we have that

$$\psi(u)=\frac{e^{-Ru}}{E[e^{-RU(T)}\mid T<\infty]}<\frac{e^{-Ru}}{E[\,1\;|T<\infty]}=e^{-Ru}.$$

Therefore,

$$\psi(u)\le e^{-Ru}.$$

□

3.5 Experimental simulations

In this section, we perform numerical studies to investigate performance of the analytical approximation of the risk model CPST. The studies are divided into three parts. The first part discussed in Section 3.5.1 introduces the statistical estimation for the ruin probability $\hat{\psi}_t(u)$ by using the Monte Carlo methods. The second part in Section 3.5.2 studies numerical approximation to the ruin probability when the amounts of claims and surrenders follows an exponential distribution by using the analytical solution such as the Cramér approximation and the Laplace transform comparing with the Monte Carlo approximation and the Lundberg's Upper bound. The third part focuses on the numerical approximation to the ruin probability when the amounts of claims and surrenders follows gamma distributions by using the De-Vylder approximation and the Laplace transform comparing with the Monte Carlo approximation and the Lundberg's Upper bound.

3.5.1 Statistical estimations for the ruin probability

In this section, we study a statistical estimate for the ruin probability $\hat{\psi}_t(u)$ derived by the direct simulation of the surplus process using the Monte Carlo methods in order to evaluate the result of the approximations suggested in this chapter.

The concept of the Monte Carlo method is to generate a lot of sample paths for the stochastic process of interest and find the average value of the interest aspect of the process. We will perform the Monte Carlo simulation described in [9]. Let N be the total number of realizations of the process $U(t)$. We can calculate the average value of the process $U(t)$ when each ruin occurs at the time point t, consequently, we obtain the corresponding statistical estimate $\hat{\psi}_t(u)$ for the ruin probability $\psi(u)$. The Monte Carlo estimation is obtained as

$$\hat{\psi}_t(u) = \frac{1}{N}\sum_{i=1}^{N} I_{\{U_i(t)<0|U_i(0)=u\}},$$

where t is a fixed time point and N is the sample size. As $N \to \infty$ and $t \to \infty$, by the law of large numbers, $\hat{\psi}_t(u)$ converges to $\psi(u)$. The time points considered here are $t = 1, 5, 50,$ and 100, and the sample size of the Monte Carlo method is $N = 200{,}000$. The parameters of the model studied in this section are as follows. The initial capital u varies in $\{0, 1, 2, 3, 5, 7, 10\}$ and the constant rate of premiums is $c = 0.2$. The parameter of the Poisson counting process of premium is $\lambda = 10$. The thinning parameters of claims and surrenders are 0.4 and 0.3, respectively.

3.5.2 Exponential distributions for the claim sizes and surrender

Let the probability density functions of the amounts of claims Y_i and the amount of surrenders Z_i are

$$g(y) = ae^{-ay} \quad \text{and} \quad h(z) = be^{-bz} \quad , y, z \geq 0,$$

where $a = 4,\ b = 6$, respectively.

For the Cramer approximation, substituting $a = 4,\ b = 6,\ c = 0.2,\ p = 0.4$, and $q = 0.3$ into the formula of r_1 and r_2 in (3.17), we get $r_1 = -5.386001$ and $r_2 = -1.113999$, respectively. Consequently, $C_1 = 0.009246$ and $C_2 = 0.740753$. Therefore, the Cramér approximation $\psi_C(u)$ is

$$\psi_C(u) = 0.009246e^{-5.386001u} + 0.740753e^{-1.113999u} \quad for\ all\ u \geq 0. \tag{3.69}$$

For the Laplace approximation, substituting $a = 4,\ b = 6,\ c = 0.2,\ p = 0.4$, and $q = 0.3$ into the formula in (3.32), we get $S = 0.73$. Consequently, $s_1 = -5.386001$, $s_2 = -1.113999$. Therefore, the Laplace approximation $\psi_{\mathscr{L}}$ is

$$\psi_{\mathscr{L}}(u) = 0.740753e^{-1.113999u} + 0.009246e^{-5.386000u} \quad for\ all\ u \geq 0. \tag{3.70}$$

For the upper bound approximation, substituting $a = 4,\ b = 6,\ c = 0.2,\ p = 0.4$, and $q = 0.3$ into $g(r)$ in Theorem 3.7 and solve for the unique positive solution $g(r) = 0$ by using the R programming to compute the value of R, we have $R = 0.831038$. Then, from Theorem 3.10, the upper bound of the ruin probability is

$$\psi(u) \leq e^{-0.831038u}. \tag{3.71}$$

The numerical approximations obtained in (3.69)–(3.71) for different values of the initial capital u is given in Table 3.1.

u	$\psi(u)$					
	Statistical estimate $\hat{\psi}(u)$				Numerical approx. $\psi_C(u)/\psi_{\mathscr{L}}(u)$	Upper bound e^{-Ru}
	$t = 1$	$t = 5$	$t = 50$	$t = 100$		
0	0.295615	0.438440	0.472175	0.472180	0.750000	1.000000
1	0.079915	0.190715	0.229500	0.229515	0.243190	0.435596
2	0.014650	0.069690	0.101090	0.101105	0.079811	0.189744
3	0.001985	0.023840	0.044430	0.044440	0.026197	0.082652
5	0.000015	0.002020	0.008170	0.008180	0.002822	0.015682
7	0.000000	0.000135	0.001595	0.001605	0.000304	0.002975
10	0.000000	0.000000	0.000130	0.000135	0.000010	0.000245

Table 3.1: Numerical approximations of the CPST risk model with exponential distributions.

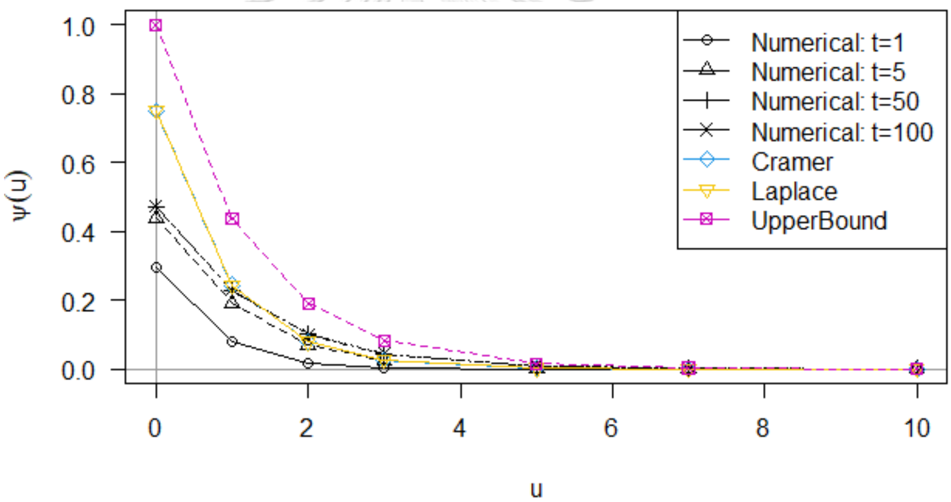


Figure 3.1: Graph of initial reserve u and the ruin probability of the CPST risk model with exponential distributions.

From Table 3.1, we can see that the approximates of ruin probability of all methods decrease when the initial capital increases. Notice that the Monte Carlo approximations $\hat{\psi}_t(u)$ do not have the same value as our approximations. Besides, we can observe that the ruin probability $\hat{\psi}_t(u)$ increases as t increases, the Monte Carlo approximation converges to our approximations, and it is less than the upper bound. Therefore, the Monte Carlo approximation is consider to be a good option. However, in practical situations, the exact value of the ruin probability is unknown; therefore, it is impossible to determine how high the value of t should be in order to make $\hat{\psi}_t(u)$ close to the exact value of the ruin probability as one desires. Consequently, our approximations are better than the Monte Carlo approximation regarding real usage.

Also, we can see that the Cramér approximations and the Laplace approximations fall between the Monte Carlo approximation of ruin probability $\hat{\psi}_t(u)$ and the upper bound which is reasonable, since the Cramér and the Laplace approximations are a type of infinite-time ruin probabilities which should be higher than any of finite-time ruin probability $\hat{\psi}_t(u)$ and should not exceed the upper bound. Moreover, we can see that the Cramér approximations (3.69) and the Laplace approximations (3.70) are equal. The reason for their equivalence is that the ruin probability formulas for both methods are equivalent to each other, yielding the same result see Remark 3.1 or derived from solving the same ODE.

The Monte Carlo simulation will be very good, if we can increase the value of t. However, it will take long computation time to do so. Therefore, a possible way to improve the Monte Carlo simulation performance is to increase the time points of interest and reduce the number of realizations of $U(t)$ instead.

3.5.3 Gamma distributions for the claim sizes and surrender

In this section, we study numerical approximations such the premium, claim sizes and surrender follow gamma distributions. Specifically, let the probability density functions of the claim sizes Y_i and surrender Z_i are

$$g(y) = \frac{\beta_Y e^{-\beta_Y y}(\beta_Y y)^{\alpha_Y - 1}}{\Gamma(\alpha_Y)} \quad \text{and} \quad h(z) = \frac{\beta_Z e^{-\beta_Z z}(\beta_Z z)^{\alpha_Z - 1}}{\Gamma(\alpha_Z)} \quad , y, z \geq 0,$$

where $\beta_Y = 8, \alpha_Y = 2, \beta_Z = 6, \alpha_Z = 1$, respectively.

For the De-Vylder approximation, substituting $\beta_Y = 8, \alpha_Y = 2, \beta_Z = 6, \alpha_Z = 1$, $c = 0.2$, $p = 0.4$, and $q = 0.3$ into the formula of r_1 and r_2 in (3.60), we get $r_1 = -11.956600$

and $r_2 = -1.415050$, respectively. Consequently, $C_1 = -0.019906$ and $C_2 = 0.769340$. Therefore, the De-Vylder approximation $\psi_{De}(u)$ is

$$\psi_{De}(u) = -0.019906e^{-11.956600u} + 0.769340e^{-1.415050u} \quad for\ all\ u \geq 0. \tag{3.72}$$

For the Laplace approximation, substituting $\beta_Y = 8, \alpha_Y = 2, \beta_Z = 6, \alpha_Z = 1$, $c = 0.2$, $p = 0.4$, and $q = 0.3$ into the formula of $\psi^*(s)$ in (3.30), then taking the inverse Laplace transform in $\psi^*(s)$ by using the MATLAB for computation. Therefore, the Laplace approximation $\psi_{\mathscr{L}}$ is

$$\psi_{\mathscr{L}}(u) = 0.769160e^{-1.415613u} - 0.020628e^{-10.812425u} \quad for\ all\ u \geq 0. \tag{3.73}$$

For the upper bound approximation, substituting $\beta_Y = 8, \alpha_Y = 2, \beta_Z = 6, \alpha_Z = 1$, $c = 0.2$, $p = 0.4$, and $q = 0.3$ into $g(r)$ in Theorem 3.7 and solve for the unique positive solution $g(r) = 0$ by using the R programming to compute the value of R, we have $R = 0.979012$. Then, from Theorem 3.10 the upper bound of the ruin probability is

$$\psi(u) \leq e^{-0.979012u}. \tag{3.74}$$

The numerical approximations obtained in (3.72)–(3.74) for different values of the initial capital u is given in Table 3.2.

u	$\psi(u)$						
	Statistical estimate $\hat{\psi}_t(u)$				$\psi_{De}(u)$	$\psi_{\mathscr{L}}(u)$	Upper bound e^{-Ru}
	$t = 1$	$t = 5$	$t = 50$	$t = 100$			
0	0.291345	0.424325	0.452320	0.452335	0.749434	0.750000	1
1	0.06555	0.162965	0.194255	0.194255	0.186883	0.186736	0.375681
2	0.008780	0.050860	0.073980	0.073995	0.045396	0.045334	0.141136
3	0.000765	0.014500	0.028220	0.028220	0.011027	0.011006	0.053022
5	0.000000	0.000730	0.003950	0.003950	0.000650	0.000648	0.007483
7	0.000000	0.000020	0.000475	0.000475	0.000038	0.000038	0.001056
10	0.000000	0.000000	0.000025	0.000025	0.000000	0.000000	0.000056

Table 3.2: Numerical approximations of the CPST risk model with gamma distributions.

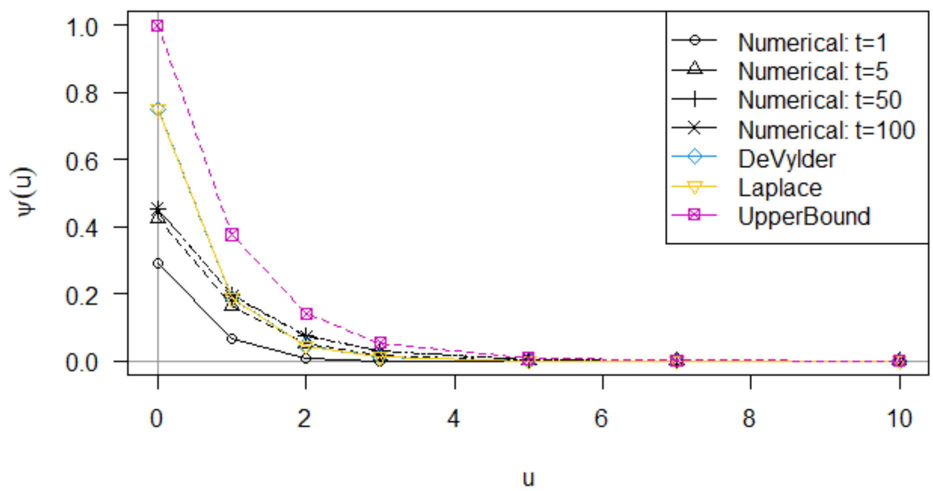


Figure 3.2: Graph of initial reserve u and the ruin probability of the CPST risk model with gamma distributions.

From Table 3.2, we can see that the approximates of ruin probability of all methods decrease when the initial capital increases. Notice that the Monte Carlo approximations $\hat{\psi}_t(u)$ do not have the same value as our approximations. Besides, we can observe that the ruin probability $\hat{\psi}_t(u)$ increases as t increases, the Monte Carlo approximation converges to our approximations, and it is less than the upper bound. Therefore, the Monte Carlo approximation is consider to be a good option. However, in practical situations, the exact value of the ruin probability is unknown; therefore, it is impossible to determine how high the value of t should be in order to make $\hat{\psi}_t(u)$ close to the exact value of the ruin probability as one desires. Consequently, our approximations are better than the Monte Carlo approximation regarding real usage.

Also, we can see that the De-Vylder approximations and the Laplace approximations fall between the Monte Carlo approximation of ruin probability $\hat{\psi}_t(u)$ and the upper bound which is reasonable, since the De-Vylder and the Laplace approximations are a type of infinite-time ruin probabilities which should be higher than any of finite-time ruin probability $\hat{\psi}_t(u)$ and should not exceed the upper bound. Moreover, we can see that the De-Vylder approximations (3.72) and the Laplace approximations (3.73) are nearby. The reason for their equivalence is that the ruin probability formulas for both methods are nearby to each other, yielding the nearby result. However, the values displayed in the Table 3.2 may differ slightly due to the rounding settings of the program.

The Monte Carlo simulation will be very good, if we can increase the value of t. However, it will take long computation time to do so. Therefore, a possible way to improve the Monte Carlo simulation performance is to increase the time points of interest and reduce the number of realizations of $U(t)$ instead.

CHAPTER IV

RISK MODEL WITH STOCHASTIC PREMIUMS AND SURRENDERS SUBJECT TO DEPENDENCE THINNING

In this chapter, we will extend our numerical approximations to the ruin probability for the risk model with surrenders when the premiums are not necessary to be constant over time. We first introduce the risk model and evaluate its properties. Then, we obtain an numerical approximation of the ruin probability by using the Cramér approximation, the Laplace transforms method, and the De Vylder Approximations. Moreover, we find the numerical approximations of the three methods and compare them with the Lundberg upper bound and the Monte Carlo approximation.

The organization of this chapter is as follows. Section 4.1 studies some properties of the risk model with stochastic premiums and surrenders subject to dependence thinning. Section 4.2 derives the analytical approximation of the ruin probability. Section 4.3 derives the Lundberg's upper bound of the ruin probability. Section 4.4 performs experimental simulations.

4.1 The risk model with stochastic premiums and surrenders subject to dependence thinning (SPST)

The risk model considered in this chapter is the risk model with stochastic premiums and surrenders subject to dependence thinning, denoted as SPST. The risk model consists of the initial capital, stochastic premiums, claims, and surrenders. In particular, the model is presented as

$$U(t) = u + \sum_{i=1}^{N(t)} X_i - \sum_{i=1}^{N(t,p)} Y_i - \sum_{i=1}^{N(t,q)} Z_i, \tag{4.1}$$

where u represents the initial capital, $N(t)$ is the Poisson process with intensity $\lambda > 0$, denoting the number of premiums up to time t. Particularly, $N(t) \sim \text{Poisson}(\lambda t)$.

The sequence $\{X_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. non-negative random variables representing the amounts of premiums with a cumulative distribution function F. $N(t,p)$, where $0 < p < 1$, is the p-thinning process of $N(t)$ denoting the number of claims up to time t. In particular, it is defined as $\sum_{i=1}^{M(t)} Q_i$ where Q_i are i.i.d. Bernoulli random variables with parameter p and $M(t)$ is independent and identically distributed with $N(t)$. The individual claim size $\{Y_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. non-negative random variables with a cumulative distribution function G. $N(t,q)$, where $0 < q < 1$, is the q-thinning process of $N(t)$ denoting the number of surrenders up to time t. The sequence of i.i.d. non-negative random variables $\{Z_i\}_{i=1}^{\infty}$ represents the amount of the i-th payment of insurance policy with a cumulative distribution function H. In addition, we suppose that $\{N(t)\}_{t\geq 0}, \{N(t,p)\}_{t\geq 0}, \{N(t,q)\}_{t\geq 0}, \{X_i\}_{i=1}^{\infty}, \{Y_i\}_{i=1}^{\infty}$, and $\{Z_i\}_{i=1}^{\infty}$ are mutually independent.

In order to ensure the insurance company's stable business, we assume that

$$E\left[\sum_{i=1}^{N(t)} X_i - \sum_{i=1}^{N(t,p)} Y_i - \sum_{i=1}^{N(t,q)} Z_i\right] > 0. \tag{4.2}$$

Since

$$\begin{aligned} E\left[\sum_{i=1}^{N(t)} X_i - \sum_{i=1}^{N(t,p)} Y_i - \sum_{i=1}^{N(t,q)} Z_i\right] &= E\left[\sum_{i=1}^{N(t)} X_i\right] - E\left[\sum_{i=1}^{N(t,p)} Y_i\right] - E\left[\sum_{i=1}^{N(t,q)} Z_i\right] \\ &= \lambda t\mu_X \ - \ \lambda pt\mu_Y - \ \lambda qt\mu_Z, \end{aligned}$$

the assumption becomes

$$\mu_X \ - \ p\mu_Y - \ q\mu_Z \ > \ 0, \tag{4.3}$$

which is called as the "**net profit condition**".

Lemma 4.1. *Define the profits process by* $\{S(t); t \geq 0\}$ *as*

$$S(t) = \sum_{i=1}^{N(t)} X_i - \sum_{i=1}^{N(t,p)} Y_i - \sum_{i=1}^{N(t,q)} Z_i.$$

Then, the profits process $S(t)$ has the following properties:

1. $S(0) = 0$,

2. $E[S(t)] = [\ \lambda\mu_X - \lambda p\mu_Y - \lambda q\mu_Z\ ]t$,

3. $Var[S(t)] = \left(\ \lambda E[X^2] + \lambda pE[Y^2] + \lambda qE[Z^2]\ \right) t$,

4. $M_{S(t)}(s) = \exp\left\{t\left[\lambda\left(M_X(s) - 1\right) + \lambda p\left(M_Y(-s) - 1\right) + \lambda q\left(M_Z(-s) - 1\right)\right]\right\}$,

5. $\{S(t)\}_{t\geq 0}$ *has stationary and independent increments.*

Proof.
(1) Since $N(t)$, $N(t,p)$, $N(t,q)$ are Poisson processes, $N(0) = 0$, $N(0,p) = 0$ and $N(0,q) = 0$. Then,

$$
\begin{aligned}
S(0) &= \sum_{i=1}^{N(0)} X_i - \sum_{i=1}^{N(0,p)} Y_i - \sum_{i=1}^{N(0,q)} Z_i \\
&= \sum_{i=1}^{0} X_i - \sum_{i=1}^{0} Y_i - \sum_{i=1}^{0} Z_i = 0 \\
&= 0.
\end{aligned}
$$

(2) By the property of expectation

$$
E[S(t)] = E\left[\sum_{i=1}^{N(t)} X_i\right] - E\left[\sum_{i=1}^{N(t,p)} Y_i\right] - E\left[\sum_{i=1}^{N(t,q)} Z_i\right],
$$

From Theorem 2.22,

$$
E[S(t)] = [\ \lambda\mu_X - \lambda p\mu_Y - \lambda q\mu_Z\ ]t.
$$

(3) By the property of variance and the independence of $X_i, Y_i, Z_i, N(t), N(t,p)$, and $N(t,q)$,

$$
Var[S(t)] = Var\left[\sum_{i=1}^{N(t)} X_i\right] + Var\left[\sum_{i=1}^{N(t,p)} Y_i\right] + Var\left[\sum_{i=1}^{N(t,q)} Z_i\right].
$$

From Theorem 2.22,

$$Var[S(t)] = \left(\ \lambda E[X^2] + \lambda p E[Y^2] + \lambda q E[Z^2]\ \right) t.$$

(4) We know that

$$M_{S(t)}(s) = E[e^{sS(t)}].$$

By the independence property of the three terms of $S(t)$,

$$M_{S(t)}(s) = E\left[e^{s\sum_{i=1}^{N(t)} X_i}\right] E\left[e^{-s\sum_{i=1}^{N(t,p)} Y_i}\right] E\left[e^{-s\sum_{i=1}^{N(t,q)} Z_i}\right].$$

The three terms are computed as follows

1) $$E\left[e^{s\sum_{i=1}^{N(t)} X_i}\right] = M_{\sum_{i=1}^{N(t)} X_i}(s) = G_{N(t)}\left[M_X(s)\right] = e^{\lambda t[M_X(s)-1]}.$$

2) $$E\left[e^{-s\sum_{i=1}^{N(t,p)} Y_i}\right] = M_{\sum_{i=1}^{N(t,p)} Y_i}(-s) = G_{N(t,p)}\left[M_Y(-s)\right] = e^{\lambda pt[M_Y(-s)-1]}.$$

3) $$E\left[e^{-s\sum_{i=1}^{N(t,q)} Z_i}\right] = M_{\sum_{i=1}^{N(t,q)} Z_i}(-s) = G_{N(t,q)}\left[M_Z(-s)\right] = e^{\lambda qt[M_Z(-s)-1]}.$$

Therefore,

$$M_{S(t)}(s) = \exp\left\{t\left[\lambda\left(M_X(s) - 1\right) + \lambda p\left(M_Y(-s) - 1\right) + \lambda q\left(M_Z(-s) - 1\right)\right]\right\}.$$

(5) Since $N(t)$ has stationary increments and $\{X_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. non-negative

random variables, we get

$$\sum_{i=1}^{N(t+h)} X_i - \sum_{i=1}^{N(t)} X_i \quad \text{is identically distributed as} \quad \sum_{i=1}^{N(t+h)-N(t)} X_i$$

and

$$\sum_{i=1}^{N(t+h)-N(t)} X_i \quad \text{is identically distributed as} \quad \sum_{i=1}^{N(s+h)-N(s)} X_i.$$

Therefore, $\sum_{i=1}^{N(t)} X_i$ has stationary increments.

To prove that the process has independent increments, let $s_1 < s_2 \leq s_3 < s_4$. Since $N(t)$ has independent increments and $\{X_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. non-negative random variables, we get

$$\sum_{i=N(s_1)+1}^{N(s_2)} X_i \quad \text{is independent with} \quad \sum_{i=N(s_3)+1}^{N(s_4)} X_i.$$

Therefore, $\sum_{i=1}^{N(t)} X_i$ has independent increments.

By the same technique, we can show that $\sum_{i=1}^{N(t,p)} Y_i$ and $\sum_{i=1}^{N(t,q)} Z_i$ have stationary and independent increments. Thus, $\{S(t); t \geq 0\}$ has stationary and independent increments.

$\square$

4.2 Approximation to the ruin probability of the risk model

In this section, we will study analytical approximation of the ruin probability for the SPST (4.1). We will start by obtaining the integral equation for the ruin probability. Then we obtain an approximation of the ruin probability using the Cramér approximation, the Laplace transforms method, and the De Vylder Approximations. To obtain the three approximations, we first obtain the integral equation stated in Theorem 4.1 below.

Theorem 4.1. *The ruin probability $\psi(u)$ for risk model* (4.1) *satisfies the integral equation*

$$\begin{aligned}(1+p+q)\psi(u) &= q[1-H(u)]+p[1-G(u)]+\int_0^\infty \psi(u+x)\,dF(x) \\ &\quad +p\int_0^u \psi(u-y)\,dG(y)+q\int_0^u \psi(u-z)\,dH(z), \quad u\geq 0,\end{aligned} \tag{4.4}$$

where F, G, and, H are cumulative distribution functions of the amounts of premiums, the individual claims sizes, and the amount of surrenders with probability density functions f, g, and h, respectively.

Proof. To compute the non-ruin probability $\phi(u)$, we consider five different possible disjoint events of the number of premiums, the number of claims, and the number of surrenders during an infinitesimal period $[0, \Delta t]$, which has been calculated in detail in the proof of Theorem 3.1 in chapter III, as follows.

Case 1:

There is no premiums, no claims, and no surrenders in the interval when $\Delta t \to 0$. The event occurs with the probability $1-\lambda\Delta t-\lambda p\Delta t-\lambda q\Delta t+o(\Delta t)$.

Case 2:

There is no premiums, no claims, and one surrender in the interval when $\Delta t \to 0$. The event occurs with the probability $\lambda q\Delta t+o(\Delta t)$.

Case 3:

There is no premiums, one claims, and no surrenders in the interval when $\Delta t \to 0$. The event occurs with the probability $\lambda p\Delta t+o(\Delta t)$.

Case 4:

There is one premium, no claims, and no surrenders in the interval when $\Delta t \to 0$. The event occurs with the probability $\lambda\Delta t+o(\Delta t)$.

Case 5:

There are more than one event of premiums, claims, and surrenders combined in the interval when $\Delta t \to 0$. The event occurs with the probability $o(\Delta t)$.

From the law of total probability for discrete random variable in Definition 2.16, it follows that

$$\begin{aligned}
\phi(u) = \quad & P[N(\Delta t)=0]P[N(\Delta t,p)=0]P[N(\Delta t,q)=0] \\
& \quad \cdot P[U(t)\geq 0, \forall t>0 | N(\Delta t)=0, N(\Delta t,p)=0, N(\Delta t,q)=0] \\
& + P[N(\Delta t)=0]P[N(\Delta t,p)=0]P[N(\Delta t,q)=1] \\
& \quad \cdot P[U(t)\geq 0, \forall t>0 | N(\Delta t)=0, N(\Delta t,p)=0, N(\Delta t,q)=1] \\
& + P[N(\Delta t)=1]P[N(\Delta t,p)=0]P[N(\Delta t,q)=0] \\
& \quad \cdot P[U(t)\geq 0, \forall t>0 | N(\Delta t)=1, N(\Delta t,p)=0, N(\Delta t,q)=0] \\
& + P[N(\Delta t)=0]P[N(\Delta t,p)=1]P[N(\Delta t,q)=0] \\
& \quad \cdot P[U(t)\geq 0, \forall t>0 | N(\Delta t)=0, N(\Delta t,p)=1, N(\Delta t,q)=0] \\
& + P[N(\Delta t)+N(\Delta t,p)+N(\Delta t,q)>1] \\
& \quad \cdot P[U(t)\geq 0, \forall t>0 | N(\Delta t)+N(\Delta t,p)+N(\Delta t,q)>1].
\end{aligned}$$

Then,

$$\begin{aligned}
\phi(u) = \quad & [1-\lambda\Delta t+o(\Delta t)][1-\lambda p\Delta t+o(\Delta t)][1-\lambda q\Delta t+o(\Delta t)] \\
& \cdot P[U(t)\geq 0, \forall t>0 | N(\Delta t)=0, N(\Delta t,p)=0, N(\Delta t,q)=0] \\
& + [1-\lambda\Delta t+o(\Delta t)][1-\lambda p\Delta t+o(\Delta t)][\lambda q\Delta t+o(\Delta t)]] \\
& \quad \cdot P[U(t)\geq 0, \forall t>0 | N(\Delta t)=0, N(\Delta t,p)=0, N(\Delta t,q)=1] \\
& + [\lambda\Delta t+o(\Delta t)][1-\lambda p\Delta t+o(\Delta t)][1-\lambda q\Delta t+o(\Delta t)] \\
& \quad \cdot P[U(t)\geq 0, \forall t>0 | N(\Delta t)=1, N(\Delta t,p)=0, N(\Delta t,q)=0] \\
& + [1-\lambda\Delta t+o(\Delta t)][\lambda p\Delta t+o(\Delta t)][1-\lambda q\Delta t+o(\Delta t)] \\
& \quad \cdot P[U(t)\geq 0, \forall t>0 | N(\Delta t)=0, N(\Delta t,p)=1, N(\Delta t,q)=0] \\
& + o(\Delta t)\cdot \; P[U(t)\geq 0, \forall t>0 | N(\Delta t)+N(\Delta t,p)+N(\Delta t,q)>1].
\end{aligned}$$

By the properties of little-oh in Theorem 2.1 for $\Delta t \to 0$ and the law of total probability for continuous random variable X_i, Y_i, and Z_i in Definition 2.17.

$$\begin{aligned}\phi(u) = \; & (1-\lambda\Delta t-\lambda p\Delta t-\lambda q\Delta t)\; P[U(t)\geq 0,\forall t>0 \mid U(\Delta t)=u]\\ & +\lambda q\Delta t \int_0^u P[U(t)\geq 0,\forall t>0 \mid U(\Delta t)=u-z]\,dH(z)\\ & +\lambda p\Delta t \int_0^u P[U(t)\geq 0,\forall t>0 \mid U(\Delta t)=u-y]\,dG(y)\\ & +\lambda\Delta t \int_0^\infty P[U(t)\geq 0,\forall t>0 \mid U(\Delta t)=u+x]\,dF(x)+o(\Delta t).\end{aligned}$$

According to the concept of stationary, we can treat Δt as a new start time. Therefore, we can express $U(\Delta t)$ as $U(0)$. This implies that we are starting a new at Δt and can use $U(0)$ as the starting point,

$$\begin{aligned}\phi(u) = \; & (1-\lambda\Delta t-\lambda p\Delta t-\lambda q\Delta t)\; P[U(t)\geq 0,\forall t>0 \mid U(0)=u]\\ & +\lambda q\Delta t \int_0^u P[U(t)\geq 0,\forall t>0 \mid U(0)=u-z]\,dH(z)\\ & +\lambda p\Delta t \int_0^u P[U(t)\geq 0,\forall t>0 \mid U(0)=u-y]\,dG(y)\\ & +\lambda\Delta t \int_0^\infty P[U(t)\geq 0,\forall t>0 \mid U(0)=u+x]\,dF(x)+o(\Delta t).\end{aligned}$$

Then, we get

$$\begin{aligned}\phi(u) \;=\; & (1-\lambda\Delta t-\lambda p\Delta t-\lambda q\Delta t)\; \phi(u)+\lambda q\Delta t\int_0^u \phi(u-z)\,dH(z)\\ & +\lambda p\Delta t\int_0^u \phi(u-y)\,dG(y)+\lambda\Delta t\int_0^\infty \phi(u+x)\,dF(x)+o(\Delta t).\end{aligned}$$

That is

$$\begin{aligned}0 \;=\; & -\lambda\Delta t\;\phi(u)-\lambda p\Delta t\;\phi(u)-\lambda q\Delta t\;\phi(u)+\lambda q\Delta t\int_0^u \phi(u-z)\,dH(z)\\ & +\lambda p\Delta t\int_0^u \phi(u-y)\,dG(y)+\lambda\Delta t\int_0^\infty \phi(u+x)\,dF(x)+o(\Delta t).\end{aligned}$$

Dividing both sides by Δt and letting Δt approach to 0, we have

$$\begin{aligned}0 \;=\; & -\lambda\;\phi(u)-\lambda p\;\phi(u)-\lambda q\;\phi(u)+\lambda q\int_0^u \phi(u-z)\,dH(z)\\ & +\lambda p\int_0^u \phi(u-y)\,dG(y)+\lambda\int_0^\infty \phi(u+x)\,dF(x).\end{aligned}$$

That is

$$\begin{aligned}(1+p+q)\phi(u) &= q\int_0^u \phi(u-z)\,dH(z) + p\int_0^u \phi(u-y)\,dG(y)\\ &\quad + \int_0^\infty \phi(u+x)\,dF(x).\end{aligned}$$

Using the property that $\phi(u) = 1 - \psi(u)$, we get

$$\begin{aligned}(1+p+q)-(1+p+q)\psi(u) &= q\int_0^u 1\,dH(z) - q\int_0^u \psi(u-z)\,dH(z)\\ &\quad +p\int_0^u 1\,dG(y) - p\int_0^u \psi(u-y)\,dG(y)\\ &\quad +\int_0^\infty 1\,dF(x) - \int_0^\infty \phi(u+x)\,dF(x), \quad u \geq 0.\end{aligned}$$

Therefore,

$$\begin{aligned}(1+p+q)-(1+p+q)\psi(u) &= qH(u) - q\int_0^u \psi(u-z)\,dH(z) + pG(u)\\ &\quad -p\int_0^u \psi(u-y)\,dG(y) + 1 - \int_0^\infty \phi(u+x)\,dF(x), \quad u \geq 0.\end{aligned}$$

Thus,

$$\begin{aligned}(1+p+q)\psi(u) &= q[1-H(u)] + p[1-G(u)] + \int_0^\infty \psi(u+x)\,dF(x)\\ &\quad +p\int_0^u \psi(u-y)\,dG(y) + q\int_0^u \psi(u-z)\,dH(z), \quad u \geq 0.\end{aligned}$$

□

4.2.1 The Cramér approximation

In this section, we obtain the Cramér approximation to the ruin probability when amounts of premiums, claims, and surrenders follow exponential distributions. In particular, the probability density functions of the premiums, claim sizes, and surrenders are

$$f(x) = ae^{-ax}, g(y) = be^{-by} \text{ and } h(z) = ce^{-cz}, x, y, z \geq 0, \tag{4.5}$$

corresponding to CDF's are F, G and H, respectively, in Theorem 4.1

Theorem 4.2. *For the risk model* (4.1) *where the amounts of premiums, claims size and surrenders follow exponential distributions with parameters a, b, and c, respective. If the net profit condition* (4.3) *is satisfied, then the Cramér approximation of the ruin probability* $\psi_C(u)$ *is*

$$\psi_C(u) = C_1 e^{r_1 u} + C_2 e^{r_2 u} \quad for\ all\ u \geq 0, \tag{4.6}$$

where $C_1, C_2, r_1,$ *and* r_2 *are as follows*

$$C_1 = \frac{C_{11}}{C_D}\ ,\ C_2 = \frac{C_{21}}{C_D}\ ,$$

$$r_1 = \frac{-\left[\dfrac{(p+q)}{bc} - \dfrac{(1+q)}{ac} - \dfrac{(1+p)}{ab}\right] + \sqrt{D}}{-2\dfrac{(1+p+q)}{abc}},$$

$$r_2 = \frac{-\left[\dfrac{(p+q)}{bc} - \dfrac{(1+q)}{ac} - \dfrac{(1+p)}{ab}\right] - \sqrt{D}}{-2\dfrac{(1+p+q)}{abc}},$$

which

$$D = \left[\frac{(p+q)}{bc} - \frac{(1+q)}{ac} - \frac{(1+p)}{ab}\right]^2 - 4\frac{(1+p+q)}{abc}\left[\frac{1}{a} - \frac{p}{b} - \frac{q}{c}\right],$$

$$C_{11} = (a - r_1)r_2(bp + cq - a(p+q)^2 + (p+q)(1+p+q)r_2),$$

$$C_{21} = (a - r_2)r_1(bp + cq - a(p+q)^2 + (p+q)(1+p+q)r_1),$$

and

$$C_D = (r_1 - r_2)[a^2(p+q)^2 + (1+p+q)^2 r_1 r_2 - a[bp + cq + (p+q)(1+p+q)(r_1 + r_2)]].$$

Proof.

Observe that CDF F and PDF f satisfy $dF(u) = f(u)du$, as mentioned in Remark 2.2, including CDF G, H and PDF g, h, respectively.

Substituting the density functions of X_i, Y_i, and Z_i into (4.4), we have

$$\begin{aligned}
&(1+p+q)\psi(u) \\
&\quad = q[e^{-cu}] + p[e^{-bu}] + \int_0^{\infty} \psi(u+x)ae^{-ax}\,dx + p\int_0^{u} \psi(u-y)be^{-by}\,dy \\
&\quad\quad + q\int_0^{u} \psi(u-z)ce^{-cz}\,dz. \qquad (4.7)
\end{aligned}$$

Differentiating the equation with respect to u, we have

$$\begin{aligned}
&(1+p+q)\psi'(u) \\
&\quad = -q[ce^{-cu}] - p[be^{-bu}] + [-a+pb+qc]\ \psi(u) + a^2\int_0^{\infty} \psi(u+x)e^{-ax}\,dx \\
&\quad\quad - pb^2\int_0^{u} \psi(u-y)e^{-by}\,dy - qc^2\int_0^{u} \psi(u-z)e^{-cz}\,dz.
\end{aligned}$$

Multiplying the equation by $\frac{1}{b}$, we have

$$\begin{aligned}
&\frac{(1+p+q)}{b}\psi'(u) \\
&\quad = -\frac{q}{b}[ce^{-cu}] - p[e^{-bu}] + \left[\frac{-a}{b} + p + \frac{qc}{b}\right]\ \psi(u) + \frac{a^2}{b}\int_0^{\infty} \psi(u+x)e^{-ax}\,dx \\
&\quad\quad - pb\int_0^{u} \psi(u-y)e^{-by}\,dy - \frac{qc^2}{b}\int_0^{u} \psi(u-z)e^{-cz}\,dz. \qquad (4.8)
\end{aligned}$$

Adding the terms of each side of (4.7) and (4.8), we have

$$\begin{aligned}
&\frac{(1+p+q)}{b}\psi'(u) + \left(1+q-\frac{qc}{b}+\frac{a}{b}\right)\psi(u) \\
&\quad = q\left[1-\frac{c}{b}\right]e^{-cu} + q\left[1-\frac{c}{b}\right]\int_0^{u} \psi(u-z)ce^{-cz}\,dz + \left[\frac{a}{b}+1\right]\int_0^{\infty} \psi(u+x)ae^{-ax}\,dx. \qquad (4.9)
\end{aligned}$$

Differentiating the equation with respect to u, we have

$$\begin{aligned}
&\frac{(1+p+q)}{b}\psi''(u) + \left(1+q-\frac{qc}{b}+\frac{a}{b}\right)\psi'(u) \\
&\quad = -qc\left[1-\frac{c}{b}\right]e^{-cu} + \left\{qc\left[1-\frac{c}{b}\right] - a\left[\frac{a}{b}+1\right]\right\}\psi(u)
\end{aligned}$$

$$-qc^2\left[1-\frac{c}{b}\right]\int_0^u \psi(u-z)e^{-cz}\,dz + a^2\left[\frac{a}{b}+1\right]\int_0^\infty \psi(u+x)e^{-ax}\,dx.$$

Multiplying the equation by $\frac{-1}{a}$, we have

$$\begin{aligned}
&-\frac{(1+p+q)}{ab}\psi''(u) - \left(\frac{1}{a}+\frac{q}{a}-\frac{qc}{ab}+\frac{1}{b}\right)\psi'(u) \\
&\quad = \frac{qc}{a}\left[1-\frac{c}{b}\right]e^{-cu} - \left\{\frac{qc}{a}\left[1-\frac{c}{b}\right] - \left[\frac{a}{b}+1\right]\right\}\psi(u) \\
&\qquad + \frac{qc^2}{a}\left[1-\frac{c}{b}\right]\int_0^u \psi(u-z)e^{-cz}\,dz - a\left[\frac{a}{b}+1\right]\int_0^\infty \psi(u+x)e^{-ax}\,dx.
\end{aligned} \tag{4.10}$$

Adding the terms of left side of (4.9) and (4.10), we have

$$\begin{aligned}
&-\frac{(1+p+q)}{ab}\psi''(u) + \left[\frac{(p+q)}{b} - \left(\frac{1}{a}+\frac{q}{a}-\frac{qc}{ab}\right)\right]\psi'(u) \\
&\quad + \left[q - \frac{qc}{b} + \frac{qc}{a} - \frac{qc^2}{ab}\right]\psi(u) \\
&\qquad = q\left[1-\frac{c}{b}\right]\left[1+\frac{c}{a}\right]e^{-cu} + qc\left[1-\frac{c}{b}\right]\left[1+\frac{c}{a}\right]\int_0^u \psi(u-z)e^{-cz}\,dz.
\end{aligned} \tag{4.11}$$

Differentiating the equation with respect to u, we have

$$\begin{aligned}
&-\frac{(1+p+q)}{ab}\psi'''(u) + \left[\frac{(p+q)}{b} - \left(\frac{1}{a}+\frac{q}{a}-\frac{qc}{ab}\right)\right]\psi''(u) \\
&\quad + \left[q - \frac{qc}{b} + \frac{qc}{a} - \frac{qc^2}{ab}\right]\psi'(u) \\
&\qquad = -qc\left[1-\frac{c}{b}\right]\left[1+\frac{c}{a}\right]e^{-cu} + qc\left[1-\frac{c}{b}\right]\left[1+\frac{c}{a}\right]\psi(u) \\
&\qquad\quad - qc^2\left[1-\frac{c}{b}\right]\left[1+\frac{c}{a}\right]\int_0^u \psi(u-z)e^{-cz}\,dz.
\end{aligned}$$

Multiplying the equation by $\frac{1}{c}$, we have

$$\begin{aligned}
&-\frac{(1+p+q)}{abc}\psi'''(u) + \left[\frac{(p+q)}{bc} - \left(\frac{1}{ac}+\frac{q}{ac}-\frac{q}{ab}\right)\right]\psi''(u) \\
&\quad + \left[\frac{q}{c} - \frac{q}{b} + \frac{q}{a} - \frac{qc}{ab}\right]\psi'(u) \\
&\qquad = -q\left[1-\frac{c}{b}\right]\left[1+\frac{c}{a}\right]e^{-cu} + q\left[1-\frac{c}{b}\right]\left[1+\frac{c}{a}\right]\psi(u) \\
&\qquad\quad - qc\left[1-\frac{c}{b}\right]\left[1+\frac{c}{a}\right]\int_0^u \psi(u-z)e^{-cz}\,dz.
\end{aligned} \tag{4.12}$$

Adding the terms of each side of (4.11) and (4.12), we have

$$-\frac{(1+p+q)}{abc}\psi'''(u)+\left[\frac{(p+q)}{bc}-\frac{(1+q)}{ac}-\frac{(1+p)}{ab}\right]\psi''(u)$$
$$+\left[\frac{p}{b}-\frac{1}{a}+\frac{q}{c}\right]\psi'(u)=0. \tag{4.13}$$

The equivalent characteristic equation is

$$-\frac{(1+p+q)}{abc}r^3+\left[\frac{(p+q)}{bc}-\frac{(1+q)}{ac}-\frac{(1+p)}{ab}\right]r^2+\left[\frac{p}{b}-\frac{1}{a}+\frac{q}{c}\right]r=0. \tag{4.14}$$

Solving the equation, we obtain the three roots as

$$r_1=\frac{-\left[\frac{(p+q)}{bc}-\frac{(1+q)}{ac}-\frac{(1+p)}{ab}\right]+\sqrt{D}}{-2\frac{(1+p+q)}{abc}},$$
$$r_2=\frac{-\left[\frac{(p+q)}{bc}-\frac{(1+q)}{ac}-\frac{(1+p)}{ab}\right]-\sqrt{D}}{-2\frac{(1+p+q)}{abc}},$$
$$r_3=0,$$

where

$$D=\left[\frac{(p+q)}{bc}-\frac{(1+q)}{ac}-\frac{(1+p)}{ab}\right]^2-4\frac{(1+p+q)}{abc}\left[\frac{1}{a}-\frac{p}{b}-\frac{q}{c}\right].$$

Therefore, the general solution of $\psi(u)$ is

$$\psi(u)=C_1e^{r_1u}+C_2e^{r_2u}+C_3. \tag{4.15}$$

Since

$$\begin{aligned}D &= \left[\frac{(p+q)}{bc}-\frac{(1+q)}{ac}-\frac{(1+p)}{ab}\right]^2-4\frac{(1+p+q)}{abc}\left[\frac{1}{a}-\frac{p}{b}-\frac{q}{c}\right]\\ &= \left[\frac{(p-q)}{bc}-\frac{(1+q)}{ac}+\frac{(1+p)}{ab}\right]^2+4\frac{(a+b)(a+c)pq}{a^2b^2c^2} > 0.\end{aligned}$$

Then, r_1 and r_2 are distinct real roots.

Since

$$\frac{(p+q)}{bc} - \frac{(1+q)}{ac} - \frac{(1+p)}{ab} = -\left[\frac{1}{b}+\frac{1}{c}\right]\left[\frac{1}{a}-\frac{p}{b}-\frac{q}{c}\right] - \frac{p}{b^2} - \frac{p}{ab} - \frac{q}{c^2} - \frac{q}{ac} \; < 0, \tag{4.16}$$

by the Vieta's theorem in Theorem 2.2 and (4.14), we get

$$r_1 r_2 = \frac{\frac{p}{b} - \frac{1}{a} + \frac{q}{c}}{-\frac{(1+p+q)}{abc}} > 0 \tag{4.17}$$

and

$$r_1 + r_2 = \frac{\frac{(p+q)}{bc} - \frac{(1+q)}{ac} - \frac{(1+p)}{ab}}{\frac{(1+p+q)}{abc}} < 0. \tag{4.18}$$

From (4.17) and the net profit condition (4.3), we can see that r_1 and r_2 have the same sign.

From (4.18) and (4.16), we get

$$r_1 < 0 \text{ and } r_2 < 0.$$

Next, once we know the values of r_1 and r_1, we will then determine the values of C_1, C_2, and C_3 for (4.15) using the initial conditions follow as,

1. $\lim_{u\to\infty} \psi(u) = 0$, since $r_1, r_2 < 0$ which yields $C_3 = 0$.
2. Letting $u = 0$ in (4.7) and using $\psi(u)$ from (4.15), we get

$$(1+p+q)(C_1 + C_2) = q + p + \int_0^\infty \psi(x) a e^{-au}\, dx.$$

Therefore,

$$p + q = C_1\left[(1+p+q) + \frac{a}{r_1 - a}\right] + C_2\left[(1+p+q) + \frac{a}{r_2 - a}\right]. \tag{4.19}$$

3. Letting $u = 0$ in (4.9) and using $\psi(u)$ from (4.15), we get

$$\begin{aligned}&\frac{(1+p+q)}{b}(C_1 r_1 + C_2 r_2) + \left(1 + q - \frac{qc}{b} + \frac{a}{b}\right)(C_1 + C_2)\\&= q\left[1 - \frac{c}{b}\right] + \left[\frac{a}{b} + 1\right]\left[\frac{C_1 a}{a - r_1} + \frac{C_2 a}{a - r_2}\right].\end{aligned}$$

Therefore,

$$\begin{aligned}&q\left[1 - \frac{c}{b}\right]\\&= C_1\left[\frac{(1+p+q)}{b} r_1 + \left(1 + q - \frac{qc}{b} + \frac{a}{b}\right) - \left[\frac{a}{b} + 1\right]\frac{a}{a - r_1}\right]\\&+ C_2\left[\frac{(1+p+q)}{b} r_2 + \left(1 + q - \frac{qc}{b} + \frac{a}{b}\right) - \left[\frac{a}{b} + 1\right]\frac{a}{a - r_2}\right].\end{aligned} \tag{4.20}$$

Solving system of (4.19) and (4.20), we get

$$C_1 = \frac{C_{11}}{C_D} \quad \text{and} \quad C_2 = \frac{C_{21}}{C_D},$$

where

$$\begin{aligned}C_{11} &= (a - r_1) r_2 (bp + cq - a(p+q)^2 + (p+q)(1+p+q) r_2),\\C_{21} &= (a - r_2) r_1 (bp + cq - a(p+q)^2 + (p+q)(1+p+q) r_1),\\C_D &= (r_1 - r_2)[a^2(p+q)^2 + (1+p+q)^2 r_1 r_2\\&\quad - a[bp + cq + (p+q)(1+p+q)(r_1 + r_2)]].\end{aligned}$$

□

To calculate the approximated ruin probability using the Cramér approximation described in (4.6), we can use the R programming for computation.

4.2.2 The Laplace transform

In this section, we obtain an approximation of ruin probability using the Laplace transforms in conjunction with integral equation of ruin probability for the SPST model (4.1).

Theorem 4.3. *The Laplace transform of ruin probability* $\psi(u)$ *for risk model* (4.1) *is*

$$\psi^*(s) = \frac{q[1-h^*(s)] + p[1-g^*(s)]}{s(1+p+q) - sp\ g^*(s) - s\ f^*(-s) - sq\ h^*(s)}. \tag{4.21}$$

where f^*, g^*, *and* h^* *are the Laplace transforms of probability density functions for the amount of premium* f, *claims size* g, *and surrender* h, *respectively.*

Proof.

Taking the Laplace transform of (4.4) and formula in Theorem 2.3, we get

$$\begin{aligned}(1+p+q)\psi^*(s) &= q\left[\frac{1}{s} - \frac{h^*(s)}{s}\right] + p\left[\frac{1}{s} - \frac{g^*(s)}{s}\right] + \ \psi^*(s)\overline{f^*(-\overline{s})}, \\ &\quad + p\ \psi^*(s)g^*(s) + q\ \psi^*(s)h^*(s).\end{aligned}$$

Multiplying both sides by s, we have

$$\begin{aligned}s(1+p+q)\psi^*(s) &= q\left[1 - h^*(s)\right] + p\left[1 - g^*(s)\right] + s\ \ \psi^*(s)\overline{f^*(-\overline{s})} \\ &\quad + s\ p\ \psi^*(s)g^*(s) + s\ q\ \psi^*(s)h^*(s).\end{aligned}$$

By the property of cross correlation and the real function $f(x)$, we get

$$\begin{aligned}s(1+p+q)\psi^*(s) &= q\left[1 - h^*(s)\right] + p\left[1 - g^*(s)\right] + s\ \ \psi^*(s)f^*(-s) \\ &\quad + s\ p\ \psi^*(s)g^*(s) + s\ q\ \psi^*(s)h^*(s).\end{aligned}$$

Therefore,

$$\begin{aligned}&q\left[1 - h^*(s)\right] + p\left[1 - g^*(s)\right] \\ &\quad = s\left[(1+p+q) - p\ g^*(s) - q\ h^*(s) - \ f^*(-s)\right]\psi^*(s).\end{aligned}$$

Thus,

$$\psi^*(s) = \frac{q[1-h^*(s)] + p[1-g^*(s)]}{s(1+p+q) - sp\ g^*(s) - s\ f^*(-s) - sq\ h^*(s)}.$$

□

Remark 4.1. Assume the risk model described in (4.1) where the amount of premium, claims size, and surrender follow exponential distributions according to (4.5), probability density functions denoted as f, g, and h, respectively, and with parameters a, b, and c. If the net profit condition given by (4.3) holds, then the Laplace transform of the ruin probability $\psi(u)$ is

$$\psi_{\mathscr{L}}(u) = \frac{acp + abq}{-bc + acp + abq} + \frac{bc(cp + bq) + a(c^2p + b^2q) + [bc(cp + bq) + a(cp + bq)]s_1}{(bc - acp - abq)(s_1 - s_2)(1 + p + q)} e^{s_1 u}$$
$$+ \frac{-bc(cp + bq) - a(c^2p + b^2q) - [bc(cp + bq) + a(cp + bq)]s_2}{(bc - acp - abq)(s_1 - s_2)(1 + p + q)} e^{s_2 u}, \tag{4.22}$$

where

$$s_1 = \frac{-c(1 + p) - b(1 + q) + a(p + q) - \sqrt{S}}{2(1 + p + q)},$$

$$s_2 = \frac{-c(1 + p) - b(1 + q) + a(p + q) + \sqrt{S}}{2(1 + p + q)},$$

and

$$S = [c(1 + p) + b(1 + q) - a(p + q)]^2 - 4(1 + p + q)(bc - acp - abq).$$

Proof. Substituting the Laplace transforms of the density functions density functions of X_i, Y_i, and Z_i with CDF's are F, G, and H, respectively, into (4.21), we have

$$\psi^*(s) = \frac{q\left[1 - \dfrac{c}{s + c}\right] + p\left[1 - \dfrac{b}{s + b}\right]}{s(1 + p + q) - sp\left(\dfrac{b}{s + b}\right) - s\left(\dfrac{a}{-s + a}\right) - sq\left(\dfrac{c}{s + c}\right)}.$$

Let $R(s) = s^2(1+p+q)+s[c(1+p)+b(1+q)-a(p+q)]+(bc-acp-abq)$ and rearrange the equation for $\psi^*(s)$, we get

$$\psi^*(s) = \frac{s^2(p + q) + s[cp + bq - a(p + q)] - a(cp + bq)}{sR(s)}. \tag{4.23}$$

Let $S = [c(1+p)+b(1+q)-a(p+q)]^2 - 4(1+p+q)(bc-acp-abq)$ tThen, $S > 0$. Factoring $R(s)$, we will obtain that

$$\psi^*(s) = \frac{s^2(p+q)+s[cp+bq-a(p+q)]-a(cp+bq)}{s(s-s_1)(s-s_2)(1+p+q)}, \tag{4.24}$$

where

$$s_1 = \frac{-c(1+p)-b(1+q)+a(p+q)-\sqrt{S}}{2(1+p+q)},$$

$$s_2 = \frac{-c(1+p)-b(1+q)+a(p+q)+\sqrt{S}}{2(1+p+q)}.$$

Since $S > 0$, then s_1 and s_2 are distinct real roots.
Since

$$c(1+p)+b(1+q)-a(p+q) = \left[\frac{1}{b}+\frac{1}{c}\right](bc-acp-abq)+cp+\frac{acp}{b}+bq+\frac{abq}{c} > 0, \tag{4.25}$$

by the Vieta's theorem in Theorem 2.2 and equation $R(s)$, we get

$$s_1 s_2 = \frac{bc-acp-abq}{1+p+q} > 0 \tag{4.26}$$

and

$$s_1 + s_2 = \frac{-[c(1+p)+b(1+q)-a(p+q)]}{1+p+q} < 0. \tag{4.27}$$

From (4.26) and the net profit condition (4.3), we can see that s_1 and s_2 have the same sign. From (4.27) and (4.25), we get

$$s_1 < 0 \text{ and } s_2 < 0.$$

Applying partial fraction decomposition to (4.24) with respect to s, we obtain

$$\begin{aligned}\psi^*(s) \quad = \quad & \frac{acp+abq}{(-bc+acp+abq)s} \\ & +\frac{bc(cp+bq)+a(c^2p+b^2q)+[bc(cp+bq)+a(cp+bq)]s_1}{(bc-acp-abq)(s_1-s_2)(s-s_1)(1+p+q)} \\ & +\frac{-bc(cp+bq)-a(c^2p+b^2q)-[bc(cp+bq)+a(cp+bq)]s_2}{(bc-acp-abq)(s_1-s_2)(s-s_2)(1+p+q)}.\end{aligned} \tag{4.28}$$

Taking the inverse Laplace transform (4.28) with respect to s, we obtain

$$\begin{aligned}\psi_{\mathscr{L}}(u) = & \frac{acp+abq}{-bc+acp+abq} \\ & +\frac{bc(cp+bq)+a(c^2p+b^2q)+[bc(cp+bq)+a(cp+bq)]s_1}{(bc-acp-abq)(s_1-s_2)(1+p+q)}e^{s_1u} \\ & +\frac{-bc(cp+bq)-a(c^2p+b^2q)-[bc(cp+bq)+a(cp+bq)]s_2}{(bc-acp-abq)(s_1-s_2)(1+p+q)}e^{s_2u}.\end{aligned}$$

□

The observation is different from Remark 3.1 in Chapter III, where $\psi_C(u) = \psi_{\mathscr{L}}(u)$. However, in this Chapter IV, $\psi_C(u) \neq \psi_{\mathscr{L}}(u)$ because the ruin probability formula for the Cramér approximation (4.6) is expressed as a sum of two exponential terms, whereas the ruin probability formula for the Laplace transforms (4.22) is expressed as a sum of three terms, with constant terms that cannot be eliminated. Also, the Laplace transform (4.22) yields a negative numerical result, as observed in Table 4.1. Therefore, it is not possible to use the Laplace transform in SPST model.

To calculate the approximated ruin probability using the Laplace transform for money amounts which follow exponential distributions described in (4.22), we can use the MATLAB commands "`partfrac`" and "`ilaplace`" for computation.

In the case that the money amounts follow gamma distributions, we also use MATLAB to calculate the approximated ruin probability. We use the general Laplace transforms for the ruin probability (4.21) with gamma distributions instead of exponential distributions.

4.2.3 The De-Vylder approximation

In this section, we consider the SPST (4.1) where premiums, claim sizes and surrenders follow other distributions rather than exponential distributions. The De-Vylder approximation is used to approximate the risk process by the classical risk model where the numbers of premiums, claims, and surrenders are exponentially distributed. Specifically, the model (4.1) is approximated by the following risk model.

$$\widetilde{U}(t) = u + \sum_{i=1}^{\widetilde{N}(t)} \widetilde{X}_i - \sum_{i=1}^{\widetilde{N}(t,\widetilde{p})} \widetilde{Y}_i - \sum_{i=1}^{\widetilde{N}(t,\widetilde{q})} \widetilde{Z}_i, \tag{4.29}$$

where $\widetilde{X}_i, \widetilde{Y}_i$, and $\widetilde{Z}_i$ have exponential distributions with parameters $\widetilde{a}, \widetilde{b}$, and $\widetilde{c}$, respectively. Also, $\widetilde{N}(t), \widetilde{N}(t,\widetilde{p})$, and $\widetilde{N}(t,\widetilde{q})$ are Poisson processes with intensities $\widetilde{\lambda}$, $\widetilde{\lambda}\widetilde{p}$, and $\widetilde{\lambda}\widetilde{q}$, respectively.

Since in this risk model the process $\{\widetilde{U}(t)\}_{t\geq 0}$ is determined by six parameters $\widetilde{\lambda}$, $\widetilde{p}$, $\widetilde{q}$, $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$, six equalities are required to determine these parameters. Therefore. we need to compute the first six moments of $\widetilde{U}(t)$ described in [17].

Theorem 4.4. *For the risk model (4.1), let $M_X(s)$, $M_Y(s)$, and $M_Z(s)$ be the moment generating functions of the random variables X, Y, and Z, respectively. Then, for any s in the domain of $M_{U(t)}$, we have*

$$M_{U(t)}(s) = \exp\left\{su + t\lambda\big(M(s) - 1 - p - q\big)\right\},$$

$$M'_{U(t)}(s) = M_{U(t)}(s)\big(u + t\lambda M'(s)\big),$$

$$M''_{U(t)}(s) = M_{U(t)}(s)\Big(\big(u + t\lambda M'(s)\big)^2 + t\lambda M''(s)\Big),$$

$$M'''_{U(t)}(s) = M_{U(t)}(s)\Big(\big(u + t\lambda M'(s)\big)^3 + t\lambda M'''(s) + 3t\lambda\big(u + t\lambda M'(s)\big)M''(s)\Big),$$

$$\begin{aligned} M^{(4)}_{U(t)}(s) = M_{U(t)}(s)\Big(&\big(u + t\lambda M'(s)\big)^4 + 6t\lambda M''(s)\big(u + t\lambda M'(s)\big)^2 \\ &+4t\lambda M'''(s)\big(u + t\lambda M'(s)\big) + 3t^2\lambda^2\left[M''(s)\right]^2 + t\lambda M^{(4)}(s)\Big), \end{aligned}$$

$$M^{(5)}_{U(t)}(s) = M_{U(t)}(s)\Big(\big(u + t\lambda M'(s)\big)^5 + 10t\lambda M''(s)\big(u + t\lambda M'(s)\big)^3$$

$$+10t\lambda M'''(s)\big(u+t\lambda M'(s)\big)^2+5t\lambda M^{(4)}(s)\big(u+t\lambda M'(s)\big)$$
$$+15t^2\lambda^2\left[M''(s)\right]^2\left(u+t\lambda M'(s)\right]\big)+10t^2\lambda^2 M''(s)M'''(s)+t\lambda M^{(5)}(s)\bigg),$$

$$M^{(6)}_{U(t)}(s)=M_{U(t)}(s)\bigg(\big(u+t\lambda M'(s)\big)^6+15t\lambda M''(s)\big(u+t\lambda M'(s)\big)^4$$
$$+20t\lambda M'''(s)\big(u+t\lambda M'(s)\big)^3+15t\lambda M^{(4)}(s)\big(u+t\lambda M'(s)\big)^2$$
$$+45t^2\lambda^2\left[M''(s)\right]^2\left(u+t\lambda M'(s)\right)^2+6t\lambda M^{(5)}(s)\big(u+t\lambda M'(s)\big)$$
$$+60t^2\lambda^2\big(u+t\lambda M'(s)\big)M''(s)M'''(s)+t\lambda M^{(6)}(s)$$
$$+15t^2\lambda^2 M''(s)M^{(4)}(s)+10t^2\lambda^2\left[M'''(s)\right]^2+15t^3\lambda^3\left[M''(s)\right]^3\bigg),$$

where $M(s)=M_X(s)+pM_Y(-s)+qM_Z(-s)$.

Proof. By the formula for the moment generating function of $S(t)$ in Lemma 4.1, we have

$$\begin{aligned} M_{U(t)}(s) &= E[e^{s(u+S(t))}] \\ &= \exp\left\{su+t\lambda\left[(M_X(s)-1)+p\left(M_Y(-s)-1\right)+q\left(M_Z(-s)-1\right)\right]\right\} \\ &= \exp\left\{su+t\lambda\left(M(s)-1-p-q\right)\right\}. \end{aligned}$$

Differentiating with respect to s on both sides of the equation, we have that

$$\begin{aligned} M'_{U(t)}(s) &= \exp\left\{su+t\lambda\big(M(s)-1-p-q\big)\right\}\cdot\big(u+t\lambda M'(s)\big) \\ &= M_{U(t)}(s)\big(u+t\lambda M'(s)\big). \end{aligned}$$

Consequently,

$$\begin{aligned} M''_{U(t)}(s) &= M_{U(t)}(s)t\lambda M''(s)+M'_{U(t)}(s)\big(u+t\lambda M'(s)\big) \\ &= M_{U(t)}(s)\Big(\big(u+t\lambda M'(s)\big)^2+t\lambda M''(s)\Big). \end{aligned}$$

Straightforwardly, we can calculate $M'''_{U(t)}(s), M^{(4)}_{U(t)}(s), M^{(5)}_{U(t)}(s)$ and $M^{(6)}_{U(t)}(s)$ and obtain the following results.

$$M'''_{U(t)}(s)=M_{U(t)}(s)\Big(\big(u+t\lambda M'(s)\big)^3+t\lambda M'''(s)+3t\lambda\big(u+t\lambda M'(s)\big)M''(s)\Big),$$

$$
\begin{aligned}
M_{U(t)}^{(4)}(s) = M_{U(t)}(s)\Big(&\big(u+t\lambda M'(s)\big)^4 + 6t\lambda M''(s)\big(u+t\lambda M'(s)\big)^2 \\
&+4t\lambda M'''(s)\big(u+t\lambda M'(s)\big) + 3t^2\lambda^2\left[M''(s)\right]^2 + t\lambda M^{(4)}(s)\Big), \\
M_{U(t)}^{(5)}(s) = M_{U(t)}(s)\Big(&\big(u+t\lambda M'(s)\big)^5 + 10t\lambda M''(s)\big(u+t\lambda M'(s)\big)^3 \\
&+10t\lambda M'''(s)\big(u+t\lambda M'(s)\big)^2 + 5t\lambda M^{(4)}(s)\big(u+t\lambda M'(s)\big) \\
&+15t^2\lambda^2\left[M''(s)\right]^2\left(u+t\lambda M'(s)]\right)+10t^2\lambda^2 M''(s)M'''(s)+t\lambda M^{(5)}(s)\Big), \\
M_{U(t)}^{(6)}(s) = M_{U(t)}(s)\Big(&\big(u+t\lambda M'(s)\big)^6 + 15t\lambda M''(s)\big(u+t\lambda M'(s)\big)^4 \\
&+20t\lambda M'''(s)\big(u+t\lambda M'(s)\big)^3 + 15t\lambda M^{(4)}(s)\big(u+t\lambda M'(s)\big)^2 \\
&+45t^2\lambda^2\left[M''(s)\right]^2\big(u+t\lambda M'(s)\big)^2 + 6t\lambda M^{(5)}(s)\big(u+t\lambda M'(s)\big) \\
&+60t^2\lambda^2\big(u+t\lambda M'(s)\big)M''(s)M'''(s) + t\lambda M^{(6)}(s) \\
&+15t^2\lambda^2 M''(s)M^{(4)}(s)+10t^2\lambda^2\left[M'''(s)\right]^2+15t^3\lambda^3\left[M''(s)\right]^3\Big). \quad \square
\end{aligned}
$$

For $k \in \{1, 2, \ldots, 6\}$, since $M_{U(t)}^{k}(s)$ is in the form of $M^k(s)$, we can find the equation for $M^k(s)$ for $k \in \{1, 2, \ldots, 6\}$ from the Remark 4.2.

Remark 4.2. For $n \in \mathbb{N}$, the n^{th} derivative of the function $M(s) = e^{sc} + pM_Y(-s) + qM_Z(-s)$ is given by

$$M^{(n)}(s) = M_X^{(n)}(s) + (-1)^n pM_Y^{(n)}(-s) + (-1)^n qM_Z^{(n)}(-s)$$

Corollary 4.1. *For the risk model (4.1), we assume that $X_i, Y_i,$ and Z_i have finite first six moments. Then, for all $t \geq 0$, we have*

$$
\begin{aligned}
E[U(t)] &= u + t\lambda\big(E[X] - pE[Y] - qE[Z]\big), \\
E[U^2(t)] &= (E[U(t)])^2 + t\lambda\left(E[X^2] + pE[Y^2] + qE[Z^2]\right), \\
E[U^3(t)] &= (E[U(t)])^3 + t\lambda\left(E[X^3] - pE[Y^3] - qE[Z^3]\right) \\
&\quad + 3t\lambda E[U(t)]\left(E[X^2] + pE[Y^2] + qE[Z^2]\right), \\
E[U^4(t)] &= (E[U(t)])^4 + 6t\lambda\left(E[X^2] + pE[Y^2] + qE[Z^2]\right)(E[U(t)])^2 \\
&\quad + 4t\lambda\left(E[X^3] - pE[Y^3] - qE[Z^3]\right)E[U(t)] \\
&\quad + 3t^2\lambda^2\left(E[X^2] + pE[Y^2] + qE[Z^2]\right)^2 \\
&\quad + t\lambda\left(E[X^4] + pE[Y^4] + qE[Z^4]\right),
\end{aligned}
$$

$$
\begin{aligned}
E[U^5(t)] &= (E[U(t)])^5 + 10t\lambda\left(E[X^2] + pE[Y^2] + qE[Z^2]\right)(E[U(t)])^3 \\
&\quad + 10t\lambda\left(E[X^3] - pE[Y^3] - qE[Z^3]\right)(E[U(t)])^2 \\
&\quad + 5t\lambda\left(E[X^4] + pE[Y^4] + qE[Z^4]\right)E[U(t)] \\
&\quad + 15t^2\lambda^2\left(E[X^2] + pE[Y^2] + qE[Z^2]\right)^2 E[U(t)] \\
&\quad + 10t^2\lambda^2\left(E[X^2] + pE[Y^2] + qE[Z^2]\right)\left(E[X^3] - pE[Y^3] - qE[Z^3]\right) \\
&\quad + t\lambda\left(E[X^5] - pE[Y^5] - qE[Z^5]\right), \\
E[U^6(t)] &= (E[U(t)])^6 + 15t\lambda\left(E[X^2] + pE[Y^2] + qE[Z^2]\right)(E[U(t)])^4 \\
&\quad + 20t\lambda\left(E[X^3] - pE[Y^3] - qE[Z^3]\right)(E[U(t)])^3 \\
&\quad + 15t\lambda\left(E[X^4] + pE[Y^4] + qE[Z^4]\right)(E[U(t)])^2 \\
&\quad + 45t^2\lambda^2\left(E[X^2] + pE[Y^2] + qE[Z^2]\right)^2(E[U(t)])^2 \\
&\quad + 6t\lambda\left(E[X^5] - pE[Y^5] - qE[Z^5]\right)E[U(t)] \\
&\quad + 60t^2\lambda^2 E[U(t)]\left(E[X^2] + pE[Y^2] + qE[Z^2]\right)\left(E[X^3] - pE[Y^3] - qE[Z^3]\right) \\
&\quad + t\lambda\left(E[X^6] + pE[Y^6] + qE[Z^6]\right) \\
&\quad + 15t^2\lambda^2\left(E[X^2] + pE[Y^2] + qE[Z^2]\right)\left[E[X^4] + pE[Y^4] + qE[Z^4]\right] \\
&\quad + 10t^2\lambda^2\left(E[X^3] - pE[Y^3] - qE[Z^3]\right)^2 \\
&\quad + 15t^3\lambda^3\left(E[X^2] + pE[Y^2] + qE[Z^2]\right)^3.
\end{aligned}
$$

Proof. Since $E[U^n(t)] = M^{(n)}_{U(t)}(0)$ and $M^{(n)}(0) = E[X^n] + (-1)^n pE[Y^n] + (-1)^n qE[Z^n]$ for all $n \in \mathbb{N}$, substituting $s = 0$ into the formulas in Theorem (4.4) yields the desired results. □

For the risk model (4.29) where $\widetilde{X}_i$, $\widetilde{Y}_i$, and $\widetilde{Z}_i$ have exponential distributions with parameters $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$ respectively, let $\widetilde{A} = \frac{1}{\widetilde{a}}$, $\widetilde{B} = \frac{1}{\widetilde{b}}$, and $\widetilde{C} = \frac{1}{\widetilde{c}}$ so that the mean of $\widetilde{X}_i$, $\widetilde{Y}_i$, and $\widetilde{Z}_i$ are $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$, respectively. We will deal with parameters $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ instead of $\widetilde{a}$, $\widetilde{b}$ and $\widetilde{c}$ for the sake of simplicity of the final formula.

Theorem 4.5. *We can approximate the process $\{U(t)\}_{t\geq 0}$ in the risk model (4.1) by a process $\{\widetilde{U}(t)\}_{t\geq 0}$ in the risk model (4.29) with parameters $\widetilde{\lambda}$, $\widetilde{p}$, $\widetilde{q}$, $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ by matching the first six moments, i.e., $E[U(t)^k] = E[\widetilde{U}(t)^k]$ for $k = 1, 2, \ldots, 6$. The desired* *parameters $\widetilde{\lambda}$, $\widetilde{p}$, $\widetilde{q}$, $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ can be solved from the system of equations:*

$$\widetilde{\lambda} = \frac{\gamma_3 + 6\gamma_1 \widetilde{B}\widetilde{C} + 3\gamma_2 \left[\widetilde{B} + \widetilde{C}\right]}{6\widetilde{A}\left(\widetilde{A} + \widetilde{B}\right)\left(\widetilde{A} + \widetilde{C}\right)},$$

$$\widetilde{q} = \frac{\gamma_2 + 2\gamma_1 \widetilde{B} - 2\widetilde{A}\left[\widetilde{A} + \widetilde{B}\right]\widetilde{\lambda}}{2\widetilde{C}\left(\widetilde{C} - \widetilde{B}\right)\widetilde{\lambda}},$$

$$\widetilde{p} = \frac{\widetilde{\lambda}\widetilde{A} - \gamma_1 - \widetilde{\lambda} q \widetilde{C}}{\widetilde{\lambda}\widetilde{B}},$$

and

$$\begin{aligned}
\widetilde{A}\widetilde{B}\widetilde{C} &= \frac{75\gamma_4^3 + 36\gamma_2\gamma_5^2 + 40\gamma_3^2\gamma_6 - 30\gamma_4\left[4\gamma_3\gamma_5 + \gamma_2\gamma_6\right]}{120\left[40\gamma_3^3 - 12\gamma_3\left(5\gamma_2\gamma_4 + \gamma_1\gamma_5\right) + 3\left(5\gamma_1\gamma_4^2 + 6\gamma_2^2\gamma_5\right)\right]}, \\
\widetilde{A}\widetilde{B} + \widetilde{A}\widetilde{C} - \widetilde{B}\widetilde{C} &= \frac{20\gamma_3^2\gamma_5 + 15\gamma_2\gamma_4\gamma_5 - 6\gamma_1\gamma_5^2 + 5\gamma_1\gamma_4\gamma_6 - 5\gamma_3\left[5\gamma_4^2 + 2\gamma_2\gamma_6\right]}{10\left[40\gamma_3^3 - 12\gamma_3\left(5\gamma_2\gamma_4 + \gamma_1\gamma_5\right) + 3\left(5\gamma_1\gamma_4^2 + 6\gamma_2^2\gamma_5\right)\right]}, \\
\widetilde{A} - \widetilde{B} - \widetilde{C} &= \frac{20\gamma_3^2\gamma_4 - 15\gamma_2\gamma_4^2 + 6\gamma_1\gamma_4\gamma_5 + 6\gamma_2^2\gamma_6 - 4\gamma_3\left[3\gamma_2\gamma_5 + \gamma_1\gamma_6\right]}{80\gamma_3^3 - 24\gamma_3\left(5\gamma_2\gamma_4 + \gamma_1\gamma_5\right) + 6\left(5\gamma_1\gamma_4^2 + 6\gamma_2^2\gamma_5\right)},
\end{aligned}$$

where

$$\begin{aligned}
\gamma_1 &= \lambda\left[E[X] - pE[Y] - qE[Z]\right], \\
\gamma_2 &= \lambda\left[E[X^2] + pE[Y^2] + qE[Z^2]\right], \\
\gamma_3 &= \lambda\left[E[X^3] - pE[Y^3] - qE[Z^3]\right], \\
\gamma_4 &= \lambda\left[E[X^4] + pE[Y^4] + qE[Z^4]\right], \\
\gamma_5 &= \lambda\left[E[X^5] - pE[Y^5] - qE[Z^5]\right], \\
\gamma_6 &= \lambda\left[E[X^6] + pE[Y^6] + qE[Z^6]\right].
\end{aligned}$$

Proof. Taking the k-th moment of the random variable that are exponentially distributed into the equation $E[U(t)^k] = E[\widetilde{U}(t)^k]$, we have the system of equations

$$\gamma_1 = \widetilde{\lambda}\widetilde{A} - \widetilde{\lambda}\widetilde{p}\widetilde{B} - \widetilde{\lambda}\widetilde{q}\widetilde{C}, \tag{4.30}$$

$$\gamma_2 = 2\widetilde{\lambda}\widetilde{A}^2 + 2\widetilde{\lambda}\widetilde{p}\widetilde{B}^2 + 2\widetilde{\lambda}\widetilde{q}\widetilde{C}^2, \tag{4.31}$$

$$\gamma_3 = 6\widetilde{\lambda}\widetilde{A}^3 - 6\widetilde{\lambda}\widetilde{p}\widetilde{B}^3 - 6\widetilde{\lambda}\widetilde{q}\widetilde{C}^3, \tag{4.32}$$

$$\gamma_4 = 24\widetilde{\lambda}\widetilde{A}^4 + 24\widetilde{\lambda}\widetilde{p}\widetilde{B}^4 + 24\widetilde{\lambda}\widetilde{q}\widetilde{C}^4, \tag{4.33}$$

$$\gamma_5 = 120\widetilde{\lambda}\widetilde{A}^5 - 120\widetilde{\lambda}\widetilde{p}\widetilde{B}^5 - 120\widetilde{\lambda}\widetilde{q}\widetilde{C}^5, \tag{4.34}$$

$$\gamma_6 = 720\widetilde{\lambda}\widetilde{A}^6 + 720\widetilde{\lambda}\widetilde{p}\widetilde{B}^6 + 720\widetilde{\lambda}\widetilde{q}\widetilde{C}^6, \tag{4.35}$$

Now our aim is to find the constants $\widetilde{\lambda}$, $\widetilde{p}$, $\widetilde{q}$, $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ from this system. From (4.30) we have $\widetilde{\lambda}\widetilde{p}\widetilde{B} = \widetilde{\lambda}\widetilde{A} - \gamma_1 - \widetilde{\lambda}\widetilde{q}\widetilde{C}$. Substituting this into (4.31)–(4.35), we get

$$\gamma_2 = -2\gamma_1\widetilde{B} + 2\widetilde{A}\left(\widetilde{A}+\widetilde{B}\right)\widetilde{\lambda} + 2\widetilde{C}\left(\widetilde{C}-\widetilde{B}\right)\widetilde{\lambda}\widetilde{q}, \tag{4.36}$$

$$\gamma_3 = 6\gamma_1\widetilde{B}^2 + 6\widetilde{A}\left(\widetilde{A}-\widetilde{B}\right)\left(\widetilde{A}+\widetilde{B}\right)\widetilde{\lambda} + 6\widetilde{C}\left(\widetilde{B}^2-\widetilde{C}^2\right)\widetilde{\lambda}\widetilde{q}, \tag{4.37}$$

$$\gamma_4 = -24\gamma_1\widetilde{B}^3 + 24\widetilde{A}\left(\widetilde{A}+\widetilde{B}\right)\left(\widetilde{A}^2-\widetilde{A}\widetilde{B}+\widetilde{B}^2\right)\widetilde{\lambda} + 24\widetilde{C}\left(\widetilde{C}^3-\widetilde{B}^3\right)\widetilde{\lambda}\widetilde{q}, \tag{4.38}$$

$$\gamma_5 = 120\gamma_1\widetilde{B}^4 + 120\widetilde{A}\left(\widetilde{A}-\widetilde{B}\right)\left(\widetilde{A}+\widetilde{B}\right)\left(\widetilde{A}^2+\widetilde{B}^2\right)\widetilde{\lambda} + 120\widetilde{C}\left(\widetilde{B}^4-\widetilde{C}^4\right)\widetilde{\lambda}\widetilde{q}, \tag{4.39}$$

$$\begin{aligned}\gamma_6 = &-720\gamma_1\widetilde{B}^5 + 720\widetilde{A}\left(\widetilde{A}+\widetilde{B}\right)\left(\widetilde{A}^4-\widetilde{A}^3\widetilde{B}+\widetilde{A}^2\widetilde{B}^2-\widetilde{A}\widetilde{B}^3+\widetilde{B}^4\right)\widetilde{\lambda}\\ &+ 720\widetilde{C}\left(\widetilde{C}^5-\widetilde{B}^5\right)\widetilde{\lambda}\widetilde{q}.\end{aligned} \tag{4.40}$$

Next, from (4.36) we have $2\widetilde{C}\left(\widetilde{C}-\widetilde{B}\right)\widetilde{\lambda}\widetilde{q} = \gamma_2 + 2\gamma_1\widetilde{B} - 2\widetilde{A}\left(\widetilde{A}+\widetilde{B}\right)\widetilde{\lambda}$. Substituting this into (4.37)–(4.40), we obtain

$$\gamma_3 = -6\gamma_1\widetilde{B}\widetilde{C} - 3\gamma_2\left(\widetilde{B}+\widetilde{C}\right) + 6\widetilde{A}\left(\widetilde{A}+\widetilde{B}\right)\left(\widetilde{A}+\widetilde{C}\right)\widetilde{\lambda}, \tag{4.41}$$

$$\begin{aligned}\gamma_4 = \ &24\gamma_1\widetilde{B}\widetilde{C}\left(\widetilde{B}+\widetilde{C}\right) + 12\gamma_2\left(\widetilde{B}^2+\widetilde{B}\widetilde{C}+\widetilde{C}^2\right)\\ &+24\widetilde{A}\left(\widetilde{A}+\widetilde{B}\right)\left(\widetilde{A}-\widetilde{B}-\widetilde{C}\right)\left(\widetilde{A}+\widetilde{C}\right)\widetilde{\lambda},\end{aligned} \tag{4.42}$$

$$\begin{aligned}\gamma_5 = \ &-120\gamma_1\widetilde{B}\widetilde{C}\left(\widetilde{B}^2+\widetilde{B}\widetilde{C}+\widetilde{C}^2\right) - 60\gamma_2\left(\widetilde{B}+\widetilde{C}\right)\left(\widetilde{B}^2+\widetilde{C}^2\right)\\ &+120\widetilde{A}\left(\widetilde{A}+\widetilde{B}\right)\left(\widetilde{A}+\widetilde{C}\right)\left(\widetilde{A}^2-\widetilde{A}\widetilde{B}+\widetilde{B}^2-\widetilde{A}\widetilde{C}+\widetilde{B}\widetilde{C}+\widetilde{C}^2\right)\widetilde{\lambda},\end{aligned} \tag{4.43}$$

$$\gamma_6 = 720\gamma_1\widetilde{B}\widetilde{C}\left(\widetilde{B}+\widetilde{C}\right)\left(\widetilde{B}^2+\widetilde{C}^2\right)+360\gamma_2\left(\widetilde{B}^4+\widetilde{B}^3\widetilde{C}+\widetilde{B}^2\widetilde{C}^2+\widetilde{B}\widetilde{C}^3+\widetilde{C}^4\right)$$
$$+720\widetilde{A}\left(\widetilde{A}+\widetilde{B}\right)\left(\widetilde{A}+\widetilde{C}\right)\left[\left(\widetilde{A}^2+\widetilde{B}^2+\widetilde{C}^2\right)\left(\widetilde{A}-\widetilde{B}-\widetilde{C}\right)+\widetilde{A}\widetilde{B}\widetilde{C}\right]\widetilde{\lambda}. \tag{4.44}$$

Next, from (4.41) we have $6\widetilde{A}\left(\widetilde{A}+\widetilde{B}\right)\left(\widetilde{A}+\widetilde{C}\right)\widetilde{\lambda}=\gamma_3+6\gamma_1\widetilde{B}\widetilde{C}+3\gamma_2\left(\widetilde{B}+\widetilde{C}\right)$. Substituting this into (4.42)–(4.44), we obtain

$$\gamma_4 = 24\gamma_1\widetilde{A}\widetilde{B}\widetilde{C}+12\gamma_2\left(\widetilde{A}\widetilde{B}+\widetilde{A}\widetilde{C}-\widetilde{B}\widetilde{C}\right)+4\gamma_3\left(\widetilde{A}-\widetilde{B}-\widetilde{C}\right), \tag{4.45}$$

$$\gamma_5 = 120\gamma_1\widetilde{A}\widetilde{B}\widetilde{C}\left(\widetilde{A}-\widetilde{B}-\widetilde{C}\right)+60\gamma_2\left(\widetilde{A}-\widetilde{B}\right)\left(\widetilde{A}-\widetilde{C}\right)\left(\widetilde{B}+\widetilde{C}\right)$$
$$+20\gamma_3\left(\widetilde{A}^2-\widetilde{A}\widetilde{B}+\widetilde{B}^2-\widetilde{A}\widetilde{C}+\widetilde{B}\widetilde{C}+\widetilde{C}^2\right), \tag{4.46}$$

$$\gamma_6 = 720\gamma_1\widetilde{A}\widetilde{B}\widetilde{C}\left(\widetilde{A}^2-\widetilde{A}\widetilde{B}+\widetilde{B}^2-\widetilde{A}\widetilde{C}+\widetilde{B}\widetilde{C}+\widetilde{C}^2\right)$$
$$+360\gamma_2\left[\widetilde{A}^3\widetilde{B}-\widetilde{A}^2\widetilde{B}^2+\widetilde{A}\widetilde{B}^3+\widetilde{A}^3\widetilde{C}-2\widetilde{A}^2\widetilde{B}\widetilde{C}+2\widetilde{A}\widetilde{B}^2\widetilde{C}-\widetilde{B}^3\widetilde{C}-\widetilde{A}^2\widetilde{C}^2\right.$$
$$\left.+2\widetilde{A}\widetilde{B}\widetilde{C}^2-\widetilde{B}^2\widetilde{C}^2+\widetilde{A}\widetilde{C}^3-\widetilde{B}\widetilde{C}^3\right]$$
$$+120\gamma_3\left[\widetilde{A}^3-\widetilde{A}^2\widetilde{B}+\widetilde{A}\widetilde{B}^2-\widetilde{B}^3-\widetilde{A}^2\widetilde{C}+\widetilde{A}\widetilde{B}\widetilde{C}-\widetilde{B}^2\widetilde{C}+\widetilde{A}\widetilde{C}^2-\widetilde{B}\widetilde{C}^2-\widetilde{C}^3\right]. \tag{4.47}$$

Multiplying (4.45) by $-5\left(\widetilde{A}-\widetilde{B}-\widetilde{C}\right)$ and adding (4.46), we get

$$\gamma_5 = 60\gamma_2\widetilde{A}\widetilde{B}\widetilde{C}+20\gamma_3\left(\widetilde{A}\widetilde{B}+\widetilde{A}\widetilde{C}-\widetilde{B}\widetilde{C}\right)+5\gamma_4\left(\widetilde{A}-\widetilde{B}-\widetilde{C}\right). \tag{4.48}$$

Multiplying (4.45) by $-30\left(\widetilde{A}^2-\widetilde{A}\widetilde{B}+\widetilde{B}^2-\widetilde{A}\widetilde{C}+\widetilde{B}\widetilde{C}+\widetilde{C}^2\right)$ and adding (4.47), we get

$$\gamma_6 = 360\gamma_2\widetilde{A}\widetilde{B}\widetilde{C}\left(\widetilde{A}-\widetilde{B}-\widetilde{C}\right)+120\gamma_3\left(\widetilde{A}-\widetilde{B}\right)\left(\widetilde{A}-\widetilde{C}\right)\left(\widetilde{B}+\widetilde{C}\right)$$
$$+30\gamma_4\left(\widetilde{A}^2-\widetilde{A}\widetilde{B}+\widetilde{B}^2-\widetilde{A}\widetilde{C}+\widetilde{B}\widetilde{C}+\widetilde{C}^2\right). \tag{4.49}$$

Multiplying (4.48) by $-6\left(\widetilde{A}-\widetilde{B}-\widetilde{C}\right)$ and adding (4.49), we get

$$\gamma_6 = 120\gamma_3\widetilde{A}\widetilde{B}\widetilde{C}+30\gamma_4\left(\widetilde{A}\widetilde{B}+\widetilde{A}\widetilde{C}-\widetilde{B}\widetilde{C}\right)+6\gamma_5\left(\widetilde{A}-\widetilde{B}-\widetilde{C}\right). \tag{4.50}$$

Hence, we get the desired system of equations. □

Theorem 4.6. ***The De-Vylder approximation***

For the risk model (4.1) under a assumptions that X_i, Y_i, and Z_i have finite six moments and that the net profit condition (4.3) hold, the De-Vylder approximation of ruin probability $\psi_{De}(u)$ is given by

$$\psi_{De}(u) = C_1 e^{r_1 u} + C_2 e^{r_2 u} \quad for\ all\ u \geq 0, \tag{4.51}$$

where

$$r_1 = \frac{A + \sqrt{D}}{B}\ ,\ r_2 = \frac{A - \sqrt{D}}{B},$$

$$,$$

$$A = -\left[(\widetilde{p} + \widetilde{q})\widetilde{B}\widetilde{C} - (1 + \widetilde{q})\widetilde{A}\widetilde{C} - (1 + \widetilde{p})\widetilde{A}\widetilde{B}\right],$$

$$B = -2(1 + \widetilde{p} + \widetilde{q})\widetilde{A}\widetilde{B}\widetilde{C},$$

$$D = \left[(\widetilde{p} + \widetilde{q})\widetilde{B}\widetilde{C} - (1 + \widetilde{q})\widetilde{A}\widetilde{C} - (1 + \widetilde{p})\widetilde{A}\widetilde{B}\right]^2 - 4(1 + \widetilde{p} + \widetilde{q})\widetilde{A}\widetilde{B}\widetilde{C}\left[\widetilde{A} - \widetilde{p}\widetilde{B} - \widetilde{q}\widetilde{C}\right],$$

$$C_1 = \frac{C_{11}}{C_D}\ ,\ C_2 = \frac{C_{21}}{C_D},$$

which

$$C_{11} = \left(\frac{1}{\widetilde{A}} - r_1\right) r_2 \left(\frac{\widetilde{p}}{\widetilde{B}} + \frac{\widetilde{q}}{\widetilde{C}} - \frac{(\widetilde{p} + \widetilde{q})^2}{\widetilde{A}} + (\widetilde{p} + \widetilde{q})(1 + \widetilde{p} + \widetilde{q})r_2\right),$$
$$C_{21} = \left(\frac{1}{\widetilde{A}} - r_2\right) r_1 \left(\frac{\widetilde{p}}{\widetilde{B}} + \frac{\widetilde{q}}{\widetilde{C}} - \frac{(\widetilde{p} + \widetilde{q})^2}{\widetilde{A}} + (\widetilde{p} + \widetilde{q})(1 + \widetilde{p} + \widetilde{q})r_1\right),$$

$$C_D$$
$$= (r_1 - r_2)\left[\frac{(\widetilde{p} + \widetilde{q})^2}{\widetilde{A}^2} + (1 + \widetilde{p} + \widetilde{q})^2 r_1 r_2 - \frac{1}{\widetilde{A}}\left(\frac{\widetilde{p}}{\widetilde{B}} + \frac{\widetilde{q}}{\widetilde{C}} + (\widetilde{p} + \widetilde{q})(1 + \widetilde{p} + \widetilde{q})(r_1 + r_2)\right)\right].$$

and the constants $\widetilde{\lambda}$, $\widetilde{p}$, $\widetilde{q}$, $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ are obtained from solving the system of equations stated in Theorem 4.5 which have the following values:

$$\widetilde{\lambda} = \frac{\gamma_3 + 6\gamma_1 \widetilde{B}\widetilde{C} + 3\gamma_2\left[\widetilde{B} + \widetilde{C}\right]}{6\widetilde{A}\left(\widetilde{A} + \widetilde{B}\right)\left(\widetilde{A} + \widetilde{C}\right)},$$

$$\widetilde{q} = \frac{\gamma_2 + 2\gamma_1 \widetilde{B} - 2\widetilde{A}\left[\widetilde{A} + \widetilde{B}\right]\widetilde{\lambda}}{2\widetilde{C}\left(\widetilde{C} - \widetilde{B}\right)\widetilde{\lambda}},$$

$$\widetilde{p} = \frac{\widetilde{\lambda}\widetilde{A} - \gamma_1 - \widetilde{\lambda} q \widetilde{C}}{\widetilde{\lambda}\widetilde{B}}.$$

and $\widetilde{A}$, $\widetilde{B}$ *and* $\widetilde{C}$ *are obtained from solving the system of equations*

$$\widetilde{A}\widetilde{B}\widetilde{C} = \frac{75\gamma_4^3 + 36\gamma_2\gamma_5^2 + 40\gamma_3^2\gamma_6 - 30\gamma_4\left[4\gamma_3\gamma_5 + \gamma_2\gamma_6\right]}{120\left[40\gamma_3^3 - 12\gamma_3\left(5\gamma_2\gamma_4 + \gamma_1\gamma_5\right) + 3\left(5\gamma_1\gamma_4^2 + 6\gamma_2^2\gamma_5\right)\right]},$$

$$\widetilde{A}\widetilde{B} + \widetilde{A}\widetilde{C} - \widetilde{B}\widetilde{C} = \frac{20\gamma_3^2\gamma_5 + 15\gamma_2\gamma_4\gamma_5 - 6\gamma_1\gamma_5^2 + 5\gamma_1\gamma_4\gamma_6 - 5\gamma_3\left[5\gamma_4^2 + 2\gamma_2\gamma_6\right]}{10\left[40\gamma_3^3 - 12\gamma_3\left(5\gamma_2\gamma_4 + \gamma_1\gamma_5\right) + 3\left(5\gamma_1\gamma_4^2 + 6\gamma_2^2\gamma_5\right)\right]},$$

$$\widetilde{A} - \widetilde{B} - \widetilde{C} = \frac{20\gamma_3^2\gamma_4 - 15\gamma_2\gamma_4^2 + 6\gamma_1\gamma_4\gamma_5 + 6\gamma_2^2\gamma_6 - 4\gamma_3\left[3\gamma_2\gamma_5 + \gamma_1\gamma_6\right]}{80\gamma_3^3 - 24\gamma_3\left(5\gamma_2\gamma_4 + \gamma_1\gamma_5\right) + 6\left(5\gamma_1\gamma_4^2 + 6\gamma_2^2\gamma_5\right)},$$

where

$$\begin{aligned}
\gamma_1 &= \lambda\left[E[X] - pE[Y] - qE[Z]\right], \\
\gamma_2 &= \lambda\left[E[X^2] + pE[Y^2] + qE[Z^2]\right], \\
\gamma_3 &= \lambda\left[E[X^3] - pE[Y^3] - qE[Z^3]\right], \\
\gamma_4 &= \lambda\left[E[X^4] + pE[Y^4] + qE[Z^4]\right], \\
\gamma_5 &= \lambda\left[E[X^5] - pE[Y^5] - qE[Z^5]\right], \\
\gamma_6 &= \lambda\left[E[X^6] + pE[Y^6] + qE[Z^6]\right].
\end{aligned}$$

To calculate the approximated ruin probability using the De-Vylder approximation described in (4.51), we can use the MATLAB commands "`solve`" for computation.

4.3 Lundberg's inequality

In this section, we will study the martingale and stopping time. This will allow us to find the adjustment coefficient equation, Lundberg's inequality for the ruin probability, which it can be used to create as Lemma, Theorem and Corollary.

Theorem 4.7. *For the profits process* $\{S(t); t \geq 0\}$,

$$E[e^{-rS(t)}] = e^{tg(r)}, \tag{4.52}$$

where

$$g(r) = -\lambda\left[1 - M_X(-r)\right] - \lambda p\left[1 - M_Y(r)\right] - \lambda q\left[1 - M_Z(r)\right]. \tag{4.53}$$

Proof. Since $X_i, Y_i, Z_i, N(t), N(t,p)$ and $N(t,q)$ are mutually independent,

$$\begin{aligned} E\left[e^{-rS(t)}\right] &= E\left[\exp\{-r\sum_{i=1}^{N(t)} X_i\} \cdot \exp\{r\sum_{i=1}^{N(t,p)} Y_i\} \cdot \exp\{r\sum_{i=1}^{N(t,q)} Z_i\}\right] \\ &= E\left[\exp\{-r\sum_{i=1}^{N(t)} X_i\}\right] \cdot E\left[\exp\{r\sum_{i=1}^{N(t,p)} Y_i\}\right] \cdot E\left[\exp\{r\sum_{i=1}^{N(t,q)} Z_i\}\right] \end{aligned}$$

By definition of MGF in Definition 2.24 and Theorem 2.22, we get

$$\begin{aligned} E\left[e^{-rS(t)}\right] &= M_{\sum_{i=1}^{N(t)} X_i}(-r) \cdot M_{\sum_{i=1}^{N(t,p)} Y_i}(r) \cdot M_{\sum_{i=1}^{N(t,q)} Z_i}(r) \\ &= e^{-\lambda t[1-M_X(-r)]} e^{-\lambda pt[1-M_Y(r)]} e^{-\lambda qt[1-M_Z(r)]} \\ &= \exp\left\{t\left(-\lambda\left[1 - M_X(-r)\right] - \lambda p\left[1 - M_Y(r)\right]\right) - \lambda q\left[1 - M_Z(r)\right]\right\}. \end{aligned}$$

Therefore,

$$E[e^{-rS(t)}] = e^{tg(r)},$$

where

$$g(r) = -\lambda\left[1 - M_X(-r)\right] - \lambda p\left[1 - M_Y(r)\right] - \lambda q\left[1 - M_Z(r)\right].$$

Then, we obtain (4.52).

□

Theorem 4.8. *Equation*

$$g(r) = 0 \tag{4.54}$$

has a unique positive solution $r = R > 0$, we can call (3.63) *is said to be an adjustment coefficient equation of the risk model* (3.5)*, with $R > 0$ is said to be an adjustment coefficient.*

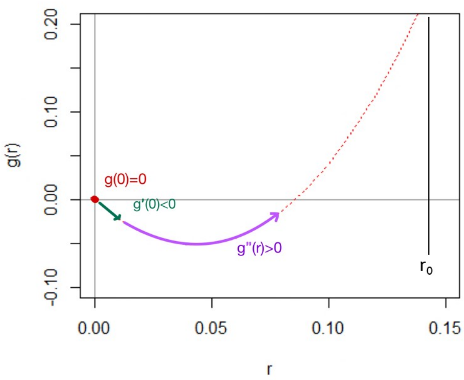
Proof. We will show that the adjustment coefficient equation has the unique positive solution, by proving the following properties of $g(r)$.

(1) $g(0) = 0$,

(2) $g'(0) < 0$,

(3) $g''(r) > 0$ for all $r > 0$,

(4) $\lim_{r \to +\infty} g(r) = \infty$.



From the definition of MGF in Definition 2.24, $M_X(0) = 1, M_Y(0) = 1$, and $M_Z(0) = 1$, we get

(1) From (4.53), then $g(0) = 0$.

(2) From (4.53), then

$$g'(r) = -\lambda E[X_i e^{-rX_i}] + \lambda p E[Y_i e^{rY_i}] + \lambda q E[Z_i e^{rZ_i}]. \tag{4.55}$$

Hence,

$$g'(0) = -\lambda E[X_i] + \lambda p E[Y_i] + \lambda q E[Z_i].$$

From net profit condition (4.3), we get

$$\begin{aligned} g'(0) &= -\lambda E[X_i] + \lambda p E[Y_i] + \lambda q E[Z_i], \\ &< -\lambda E[X_i] + \lambda \mu_X = 0. \end{aligned}$$

Therefore, $g'(0) < 0$.

(3) Let $r > 0$. Due to the explanation of $g'(r)$ in (4.55), we have that

$$g''(r) = \lambda E[X_i^2 e^{-rX_i}] + \lambda p E[Y_i^2 e^{rY_i}] + \lambda q E[Z_i^2 e^{rZ}].$$

Since X_i, Y_i, Z_i are non-negative random variables and $r > 0$,
$E[X_i^2 e^{-rX_i}] > 0$, $E[Y_i^2 e^{rY_i}] \geq 0$ and $E[Z_i^2 e^{rZ_i}] \geq 0$.
Hence,

$$g''(r) = \lambda E[X^2 e^{-rX}] + \lambda p E[Y^2 e^{rY}] + \lambda q E[Z^2 e^{rZ}] > 0.$$

Therefore,

$$g''(r) > 0 \quad \text{for all} \quad r > 0.$$

(4) Since (4.53) and the definition of MGF in Definition 2.24,

$$\begin{aligned} \lim_{r \to +\infty} g(r) &= -\lim_{r \to +\infty} \lambda \left[1 - M_X(-r)\right] - \lim_{r \to +\infty} \lambda p \left[1 - M_Y(r)\right] \\ &\quad - \lim_{r \to +\infty} \lambda q \left[1 - M_Z(r)\right]. \end{aligned}$$

Since $\lim_{r \to +\infty} M_X(-r) = 0$, $\lim_{r \to +\infty} M_Y(r) = \infty$, and $\lim_{r \to +\infty} M_Z(r) = \infty$,

$$\lim_{r \to +\infty} g(r) = \infty.$$

□

To determine the value of R based in the theory, it can be obtained as the unique positive solution of $g(r) = 0$, as indicated in Theorem 4.8. The function $g(r)$ is determined by (4.53). In practical applications, we will use the R command "`uniroot`" to compute the value of adjustment coefficient R.

For the profits process $\{S(t);\ t \geq 0\}$, let $F_t^S = \sigma\{S(v);\ v \leq t\}$ be a filtration.

Theorem 4.9. *The random process* $\{H_u(t);\ F_t^S;\ t \geq 0\}$ *is a martingale, where* $H_u(t) = \dfrac{e^{-r(u+S(t))}}{e^{tg(r)}}$.

Proof. Let $v < t$, we will show that $E[H_u(t) \mid F_v^S] = H_u(v)$.

Consider

$$\begin{aligned} E[H_u(t) \mid F_v^S] &= E\left[\frac{e^{-r(u+S(t))}}{e^{tg(r)}} \,\middle|\, F_v^S\right] \\ &= E\left[\frac{e^{-r(u+S(t))}}{e^{tg(r)}} \cdot \frac{e^{-rS(v)+rS(v)}}{e^{vg(r)-vg(r)}} \,\middle|\, F_v^S\right] \\ &= E\left[\frac{e^{-r(u+S(v))}}{e^{vg(r)}} \cdot \frac{e^{-r(S(t)-S(v))}}{e^{(t-v)g(r)}} \,\middle|\, F_v^S\right] \\ &= \frac{e^{-r(u+S(v))}}{e^{vg(r)}} \cdot \frac{1}{e^{(t-v)g(r)}} \cdot E[e^{-r(S(t)-S(v))} \mid F_v^S]. \end{aligned} \tag{4.56}$$

Consider

$$S(t) - S(v) = \sum_{i=N(v)+1}^{N(t)} X_i - \sum_{i=N(v,p)+1}^{N(t,p)} Y_i - \sum_{i=N(v,q)+1}^{N(t,q)} Z_i.$$

Since X_i, Y_i, Z_i are i.i.d. and $N(t)$ is stationary increment, we get

$$\begin{aligned} S(t) - S(v) &\overset{d}{\cong} \sum_{i=1}^{N(t)-N(v)} X_i - \sum_{i=1}^{N(t,p)-N(v,p)} Y_i - \sum_{i=1}^{N(t,q)-N(v,q)} Z_i \\ &\overset{d}{\cong} \sum_{i=1}^{N(t-v)} X_i - \sum_{i=1}^{N(t-v,p)} Y_i - \sum_{i=1}^{N(t-v,q)} Z_i \\ &= S(t-v). \end{aligned}$$

Therefore, (4.56) is become

$$E[H_u(t)|F_v^S] = \frac{e^{-r(u+S(v))}}{e^{vg(r)}} \cdot \frac{1}{e^{(t-v)g(r)}} \cdot E[e^{-r(S(t-v))}|F_v^S].$$

Since $S(t-v)$ and F_v^S are mutually independent and Theorem 2.18,

$$E[H_u(t)|F_v^S] = \frac{e^{-r(u+S(v))}}{e^{vg(r)}} \cdot \frac{1}{e^{(t-v)g(r)}} \cdot E[e^{-r(S(t-v))}].$$

From Theorem 4.7, we get

$$\begin{aligned} E[H_u(t)|F_v^S] &= \frac{e^{-r(u+S(v))}}{e^{vg(r)}} \cdot \frac{1}{e^{(t-v)g(r)}} \cdot e^{(t-v)g(r)}, \\ &= H_u(v). \end{aligned}$$

Therefore,

$$E[H_u(t)|F_v^S] = H_u(v),$$

i.e., the random process $H_u(t)$ is a martingale. □

Lemma 4.2. *The ruin time T is the stopping time of F_t^S.*

Proof. Let T be the ruin time where $U(T) < 0$ and $F_t^S = \sigma\{S(v);\ v \leq t\}$.
From (4.1) and Lemma 4.1, then

$$U(t) = u + S(t).$$

Since F_t^S or $\sigma - algebra$ generated by random process $S(v)$ from time 0 to t occurs, it gives information $S(t)$ from time 0 to t. Hence, event $\{T \leq t\}$ is a member of F_t^S.
Therefore,

$$T \text{ is the stopping time of } F_t^S.$$

□

Theorem 4.10. *For the surplus process $\{U(t); t \geq 0\}$, the ruin probability $\psi(u)$ satisfies Lundberg inequality:*

$$\psi(u) \leq e^{-Ru} \quad , u \geq 0, \tag{4.57}$$

where R is adjustment coefficient.

Proof. Let T be the ruin time, $t_0 > 0$ be a fixed time and $t_0 \wedge T = min(t_0, T)$, then $t_0 \wedge T$ is a stopping time.

Therefore, $t_0 \wedge T$ is a bounded stopping time.

From, $H_u(t) = \dfrac{e^{-r(u+S(t))}}{e^{tg(r)}}$ and Lemma 4.1, then

$$e^{-ru} = E[H_u(0)].$$

By Theorem 2.24 (The Martingale Stopping Time Theorem), we have that

$$E[H_u(0)] = E[H_u(T \wedge t_0)].$$

Therefore,

$$e^{-ru} = E[H_u(T \wedge t_0)].$$

Since

$$T \wedge t_0 = \; min(T, t_0) \; = \begin{cases} T, & \text{if } T \leq t_0, \\ t_0, & \text{if } T > t_0, \end{cases} ,$$

$$\begin{aligned} E[H_u(T \wedge t_0)] &= E[H_u(T \wedge t_0) \cdot [\mathbb{1}_{T \leq t_0} + \mathbb{1}_{T > t_0}]] \\ &= E[H_u(T \wedge t_0) \cdot \mathbb{1}_{T \leq t_0}] + E[H_u(T \wedge t_0) \cdot \mathbb{1}_{T > t_0}] \\ &= E[H_u(T) | T \leq t_0] \cdot P(T \leq t_0) + E[H_u(t_0) | T > t_0] \cdot P(T > t_0). \end{aligned}$$

Let $r = R$. Therefore,

$$e^{-Ru} = E[e^{-RU(T)} | T \leq t_0] \cdot P(T \leq t_0) + E[e^{-RU(t_0)} | T > t_0] \cdot P(T > t_0). \tag{4.58}$$

By the fact that $0 \leq E[e^{-RU(t_0)} | T > t_0] \leq 1$ and Theorem 2.20 (Markov's inequality), we obtain

$$\lim_{t_0 \to \infty} \Big[E[e^{-RU(t_0)} | T > t_0] \cdot P(T > t_0) \Big] = 0.$$

From (4.58), we get that

$$\begin{aligned}
\lim_{t_0\to\infty} e^{-Ru} &= \lim_{t_0\to\infty}\Big[E[e^{-RU(T)}|T\le t_0]\cdot P(T\le t_0)+E[e^{-RU(t_0)}|T>t_0]\cdot P(T>t_0)\Big]\\
&= \lim_{t_0\to\infty}\Big[E[e^{-RU(T)}|T\le t_0]\cdot P(T\le t_0)\Big]+\lim_{t_0\to\infty}\Big[E[e^{-RU(t_0)}|T>t_0]\cdot P(T>t_0)\Big]\\
&= \lim_{t_0\to\infty}\Big[E[e^{-RU(T)}|T\le t_0]\cdot P(T\le t_0)\Big].
\end{aligned}$$

Then,

$$\begin{aligned}
e^{-Ru} &= E[e^{-RU(T)}|T\le\infty]\cdot P(T\le\infty)\\
&= E[e^{-RU(T)}|T\le\infty]\cdot\psi(u).
\end{aligned}$$

Therefore,

$$\psi(u)=\frac{e^{-Ru}}{E[e^{-RU(T)}|T\le\infty]}. \tag{4.59}$$

Since $U(T)<0$, we have that $\dfrac{1}{e^{-RU(T)}}<1$.

From (4.59), we have that

$$\psi(u)=\frac{e^{-Ru}}{E[e^{-RU(T)}\mid T<\infty]}<\frac{e^{-Ru}}{E[\,1\ |T<\infty]}=e^{-Ru}.$$

Therefore,

$$\psi(u)\le e^{-Ru}.$$

□

4.4 Experimental simulations

In this section, we perform numerical studies to investigate performance of the analytical approximation of the risk model with surrender under the thinning dependence. The studies are divided into three parts. The first part discussed in Section 4.4.1 introduces the statistical estimation for the ruin probability $\hat{\psi}_t(u)$ by using the Monte Carlo methods. The second part in Section 4.4.2 studies numerical approximation to the ruin probability when the amounts of premiums, claims, and surrenders follows an exponential distribution by using the analytical solution such as the Cramér approximation and the Laplace transform comparing with the Monte Carlo approximation and the Lundberg's

Upper bound. The third part focuses on the numerical approximation to the ruin probability when the amounts of premiums, claims, and surrenders follows gamma distribution by using the De-Vylder approximation and the Laplace transform comparing with the Monte Carlo approximation and the Lundberg's Upper bound.

4.4.1 Statistical estimations for the ruin probability

In this section, we study a statistical estimate for the ruin probability $\hat{\psi}_t(u)$ derived by the direct simulation of the surplus process using the Monte Carlo methods in order to evaluate the result of the approximations suggested in this chapter.

Let N be the total number of realizations of the process $U(t)$. We can calculate the average value of the process $U(t)$ when each ruin occurs at the time point t, consequently, we obtain the corresponding statistical estimate $\hat{\psi}_t(u)$ for the ruin probability $\psi(u)$. The Monte Carlo estimations is obtained as

$$\hat{\psi}_t(u) = \frac{1}{N}\sum_{i=1}^{N} I_{\{U_i(t)<0|U_i(0)=u\}},$$

where t is a fixed time point and N is the sample size. As $N \to \infty$ and $t \to \infty$, by the law of large numbers, $\hat{\psi}_t(u)$ converges to $\psi(u)$. The time points considered here are $t = 1, 5, 50$, and 100, and the sample size of the Monte Carlo method is $N = 200{,}000$. The parameters of the model studied in this section are as follows. The initial capital u varies in $\{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 2, 3, 5\}$. The parameter of the Poisson counting process of premium is $\lambda = 10$. The thinning parameters of claims and surrenders are 0.4 and 0.3, respectively.

4.4.2 Exponential distributions for the premium, claim sizes and surrender

Let the probability density functions of the amounts of premiums X_i, the amounts of claims Y_i, and the amounts of surrenders Z_i are

$$f(x) = ae^{-ax}, \quad g(y) = be^{-by} \quad \text{and} \quad h(z) = ce^{-cz} \quad , x, y, z \geq 0,$$

where $a = 5, b = 4$ and $c = 6$, respectively.

For the Cramer approximation in, substituting $a = 5$, $b = 4$, $c = 6$, $p = 0.4$, and $q = 0.3$ into the formula of r_1 and r_2 in (4.6), we get $r_1 = -5.271671$ and $r_2 = -0.669505$, respectively. Consequently, $C_1 = 0.005614$ and $C_2 = 0.847327$. Therefore, the Cramér approximation $\psi_C(u)$ is

$$\psi_C(u) = 0.005614e^{-5.271671u} + 0.847327e^{-0.669505u} \quad for\ all\ u \geq 0. \tag{4.60}$$

For the Laplace approximation, substituting $a = 5$, $b = 4$, $c = 6$, $p = 0.4$, and $q = 0.3$ into the formula in (4.51), we get $S = 61.21$. Consequently, $s_1 = -5.271671$, $s_2 = -0.669505$. Therefore, the Laplace approximation $\psi_{\mathscr{L}}$ is Consequently, the Laplace approximation $\psi_{\mathscr{L}}$ is

$$\psi_{\mathscr{L}}(u) = 0.022457e^{-5.271671u} + 3.389308e^{-0.669505u} - 3.0 \quad for\ all\ u \geq 0. \tag{4.61}$$

For the upper bound approximation, substituting $a = 5$, $b = 4$, $c = 6$, $p = 0.4$, and $q = 0.3$ into $g(r)$ in Theorem 4.7 and solve for the unique positive solution $g(r) = 0$ by using the R programming to compute the value of R, we have $R = 0.669505$. Then, from Theorem 4.10, the upper bound of the ruin probability is

$$\psi(u) \leq e^{-0.669505u}. \tag{4.62}$$

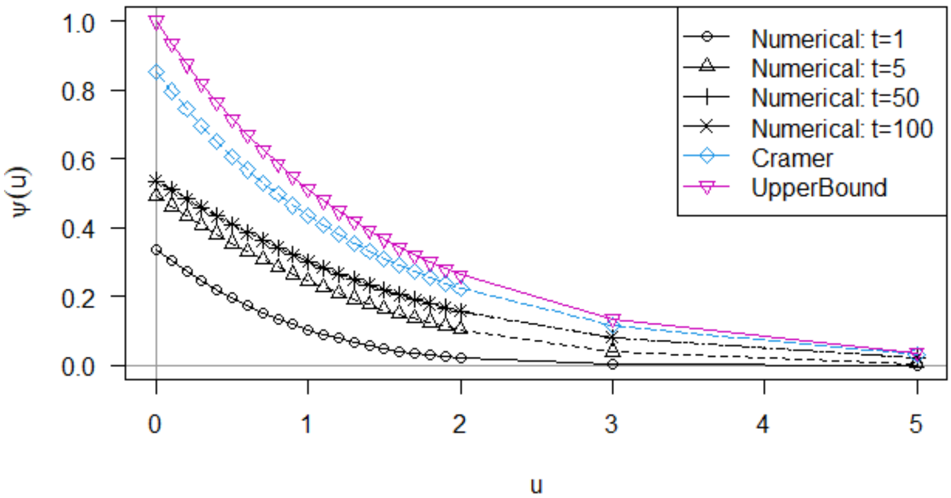


Figure 4.1: Graph of initial reserve u and the ruin probability of the SPST risk model with exponential distributions.

From Table 4.1, we can see that the approximates of ruin probability of all methods decrease when the initial capital increases, except for the Laplace transforms. Notice that the Monte Carlo approximations $\hat{\psi}_t(u)$ do not have the same value as our approximations. Besides, we can observe that the ruin probability $\hat{\psi}_t(u)$ increases as t increases, the Monte Carlo approximation converges to our approximations, and it is less than the upper bound. Therefore, the Monte Carlo approximation is consider to be a good option. However, in practical situations, the exact value of the ruin probability is unknown; therefore, it is impossible to determine how high the value of t should be in order to make $\hat{\psi}_t(u)$ close to the exact value of the ruin probability as one desires. Consequently, the Cramér approximations are better than the Monte Carlo approximation regarding real usage.

Also, we can see that the Cramér approximates of the ruin probability $\psi_C(u)$ have values between the Monte Carlo approximation of ruin probability $\hat{\psi}_t(u)$ and the Lundberg upper bound which is reasonable, since the Cramér and the Laplace approximations are a type of infinite-time ruin probabilities which should be higher than any of finite-time ruin probability $\hat{\psi}_t(u)$ and should not exceed the upper bound. In contrast, the Laplace approximate $\psi_{\mathscr{L}}(u)$ are negative for high values of initial capitals which shows bad performance of the Laplace approximation in this case. Therefore, we investigate the reason of such phenomenon mathematically and found that $\lim_{u\to\infty} \psi_{\mathscr{L}}(u) = \dfrac{acp + abq}{-bc + acp + abq} < 0$. Thus, based on the previous reason, it is not possible to use the Laplace transform of the ruin probability where the money amounts follow exponential distributions in SPST model.

The Monte Carlo simulation will be very good, if we can increase the value of t. However, it will take long computation time to do so. Therefore, a possible way to improve the Monte Carlo simulation performance is to increase the time points of interest and reduce the number of realizations of $U(t)$ instead. Furthermore, we have attempted to change the type of transformation from the Laplace transform, into the Fourier transform. This may help us fix the problem that the approximated ruin probabilities of the Laplace transform are negative because the Fourier transform deals with integrals of complex functions. We can do this by replacing the variable s in the formula of the approximated ruin probability of the Laplace transform by is where i is the imaginary number. Unfortunately, the numerical results obtained from this transformed formula for the ruin probability of Fourier transforms still yield negative values.

4.4.3 Gamma distributions for the premium, claim sizes and surrender

In this section, we study numerical approximations such the premium, claim sizes and surrender follow gamma distributions. Specifically, letthe probability density functions of the premium X_i, the claim sizes Y_i and the surrender Z_i are

$$f(x) = \frac{\beta_X e^{-\beta_X x}(\beta_X x)^{\alpha_X - 1}}{\Gamma(\alpha_X)}, \quad g(y) = \frac{\beta_Y e^{-\beta_Y y}(\beta_Y y)^{\alpha_Y - 1}}{\Gamma(\alpha_Y)}, \quad \text{and}$$

$$h(z) = \frac{\beta_Z e^{-\beta_Z z}(\beta_Z z)^{\alpha_Z - 1}}{\Gamma(\alpha_Z)},$$

for $x, y, z \geq 0$, where $\beta_X = 5, \alpha_X = 1, \beta_Y = 8, \alpha_Y = 2, \beta_Z = 12, \alpha_Z = 2$, respectively.

For the De-Vylder approximation, substituting $\beta_X = 5, \alpha_X = 1, \beta_Y = 8, \alpha_Y = 2, \beta_Z = 12, \alpha_Z = 2$, $p = 0.4$, and $q = 0.3$ into the formula of r_1 and r_2 in (4.51), we get $r_1 = -15.675242$ and $r_2 = -0.793073$, respectively. Consequently, $C_1 = -0.019467$ and $C_2 =$

0.874847. Therefore, the De-Vylder approximation $\psi_{De}(u)$ is

$$\psi_{De}(u) = -0.019467e^{-15.675242u} + 0.874847e^{-0.793072u} \quad for\ all\ u \geq 0. \tag{4.63}$$

For the Laplace approximation, substituting $\beta_X = 5, \alpha_X = 1, \beta_Y = 8, \alpha_Y = 2, \beta_Z = 12, \alpha_Z = 2$, $p = 0.4$, and $q = 0.3$ into the formula of $\psi^*(s)$ in (4.21), then taking the inverse Laplace transform in $\psi^*(s)$ by using the MATLAB for computation. Therefore, the Laplace approximation $\psi_{\mathscr{L}}$ is

$$\begin{aligned}\psi_{\mathscr{L}}(u) =& -3.0 + 3.498836e^{-0.794286u} - 0.092109e^{-17.943008u} \\ &+ (0.002519 - 0.010671i)e^{(-9.601941-1.702506i)u} \\ &+ (0.002519 + 0.010671i)e^{(-9.601941+1.702506i)u} \quad \text{for all } u \geq 0.\end{aligned}$$

Since $z_1 e^{z_2 u} + \overline{z_1} e^{\overline{z_2} u} = 2Re(z_1 e^{z_2 u})$, for all $u \geq 0$

$$\begin{aligned}\psi_{\mathscr{L}}(u) =& -3.0 + 3.498836e^{-0.794286u} - 0.092109e^{-17.943008u} \\ &+ 0.005038e^{-9.601941u}\cos(1.702506) - 0.021254e^{-9.601941u}\sin(1.702506)\end{aligned} \tag{4.64}$$

For the upper bound approximation, substituting $\beta_X = 5, \alpha_X = 1, \beta_Y = 8, \alpha_Y = 2, \beta_Z = 12, \alpha_Z = 2$, $p = 0.4$, and $q = 0.3$ into $g(r)$ in Theorem 4.7 and solve for the unique positive solution $g(r) = 0$ by using the R programming to compute the value of R, we have $R = 0.794286$. Then, from Theorem 4.10, the upper bound of the ruin probability is

$$\psi(u) \leq e^{-0.794286u}. \tag{4.65}$$

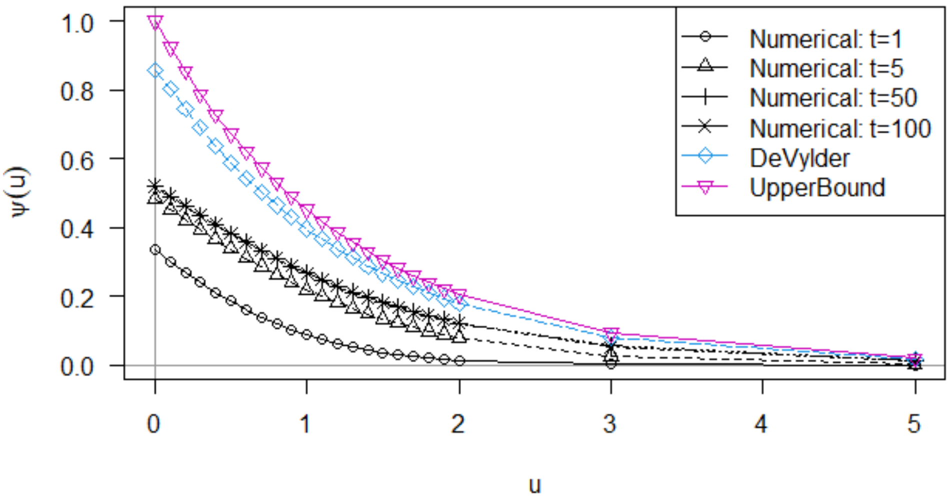


Figure 4.2: Graph of initial reserve u and the ruin probability of the SPST risk model with gamma distributions.

From Table 4.2, we can see that the approximates of ruin probability of all methods decrease when the initial capital increases, except for the Laplace transforms. Notice that the Monte Carlo approximations $\hat{\psi}_t(u)$ do not have the same value as our approximations. Besides, we can observe that the ruin probability $\hat{\psi}_t(u)$ increases as t increases, the Monte Carlo approximation converges to our approximations, and it is less than the upper bound. Therefore, the Monte Carlo approximation is consider to be a good option. However, in practical situations, the exact value of the ruin probability is unknown; therefore, it is impossible to determine how high the value of t should be in order to make $\hat{\psi}_t(u)$ close to the exact value of the ruin probability as one desires. Consequently, the De-Vyder approximations are better than the Monte Carlo approximation regarding real usage.

Also, we can see that the De-Vyder approximates of the ruin probability $\psi_{De}(u)$ have values between the Monte Carlo approximation of ruin probability $\hat{\psi}_t(u)$ and the Lundberg upper bound which is reasonable, since the De-Vyder and the Laplace approximations are a type of infinite-time ruin probabilities which should be higher than any of finite-time ruin probability $\hat{\psi}_t(u)$ and should not exceed the upper bound. In contrast, the Laplace approximate $\psi_{\mathscr{L}}(u)$ are negative for high values of initial capitals which shows bad performance of the Laplace approximation in this case. Therefore, we investigate the reason of such phenomenon mathematically and from (4.64) found that $\lim_{u\to\infty} \psi_{\mathscr{L}}(u) = -3 < 0$. Thus, based on the previous reason, it is not possible to use the Laplace transform of the ruin probability where the money amounts follow gamma distributions in SPST model.

The Monte Carlo simulation will be very good, if we can increase the value of t. However, it will take long computation time to do so. Therefore, a possible way to improve the Monte Carlo simulation performance is to increase the time points of interest and reduce the number of realizations of $U(t)$ instead. Furthermore, we have attempted to change the type of transformation from the Laplace transform, into the Fourier transform. This may help us fix the problem that the approximated ruin probabilities of the Laplace transform are negative because the Fourier transform deals with integrals of complex functions. We can do this by replacing the variable s in the formula of the approximated ruin probability of the Laplace transform by is where i is the imaginary number. Unfortunately, the numerical results obtained from this transformed formula for the ruin probability of Fourier transforms still yield negative values.

The numerical approximations obtained in (4.60)–(4.62) for different values of the initial capital u is given in Table 4.1.

u	$\psi(u)$						
	Statistical estimate $\hat{\psi}_t(u)$				$\psi_C(u)$	$\psi_{\mathscr{L}}(u)$	Upper bound e^{-Ru}
	$t=1$	$t=5$	$t=50$	$t=100$			
0	0.334325	0.490920	0.535130	0.535215	0.852941	0.411765	1.000000
0.1	0.303330	0.462225	0.508595	0.508705	0.795769	0.183077	0.935241
0.2	0.273110	0.433905	0.482160	0.482265	0.743093	-0.027627	0.874677
0.3	0.245010	0.406690	0.456810	0.456920	0.694297	-0.222813	0.818034
0.4	0.219055	0.380455	0.432245	0.432365	0.648937	-0.404253	0.765059
0.5	0.195000	0.354790	0.407845	0.407965	0.606678	-0.573290	0.715515
0.6	0.172885	0.330065	0.384190	0.384315	0.567251	-0.730995	0.669179
0.7	0.153085	0.307350	0.362150	0.362255	0.530435	-0.878260	0.625844
0.8	0.134950	0.285445	0.340895	0.341005	0.496036	-1.015854	0.585316
0.9	0.117805	0.264290	0.320260	0.320360	0.463885	-1.144459	0.547411
1	0.102915	0.245045	0.301360	0.301485	0.433828	-1.264688	0.511962
1.1	0.089255	0.226045	0.282665	0.282780	0.405724	-1.377104	0.478808
1.2	0.077220	0.208870	0.265405	0.265520	0.379444	-1.482224	0.447801
1.3	0.066320	0.191790	0.248300	0.248435	0.354868	-1.580527	0.418802
1.4	0.056875	0.176660	0.232910	0.233065	0.331885	-1.672458	0.391681
1.5	0.048515	0.161845	0.217595	0.217750	0.310392	-1.758433	0.366316
1.6	0.041245	0.148255	0.203490	0.203635	0.290291	-1.838838	0.342594
1.7	0.035180	0.135870	0.190560	0.190710	0.271491	-1.914034	0.320408
1.8	0.029710	0.124460	0.178585	0.178730	0.253910	-1.984361	0.299659
1.9	0.025200	0.113740	0.167040	0.167180	0.237467	-2.050133	0.280254
2	0.021080	0.104095	0.156500	0.156640	0.222089	-2.111645	0.262105
3	0.002980	0.040245	0.080515	0.080635	0.113701	-2.545197	0.134188
5	0.000020	0.004710	0.021420	0.021495	0.029802	-2.880794	0.035171

Table 4.1: Numerical approximations of the SPST risk model with exponential distributions.

The numerical approximations obtained in (4.63)–(4.65) for different values of the initial capital u is given in Table 4.2.

u	$\psi(u)$						
	Statistical estimate $\hat{\psi}_t(u)$				$\psi_{De}(u)$	$\psi_{\mathscr{L}}(u)$	Upper bound e^{-Ru}
	$t = 1$	$t = 5$	$t = 50$	$t = 100$			
0	0.333420	0.481655	0.519530	0.519545	0.855380	0.411765	1.000000
0.1	0.300120	0.450950	0.490465	0.490480	0.804085	0.216882	0.923644
0.2	0.268845	0.420695	0.462025	0.462040	0.745682	-0.017974	0.853118
0.3	0.239060	0.391785	0.434355	0.434370	0.689434	-0.243758	0.787977
0.4	0.211670	0.364330	0.40810	0.408120	0.636995	-0.453785	0.727811
0.5	0.186045	0.337270	0.381990	0.382030	0.588454	-0.648067	0.672238
0.6	0.161895	0.310735	0.356555	0.356595	0.543594	-0.827594	0.620908
0.7	0.140015	0.286120	0.332745	0.332785	0.502149	-0.993446	0.573498
0.8	0.120315	0.262140	0.309550	0.309575	0.463864	-1.146647	0.529708
0.9	0.102590	0.239515	0.287405	0.287430	0.428497	-1.288157	0.489262
1	0.086860	0.218145	0.266150	0.266185	0.395827	-1.418865	0.451904
1.1	0.073535	0.198885	0.246975	0.247025	0.365647	-1.539593	0.417398
1.2	0.061370	0.180590	0.22860	0.228640	0.337769	-1.651104	0.385527
1.3	0.051570	0.164055	0.211925	0.211975	0.312016	-1.754100	0.35609
1.4	0.043235	0.148725	0.196285	0.196320	0.288227	-1.849232	0.328900
1.5	0.035735	0.134470	0.181335	0.181370	0.266251	-1.937100	0.303787
1.6	0.029305	0.121365	0.167415	0.167445	0.245951	-2.018259	0.280591
1.7	0.023600	0.109420	0.154635	0.154665	0.227199	-2.093221	0.259166
1.8	0.019015	0.098505	0.142770	0.142800	0.209876	-2.162459	0.239377
1.9	0.015210	0.088795	0.131935	0.131975	0.193874	-2.226410	0.221099
2	0.012295	0.079505	0.121715	0.121755	0.179093	-2.285478	0.204217
3	0.001080	0.026025	0.054895	0.054945	0.081031	-2.677105	0.092286
5	0.000005	0.002120	0.011390	0.011420	0.016588	-2.934059	0.018846

Table 4.2: Numerical approximations of the SPST risk model with gamma distributions.

CHAPTER V

RENEWAL RISK MODEL WITH CONSTANT PREMIUMS AND SURRENDERS

In this chapter, we study numerical approximations of renewal risk model with constant premiums and surrenders when the arrival times of premiums, claims, and surrenders follow generalized exponential distributions. We first introduce the risk model. Then, we derive formula for different approximation method of the ruin probability which are the Cramér approximation and the Laplace transforms method. Moreover, we perform numerical studies to investigate performance of the two methods and compare them with the Monte Carlo approximation.

The organization of this chapter is as follows. Section 5.1 studies some properties of the renewal risk model with constant premiums and surrenders. Section 5.2 derives the analytical approximation of the ruin probability. Section 5.3 performs experimental simulations.

5.1 The renewal risk model with constant premiums and surrenders

In this section, the generalized exponential distribution has been introduced in [16]. A random variable X has the generalized exponential distribution with parameters η and λ, if it has distribution function $F(x) = (1 - e^{-\lambda x})^{\eta}$ for $x > 0, \lambda > 0, \eta > 0$, with corresponding density function $f(x) = \eta\lambda(1 - e^{-\lambda x})^{\eta-1}e^{-\lambda x}$ for $x > 0, \lambda > 0, \eta > 0$.

We introduce the renewal risk model with constant premiums and surrenders. The risk model consists of the initial capital, premiums, claims, and surrenders. In particular, the surplus at time t, $U(t)$, is defined as

$$U(t) = u + cM(t) - \sum_{i=1}^{N(t)} Y_i - \sum_{i=1}^{K(t)} Z_i, \tag{5.1}$$

where u represents the initial capital, c is the constant rate of premium, $M(t)$ denoting the number of premiums up to time t where the inter-premium times follow generalized exponential distribution with shape parameter $\eta_1 = 2$ and scale parameter λ_1. $N(t)$ denoting the number of claims up to time t where the inter-claim times follow generalized exponential distribution with parameter $\eta_2 = 2$ and λ_2. The individual claim size $\{Y_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. non-negative random variables with a cumulative distribution function G. $K(t)$ denoting the number of surrenders up to time t where the inter-surrender times follow generalized exponential distribution with parameter $\eta_3 = 2$ and λ_3. The sequence of i.i.d. non-negative random variables $\{Z_i\}_{i=1}^{\infty}$ represents the amount of the i-th payment of insurance policy with a cumulative distribution function H. In addition, we suppose that $\{Y_i\}_{i=1}^{\infty}$, $\{Z_i\}_{i=1}^{\infty}$, $\{M(t)\}_{t\geq 0}$, $\{N(t)\}_{t\geq 0}$, and $\{K(t)\}_{t\geq 0}$ are also mutually independent.

5.2 Approximation to the ruin probability of the risk model

In this section, we will study analytical approximation of the ruin probability for the renewal risk model with constant premiums and surrenders. We will start by obtaining the integral equation for the ruin probability. Then we obtain an approximation of the ruin probability using the Cramér approximation, and the Laplace transforms method. To obtain the three approximations, we first obtain the integro-differential equations stated in Theorem 5.2 below.

Define the sequence of i.i.d. random variables $\{I_i\}_{i=1}^{\infty}$ represents the inter-arrival times of i^{th} premium. The sequence of i.i.d. random variables $\{J_i\}_{i=1}^{\infty}$ represents the inter-arrival times of i^{th} claim. The sequence of i.i.d. random variables $\{K_i\}_{i=1}^{\infty}$ represents the inter-arrival times of i^{th} surrender. In particular, the probability density functions of I_i, J_i, and K_i are

$$f(x) = 2\lambda_1(1 - e^{-\lambda_1 x})e^{-\lambda_1 x}, \quad g(y) = 2\lambda_2(1 - e^{-\lambda_2 y})e^{-\lambda_2 y}, \quad \text{and}$$
$$h(z) = 2\lambda_3(1 - e^{-\lambda_3 z})e^{-\lambda_3 z},$$

for $x, y, z \geq 0$, respectively.

Theorem 5.1. *The random variable* $T_1 = min(I_1, J_1, K_1)$ *has the probability density function defined as*

$$f_{T_1}(x) = 2e^{-2x(\lambda_1+\lambda_2+\lambda_3)} \Big[(e^{\lambda_1 x} - 1)(2e^{\lambda_2 x} - 1)(2e^{\lambda_3 x} - 1)\lambda_1 + (2e^{\lambda_1 x} - 1)(e^{\lambda_2 x} - 1)(2e^{\lambda_3 x} - 1)\lambda_2 + (2e^{\lambda_1 x} - 1)(2e^{\lambda_2 x} - 1)(e^{\lambda_3 x} - 1)\lambda_3\Big], \qquad (5.2)$$

where $I_1, J_1,$ *and* K_1 *are the interarrival times of first premium, first claim, and first surrender, respectively.*

Proof. Let $T_1 = \min(I_1, J_1, K_1)$.
Since the interarrival time I_1, J_1, and K_1 are mutually independent,

$$\begin{aligned} P[T_1 > x] &= P[I_1 > x \text{ and } J_1 > x \text{ and } K_1 > x] \\ &= P[I_1 > x] \;\; P[J_1 > x] \;\; P[K_1 > x] \\ &= (2e^{-\lambda_1 x} - e^{-2\lambda_1 x})(2e^{-\lambda_2 x} - e^{-2\lambda_2 x})(2e^{-\lambda_3 x} - e^{-2\lambda_3 x}) \\ &= e^{-2x(\lambda_1+\lambda_2+\lambda_3)}(2e^{\lambda_1 x} - 1)(2e^{\lambda_2 x} - 1)(2e^{\lambda_3 x} - 1). \end{aligned}$$

Therefore

$$P[T_1 \le x] = 1 - e^{-2x(\lambda_1+\lambda_2+\lambda_3)}(2e^{\lambda_1 x} - 1)(2e^{\lambda_2 x} - 1)(2e^{\lambda_3 x} - 1).$$

Thus, the density of T_1 is

$$\begin{aligned} f_{T_1}(x) &= \frac{d}{dx} P[T_1 \le x] \\ &= 2e^{-2x(\lambda_1+\lambda_2+\lambda_3)} \Big[(e^{\lambda_1 x} - 1)(2e^{\lambda_2 x} - 1)(2e^{\lambda_3 x} - 1)\lambda_1 \\ &\quad + (2e^{\lambda_1 x} - 1)(e^{\lambda_2 x} - 1)(2e^{\lambda_3 x} - 1)\lambda_2 \\ &\quad + (2e^{\lambda_1 x} - 1)(2e^{\lambda_2 x} - 1)(e^{\lambda_3 x} - 1)\lambda_3\Big]. \end{aligned}$$

□

Lemma 5.1. *The probability that the random variable* $T_1 = min(I_1, J_1, K_1)$ *is equal to each of its components is*

$$P[T_1 = I_1] = \alpha_1 \ , \ P[T_1 = J_1] = \alpha_2 \ , \ and \ P[T_1 = K_1] = \alpha_3,$$

respectively, where

$$\begin{aligned}
\alpha_1 = &\quad \lambda_1 \left[\frac{7}{\lambda_1 + \lambda_2 + \lambda_3} - \frac{4}{2\lambda_2 + \lambda_3 + \lambda_1} - \frac{4}{\lambda_1 + 2\lambda_3 + \lambda_2} - \frac{8}{\lambda_2 + \lambda_3 + 2\lambda_1} \right.\\
&\left. + \frac{2}{2\lambda_2 + 2\lambda_3 + \lambda_1} + \frac{4}{2\lambda_1 + \lambda_3 + 2\lambda_2} + \frac{4}{\lambda_2 + 2\lambda_3 + 2\lambda_1} \right],\\
\alpha_2 = &\quad \lambda_2 \left[\frac{7}{\lambda_1 + \lambda_2 + \lambda_3} - \frac{4}{2\lambda_1 + \lambda_3 + \lambda_2} - \frac{4}{\lambda_1 + 2\lambda_3 + \lambda_2} - \frac{8}{\lambda_1 + \lambda_3 + 2\lambda_2} \right.\\
&\left. + \frac{2}{2\lambda_1 + 2\lambda_3 + \lambda_2} + \frac{4}{2\lambda_1 + \lambda_3 + 2\lambda_2} + \frac{4}{\lambda_1 + 2\lambda_3 + 2\lambda_2} \right],\\
\alpha_3 = &\quad \lambda_3 \left[\frac{7}{\lambda_1 + \lambda_2 + \lambda_3} - \frac{4}{2\lambda_1 + \lambda_3 + \lambda_2} - \frac{4}{\lambda_1 + 2\lambda_2 + \lambda_3} - \frac{8}{\lambda_1 + \lambda_2 + 2\lambda_3} \right.\\
&\left. + \frac{2}{2\lambda_1 + 2\lambda_2 + \lambda_3} + \frac{4}{2\lambda_1 + \lambda_2 + 2\lambda_3} + \frac{4}{\lambda_1 + 2\lambda_3 + 2\lambda_2} \right],
\end{aligned}$$

and

$$\alpha_1 + \alpha_2 + \alpha_3 = 1.$$

Proof. Let $T_1 = \min(I_1, J_1, K_1)$. Then

$$\begin{aligned}
&P[T_1 = I_1]\\
&= \int_0^\infty P(I_1 = x, J_1 \geq x, K_1 \geq x)\, dx\\
&= \int_0^\infty P(I_1 = x) P(J_1 \geq x) P(K_1 \geq x)\, dx\\
&= \int_0^\infty 2\lambda_1 (1 - e^{-\lambda_1 x}) e^{-\lambda_1 x} \cdot (2e^{-\lambda_2 x} - e^{-2\lambda_2 x}) \cdot (2e^{-\lambda_3 x} - e^{-2\lambda_3 x})\, dx\\
&= \lambda_1 \left[\frac{7}{\lambda_1 + \lambda_2 + \lambda_3} - \frac{4}{2\lambda_2 + \lambda_3 + \lambda_1} - \frac{4}{\lambda_2 + 2\lambda_3 + \lambda_1} \right.\\
&\quad - \frac{8}{\lambda_2 + \lambda_3 + 2\lambda_1} + \frac{2}{2\lambda_2 + 2\lambda_3 + \lambda_1} + \frac{4}{2\lambda_2 + \lambda_3 + 2\lambda_1}\\
&\quad \left. + \frac{4}{\lambda_2 + 2\lambda_3 + 2\lambda_1} \right]\\
&= \alpha_1.
\end{aligned}$$

By the same technique, we can show that $P[T_1 = J_1] = \alpha_2$ and $P[T_1 = K_1] = \alpha_3$.

□

Theorem 5.2. *The ruin probability* $\psi(u)$ *for risk model* (5.1) *satisfies the integro-differential equation*

$$\begin{aligned}\alpha_1\psi'(u) &= \frac{(\alpha_2+\alpha_3)}{c}\psi(u) - \frac{\alpha_2}{c}[1-G(u)] - \frac{\alpha_3}{c}[1-H(u)] \\ &\quad -\frac{\alpha_2}{c}\int_0^u \psi(u-y)\,dG(y) - \frac{\alpha_3}{c}\int_0^u \psi(u-z)\,dH(z), \quad u \geq 0,\end{aligned} \tag{5.3}$$

where G *and* H *are cumulative distribution functions of the individual claims sizes and the amount of surrenders with probability density functions* g *and* h*, respectively.*

Proof. To compute the non-ruin probability $\phi(u)$, we consider non-ruin probability $\phi(u)$ and distinguish according to whether there are disjoint events possible of the first occurrence of any event among the three events - the first time of premium payment, the first time of claim payment, and the first time of surrender payment during infinitesimal time t. Particularly,

$$\phi(u) = \int_0^\infty P[\ U(s) \geq 0\ \forall\ s \geq 0 \mid U(0) = u\ ,\ T_1 = t\ ]\ f_{T_1}(t)\,dt.$$

By the law of total probability of T_1,

$$\begin{aligned}f_{T_1}(t) &= f_{T_1}(t \mid T_1 = I_1)P(T_1 = I_1) + f_{T_1}(t \mid T_1 = J_1)P(T_1 = J_1) \\ &\quad + f_{T_1}(t \mid T_1 = K_1)P(T_1 = K_1).\end{aligned}$$

Thus,

$$\begin{aligned}\phi(u) &= \int_0^\infty f_{T_1}(t)\cdot\Big\{P[\ U(s) \geq 0\ \forall s \geq 0 \mid U(0) = u, I_1 = t\ ]\ P[T_1 = I_1] \\ &\quad + P[\ U(s) \geq 0\ \forall s \geq 0 \mid U(0) = u, J_1 = t\ ]\ P[T_1 = J_1] \\ &\quad + P[\ U(s) \geq 0\ \forall s \geq 0 \mid U(0) = u, K_1 = t\ ]\ P[T_1 = K_1]\Big\}\,dt.\end{aligned}$$

Therefore,

$$\phi(u) = \int_0^\infty f_{T_1}(t) \cdot \Bigg\{ \alpha_1 P[U(s) \geq 0 \;\forall s \geq 0 \mid N(I_1) = 1] + \alpha_2 P[U(s) \geq 0 \;\forall s \geq 0 \mid M(J_1) = 1] + \alpha_3 P[U(s) \geq 0 \;\forall s \geq 0 \mid K(K_1) = 1] \Bigg\} \, dt.$$

Using the property that $\displaystyle\int_0^\infty f_{T_1}(x)\,dx = 1$, we get

$$\phi(u) = \alpha_1 P[U(s) \geq 0 \;\forall s \geq 0 \mid N(I_1) = 1] + \alpha_2 P[U(s) \geq 0 \;\forall s \geq 0 \mid M(J_1) = 1] + \alpha_3 P[U(s) \geq 0 \;\forall s \geq 0 \mid K(K_1) = 1].$$

In particular, from the law of total probability, the non-ruin probability can be computed as

$$\begin{aligned}\phi(u) &= \alpha_1 P[U(s) \geq 0 \;\forall s \geq 0 \mid U(I_1) = u + c] \\ &\quad + \alpha_2 \int_0^u P[U(s) \geq 0 \;\forall s \geq 0 \mid U(J_1) = u - y]\, dG(y) \\ &\quad + \alpha_3 \int_0^u P[U(s) \geq 0 \;\forall s \geq 0 \mid U(K_1) = u - z]\, dH(z).\end{aligned}$$

According to the concept of stationary, we can treat any interarrival time as a new start time. Therefore, we can express any interarrival time as $U(0)$. This implies that we are starting a new at any interarrival time and can use $U(0)$ as the starting point,

$$\begin{aligned}\phi(u) &= \alpha_1 P[U(s) \geq 0 \;\forall s \geq 0 \mid U(0) = u + c] \\ &\quad + \alpha_2 \int_0^u P[U(s) \geq 0 \;\forall s \geq 0 \mid U(0) = u - y]\, dG(y) \\ &\quad + \alpha_3 \int_0^u P[U(s) \geq 0 \;\forall s \geq 0 \mid U(0) = u - z]\, dH(z).\end{aligned}$$

Then, we get that

$$\phi(u) = \alpha_1 \phi(u + c) + \alpha_2 \int_0^u \phi(u - y)\, dG(y) + \alpha_3 \int_0^u \phi(u - z)\, dH(z).$$

By the Taylor series expansion in $\phi(u + c)$ around $x_0 = u$, particularly, $\phi(u + c) =$

$\phi(u) + c\phi'(u) + o(\Delta t)$ for $\Delta t \to 0$, we get

$$-c\alpha_1\phi'(u) \quad = \quad \alpha_2 \int_0^u \phi(u-y)\, dG(y) + (\alpha_1 - 1)\phi(u) + \alpha_3 \int_0^u \phi(u-z)\, dH(z).$$

Dividing both sides by $-c$ and using the property $\alpha_1 + \alpha_2 + \alpha_3 = 1$, we have

$$\alpha_1\phi'(u) = \frac{(\alpha_2 + \alpha_3)}{c}\phi(u) - \frac{\alpha_2}{c}\int_0^u \phi(u-y)\, dG(y) - \frac{\alpha_3}{c}\int_0^u \phi(u-z)\, dH(z). \tag{5.4}$$

Using the property that $\phi(u) = 1 - \psi(u)$, we get

$$\begin{aligned} -\alpha_1\psi'(u) \quad = \quad & \frac{(\alpha_2 + \alpha_3)}{c} - \frac{(\alpha_2 + \alpha_3)}{c}\psi(u) - \frac{\alpha_2}{c}\int_0^u 1\, dG(y) + \frac{\alpha_2}{c}\int_0^u \psi(u-y)\, dG(y) \\ & -\frac{\alpha_3}{c}\int_0^u 1\, dH(z) + \frac{\alpha_3}{c}\int_0^u \psi(u-z)\, dH(z). \end{aligned}$$

Therefore,

$$\begin{aligned} -\alpha_1\psi'(u) \quad = \quad & \frac{(\alpha_2 + \alpha_3)}{c} - \frac{(\alpha_2 + \alpha_3)}{c}\psi(u) - \frac{\alpha_2}{c}G(u) + \frac{\alpha_2}{c}\int_0^u \psi(u-y)\, dG(y) \\ & -\frac{\alpha_3}{c}H(u) + \frac{\alpha_3}{c}\int_0^u \psi(u-z)\, dH(z). \end{aligned}$$

Thus,

$$\begin{aligned} \alpha_1\psi'(u) \quad = \quad & \frac{(\alpha_2 + \alpha_3)}{c}\psi(u) - \frac{\alpha_2}{c}\int_0^u \psi(u-y)\, dG(y) - \frac{\alpha_3}{c}\int_0^u \psi(u-z)\, dH(z) \\ & -\frac{\alpha_2}{c}[1 - G(u)] - \frac{\alpha_3}{c}[1 - H(u)]. \end{aligned}$$

□

Corollary 5.1. *For risk model* (5.1)*,*

$$\psi(0) = \frac{\alpha_2}{c\alpha_1}E[Y] + \frac{\alpha_3}{c\alpha_1}E[Z]. \tag{5.5}$$

Proof. Integrate the integro-differential equations (5.4) over the interval $(0, t)$ on u yields

$$\alpha_1 \int_0^t \phi'(u)\, du = \frac{(\alpha_2 + \alpha_3)}{c}\int_0^t \phi(u)\, du - \frac{\alpha_2}{c}\int_0^t \int_0^u \phi(u-y)\, dG(y)\, du$$

$$- \frac{\alpha_3}{c} \int_0^t \int_0^u \phi(u-z)\, dH(z)\, du. \tag{5.6}$$

Consider $-\frac{\alpha_3}{c} \int_0^t \int_0^u \phi(u-z)\, dH(z)\, du$ and the property of CDF H, we can show that

$$\begin{aligned} -\frac{\alpha_3}{c} \int_0^t \int_0^u \phi(u-z)\, dH(z)\, du &= \frac{\alpha_3}{c} \int_0^t \int_0^u \phi(u-z)\, d[1-H(z)]\, du, \\ &= \frac{\alpha_3}{c} \int_0^t \Big[\phi(0)(1-H(u)) - \phi(u) \\ &\quad + \int_0^u (1-H(z))\phi'(u-z)\, dz \Big] du. \end{aligned}$$

Then,

$$\begin{aligned} -\frac{\alpha_3}{c} \int_0^t \int_0^u \phi(u-z)\, dH(z)\, du = &\frac{\alpha_3}{c} \int_0^t \phi(0)(1-H(u))\, du - \frac{\alpha_3}{c} \int_0^t \phi(u)\, du \\ &+ \frac{\alpha_3}{c} \int_0^t \int_0^u (1-H(z))\phi'(u-z)\, dz\, du. \end{aligned} \tag{5.7}$$

By the same technique, we can show that

$$\begin{aligned} -\frac{\alpha_2}{c} \int_0^t \int_0^u \phi(u-y)\, dG(y)\, du = &\frac{\alpha_2}{c} \int_0^t \phi(0)(1-G(u))\, du - \frac{\alpha_2}{c} \int_0^t \phi(u)\, du \\ &+ \frac{\alpha_2}{c} \int_0^t \int_0^u (1-G(y))\phi'(u-y)\, dy\, du. \end{aligned} \tag{5.8}$$

Substituting (5.7) and (5.8) into (5.6), we get

$$\begin{aligned} \alpha_1 \int_0^t \phi'(u)\, du = &\frac{\alpha_3}{c} \int_0^t \phi(0)(1-H(u))\, du + \frac{\alpha_2}{c} \int_0^t \phi(0)(1-G(u))\, du \\ &+ \frac{\alpha_3}{c} \int_0^t \int_0^u (1-H(z))\phi'(u-z)\, dz\, du \\ &+ \frac{\alpha_2}{c} \int_0^t \int_0^u (1-G(y))\phi'(u-y)\, dy\, du. \end{aligned} \tag{5.9}$$

Consider $\frac{\alpha_3}{c} \int_0^t \int_0^u (1-H(z))\phi'(u-z)\, dz\, du$, we can show that

$$\begin{aligned} \frac{\alpha_3}{c} \int_0^t \int_0^u (1-H(z))\phi'(u-z)\, dz\, du &= \frac{\alpha_3}{c} \int_0^t \int_z^t (1-H(z))\phi'(u-z)\, du\, dz, \\ &= \frac{\alpha_3}{c} \int_0^t (1-H(z)) \int_z^t \phi'(u-z)\, d(u-z)\, dz. \end{aligned}$$

Then,

$$\frac{\alpha_3}{c}\int_0^t\int_0^u (1-H(z))\phi'(u-z)\,dz\,du = \frac{\alpha_3}{c}\int_0^t (1-H(z))\phi(t-z)\,dz - \frac{\alpha_3}{c}\int_0^t (1-H(z))\phi(0)\,dz. \tag{5.10}$$

By the same technique, we can show that

$$\frac{\alpha_2}{c}\int_0^t\int_0^u (1-G(y))\phi'(u-y)\,dy\,du = \frac{\alpha_2}{c}\int_0^t (1-G(y))\phi(t-y)\,dy - \frac{\alpha_2}{c}\int_0^t (1-G(y))\phi(0)\,dy. \tag{5.11}$$

Substituting (5.10) and (5.11) into (5.9), we get

$$\alpha_1\phi(t) - \alpha_1\phi(0) = \frac{\alpha_3}{c}\int_0^t (1-H(z))\phi(t-z)\,dz + \frac{\alpha_2}{c}\int_0^t (1-G(y))\phi(t-y)\,dy.$$

Letting t approach to ∞ and using the property that $\lim_{u\to\infty}\phi(u) = 1$, we get,

$$\alpha_1 - \alpha_1\phi(0) = \frac{\alpha_3}{c}\int_0^\infty (1-H(z))\,dz + \frac{\alpha_2}{c}\int_0^\infty (1-G(y))\,dy.$$

Since $\int_0^\infty (1-H(z))\,dz = E[Z]$ and $\int_0^\infty (1-G(y))\,dy = E[Y]$, therefore,

$$\alpha_1 - \alpha_1\phi(0) = \frac{\alpha_3}{c}E[Z] + \frac{\alpha_2}{c}E[Y].$$

Using the property that $\phi(u) = 1 - \psi(u)$, we get

$$\psi(0) = \frac{\alpha_3}{c\alpha_1}E[Z] + \frac{\alpha_2}{c\alpha_1}E[Y].$$

□

5.2.1 The Cramér approximation

In this section, we obtain the Cramér approximation to the ruin probability when amounts of claims and surrenders follow exponential distributions. In particular, the probability density functions of the claim sizes and surrenders are

$$g(y) = ae^{-ay} \quad \text{and} \quad h(z) = be^{-bz}, y, z \geq 0, \tag{5.12}$$

corresponding to CDF's are G and H, respectively, in Theorem 5.2.

Theorem 5.3. *For the risk model* (5.1) *where the amounts of claims size and surrenders follow exponential distributions with parameters* a *and* b*, respective. If* $c\alpha_1 - \frac{\alpha_2}{a} - \frac{\alpha_3}{b} > 0$ *and* $\alpha_1, \alpha_2, \alpha_3 > 0$*, then the Cramér approximation of the ruin probability* $\psi_C(u)$ *is*

$$\psi_C(u) = C_1 e^{r_1 u} + C_2 e^{r_2 u} \quad \textit{for all } u \geq 0, \tag{5.13}$$

where $C_1, C_2, r_1,$ *and* r_2 *are as follows*

$$C_1 = \frac{b\alpha_2^2 + a\alpha_3^2 - abc\alpha_2\alpha_1 - abc\alpha_1\alpha_3 + b\alpha_2\alpha_3 + a\alpha_2\alpha_3 - c\alpha_1(b\alpha_2 + a\alpha_3)r_2}{abc^2\alpha_1^2(r_1 - r_2)},$$

$$C_2 = \frac{-b\alpha_2^2 - a\alpha_3^2 + abc\alpha_2\alpha_1 + abc\alpha_1\alpha_3 - b\alpha_2\alpha_3 - a\alpha_2\alpha_3 + c\alpha_1(b\alpha_2 + a\alpha_3)r_1}{abc^2\alpha_1^2(r_1 - r_2)},$$

$$r_1 = \frac{-\left[\frac{1}{a} + \frac{1}{b} - \frac{(\alpha_2 + \alpha_3)}{abc\alpha_1}\right] - \sqrt{D}}{\frac{2}{ab}},$$

$$r_2 = \frac{-\left[\frac{1}{a} + \frac{1}{b} - \frac{(\alpha_2 + \alpha_3)}{abc\alpha_1}\right] + \sqrt{D}}{\frac{2}{ab}},$$

which

$$D = \left[\frac{1}{a} + \frac{1}{b} - \frac{(\alpha_2 + \alpha_3)}{abc\alpha_1}\right]^2 - \frac{4}{ab}\left[1 - \frac{\alpha_2}{ac\alpha_1} - \frac{\alpha_3}{bc\alpha_1}\right].$$

Proof.

Observe that CDF G and PDF g satisfy $dG(u) = g(u)du$, as mentioned in Remark 2.2, including CDF H and PDF h.

Substituting the density functions of Y_i and Z_i into (5.3), we have

$$\psi'(u) = \frac{(\alpha_2+\alpha_3)}{c\alpha_1}\psi(u) - \frac{\alpha_2}{c\alpha_1}e^{-au} - \frac{\alpha_3}{c\alpha_1}e^{-bu} \\ -\frac{\alpha_2}{c\alpha_1}\int_0^u \psi(u-y)ae^{-ay}\,dy - \frac{\alpha_3}{c\alpha_1}\int_0^u \psi(u-z)be^{-bz}\,dz. \quad (5.14)$$

Differentiating the equation with respect to u, we have

$$\psi''(u) = \frac{(\alpha_2+\alpha_3)}{c\alpha_1}\psi'(u) + \frac{a\alpha_2}{c\alpha_1}e^{-au} + \frac{b\alpha_3}{c\alpha_1}e^{-bu} - \frac{a\alpha_2}{c\alpha_1}\psi(u) - \frac{b\alpha_3}{c\alpha_1}\psi(u) \\ +\frac{a\alpha_2}{c\alpha_1}\int_0^u \psi(u-y)ae^{-ay}\,dy + \frac{b\alpha_3}{c\alpha_1}\int_0^u \psi(u-z)be^{-bz}\,dz.$$

Multiplying the equation by $\frac{1}{a}$, we have

$$\frac{\psi''(u)}{a} = \frac{(\alpha_2+\alpha_3)}{ac\alpha_1}\psi'(u) + \frac{\alpha_2}{c\alpha_1}e^{-au} + \frac{b\alpha_3}{ac\alpha_1}e^{-bu} - \frac{\alpha_2}{c\alpha_1}\psi(u) - \frac{b\alpha_3}{ac\alpha_1}\psi(u) \\ +\frac{\alpha_2}{c\alpha_1}\int_0^u \psi(u-y)ae^{-by}\,dy + \frac{b\alpha_3}{ac\alpha_1}\int_0^u \psi(u-z)be^{-bz}\,dz. \quad (5.15)$$

Adding (5.14) and (5.15), we have

$$\frac{\psi''(u)}{a} = -\left[1-\frac{(\alpha_2+\alpha_3)}{ac\alpha_1}\right]\psi'(u) + \left[\frac{\alpha_3}{c\alpha_1} - \frac{b\alpha_3}{ac\alpha_2}\right]\psi(u) - \frac{b\alpha_3}{ac\alpha_1}e^{-bu} - \frac{\alpha_3}{c\alpha_1}e^{-bu} \\ + \left[\frac{b\alpha_3}{ac\alpha_1} - \frac{\alpha_3}{c\alpha_1}\right]\int_0^u \psi(u-z)be^{-bz}\,dz. \quad (5.16)$$

Differentiating the term with respect to u, we have

$$\frac{\psi'''(u)}{a} = -\left[1-\frac{(\alpha_2+\alpha_3)}{ac\alpha_1}\right]\psi''(u) + \left[\frac{\alpha_3}{c\alpha_1} - \frac{b\alpha_3}{ac\alpha_1}\right]\psi'(u) + \left[\frac{b^2\alpha_3}{ac\alpha_1} - \frac{b\alpha_3}{c\alpha_1}\right]\psi(u) \\ +\frac{b^2\alpha_3}{ac\alpha_1}e^{-bu} + \frac{b\alpha_3}{c\alpha_1}e^{-au} - \left[\frac{b^2\alpha_3}{ac\alpha_1} - \frac{b\alpha_3}{c\alpha_1}\right]\int_0^u \psi(u-z)be^{-bz}\,dz.$$

Multiplying the equation by $\frac{1}{b}$, we have

$$\frac{\psi'''(u)}{ab} = -\left[\frac{1}{b} - \frac{(\alpha_2+\alpha_3)}{abc\alpha_1}\right]\psi''(u) + \left[\frac{\alpha_3}{bc\alpha_1} - \frac{\alpha_3}{ac\alpha_1}\right]\psi'(u) + \left[\frac{b\alpha_3}{ac\alpha_1} - \frac{\alpha_3}{c\alpha_1}\right]\psi(u)$$
$$+\frac{b\alpha_3}{ac\alpha_1}e^{-bu} + \frac{\alpha_3}{c\alpha_1}e^{-bu} - \left[\frac{b\alpha_3}{ac\alpha_1} - \frac{\alpha_3}{c\alpha_1}\right]\int_0^u \psi(u-z)be^{-bz}\,dz. \qquad (5.17)$$

Adding the terms of each side of (5.16) and (5.17), we have

$$\frac{\psi'''(u)}{ab} + \left[\frac{1}{a} + \frac{1}{b} - \frac{(\alpha_2+\alpha_3)}{abc\alpha_1}\right]\psi''(u) + \left[1 - \frac{\alpha_2}{ac\alpha_1} - \frac{\alpha_3}{bc\alpha_1}\right]\psi'(u) = 0. \qquad (5.18)$$

The equivalent characteristic equation is

$$\frac{r^3}{ab} + \left[\frac{1}{a} + \frac{1}{b} - \frac{(\alpha_2+\alpha_3)}{abc\alpha_1}\right]r^2 + \left[1 - \frac{\alpha_2}{ac\alpha_1} - \frac{\alpha_3}{bc\alpha_1}\right]r = 0. \qquad (5.19)$$

Solving the equation, we obtain the three roots as

$$r_2 = \frac{-\left[\frac{1}{a} + \frac{1}{b} - \frac{(\alpha_2+\alpha_3)}{abc\alpha_1}\right] + \sqrt{D}}{\frac{2}{ab}},$$

$$r_1 = \frac{-\left[\frac{1}{a} + \frac{1}{b} - \frac{(\alpha_2+\alpha_3)}{abc\alpha_1}\right] - \sqrt{D}}{\frac{2}{ab}},$$

$$r_3 = 0,$$

where

$$D = \left[\frac{1}{a} + \frac{1}{b} - \frac{(\alpha_2+\alpha_3)}{abc\alpha_1}\right]^2 - \frac{4}{ab}\left[1 - \frac{\alpha_2}{ac\alpha_1} - \frac{\alpha_3}{bc\alpha_1}\right].$$

Therefore, the general solution of $\psi(u)$ is

$$\psi(u) = C_1e^{r_1u} + C_2e^{r_2u} + C_3. \qquad (5.20)$$

Since

$$\begin{aligned} D &= \left[\frac{1}{a}+\frac{1}{b}-\frac{(\alpha_2+\alpha_3)}{abc\alpha_1}\right]^2-\frac{4}{ab}\left[1-\frac{\alpha_2}{ac\alpha_1}-\frac{\alpha_3}{bc\alpha_1}\right] \\ &= \left[\frac{1}{b}-\frac{1}{a}-\frac{(\alpha_2-\alpha_3)}{abc\alpha_1}\right]^2+\frac{4\alpha_2\alpha_3}{a^2b^2c^2\alpha_1^2} \ > \ 0. \end{aligned}$$

Then, r_1 and r_2 are distinct real roots.

Since

$$\frac{1}{a}+\frac{1}{b}-\frac{(\alpha_2+\alpha_3)}{abc\alpha_1}=\left[\frac{1}{bc\alpha_1}+\frac{1}{ac\alpha_1}\right]\left[c\alpha_1-\frac{\alpha_2}{a}-\frac{\alpha_3}{b}\right]+\frac{\alpha_2}{a^2c\alpha_1}+\frac{\alpha_3}{b^2c\alpha_1} \ > \ 0, \quad (5.21)$$

by the Vieta's theorem in Theorem 2.2 and (5.19), we get

$$r_1r_2=\frac{1-\dfrac{\alpha_2}{ac\alpha_1}-\dfrac{\alpha_3}{bc\alpha_1}}{\dfrac{1}{ab}}>0 \quad (5.22)$$

and

$$r_1+r_2=\frac{-\left[\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{(\alpha_2+\alpha_3)}{abc\alpha_1}\right]}{\dfrac{1}{ab}}<0. \quad (5.23)$$

From (5.22) and $c\alpha_1-\dfrac{\alpha_2}{a}-\dfrac{\alpha_3}{b}>0$, we can see that r_1 and r_2 have the same sign.
From (5.23) and (5.21), we get

$$r_1<0 \text{ and } r_2<0.$$

Next, once we know the values of r_1 and r_1, we will then determine the values of C_1, C_2, and C_3 for (5.20) using the initial conditions follow as,

1. $\lim\limits_{u\to\infty}\psi(u)=0$, since $r_1, r_2<0$ which yields $C_3=0$.
2. Substituting, $\phi(0)=\dfrac{\alpha_3}{c\alpha_1 b}+\dfrac{\alpha_2}{c\alpha_1 a}$ in (5.20), we get

$$C_1+C_2=\frac{\alpha_2}{c\alpha_1}\left(\frac{1}{a}\right)+\frac{\alpha_3}{c\alpha_1}\left(\frac{1}{b}\right). \quad (5.24)$$

3. Letting $u = 0$ in (5.14) and using $\psi(u)$ from (5.20), we get

$$C_1 r_1 + C_2 r_2 = \frac{(\alpha_2 + \alpha_3)}{c\alpha_1}\left[\frac{\alpha_2}{c\alpha_1}\left(\frac{1}{a}\right) + \frac{\alpha_3}{c\alpha_1}\left(\frac{1}{b}\right)\right] - \frac{\alpha_2}{c\alpha_1} - \frac{\alpha_3}{c\alpha_1}. \qquad (5.25)$$

Solving the system of equations (5.24) and (5.25), we get

$$C_1 = \frac{b\alpha_2^2 + a\alpha_3^2 - abc\alpha_1\alpha_2 - abc\alpha_1\alpha_3 + b\alpha_2\alpha_3 + a\alpha_2\alpha_3 - c\alpha_1(b\alpha_2 + a\alpha_3)r_2}{abc^2\alpha_1^2(r_1 - r_2)}$$

and

$$C_2 = \frac{-b\alpha_2^2 - a\alpha_3^2 + abc\alpha_1\alpha_2 + abc\alpha_1\alpha_3 - b\alpha_2\alpha_3 - a\alpha_2\alpha_3 + c\alpha_1(b\alpha_2 + a\alpha_3)r_1}{abc^2\alpha_1^2(r_1 - r_2)}.$$

□

To calculate the approximated ruin probability using the Cramér approximation described in (5.13), we can use the R programming for computation.

5.2.2 The Laplace transform

In this section, we obtain an approximation of ruin probability using the Laplace transforms in conjunction with integral equation of ruin probability for the the renewal risk model with constant premiums and surrenders.

Theorem 5.4. *The Laplace transform of ruin probability $\psi(u)$ for risk model* (5.1) *is*

$$\psi^*(s) = \frac{c\alpha_1 s\psi(0) - \alpha_2[1 - g^*(s)] - \alpha_3[1 - h^*(s)]}{s(c\alpha_1 s - \alpha_2[1 - g^*(s)] - \alpha_3[1 - h^*(s))}, \qquad (5.26)$$

where $\psi(0) = \frac{\alpha_2}{c\alpha_1}E[Y] + \frac{\alpha_3}{c\alpha_1}E[Z]$ *and* g^*, h^* *are the Laplace transforms of probability density functions for the amount of claims size* g *and surrender* h*, respectively.*

Proof. Taking the Laplace transform of (5.3) and formula in Theorem 2.3, we get

$$\begin{aligned}\alpha_1 s\psi^*(s) - \alpha_1\psi(0) &= \frac{(\alpha_2+\alpha_3)}{c}\psi^*(s) - \frac{\alpha_2}{c}\left[\frac{1}{s} - \frac{g^*(s)}{s}\right] - \frac{\alpha_3}{c}\left[\frac{1}{s} - \frac{h^*(s)}{s}\right] \\ &\quad -\frac{\alpha_2}{c}\psi^*(s)g^*(s) - \frac{\alpha_3}{c}\psi^*(s)h^*(s).\end{aligned}$$

Multiplying both sides by $-cs$, we have

$$\begin{aligned}-c\alpha_1 s^2\psi^*(s) + cs\alpha_1\psi(0) &= -s(\alpha_2+\alpha_3)\psi^*(s) + \alpha_2\left[1-g^*(s)\right] + \alpha_3\left[1-h^*(s)\right] \\ &\quad + s\alpha_2\psi^*(s)g^*(s) + s\alpha_3\psi^*(s)h^*(s).\end{aligned}$$

Therefore,

$$\begin{aligned}&-\alpha_2\left[1-g^*(s)\right] - \alpha_3\left[1-h^*(s)\right] \\ &\quad = \psi^*(s)\left[c\alpha_1 s^2 - s(\alpha_2+\alpha_3) + s\alpha_2 g^*(s) + s\alpha_3 h^*(s)\right] - cs\alpha_1\psi(0).\end{aligned}$$

Thus,

$$\psi^*(s) = \frac{cs\alpha_1\psi(0) - \alpha_2\left[1-g^*(s)\right] - \alpha_3\left[1-h^*(s)\right]}{s\left[c\alpha_1 s - \alpha_2\left[1-g^*(s)\right] - \alpha_3\left[1-h^*(s)\right]\right]}.$$

□

Corollary 5.2. *For the renewal risk model defined in* (5.1) *where the amount of claims size and surrender follow exponential distributions according to* (5.12)*, probability density functionss denoted as g and h, respectively, and with parameters a and b. If $c\alpha_1 - \frac{\alpha_2}{a} - \frac{\alpha_3}{b} > 0$ and $\alpha_1, \alpha_2, \alpha_3 > 0$, then the Laplace transform of the ruin probability $\psi(u)$ is*

$$\psi_{\mathscr{L}}(u) = \frac{b^2\alpha_2 + a^2\alpha_3 + (b\alpha_2 + a\alpha_3)s_1}{abc\alpha_1(s_1 - s_2)}e^{s_1 u} + \frac{-b^2\alpha_2 - a^2\alpha_3 - (b\alpha_2 + a\alpha_3)s_2}{abc\alpha_1(s_1 - s_2)}e^{s_2 u} \tag{5.27}$$

where

$$s_1 = \frac{\alpha_2 - ac\alpha_1 - bc\alpha_1 + \alpha_3 - \sqrt{S}}{2c\alpha_1},$$

$$s_2 = \frac{\alpha_2 - ac\alpha_1 - bc\alpha_1 + \alpha_3 + \sqrt{S}}{2c\alpha_1},$$

and

$$S = (-\alpha_2 + ac\alpha_1 + bc\alpha_1 - \alpha_3)^2 - 4c\alpha_1(-b\alpha_2 + abc\alpha_1 - a\alpha_3).$$

Proof.

Substituting the Laplace transforms of the density functions density functions of Y_i and Z_i with CDF's are G and H, respectively, into (5.26), we have

$$\psi^*(s) = \frac{cs\left[\frac{\alpha_2}{ca} + \frac{\alpha_3}{cb}\right] - \alpha_2\left[1 - \frac{a}{s+a}\right] - \alpha_3\left[1 - \frac{b}{s+b}\right]}{s\left(c\alpha_1 s - \alpha_2\left[1 - \frac{a}{s+a}\right] - \alpha_3\left[1 - \frac{b}{s+b}\right]\right)}.$$

Let $R(s) = -b\alpha_2 + abc\alpha_1 - a\alpha_3 + (-\alpha_2 + ac\alpha_1 + bc\alpha_1 - \alpha_3)s + c\alpha_1 s^2$ and rearrange the equation for $\psi^*(s)$, we get

$$\psi^*(s) = \frac{b^2\alpha_2 + a^2\alpha_3 + (b\alpha_2 + a\alpha_3)s}{abR(s)}. \tag{5.28}$$

Let $S = (-\alpha_2 + ac\alpha_1 + bc\alpha_1 - \alpha_3)^2 - 4c\alpha_1(-b\alpha_2 + abc\alpha_1 - a\alpha_3)$. Then, $S > 0$. Factoring $R(s)$, we will obtain that

$$\psi^*(s) = \frac{b^2\alpha_2 + a^2\alpha_3 + (b\alpha_2 + a\alpha_3)s}{ab(s - s_1)(s - s_2)\alpha_1 c}, \tag{5.29}$$

where

$$s_1 = \frac{\alpha_2 - ac\alpha_1 - bc\alpha_1 + \alpha_3 - \sqrt{S}}{2c\alpha_1},$$

$$s_2 = \frac{\alpha_2 - ac\alpha_1 - bc\alpha_1 + \alpha_3 + \sqrt{S}}{2c\alpha_1}.$$

Since $S > 0$, then s_1 and s_2 are distinct real roots.

Since

$$\alpha_2 - ac\alpha_1 - bc\alpha_1 + \alpha_3 = -\left[\frac{1}{b} + \frac{1}{a}\right](abc\alpha_1 - b\alpha_2 - a\alpha_3) - \frac{b\alpha_2}{a} - \frac{a\alpha_3}{b} < 0, \tag{5.30}$$

by the Vieta's theorem in Theorem 2.2 and equation $R(s)$, we get

$$s_1 s_2 = \frac{-b\alpha_2 + abc\alpha_1 - a\alpha_3}{c\alpha_1} > 0 \tag{5.31}$$

and

$$s_1 + s_2 = \frac{\alpha_2 - ac\alpha_1 - bc\alpha_1 + \alpha_3}{c\alpha_1} < 0. \tag{5.32}$$

From (5.31) and $c\alpha_1 - \frac{\alpha_2}{a} - \frac{\alpha_3}{b} > 0$, we can see that s_1 and s_2 have the same sign. From (5.32) and (5.30), we get

$$s_1 < 0 \text{ and } s_2 < 0.$$

Applying partial fraction decomposition to (5.29) with respect to s, we obtain

$$\psi^*(s) = \frac{b^2\alpha_2 + a^2\alpha_3 + (b\alpha_2 + a\alpha_3)s_1}{abc\alpha_1(s_1 - s_2)(s - s_1)} + \frac{-b^2\alpha_2 - a^2\alpha_3 - (b\alpha_2 + a\alpha_3)s_2}{abc\alpha_1(s_1 - s_2)(s - s_2)}. \tag{5.33}$$

Taking the inverse Laplace transform (5.33) with respect to s, we obtain

$$\psi_{\mathscr{L}}(u) = \frac{b^2\alpha_2 + a^2\alpha_3 + (b\alpha_2 + a\alpha_3)s_1}{abc\alpha_1(s_1 - s_2)} e^{s_1 u} + \frac{-b^2\alpha_2 - a^2\alpha_3 - (b\alpha_2 + a\alpha_3)s_2}{abc\alpha_1(s_1 - s_2)} e^{s_2 u}.$$

□

It can be observed that the ruin probability of the Cramér approximation in (5.13) Theorem 5.3 and the Laplace transforms in (5.27) Theorem 5.2 are equal. This can be proven by showing that the formulas of both approximations yield the same value, as mentioned in Remark 5.1.

Remark 5.1. For the amount of claims size and surrender follow exponential distributions according to (5.12), probability density functions denoted as g and h, respectively, and with parameters a and b. The ruin probability of the Cramér approximation $\psi_C(u)$ (5.13) and the Laplace transforms $\psi_{\mathscr{L}}(u)$ (5.27) yield the same value, for all $u \geq 0$

$$\begin{aligned}
\psi_C(u) &= C_1 e^{r_1 u} + C_2 e^{r_2 u}, \\
\psi_{\mathscr{L}}(u) &= \frac{b^2\alpha_2 + a^2\alpha_3 + (b\alpha_2 + a\alpha_3)s_1}{abc\alpha_1(s_1 - s_2)} e^{s_1 u} + \frac{-b^2\alpha_2 - a^2\alpha_3 - (b\alpha_2 + a\alpha_3)s_2}{abc\alpha_1(s_1 - s_2)} e^{s_2 u},
\end{aligned}$$

where C_1, C_2, r_1, r_2, s_1 and s_2 are as follows

$$C_1 = \frac{b\alpha_2^2 + a\alpha_3^2 - abc\alpha_2\alpha_1 - abc\alpha_1\alpha_3 + b\alpha_2\alpha_3 + a\alpha_2\alpha_3 - c\alpha_1(b\alpha_2 + a\alpha_3)r_2}{abc^2\alpha_1^2(r_1 - r_2)},$$

$$C_2 = \frac{-b\alpha_2^2 - a\alpha_3^2 + abc\alpha_2\alpha_1 + abc\alpha_1\alpha_3 - b\alpha_2\alpha_3 - a\alpha_2\alpha_3 + c\alpha_1(b\alpha_2 + a\alpha_3)r_1}{abc^2\alpha_1^2(r_1 - r_2)},$$

$$r_1 = \frac{-\left[\frac{1}{a} + \frac{1}{b} - \frac{(\alpha_2 + \alpha_3)}{abc\alpha_1}\right] - \sqrt{\left[\frac{1}{a} + \frac{1}{b} - \frac{(\alpha_2 + \alpha_3)}{abc\alpha_1}\right]^2 - \frac{4}{ab}\left[1 - \frac{\alpha_2}{ac\alpha_1} - \frac{\alpha_3}{bc\alpha_1}\right]}}{\frac{2}{ab}},$$

$$r_2 = \frac{-\left[\frac{1}{a} + \frac{1}{b} - \frac{(\alpha_2 + \alpha_3)}{abc\alpha_1}\right] + \sqrt{\left[\frac{1}{a} + \frac{1}{b} - \frac{(\alpha_2 + \alpha_3)}{abc\alpha_1}\right]^2 - \frac{4}{ab}\left[1 - \frac{\alpha_2}{ac\alpha_1} - \frac{\alpha_3}{bc\alpha_1}\right]}}{\frac{2}{ab}},$$

$$s_1 = \frac{\alpha_2 - ac\alpha_1 - bc\alpha_1 + \alpha_3 - \sqrt{(-\alpha_2 + ac\alpha_1 + bc\alpha_1 - \alpha_3)^2 - 4c\alpha_1(-b\alpha_2 + abc\alpha_1 - a\alpha_3)}}{2c\alpha_1},$$

$$s_2 = \frac{\alpha_2 - ac\alpha_1 - bc\alpha_1 + \alpha_3 + \sqrt{(-\alpha_2 + ac\alpha_1 + bc\alpha_1 - \alpha_3)^2 - 4c\alpha_1(-b\alpha_2 + abc\alpha_1 - a\alpha_3)}}{2c\alpha_1}.$$

Proof. We want to show that the various coefficients and constants have the same value demonstrated as follows.

$$\begin{aligned} r_1 &= \frac{-\left[\frac{1}{a} + \frac{1}{b} - \frac{(\alpha_2 + \alpha_3)}{abc\alpha_1}\right] - \sqrt{\left[\frac{1}{a} + \frac{1}{b} - \frac{(\alpha_2 + \alpha_3)}{abc\alpha_1}\right]^2 - \frac{4}{ab}\left[1 - \frac{\alpha_2}{ac\alpha_1} - \frac{\alpha_3}{bc\alpha_1}\right]}}{\frac{1}{ab}} \\ &= \frac{\alpha_2 - ac\alpha_1 - bc\alpha_1 + \alpha_3 - \sqrt{(-\alpha_2 + ac\alpha_1 + bc\alpha_1 - \alpha_3)^2 - 4c\alpha_1(-b\alpha_2 + abc\alpha_1 - a\alpha_3)}}{2c\alpha_1} \\ &= s_1, \end{aligned}$$

$$\begin{aligned} r_2 &= \frac{-\left[\frac{1}{a} + \frac{1}{b} - \frac{(\alpha_2 + \alpha_3)}{abc\alpha_1}\right] + \sqrt{\left[\frac{1}{a} + \frac{1}{b} - \frac{(\alpha_2 + \alpha_3)}{abc\alpha_1}\right]^2 - \frac{4}{ab}\left[1 - \frac{\alpha_2}{ac\alpha_1} - \frac{\alpha_3}{bc\alpha_1}\right]}}{\frac{2}{ab}} \\ &= \frac{\alpha_2 - ac\alpha_1 - bc\alpha_1 + \alpha_3 + \sqrt{(-\alpha_2 + ac\alpha_1 + bc\alpha_1 - \alpha_3)^2 - 4c\alpha_1(-b\alpha_2 + abc\alpha_1 - a\alpha_3)}}{2c\alpha_1} \\ &= s_2, \end{aligned}$$

and

$$
\begin{aligned}
C_1 &= \frac{b\alpha_2^2 + a\alpha_3^2 - abc\alpha_2\alpha_1 - abc\alpha_1\alpha_3 + b\alpha_2\alpha_3 + a\alpha_2\alpha_3 - c\alpha_1(b\alpha_2 + a\alpha_3)r_2}{abc^2\alpha_1^2(r_1 - r_2)} \\
&= \frac{b^2\alpha_2 + a^2\alpha_3 + (b\alpha_2 + a\alpha_3)s_1}{abc\alpha_1(s_1 - s_2)},
\end{aligned}
$$

$$
\begin{aligned}
C_2 &= \frac{-b\alpha_2^2 - a\alpha_3^2 + abc\alpha_2\alpha_1 + abc\alpha_1\alpha_3 - b\alpha_2\alpha_3 - a\alpha_2\alpha_3 + c\alpha_1(b\alpha_2 + a\alpha_3)r_1}{abc^2\alpha_1^2(r_1 - r_2)} \\
&= \frac{-b^2\alpha_2 - a^2\alpha_3 - (b\alpha_2 + a\alpha_3)s_2}{abc\alpha_1(s_1 - s_2)}.
\end{aligned}
$$

Therefore,

$$\psi_C(u) = \psi_{\mathscr{L}}(u).$$

□

To calculate the approximated ruin probability using the Laplace transform for money amounts which follow exponential distributions described in (5.27), we can use the MATLAB commands "`partfrac`" and "`ilaplace`" for computation.

5.3 Experimental simulations

In this section, we perform numerical studies to investigate performance of the analytical approximation of the renewal risk model with constant premiums and surrenders, we focuses on the numerical approximation to the ruin probability when the amounts of claims, and surrenders follows an exponential distribution by using the analytical solution such as the Cramér approximation and the Laplace transform comparing with the Monte Carlo approximation.

5.3.1 Statistical estimations for the ruin probability

In this section, we study a statistical estimate for the ruin probability $\hat{\psi}_t(u)$ derived by the direct simulation of the surplus process using the Monte Carlo methods in order to evaluate the result of the approximations suggested in this chapter.

Let N be the total number of realizations of the process $U(t)$. We can calculate the average value of the process $U(t)$ when each ruin occurs at the time point t, consequently, we obtain the corresponding statistical estimate $\hat{\psi}_t(u)$ for the ruin probability $\psi(u)$. The Monte Carlo estimations is obtained as

$$\hat{\psi}_t(u) = \frac{1}{N}\sum_{i=1}^{N} I_{\{U_i(t)<0|U_i(0)=u\}},$$

where t is a fixed time point and N is the sample size. As $N \to \infty$ and $t \to \infty$, by the law of large numbers, $\hat{\psi}_t(u)$ converges to $\psi(u)$. The time points considered here are $t = 1, 5, 50$, and 100, and the sample size of the Monte Carlo method is $N = 200{,}000$. The parameters of the model studied in this section are as follows. The initial capital u varies in $\{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 2, 3\}$ and the constant rate of premiums is $c = 7$. The parameter of the inter-arrival times of premium is $\lambda_1 = 4.5$. The parameter of the inter-arrival times of claim is $\lambda_2 = 6$. The parameter of the inter-arrival times of surrender is $\lambda_3 = 1$.

5.3.2 Exponential distributions for the claim sizes and surrender

Let the probability density function Y_i and Z_i be

$$g(y) = ae^{-ay} \quad \text{and} \quad h(z) = be^{-bz} \quad , y, z \geq 0,$$

where $a = 0.33$, $b = 0.238$, respectively.

For the Cramer approximation, substituting $a = 0.33$, $b = 0.238$, $c = 7$, $\lambda_1 = 4.5$, $\lambda_2 = 6$ and $\lambda_3 = 1$ into the formula of r_1 and r_2 in (5.13), we get $r_1 = -0.246214$ and $r_2 = -0.096096$, respectively. Consequently, $C_1 = 0.005705$ and $C_2 = 0.696173$. Therefore, the Cramér approximation $\psi_C(u)$ is

$$\psi_C(u) = 0.005705e^{-0.246214u} + 0.696173e^{-0.096096u} \quad for\ all\ u \geq 0. \tag{5.34}$$

For the Laplace approximation, substituting $a = 0.33$, $b = 0.238$, $c = 7$, $\lambda_1 = 4.5$, $\lambda_2 = 6$ and $\lambda_3 = 1$ into the formula in (5.27), we get $S = 28$. Consequently, $s_1 = -0.246214$ and $s_2 = -0.096096$. Therefore, the Laplace approximation $\psi_{\mathscr{L}}$ is

$$\psi_{\mathscr{L}}(u) = 0.696173e^{-0.096096u} + 0.005705e^{-0.246214u} \quad for\ all\ u \geq 0. \tag{5.35}$$

The numerical approximations obtained in (5.34)–(5.35) for different values of the initial capital u is given in Table 5.1

u	$\psi(u)$				
	Statistical estimate $\hat{\psi}(u)$				Numerical approx. $\psi_C(u)/\psi_{\mathscr{L}}(u)$
	$t=1$	$t=5$	$t=50$	$t=100$	
0	0.673005	0.692920	0.692975	0.692975	0.701879
0.1	0.667260	0.687465	0.687515	0.687515	0.695082
0.2	0.661650	0.682095	0.682145	0.682145	0.688352
0.3	0.655915	0.676540	0.676590	0.676590	0.681689
0.4	0.650170	0.671075	0.671120	0.671120	0.675091
0.5	0.644290	0.665390	0.665435	0.665435	0.668559
0.6	0.638650	0.660030	0.660085	0.660085	0.662090
0.7	0.632830	0.654465	0.654520	0.654520	0.655686
0.8	0.626515	0.648545	0.648600	0.648600	0.649344
0.9	0.620535	0.642780	0.642835	0.642835	0.643065
1	0.614885	0.637340	0.637400	0.637400	0.636847
1.1	0.608835	0.631570	0.631630	0.631630	0.630691
1.2	0.603075	0.626085	0.626145	0.626145	0.624595
1.3	0.597760	0.620965	0.621025	0.621025	0.618559
1.4	0.592015	0.615590	0.615660	0.615660	0.612582
1.5	0.586200	0.610005	0.610075	0.610075	0.606664
1.6	0.580230	0.604330	0.604395	0.604395	0.600804
1.7	0.574200	0.598435	0.598500	0.598500	0.595001
1.8	0.568010	0.592435	0.592500	0.592500	0.589255
1.9	0.562120	0.586625	0.586690	0.58669	0.583566
2	0.556255	0.580895	0.580965	0.580965	0.577932
3	0.498995	0.524915	0.524980	0.524980	0.524538

Table 5.1: Numerical approximations of the renewal risk model with exponential distributions.

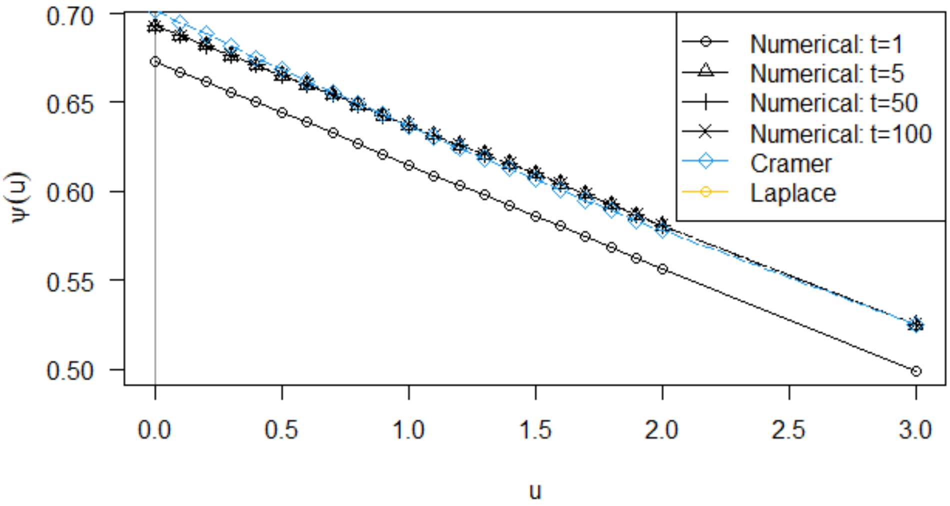


Figure 5.1: Graph of initial reserve u and the ruin probability of the renewal risk model with exponential distributions.

From Table 5.1, we can see that the approximates of ruin probability of all methods decrease when the initial capital increases. Besides, we can observe that the ruin probability $\hat{\psi}_t(u)$ increases as t increases and the Monte Carlo approximation converges to our approximations. Therefore, the Monte Carlo approximation is consider to be a good option.

Also, we can see that the Cramér approximations and the Laplace approximations are nearby the Monte Carlo approximation of ruin probability $\hat{\psi}_t(u)$. In addition, the Monte Carlo approximation $\hat{\psi}_t(u)$ should converges to the exact value of the ruin probability when $t \to \infty$. Moreover, we can see that the Cramér approximations (5.34) and the Laplace approximations (5.35) are equal. The reason for their equivalence is that the ruin probability formulas for both methods are equivalent to each other, yielding the same result see Remark 5.1 or derived from solving the same ODE.

The Monte Carlo simulation will be very good, if we can increase the value of t. However, it will take long computation time to do so. Therefore, a possible way to improve the Monte Carlo simulation performance is to increase the time points of interest and reduce the number of realizations of $U(t)$ instead.

CHAPTER VI

CONCLUSION

6.1 Conclusions and Discussions

In chapter III, we studied suitable analytical approximations of the ruin probability for the risk model CPST by using the Cramér approximation in Theorem 3.2, the Laplace transform in Theorem 3.3, the De-Vylder approximation in Theorem 3.6 and the Lundberg upper bound in Theorem 3.10. Moreover, numerical methods are used to assist in solving systems of equations or finding the inverse Laplace transform in situations where manual computation is not feasible. Moreover, we performed experimental simulations to study their performance. The computation results presented in Tables 3.1 and 3.2 indicate that the Cramér approximation in Tables 3.1 and the De-Vylder approximation in Tables 3.2 have the near values of ruin probability $\hat{\psi}_t(u)$ from Monte Carlo approximation when u has a large value, and both yield ruin probabilities no more than the upper bound. Also, the computation results presented in Tables 3.1 indicate that the Cramér approximation and the Laplace transform have the same value of ruin probability. Similarly, the results in Tables 3.2 indicate that the De-Vylder approximation and the Laplace transform have approximately the same value of ruin probability as explained in chapter III.

In chapter IV, we studied a suitable analytical approximation of the ruin probability for the risk model SPST by using the Cramér approximation in Theorem 4.2, the Laplace transform in Theorem 4.3, the De-Vylder approximation in Theorem 4.6 and the Lundberg upper bound in Theorem 4.10. Moreover, we perform experimental simulation to study its performance. Numerical methods are used to assist in solving systems of equations or finding the inverse Laplace transform in situations where manual computation is not feasible. The results of computations presented in Tables 4.1 and 4.2 indicate that the Cramér approximation in Tables 4.1 and the De-Vylder approximation in Tables 4.2 have the near value of ruin probability $\hat{\psi}_t(u)$ from Monte Carlo approximation when u has a large value, and both yield ruin probability no more than upper bound as explained in the chapter IV.

In chapter V, we studies a suitable analytical approximation of the ruin probability for the renewal risk model with constant premiums and surrenders by using the Cramér approximation in Theorem 5.3 and the Laplace transform in Theorem 5.4. Moreover, we perform experimental simulation to study its performance. Numerical methods are used to assist in solving systems of equations or finding the inverse Laplace transform in situations where manual computation is not feasible. The results of computations presented in Tables 5.1 indicate that the Cramér approximation and the Laplace transform have the near value of ruin probability $\hat{\psi}_t(u)$ from Monte Carlo approximation when u has a large value. As well as, the results of computations presented in Tables 5.1 indicate that the Cramér approximation and the Laplace transform have approximately the same value of ruin probability as explained in the chapter V.

Moreover, the statistical estimate $\hat{\psi}_t(u)$ for the ruin probability $\psi(u)$, once time t reaches a certain point, the probability of ruin $\psi(u)$ after that point will remain constant.

Finally, noted that while the numerical example discussed above is insufficient to draw conclusions about the accuracy of the commonly recommended estimation method, and may not be reflective of the actual situation of an insurance company, it is highly desirable to have tools to control the accuracy of parameter estimates. Nevertheless, these estimates can help us to draw some general conclusions. Several extensions of our study can be done such as to investigate another approximation method for the model or to extend the numerical approximation of the ruin probability for more general risk models to accommodate other features of risk models.

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