การประมาณค่าด้วยการแจกแจงปกติสำหรับตราสารที่มีหนี้เป็นหลักประกันซึ่งไม่อิสระเฉพาะที่

นางสาวสุพร จงปรีชาหาญ

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2562

NORMAL APPROXIMATION FOR LOCALLY DEPENDENT COLLATERALIZED DEBT OBLIGATION

Miss Suporn Jongpreechaharn

A Dissertation Submitted in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy Program in Mathematics
Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Academic Year 2019

Thesis Title	NORMAL APPROXIMATION FOR LOCALLY DEPENDENT COLLATERALIZED DEBT OBLIGATION
By	Miss Suporn Jongpreechaharn
Field of Study	Mathematics
Thesis Advisor	Professor Kritsana Neammanee, Ph.D.
Thesis Co-Advisor	Associate Professor Nattakarn Chaidee, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Doctoral Degree

.... Dean of the Faculty of Science

(Professor Polkit Sangvanich, Ph.D.)

THESIS COMMITTEE

.... Chairman

(Associate Professor Songkiat Sumetkijakan, Ph.D.)

.... Thesis Advisor

(Professor Kritsana Neammanee, Ph.D.)

.... Thesis Co–Advisor

(Associate Professor Nattakarn Chaidee, Ph.D.)

.... Examiner

(Assistant Professor Jiraphan Suntornchost, Ph.D.)

.... Examiner

(Raywat Tanadkithirun, Ph.D.)

.... External Examiner

(Assistant Professor Dawud Thongtha, Ph.D.)

สุพร จงปรีชาหาญ: การประมาณค่าด้วยการแจกแจงปกติสำหรับตราสารที่มีหนี้เป็นหลักประกันซึ่งไม่อิสระเฉพาะที่. (NORMAL APPROXIMATION FOR LOCALLY DEPENDENT COLLATERALIZED DEBT OBLIGATION) อ.ที่ปรึกษาวิทยานิพนธ์หลัก: ศ. ดร.กฤษณะ เนียมมณี, อ.ที่ปรึกษาวิทยานิพนธ์ร่วม: รศ. ดร.ณัฐกาญจน์ ใจดี, 70 หน้า.

วิทยานิพนธ์ฉบับนี้สนใจในตราสารที่มีหนี้เป็นหลักประกันเนื่องจากตราสารนี้เป็นสาเหตุของการเกิดวิกฤติเศรษฐกิจระหว่างปี 2007 ถึง 2008 จากความเสียหายในวิกฤติครั้งนี้ทำให้นักวิจัยหลายท่านพยายามประมาณค่าความสูญเสียในแต่ละชั้นของตราสารที่มีหนี้เป็นหลักประกัน เช่น ในปี 2009 คารูอิและเจียวใช้ตัวแปรสุ่มปกติในการประมาณค่าความสูญเสียในแต่ละชั้นของตราสารที่มีหนี้เป็นหลักประกันซึ่งประกอบด้วยสินทรัพย์ที่อิสระต่อกัน อย่างไรก็ตามในงานนี้เราสนใจโครงสร้างความสัมพันธ์ระหว่างสินทรัพย์ในตราสารที่มีหนี้เป็นหลักประกัน เรานำเสนอเงื่อนไขความไม่อิสระสองแบบคือความไม่อิสระเฉพาะที่และความไม่อิสระเฉพาะที่ที่ไม่มีส่วนร่วม เงื่อนไขนี้มีความหมายอย่างคร่าว ๆ คือการผิดชำระหนี้ของสินทรัพย์หนึ่งจะมีอิทธิพลต่อการผิดชำระหนี้ของสินทรัพย์อื่น แต่สินทรัพย์บางอย่างอาจจะไม่มีความสัมพันธ์กัน ค่าเฉลี่ยของความสูญเสียในแต่ละชั้นของตราสารที่มีหนี้เป็นหลักประกันถูกประมาณโดยค่าเฉลี่ยของฟังก์ชันคอลสำหรับตัวแปรสุ่มปกติมาตรฐาน ขอบเขตการประมาณค่าแบบเอกรูปและแบบไม่เอกรูปถูกเสนอภายใต้เงื่อนไขความไม่อิสระเฉพาะที่และความไม่อิสระเฉพาะที่ที่ไม่มีส่วนร่วมโดยใช้วิธีของสไตน์ นอกจากนี้เรายังได้แสดงตัวอย่างสองตัวอย่างภายใต้ความไม่อิสระเฉพาะที่ที่ไม่มีส่วนร่วมและได้เสนอขอบเขตการประมาณค่าเชิงตัวเลข

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์	ลายมือชื่อนิสิต ....
สาขาวิชา คณิตศาสตร์	ลายมือชื่อ อ.ที่ปรึกษาหลัก ....
ปีการศึกษา 2562	ลายมือชื่อ อ.ที่ปรึกษาร่วม ....

5972881623 : MAJOR MATHEMATICS
KEYWORDS : CDO/ NORMAL APPROXIMATION/ LOCAL DEPENDENCE/ DISJOINT LOCAL DEPENDENCE/ UNIFORM BOUND/ NON–UNIFORM BOUND/ STEIN'S METHOD

SUPORN JONGPREECHAHARN : NORMAL APPROXIMATION FOR LOCALLY DEPENDENT COLLATERALIZED DEBT OBLIGATION. ADVISOR : PROF. KRITSANA NEAMMANEE, Ph.D. CO-ADVISOR : ASSOC. PROF. NATTAKARN CHAIDEE, Ph.D., 70 pp.

This dissertation sheds light on a collateralized debt obligation (CDO) since it resulted in the financial crisis between 2007 and 2008. Because of damage of this crisis, several authors have attempted to approximate loss on a tranche of a CDO. For instance, in 2009, Karoui and Jiao used a normal random variable to approximate the loss on a tranche of a CDO containing independent assets. However, in this work, we are attentive to dependence structure among assets in a CDO. We present two types of dependent condition: local dependence (LD) and disjoint local dependence (DLD). They roughly mean that defaults of some assets may influence defaults of other assets in their neighborhood but some assets are not correlated. An average loss on a tranche of a CDO is approximated by an average of a call function for the standard normal random variable. The uniform and non–uniform bounds are presented under the LD and DLD conditions by using the Stein's method. Moreover, we illustrate two examples under the DLD condition and propose numerical bounds.

Department : Mathematics and Computer Science Student's Signature ....

Field of Study : Mathematics Advisor's Signature ....

Academic Year : 2019 Co-Advisor's Signature ....

ACKNOWLEDGEMENTS

I am greatly indepted to my thesis advisor, Professor Dr.Kritsana Neammanee and my thesis co–advisor, Associate Professor Dr.Nattakarn Chaidee for their generous guidance and helpful suggestion throughout this research. I would like to express my special thanks to Associate Professor Dr.Songkiat Sumetkijakan, Assistant Professor Dr.Jiraphan Suntornchost, Dr.Raywat Tanadkithirun and Assistant Professor Dr.Dawud Thongtha, my thesis committees, for their beneficial suggestion and recommendation. I would like to express my deep gratitude to Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University for providing valuable knowledge and facilities throughout the course of study.

In particular, I wish to express my thankfulness to the Human Resource Development in Science Project (Science Achievement Scholarship of Thailand, SAST) for financial supports. I appreciatively thank to my dear friends for giving me good experiences, good wish and help during the course of my research.

Finally, I would like to express my deep gratitude to my beloved family for their love and encouragement throughout my graduate study.

CONTENTS

CHAPTER I
INTRODUCTION

A collateralized debt obligation (CDO) is a family of reference assets which are ranked by a credit rating agency and assigned to a specific class called tranche. A CDO roughly consists of senior, mezzanine and equity tranches. A senior tranche contains low-risk assets with low return while the mezzanine tranche contains assets with moderate risk and moderate return. On the other hand, assets with high risk and high return are contained in an equity tranche. The priority of payment and interest is arranged from the senior, mezzanine and equity tranches, respectively. Conversely, the equity tranche first bears risk and loss at the expense of high interest rate until the loss reaches equity tranche's limit called a detachment point (DP). The remaining loss passes through the mezzanine and senior tranches,

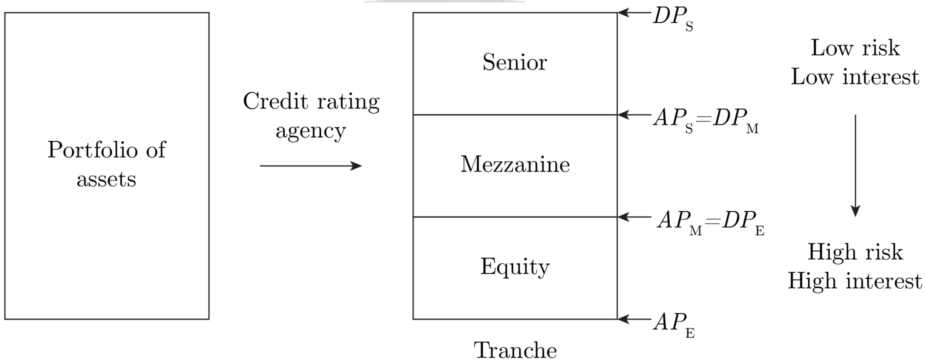


Figure 1.1: The structure of CDO

respectively, due to their starting points of lost absorbing called attachment points (AP). The structure of CDO is shown in Figure 1.1, where the notations AP_{S}, AP_{M} and AP_{E} stand for the attachment point of senior, mezzanine and equity tranches, respectively and DP_{S}, DP_{M} and DP_{E} stand for detachment point of senior, mezzanine and equity tranches, respectively. Note that $AP_{\mathrm{E}} = 0$, $AP_{\mathrm{M}} = DP_{\mathrm{E}}$, $AP_{\mathrm{S}} = DP_{\mathrm{M}}$ and $DP_{\mathrm{S}} = 1$. Consequently, from this structure, we concentrate on loss on a tranche of a CDO.

A CDO is popular and crucial because it was a cause of the financial crisis between 2007 and 2008 which made an extensive damage to the world. After this crisis, financial institutions have seriously discovered a way to handle future crises. On the one hand, many countries have restricted the proportion of investment in CDO. However, the possible return from CDO attracts investors to manage a risk instead of limiting an investment. One way to hedge a risk is to know an average loss on a tranche of a CDO before investing. Thus, we need to predict a default time of each asset and find a total loss on a CDO. For example, [2], [3] and [10] used one factor Gaussian copula model to determine the probability of default for each asset. Then, correlation structure among assets was concentrated to deduce the loss distribution. This model was also used to treat default correlation among assets ([4]). After that, the expected loss on each tranche of CDO was investigated by using Monte Carlo and analytic methods. Moreover, the model was extended to multifactor copula model ([9], [11]) which was used to establish a distribution of default. Hull and White ([11]) calculated loss distribution by using recurrence relation and probability bucketing. Furthermore, [10] and [18] used the Archimedean copula process to model dependence between default time while [14] used one factor normal inverse Gaussian copula model. In one factor copula model, parameters are generally deterministic, but [1] and [2] investigated a stochastic correlation model which is a slight extension of one factor Gaussian copula model.

Nowadays, an approximation approach is famous and suitable for this problem. For instance, in [19], Yang, Hurd and Zhang used saddle point approximation method to compute loss distribution. The total loss on a CDO was represented by a series of random walk which is a series of independent assets by Pagés and Wilbertz ([15]). They reduced the number of steps for random walk by using the dual quantization method and approximated the loss on a tranche of CDO by the optimal dual quantization. Moreover, in 1972, Stein ([16]) introduced the Stein's method which is a powerful and brilliant method for approximation since the method can be applied in a problem with dependent structure and the error in the approximation can be obtained.

Karoui and Jiao ([7], [8]) examined the total loss of a CDO to be a series of loss for independent assets. They used the Stein's method to approximate loss on a tranche of a CDO by a normally distributed random variable and obtained bounds of the approximation. After that, Jongpreechaharn and Neammanee ([12], [13]) improved the bound under the same condition. The results from [13] are stated below.

$$\sup_{k>0} \left| E(W-k)^+ - E(Z-k)^+ \right| \leq 3 \sum_{i=1}^{n} E\left|X_i\right|^3$$

and

$$\begin{aligned} &\left| E(W-k)^+ - E(Z-k)^+ \right| \\ &\leq 2.86 e^{-3k^2/8} \sum_{i=1}^{n} E\left|X_i\right|^3 + \frac{6.54}{k} \left[\frac{1}{\sqrt{3}} \left(\sum_{i=1}^{n} EX_i^4 + 3 \right)^{1/2} + 1 \right] \left(\sum_{i=1}^{n} EX_i^4 \right)^{1/2}, \end{aligned}$$

where $x^+ = \max\{x, 0\}$ for any real number x, $W = \displaystyle\sum_{i=1}^{n} X_i$ with zero mean and unit variance, Z is the standard normal random variable and k is a positive real number.

However, assets in a CDO may be correlated, i.e., a default of one asset may induce defaults of relevant assets. Meanwhile, some assets are independent. For instance, if a restaurant owner runs out of money, then the restaurant is closed due to a shortage of materials. Consequently, all staff are unemployed and hence, they default. Employees in the same industry who are in debt may simultaneously default, if the industry goes bankrupt. An agriculturist in the red may default when a flood ruins crops. This damage leads to a default of a processing factory because of no raw material. Since the lack of export goods, the loan of an export company defaults. But, defaults of employees who are unemployed do not affect the product of the processing factory or we can say that they do not influence the

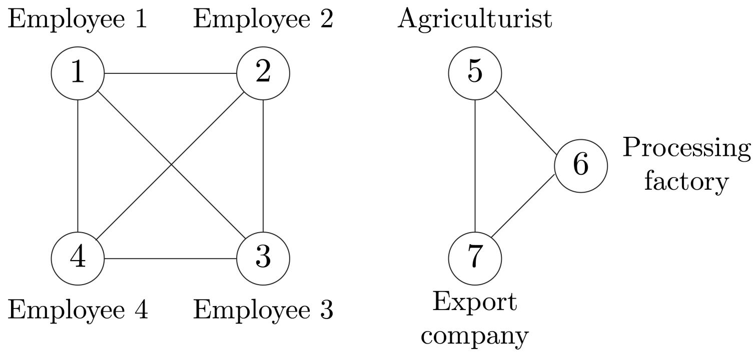


Figure 1.2: Example of locally dependent assets

default of the processing factory (Figure 1.2).

Remark that an edge in Figure 1.2 represents the correlation among two assets, and the assets that have no link are independent.

From the above example, we can see that some assets are relative in the sense that they have some common structures such as they are in the same organization or they have common resources. If an asset defaults, other correlated assets may default as well, but the other assets that have no connection with the original asset will not default.

Consequently, in this dissertation, we concentrate on dependent structures among assets including local dependence (LD) and disjoint local dependence (DLD). Note that the dependence structures in this work are motivated by Chen and Shao ([6]).

Before proposing the definition of LD and DLD conditions, we first define notations used in this work. Consider a standard CDO with n underlying assets. The i^{th} asset is assumed to have a deterministic recovery rate R_i and a default time τ_i. We can obtain the total loss on the portfolio at the time T by

$$L(T) = \frac{1}{n}\sum_{i=1}^{n}(1 - R_i)\mathbb{I}(\tau_i \leq T),$$

where $\mathbb{I}(A)$ is the indicator function of a set A. Note that in the real situation, we do not know the value of the default time τ_i. Hence, the key of hedging the risk in CDO is to compute an average loss on a tranche of the CDO defined by the difference of averages for call functions

$$E(L(T) - AP)^{+} - E(L(T) - DP)^{+}.$$

Therefore, our problem is approximating

$$E\left(L(T) - \tilde{k}\right)^{+},$$

where $\tilde{k}$ is a positive real number and $0 < \tilde{k} \leq 1$. Note that, when $\tilde{k} = 0, EL(T)^{+}$ is easily calculated.

Let Z be a standard normal random variable. To approximate $E\left(L(T) - \tilde{k}\right)^{+}$ by a call function of the standard normal random variable, we need to normalize $L(T)$. Let

$$X_i = \frac{(1 - R_i)\left[\mathbb{I}(\tau_i \leq T) - p_i\right]}{n\sqrt{\operatorname{Var} L(T)}},$$

where $p_i = P\left(\mathbb{I}(\tau_i \leq T) = 1\right)$ and let $W = \sum_{i=1}^{n} X_i$. Then

$$W = \frac{L(T) - EL(T)}{\sqrt{\operatorname{Var} L(T)}}$$

with

$$EW = 0 \quad \text{and} \quad \operatorname{Var} W = 1.$$

Notice that, to determine the rate of convergence of $\operatorname{Var} L(T)$, we assume that $\{\mathbb{I}(\tau_i \leq T)\}_{i=1}^{n}$ are independent and identically distributed. Then, by setting $R = R_i, p = p_i$ for $i = 1, 2, 3, \ldots, n$ and $q = 1 - p$, we obtain that

$$\operatorname{Var} L(T) = \frac{(1 - R)pq}{n} = \mathcal{O}\left(\frac{1}{n}\right).$$

Next, we let $k = \dfrac{\tilde{k} - EL(T)}{\sqrt{\operatorname{Var} L(T)}}$. Then,

$$
\begin{aligned}
&\left|E\left(L(T)-\tilde{k}\right)^{+}-\sqrt{\operatorname{Var} L(T)}E\left(Z-k\right)^{+}\right| \\
&\qquad=\sqrt{\operatorname{Var} L(T)}\left|E(W-k)^{+}-E(Z-k)^{+}\right|.
\end{aligned}
$$

Hence, the problem is transformed to find a bound for $|E(W-k)^{+}-E(Z-k)^{+}|$. Notice that, we assume $k>0$ in this work. Let

$$\delta(n,k)=\left|E(W-k)^{+}-E(Z-k)^{+}\right|$$

and

$$\delta(n)=\sup_{k>0}\left|E(W-k)^{+}-E(Z-k)^{+}\right|.$$

We next introduce the LD condition which is taken from [6]. For $A\subset\{1,2,3,\ldots,n\}$, let X_A denote $\{X_i, i\in A\}$.

Definition 1.1 (LD condition)**.** We say that random variables $X_1,X_2,X_3,\ldots,X_n$ satisfy the *local dependence* condition if for each $i=1,2,3,\ldots,n$, there exist $A_i\subseteq B_i\subseteq C_i\subseteq\{1,2,3,\ldots,n\}$ such that X_i is independent of $X_{A_i^c}$, X_{A_i} is independent of $X_{B_i^c}$ and X_{B_i} is independent of $X_{C_i^c}$.

The following is the uniform bound on normal approximation for LD CDO.

Theorem 1.2 (Uniform Bound)**.** *Under LD condition, we have*

$$\delta(n)=\mathcal{O}\left(\frac{1}{n^2\left(\operatorname{Var} L(T)\right)^{3/2}}\right)+\mathcal{O}\left(\frac{1}{n^{3/2}\operatorname{Var} L(T)}\right).$$

Furthermore, if $\operatorname{Var} L(T)=\mathcal{O}\left(\frac{1}{n}\right)$, *then*

$$\delta(n)=\mathcal{O}\left(\frac{1}{\sqrt{n}}\right).$$

To present a non–uniform bound for LD CDO, we let the following notations throughout this work:

1. $Y_i = \sum_{j \in A_i} X_j$;

2. $p_i = P\left(\mathbb{I}(\tau_i \leq T) = 1\right), q_i = 1 - p_i$ and $p_{ij} = P(\mathbb{I}(\tau_i \leq T) = 1, \mathbb{I}(\tau_j \leq T) = 1)$;

3. $|A| = \max_{1 \leq i \leq n} |A_i|$ and $|B| = \max_{1 \leq i \leq n} |B_i|$;

4. $\kappa_1 = \max_{1 \leq i \leq n} \max\{|C_i|, |C_i^{-1}|\}$ where $C_i^{-1} = \{j \mid i \in C_j\}$;

5. $\kappa_2 = \max_{1 \leq i \leq n} \{|N(B_i)|\}$, where $N(B_i) = \{j \mid B_j \cap B_i \neq \varnothing\}$;

6. $\kappa_3 = \max_{1 \leq i \leq n} \max\{|B_i|, |B_i^{-1}|\}$, where $B_i^{-1} = \{j \mid i \in B_j\}$;

7. $\kappa = \max\{\kappa_1, \kappa_2, \kappa_3\}$.

Remark 1.3. We observe from the definition of κ that, when κ is large, it means that there are many correlated assets in a CDO. Hence, when an asset in this CDO defaults, the whole CDO may be defaulting. Therefore, the CDO manager should limit the number of correlated assets in the CDO to hedging risk. Consequently, we assume in this work that $\kappa = \max\{\kappa_1, \kappa_2, \kappa_3\}$ does not depend on n.

Notice in the case of independent random variables that they satisfy the LD condition with $|A_i| = |B_i| = |B_i^{-1}| = |C_i| = |C_i^{-1}| = |N(B_i)| = 1$ for every i. Consequently, $\kappa = 1$ does not depend on n.

From the above notations and assumption in Remark 1.3, we have the non–uniform bound on normal approximation for the LD CDO as shown.

Theorem 1.4 (Non-uniform Bound). *Under LD condition and for $k \geq 2$, we have*

$$\delta(n,k) = C_1(k,\kappa)\mathcal{O}\left(\frac{1}{n^2\left(\operatorname{Var} L(T)\right)^{3/2}}\right) + C_2(k,\kappa)\mathcal{O}\left(\frac{1}{n^{3/2}\operatorname{Var} L(T)}\right)$$
$$+ \frac{1}{k}\mathcal{O}\left(\frac{1}{n^2\left(\operatorname{Var} L(T)\right)^{3/2}}\right),$$

where $$C_1(k,\kappa) = \left(2 + \frac{2\kappa}{3}\right)\left(\frac{e^{-k^2/2}}{\sqrt{2\pi}k^2} + \frac{1}{k}\right) + \frac{1}{3k^2}$$

and $$C_2(k,\kappa) = \sqrt{\kappa}\left(\frac{e^{-k^2/2}}{\sqrt{2\pi}k^2} + \frac{1}{k}\right).$$

Furthermore, if $\operatorname{Var} L(T) = \mathcal{O}\left(\frac{1}{n}\right)$, *then*

$$\delta(n,k) = \left(C_1(k,\kappa) + C_2(k,\kappa) + \frac{1}{k}\right)\mathcal{O}\left(\frac{1}{\sqrt{n}}\right).$$

Next, we define a special and realistic case of the LD condition, called disjoint local dependence (DLD) condition.

Definition 1.5 (DLD condition)**.** We say that random variables $X_1, X_2, X_3, \dots, X_n$ satisfy the *disjoint local dependence* condition if there exists a partition $\{A_i\}_{i=1}^d$ of $\{1, 2, 3, \dots, n\}$, where $d \leq n$ such that for each $i = 1, 2, 3, \dots, d$, X_{A_i} is independent of $X_{A_i^c}$.

Notice that the DLD condition is a special case of the LD condition when $A_i = B_i = C_i$ and $\{A_i\}$ is a partition of $\{1, 2, 3, \dots, n\}$. Although we can directly apply Theorem 1.2 and Theorem 1.4 under LD condition to obtain error bounds for DLD condition, a direct proof for DLD condition gives sharper bounds than those obtained from the LD condition.

From the structure of DLD condition, we can classify assets due to their relation. Hence, assume that the n assets can be split into d groups and the i^{th} company has $m_i - m_{i-1}$ indebted personnel (for $i = 1, 2, \dots, d$ when $m_0 = 0$ and $m_d = n$) as shown in Figure 1.3.

From this structure, we have the uniform and non–uniform bounds as follows.

Theorem 1.6 (Uniform Bound)**.** *Under the DLD condition, we have*

$$\delta(n) \leq 24.97 \sum_{i=1}^{d} E|Y_i|^3 + 0.8\left(\sum_{i=1}^{d} EY_i^4\right)^{1/2} + \left(dEW^4 \sum_{i=1}^{d} EY_i^6\right)^{1/2}.$$

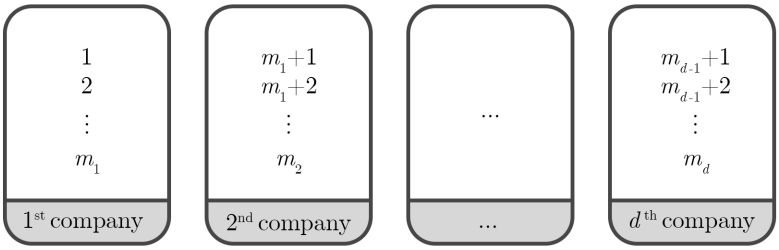


Figure 1.3: Classification of assets in a DLD CDO

Furthermore, if we use the fact that

$$|Y_i| \leq \frac{|A_i|}{n\sqrt{\operatorname{Var} L(T)}} \quad \textit{and} \quad EW^4 \leq 3 + \frac{d|A|^4}{n^4\left(\operatorname{Var} L(T)\right)^2},$$

we have

$$\begin{aligned}\delta(n) \leq & \frac{24.97d|A|^3}{n^3\left(\operatorname{Var} L(T)\right)^{3/2}} + \frac{0.8\sqrt{d}|A|^2}{n^2 \operatorname{Var} L(T)} \\ & + \left(3 + \frac{d|A|^4}{n^4\left(\operatorname{Var} L(T)\right)^2}\right)^{1/2} \frac{d|A|^3}{n^3\left(\operatorname{Var} L(T)\right)^{3/2}}.\end{aligned}$$

Theorem 1.7 (Non-uniform Bound)**.** *Under the DLD condition with $k \geq 2$, we have*

$$\delta(n,k) \;\leq\; C_1(k)\sum_{i=1}^{d} E|Y_i|^3 + C_2(k)\left(\sum_{i=1}^{d} EY_i^4\right)^{1/2} + C_3(k)\left(d\sum_{i=1}^{d} EY_i^6\right)^{1/2},$$

where
$$C_1(k) = \frac{5.5e^{-k^2/2}}{\sqrt{2\pi}k^2} + \frac{5.5}{k} + \frac{1}{2k^2},$$
$$C_2(k) = \frac{e^{-k^2/2}}{\sqrt{2\pi}k^2} + \frac{1}{k}$$
and
$$C_3(k) = \frac{1}{k}\left(3\sqrt{EW^6} + 15.69\sqrt{EW^4} + 18.24\right).$$

Furthermore, if we use the fact that

$$|Y_i| \leq \frac{|A_i|}{n\sqrt{\operatorname{Var} L(T)}},$$

$$EW^4 \leq 3 + \frac{d|A|^4}{n^4 \left(\operatorname{Var} L(T)\right)^2}$$

and

$$EW^6 \leq 15 + \frac{(1+10d)\, d|A|^6}{n^6 \left(\operatorname{Var} L(T)\right)^3} + \frac{15d|A|^4}{n^4 \left(\operatorname{Var} L(T)\right)^2},$$

we have

$$\delta(n,k) \leq \frac{C_1(k) d|A|^3}{n^3 \left(\operatorname{Var} L(T)\right)^{3/2}} + \frac{C_2(k)\sqrt{d}|A|^2}{n^2 \operatorname{Var} L(T)} + \frac{\overline{C_3(k)} d|A|^3}{n^3 (\operatorname{Var} L(T))^{3/2}},$$

where

$$\overline{C_3(k)} = \frac{1}{k}\left[3\left(15 + \frac{(1+10d)\, d|A|^6}{n^6 \left(\operatorname{Var} L(T)\right)^3} + \frac{15d|A|^4}{n^4 \left(\operatorname{Var} L(T)\right)^2}\right)^{1/2} + 15.69\left(3 + \frac{d|A|^4}{n^4 \left(\operatorname{Var} L(T)\right)^2}\right)^{1/2} + 18.24\right].$$

Moreover, we give two examples for Theorem 1.6 and Theorem 1.7 under the DLD condition where we classify assets in the CDO according to their workplace. In the first situation, we consider a CDO containing bankrupt assets. The assets correspond with loans of personnel from d companies. Each company tends to face bankruptcy due to the global crisis. If a company goes bankrupt, then all personnel in the company are unemployed. Consequently, they default. In other words, when an asset defaults, then other assets in the same company also default. In addition, bankruptcy of a company does not affect other companies. Let p_{m_i} be the probability that the i^{th} company defaults. The following are our results.

Example 1.8. Under the bankrupt assets situation, we have

1. the uniform bound for loss on a tranche of CDO containing bankrupt assets is

$$\delta(n) \leq 24.97\gamma_{d,3} + 0.8\gamma_{d,4}^{1/2} + \left[d\gamma_{d,6}\left(3 + \gamma_{d,4}\right)\right]^{1/2};$$

2. for $k \geq 2$, the non-uniform bound for loss on a tranche of CDO containing

bankrupt assets is

$$\delta(n,k) \le C_1(k)\gamma_{d,3} + C_2(k)\gamma_{d,4}^{1/2} + C_3(k)\left(d\gamma_{d,6}\right)^{1/2},$$

where $$\gamma_{d,r} = \frac{1}{n^r(\operatorname{Var} L(T))^{r/2}} \sum_{i=1}^{d} p_{m_i} q_{m_i} (p_{m_i}^{r-1} + q_{m_i}^{r-1}) \left(\sum_{j \in A_i} (1 - R_j) \right)^r,$$

$$EW^4 \le 3 + \gamma_{d,4},$$

$$EW^6 \le 15 + \gamma_{d,6} + 15\gamma_{d,4} + 10\gamma_{d,3}^2$$

and $$\operatorname{Var} L(T) = \frac{1}{n^2} \sum_{i=1}^{d} p_{m_i} q_{m_i} \left(\sum_{j \in A_i} (1 - R_j) \right)^2.$$

When we set parameters: $d = n/2, p = p_i, R = R_i$ and $m_i - m_{i-1} = 2$, we have the bounds for loss on a tranche of the CDO with bankrupt assets for $k \ge 2$ are

$$\delta(n) \le \frac{24.97\sqrt{2}(p^2+q^2)}{\sqrt{npq}} + \frac{0.8\sqrt{2(p^3+q^3)}}{\sqrt{npq}} + \sqrt{\frac{2(p^5+q^5)}{np^2q^2}\left(3 + \frac{2(p^3+q^3)}{npq}\right)}$$

and

$$\delta(n,k) \le \frac{\sqrt{2}(p^2+q^2)C_1(k)}{\sqrt{npq}} + \frac{\sqrt{2(p^3+q^3)}C_2(k)}{\sqrt{npq}} + \sqrt{\frac{2(p^5+q^5)}{np^2q^2}}C_3(k),$$

where $$EW^4 \le 3 + \frac{2(p^3+q^3)}{npq}$$

and $$EW^6 \le 15 + \frac{4(p^5+q^5)}{(npq)^2} + \frac{30(p^3+q^3) + 20(p^2+q^2)^2}{npq}.$$

Moreover, if we set $R = 0.7$ and $p = 0.5$, then we have the result shown in Figure 1.4.

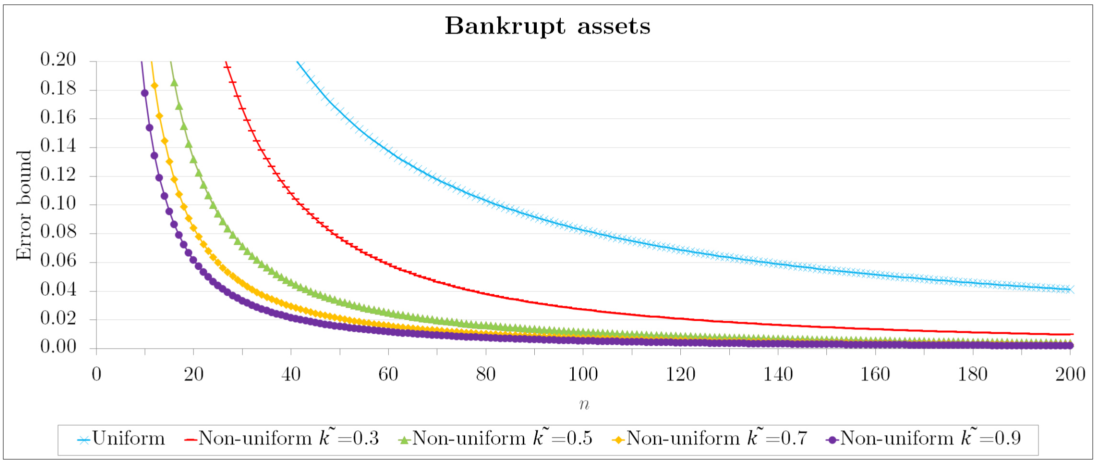


Figure 1.4: Uniform and non-uniform bounds for loss on a tranche of CDO with bankrupt assets

In the second situation, we consider a CDO containing laid–off assets. Under an economic contraction around the world, many companies must manage their financial status. One of many solutions to reduce the exceeding cost is a layoff. The n assets in the CDO are split into a number of groups, and each group represents a company or a department. We suppose that each organization plans to lay off at most one employee. Hence, if our colleague is laid off, then we are still employed. On the other hand, the layoff of other companies does not affect our company. Moreover, it is possible that no coworker in the same company are laid off.

Example 1.9. Under the laid–off assets situation, we have

1. the uniform bound for loss on a tranche of CDO containing laid-off assets is

$$\delta(n) \leq 24.97\beta_{d,3} + 0.8\beta_{d,4}^{1/2} + \left[d\beta_{d,6}\left(3 + \beta_{d,4}\right)\right]^{1/2};$$

2. for $k \geq 2$, the non-uniform bound for loss on a tranche of CDO containing laid-off assets is

$$\delta(n,k) \leq C_1(k)\beta_{d,3} + C_2(k)\beta_{d,4}^{1/2} + C_3(k)\left(d\beta_{d,6}\right)^{1/2},$$

where $$EW^4 \leq 3 + \beta_{d,4},$$

$$EW^6 \leq 15 + \beta_{d,6} + 15\beta_{d,4} + 10\beta_{d,3}^2,$$

$$\beta_{d,r} = \frac{1}{n^r(\operatorname{Var} L(T))^{r/2}} \sum_{i=1}^{d} \left[\sum_{j\in A_i} p_j \left| 1 - R_j - \sum_{l\in A_i}(1-R_l)p_l \right|^r + (1-p_{A_i}) \left(\sum_{j\in A_i}(1-R_j)p_j \right)^r \right]$$

and $$\operatorname{Var} L(T) = \frac{1}{n^2} \sum_{i=1}^{d} \left[\sum_{j\in A_i}(1-R_j)^2 p_j - \left(\sum_{j\in A_i}(1-R_j)p_j \right)^2 \right].$$

Next, we compare the bounds by applying Example 1.9 with the following parameters: $d = n/2, p = p_i$ and $m_i - m_{i-1} = 2$. We obtain that

$$\delta(n) \leq \frac{24.97\sqrt{2}(\bar{p}^2+\bar{q}^2)}{\sqrt{n\bar{p}\bar{q}}} + \frac{0.8\sqrt{2}(\bar{p}^3+\bar{q}^3)^{1/2}}{\sqrt{n\bar{p}\bar{q}}} + \sqrt{\frac{2(\bar{p}^5+\bar{q}^5)}{n\bar{p}^2\bar{q}^2}\left(3 + \frac{2(\bar{p}^3+\bar{q}^3)}{n\bar{p}\bar{q}}\right)}$$

and

$$\delta(n,k) \leq \frac{\sqrt{2}(\bar{p}^2+\bar{q}^2)C_1(k)}{\sqrt{n\bar{p}\bar{q}}} + \frac{\sqrt{2}(\bar{p}^3+\bar{q}^3)^{1/2}C_2(k)}{\sqrt{n\bar{p}\bar{q}}} + \sqrt{\frac{2(\bar{p}^5+\bar{q}^5)}{n\bar{p}^2\bar{q}^2}}C_3(k),$$

where $$\bar{p} = 2p,$$

$$EW^4 \leq 3 + \frac{2(\bar{p}^3+\bar{q}^3)}{n\bar{p}\bar{q}}$$

and $$EW^6 \leq 15 + \frac{4(\bar{p}^5+\bar{q}^5)}{(n\bar{p}\bar{q})^2} + \frac{30(\bar{p}^3+\bar{q}^3) + 20(\bar{p}^2+\bar{q}^2)^2}{n\bar{p}\bar{q}}.$$

By setting additional parameters $R = 0.7$ and $p = 0.4$, we obtain uniform and non–uniform bounds as shown in Figure 1.5.

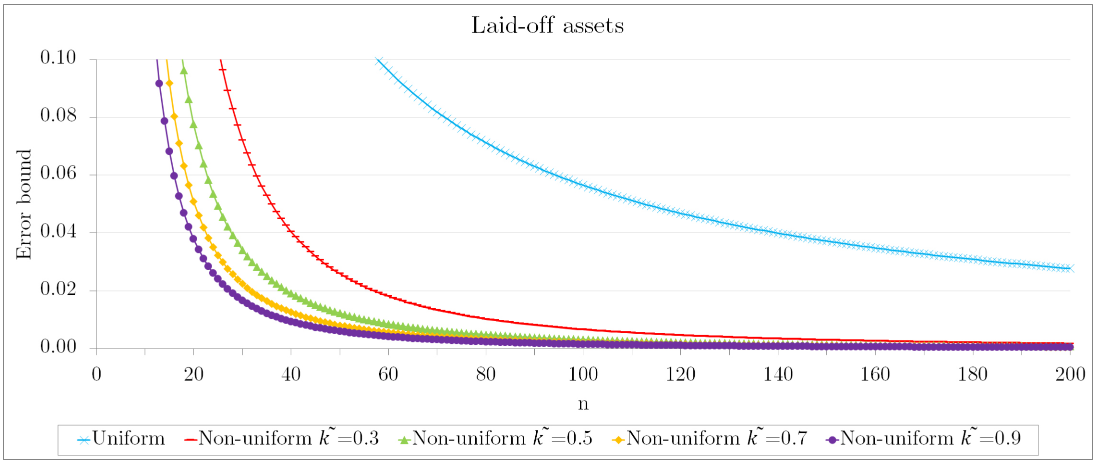


Figure 1.5: Uniform and non-uniform bounds for loss on a tranche of the CDO with laid-off assets

This dissertation is organized as follows. In Chapter II, we introduce the Stein's method on normal approximation for the call function. We also provide properties of the Stein solution and its derivative which are useful and important in this work. Next, the definitions of LD and DLD conditions are proposed in Chapter III together with some examples. In each condition, we determine $\operatorname{Var} L(T)$ and upper bounds for the forth and the sixth moments of W that are contained in our results. Moreover, under DLD condition, we illustrate two examples to provide the exact value of $\operatorname{Var} L(T)$ and the absolute moments of Y_i. In Chapter IV, we establish uniform and non–uniform bounds on normal approximation for the LD CDO while uniform and non–uniform bounds on normal approximation for the DLD CDO are presented in Chapter V. Furthermore, we illustrate bounds for CDO containing bankrupt assets and laid–off assets in Chapter V together with the numerical bounds under some specific parameters. Finally, we propose some further research in Chapter VI.

CHAPTER II
STEIN'S METHOD ON NORMAL APPROXIMATION FOR CALL FUNCTION

In this chapter, we introduce a powerful and brilliant method for obtaining a bound and the rate of convergence on normal approximation discovered by Stein ([16]) in 1972, called the Stein's method. We consider the solution of the Stein equation for the call function and obtain the bounds for the solution.

Let Z be a standard normal random variable and $f : \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function with $E\,|f'(Z)| < \infty$. The Stein's method begins with the characterization of standard normal random variable Z,

$$EZf(Z) = Ef'(Z).$$

From this characterization, we have the Stein equation on normal approximation for a given function h as follows:

$$xf(x) - f'(x) = h(x) - Eh(Z).$$

To apply the Stein's method with the CDO tranche pricing problem, we concentrate on a call function

$$h(x) = (x - k)^+$$

for a fixed positive real number k where $(x-k)^+ = \max\{x-k, 0\}$. Thus, we obtain the Stein equation on normal approximation for the call function

$$xf_k(x) - f_k'(x) = (x - k)^+ - E(Z - k)^+. \tag{2.1}$$

From this equation, substituting x by a random variable W and taking expectation on both sides of the equation, we obtain

$$EWf_k(W) - Ef_k'(W) = E(W-k)^+ - E(Z-k)^+, \tag{2.2}$$

where f_k is the solution of (2.1). Notice that, a bound from approximating $E(W-k)^+$ by $E(Z-k)^+$ is obtained by computing the term $|EWf_k(W) - Ef_k'(W)|$. This is a core of the Stein's method for normal approximation.

In order to bound $|EWf_k(W) - Ef_k'(W)|$, the properties of Stein solution f_k and its derivative f_k' are essential. Notice that

$$f_k(x) = \begin{cases} \sqrt{2\pi}e^{x^2/2}E(Z-k)^+\Phi(x) & \text{if } x \le k; \\ 1 - \sqrt{2\pi}e^{x^2/2}[k + E(Z-k)^+]\Phi(-x) & \text{if } x > k \end{cases} \tag{2.3}$$

and

$$f_k'(x) = \begin{cases} E(Z-k)^+\left(1 + \sqrt{2\pi}x\Phi(x)e^{x^2/2}\right) & \text{if } x < k; \\ [k + E(Z-k)^+]\left(1 - \sqrt{2\pi}x\Phi(-x)e^{x^2/2}\right) & \text{if } x > k, \end{cases}$$

(see [12]), where $\Phi(x) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{x} e^{-t^2/2}dt$, for $x \in \mathbb{R}$, is the cumulative distribution function of Z. Observe that, f_k is not differentiable only at the point k. Hence, to make (2.1) valid, we conventionally let

$$f_k'(k) = kf_k(k) + E(Z-k)^+ = E(Z-k)^+\left(1 + \sqrt{2\pi}k\Phi(k)e^{k^2/2}\right).$$

This implies that

$$f_k'(x) = \begin{cases} E(Z-k)^+\left(1 + \sqrt{2\pi}x\Phi(x)e^{x^2/2}\right) & \text{if } x \le k; \\ [k + E(Z-k)^+]\left(1 - \sqrt{2\pi}x\Phi(-x)e^{x^2/2}\right) & \text{if } x > k. \end{cases} \tag{2.4}$$

In the following propositions, we give some properties of f_k and f_k' which are used in our work.

Proposition 2.1. *For $k \geq 1$, we have $|f_k(x)| \leq \dfrac{1}{k^2}$ for all $x \leq k$.*

Proof. Let $k \geq 1$ and $x \leq k$. We note that $f_k(x) \geq 0$. If $|x| \leq k$, then by the fact that

$$E(Z-k)^+ \leq \frac{e^{-k^2/2}}{\sqrt{2\pi}k^2} \quad \text{for } k \geq 1 \tag{2.5}$$

(see [12], p.116), we have

$$f_k(x) \leq \frac{1}{k^2}.$$

Suppose that $x \leq -k$. By the fact that

$$\Phi(-a) \leq \frac{e^{-a^2/2}}{\sqrt{2\pi}a} \quad \text{for } a > 0 \tag{2.6}$$

(see [17], p.23) and (2.5), we have

$$f_k(x) \leq \frac{E(Z-k)^+}{-x} \leq \frac{e^{-k^2/2}}{\sqrt{2\pi}k^3} \leq \frac{1}{k^2}.$$

Hence,

$$|f_k(x)| \leq \frac{1}{k^2} \quad \text{for } x \leq k.$$

□

Before proving the next proposition, we let $\|g\| = \sup_{x\in\mathbb{R}} |g(x)|$ for any real valued function g on $\mathbb{R}$.

Proposition 2.2. *For $k \geq 1$, we have $\|f_k'\| \leq \dfrac{e^{-k^2/2}}{\sqrt{2\pi}k^2} + \dfrac{1}{k}$.*

Proof. By the expression of f_k' in (2.4), we divide the proof into 2 cases. The first case is $x \leq k$. If $x < 0$, then we have

$$0 \leq 1 + \sqrt{2\pi}x\Phi(x)e^{x^2/2} \leq 1$$

where we use (2.6) in the first inequality. By (2.5), we have

$$0 \leq f_k'(x) \leq \frac{e^{-k^2/2}}{\sqrt{2\pi}k^2} \quad \text{for } x < 0.$$

Suppose that $x \geq 0$. Then, $f_k'(x) \geq 0$. By (2.5), we have

$$0 \leq f_k'(x) \leq E(Z-k)^+ \left(1+\sqrt{2\pi}ke^{k^2/2}\right) \leq \frac{e^{-k^2/2}}{\sqrt{2\pi}k^2} + \frac{1}{k} \quad \text{for } 0 \leq x \leq k.$$

Therefore,

$$0 \leq f_k'(x) \leq \frac{e^{-k^2/2}}{\sqrt{2\pi}k^2} + \frac{1}{k} \quad \text{for } x \leq k.$$

The second case is $x > k$. By (2.6), we obtain $\sqrt{2\pi}x\Phi(-x)e^{x^2/2} \leq 1$. Then, $f_k'(x) \geq 0$. On the other hand, note that

$$\Phi(-a) \geq \frac{e^{-a^2/2}}{\sqrt{2\pi}}\left(\frac{1}{a} - \frac{1}{a^3}\right) \quad \text{for } a > 0 \tag{2.7}$$

(see [12], p.116). Thus,

$$\sqrt{2\pi}x\Phi(-x)e^{x^2/2} \geq 1 - \frac{1}{x^2} > 1 - \frac{1}{k^2}.$$

From this fact and (2.5), we obtain

$$f_k'(x) < \frac{k + E(Z-k)^+}{k^2} \leq \frac{1}{k} + \frac{e^{-k^2/2}}{\sqrt{2\pi}k^4}.$$

Hence,

$$0 \leq f_k'(x) < \frac{e^{-k^2/2}}{\sqrt{2\pi}k^4} + \frac{1}{k} \quad \text{for } x > k.$$

Combining 2 cases, we obtain

$$\|f_k'\| \leq \frac{e^{-k^2/2}}{\sqrt{2\pi}k^2} + \frac{1}{k} \quad \text{for } k \geq 1. \qquad \square$$

Proposition 2.3. *For real numbers x, t with $|t| \leq 1$ and a positive real number k, we have*

$$|f_k'(x+t) - f_k'(x)| \leq 2x^2|t| + 10.46|x||t| + 12.16|t|.$$

Proof. From (2.1), we have

$$
\begin{aligned}
&f_k'(x+t) - f_k'(x) \\
&= (x+t)f_k(x+t) - xf_k(x) - (x+t-k)^+ + (x-k)^+ \\
&= \begin{cases}
(x+t)f_k(x+t) - xf_k(x) - t & \text{if } x > k \text{ and } x+t > k; \\
(x+t)f_k(x+t) - xf_k(x) & \text{if } x \le k \text{ and } x+t \le k; \\
(x+t)f_k(x+t) - xf_k(x) + (x-k) & \text{if } x > k \text{ and } x+t \le k; \\
(x+t)f_k(x+t) - xf_k(x) + (x+t-k) & \text{if } x \le k \text{ and } x+t > k.
\end{cases}
\end{aligned}
$$

Case 1: $x > k$ and $x+t > k$. We note from Lemma 2.4 in [5], p.16 that

$$\|f_k\| \le 2 \tag{2.8}$$

and

$$\|f_k'\| \le \sqrt{\frac{2}{\pi}}. \tag{2.9}$$

Since f_k is continuous on (k, ∞), we can use the mean value theorem, (2.8) and (2.9) to show that

$$
\begin{aligned}
|f_k'(x+t) - f_k'(x)| &= |x\left[f_k(x+t) - f_k(x)\right] + tf_k(x+t) - t| \\
&\le |x||f_k(x+t) - f_k(x)| + |t|(|f_k(x+t)| + 1) \\
&\le |x|\|f_k'\||t| + 3|t| \\
&\le 0.8|x||t| + 3|t|.
\end{aligned}
$$

Case 2: $x \le k$ and $x+t \le k$. By the same argument as shown in Case 1 with the fact that f_k is a continuous function on $(-\infty, k]$, we can conclude that

$$|f_k'(x+t) - f_k'(x)| \le |x||f_k(x+t) - f_k(x)| + |t||f_k(x+t)| \le 0.8|x||t| + 2|t|.$$

Case 3: $k < x \le k - t$. We note from (2.4) that

$$f_k'(x+t) - f_k'(x) = B_1 + B_2 + B_3,$$

where $$B_1 = (x+t)e^{(x+t)^2/2-k^2/2} - k,$$
$$B_2 = (x+t)\sqrt{2\pi}ke^{(x+t)^2/2}\left[\Phi(k)-\Phi(x+t)\right],$$
$$B_3 = \sqrt{2\pi}\left(k+E(Z-k)^+\right)\left[g_1(x)-g_1(x+t)\right]$$
and $$g_1(s) = se^{s^2/2}\Phi(-s).$$

Note that

$$\begin{aligned}
B_1 &= (x+t)e^{(x+t)^2/2-k^2/2}\mathbb{I}(x+t\geq 0)+(x+t)e^{(x+t)^2/2-k^2/2}\mathbb{I}(x+t<0)-k\\
&\leq (x+t)-k\\
&\leq 0.
\end{aligned}$$

To find a lower bound for B_1, we separate B_1 into 3 cases including $-k < x+t \leq 0$, $-1 \leq x+t \leq -k$ and $0 < x+t \leq k$. If $-k < x+t \leq 0$, then

$$B_1 \geq x+t-k > -|t|.$$

Thus,

$$-|t| < B_1 \leq 0 \quad \text{for } -k < x+t \leq 0. \tag{2.10}$$

If $-1 \leq x+t \leq -k$, then $(x+t)^2-k^2 \geq 0$ and $k \leq 1$. Note that

$$e^{(x+t)^2/2-k^2/2} = 1+\left(\frac{(x+t)^2}{2}-\frac{k^2}{2}\right)e^{x_1}$$

for some $0 \leq x_1 \leq \dfrac{(x+t)^2}{2}-\dfrac{k^2}{2}$. By using the fact that $-1 \leq x+t+k \leq 0$ and $-1 \leq t < x+t-k \leq 0$, we have $0 \leq (x+t)^2-k^2 \leq |t| \leq 1$. Hence,

$$\begin{aligned}
B_1 &= (x+t)\left[1+\left(\frac{(x+t)^2}{2}-\frac{k^2}{2}\right)e^{x_1}\right]-k\\
&\geq x+t+\frac{\sqrt{e}(x+t)|t|}{2}-k\\
&\geq -|t|-\frac{\sqrt{e}|t|}{2}
\end{aligned}$$

$$\geq -1.83|t| \quad \text{for } -1 \leq x+t \leq -k.$$

Therefore,

$$-1.83|t| \leq B_1 \leq 0 \quad \text{for } -1 \leq x+t \leq -k. \tag{2.11}$$

Suppose that $0 < x+t \leq k$. Let $x_0 = \frac{(x+t)^2}{2} - \frac{k^2}{2}$. Then, $x_0 < 0$. By the mean value theorem, we have $1 - e^{x_0} = -x_0 e^{x_1}$, i.e., $e^{x_0} = 1 + x_0 e^{x_1}$ for some $x_1 < 0$. Note that $x_0 = \frac{x^2}{2} + xt + \frac{t^2}{2} - \frac{k^2}{2} \geq xt + \frac{t^2}{2}$. Hence,

$$\begin{aligned}
B_1 &= (x+t)\left(1 + x_0 e^{x_1}\right) - k \\
&\geq x + t + (x+t)\left(xt + \frac{t^2}{2}\right) - k \\
&\geq -|t| - x^2|t| - \frac{|t|^3}{2} \\
&\geq -x^2|t| - 1.5|t| \quad \text{for } 0 < x+t \leq k.
\end{aligned}$$

Thus,

$$-x^2|t| - 1.5|t| \leq B_1 \leq 0 \quad \text{for } 0 < x+t \leq k. \tag{2.12}$$

By (2.10)–(2.12), we obtain

$$|B_1| \leq x^2|t| + 1.83|t| \quad \text{for } k < x \leq k - t.$$

Next, we consider B_2. If $x+t < 0$, then $B_2 < 0$. By the mean value theorem, there exists $c \in (x+t, k)$ such that

$$\Phi(k) - \Phi(x+t) = \Phi'(c)(k - x - t) \leq \frac{|t|}{\sqrt{2\pi}}.$$

Since $x+t < 0, k < x < -t \leq 1$. By this fact and $-1 \leq t < k+t < x+t < 0$, we have

$$B_2 \geq (x+t)ke^{(x+t)^2/2}|t| > -\sqrt{e}|t| \geq -1.65|t| \quad \text{for } x+t < 0.$$

Then,

$$-1.65|t| \leq B_2 < 0 \quad \text{for } x+t<0.$$

Suppose that $x+t \geq 0$. Then, $B_2 \geq 0$. Note that

$$\Phi(b)-\Phi(a) = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-s^2/2}ds \leq \frac{e^{-a^2/2}}{\sqrt{2\pi}}(b-a) \quad \text{for } 0 \leq a \leq b. \tag{2.13}$$

From (2.13) and the fact that $x+t \leq k < x$, we have

$$0 \leq B_2 \leq (x+t)k|t| \leq kx|t| \leq x^2|t| \quad \text{for } x+t \geq 0.$$

Hence,

$$|B_2| \leq x^2|t| + 1.65|t| \ \ \text{for } x+t \in \mathbb{R}.$$

To bound B_3, we first show that

$$|g_1'(s)| \leq 3.18 \quad \text{for } s \geq -1. \tag{2.14}$$

To show (2.14), we note that $g_1'(s) = -\frac{s}{\sqrt{2\pi}} + \Phi(-s)e^{s^2/2}(s^2+1)$. If $|s| \leq 1$, then $|g_1'(s)| \leq \frac{1}{\sqrt{2\pi}} + 2\Phi(1)\, e^{1/2} \leq 3.18$. Suppose that $s > 1$. By (2.6) and (2.7), we have $|g_1'(s)| \leq \frac{1}{\sqrt{2\pi}}$. Hence,

$$|g_1'(s)| \leq 3.18 \quad \text{for } s \geq -1.$$

By (2.14) and the fact that

$$E(Z-k)^+ \leq \frac{e^{-k^2/2}}{\sqrt{2\pi}} \tag{2.15}$$

(see [12], p.116), we have

$$|B_3| \leq \sqrt{2\pi}\left(x + \frac{1}{\sqrt{2\pi}}\right)|g_1'(s)||t| \leq 7.98|x||t| + 3.18|t|$$

for some $s \in (x+t, x) \subseteq [-1, \infty)$.

Consequently,

$$|f_k'(x+t) - f_k'(x)| \le 2x^2|t| + 7.98|x||t| + 6.66|t| \quad \text{for } k < x \le k-t.$$

Case 4: $k - t < x \le k$. By (2.3), we obtain

$$f_k'(x+t) - f_k'(x) = C_1 + C_2 + C_3,$$

where $\quad C_1 = \sqrt{2\pi}\left[k + E(Z-k)^+\right][g_1(x) - g_1(x+t)],$

$\quad C_2 = \sqrt{2\pi}kxe^{x^2/2}[\Phi(x) - \Phi(k)]$

and $\quad C_3 = -xe^{x^2/2 - k^2/2} + k.$

By (2.14), (2.15) and the fact that $0 < k < x + t \le x + 1$, we can deduce that

$$|C_1| \le \sqrt{2\pi}\left(|x| + 1 + \frac{1}{\sqrt{2\pi}}\right)|g_1'(s)||t| \le 7.98|x||t| + 11.16|t|$$

for some $s \in (x, x+t) \subseteq [-1, \infty)$.

If $x \ge 0$, then $C_2 \le 0$ and by (2.13), we have

$$-C_2 = \sqrt{2\pi}kxe^{x^2/2}[\Phi(k) - \Phi(x)] \le kx(k-x) \le (|x|+1)|x||t| = x^2|t| + |x||t|.$$

Hence,

$$-x^2|t| - |x||t| \le C_2 \le 0 \quad \text{for } x \ge 0. \tag{2.16}$$

Suppose that $x < 0$. Then, $C_2 > 0$. Since $k - t < x < 0, k < t \le 1$. By this fact and $-1 < k - t < x < 0$, we have

$$\begin{aligned} C_2 &= kxe^{x^2/2}\int_k^x e^{-s^2/2}ds \\ &= k|x|e^{x^2/2}\int_x^k e^{-s^2/2}ds \end{aligned}$$

$$\leq |x|e^{x^2/2}(k-x)$$
$$\leq \sqrt{e}|x||t|$$
$$\leq 1.65|x||t|.$$

Hence,

$$0 < C_2 \leq 1.65|x||t| \quad \text{for } x < 0. \tag{2.17}$$

By (2.16) and (2.17), we obtain

$$|C_2| \leq x^2|t| + 1.65|x||t| \quad \text{for } x \in \mathbb{R}.$$

Consider C_3. If $x \leq 0$, then $C_3 \geq 0$. If $x > 0$, then $0 < x \leq k$. This implies that $C_3 \geq -x + k \geq 0$. Hence,

$$C_3 \geq 0 \quad \text{for } x \in \mathbb{R}. \tag{2.18}$$

Next, we give an upper bound for C_3. To do this, we consider the possible value of x in 3 cases: $-k \leq x \leq 0, -1 \leq x \leq -k$ and $0 \leq x < k$.

If $-k \leq x < 0$, then

$$C_3 \leq -x + k \leq |t|. \tag{2.19}$$

If $-1 \leq x \leq -k$, then $x^2 - k^2 > 0$ and $k \leq 1$. By the mean value theorem, we have $e^{x^2/2-k^2/2} - 1 = \left(\frac{x^2}{2} - \frac{k^2}{2}\right)e^{x_0}$, that is $e^{x^2/2-k^2/2} = 1 + \left(\frac{x^2}{2} - \frac{k^2}{2}\right)e^{x_0}$ for some $0 \leq x_0 \leq \frac{x^2}{2} - \frac{k^2}{2}$. By the fact that $-1 \leq x+k \leq 0$ and $-1 \leq -t \leq x-k \leq 0$, we have $0 \leq x^2 - k^2 \leq |t| \leq 1$. Hence,

$$\begin{aligned} C_3 &= -x\left[1 + \left(\frac{x^2}{2} - \frac{k^2}{2}\right)e^{x_0}\right] + k \\ &\leq -x - \frac{\sqrt{e}x|t|}{2} + x + t \\ &\leq 0.83|x||t| + |t| \quad \text{for } -1 \leq x \leq -k. \end{aligned} \tag{2.20}$$

Suppose that $0 \leq x < k$. Then, we have $x^2 - k^2 < 0$. By the mean value theorem, we have $e^{x^2/2-k^2/2} = 1 + \left(\frac{x^2}{2} - \frac{k^2}{2}\right)e^{x_1}$ for some $x_1 < 0$. Note that

$0 \le k^2 - x^2 = (k-x)(k+x) \le t(2x+t) \le t(2x+1)$. This implies that

$$\begin{aligned} C_3 &= -x\left[1 + \left(\frac{x^2}{2} - \frac{k^2}{2}\right)e^{x_1}\right] + k \\ &\le -x + x\left(\frac{k^2}{2} - \frac{x^2}{2}\right)e^{x_1} + x + t \\ &\le t + x\left(\frac{k^2}{2} - \frac{x^2}{2}\right) \\ &\le t + x\left[\frac{t}{2}(2x+1)\right] \\ &\le x^2|t| + 0.5|x||t| + |t| \quad \text{for } 0 \le x < k. \end{aligned} \tag{2.21}$$

From (2.18)–(2.21), we have

$$0 \le C_3 \le x^2|t| + 0.83|x||t| + |t| \quad \text{for } k - t < x \le k.$$

Consequently,

$$|f_k'(x+t) - f_k'(x)| \le 2x^2|t| + 10.46|x||t| + 12.16|t| \quad \text{for } k - t < x \le k.$$

From Cases 1–4, we can conclude that

$$|f_k'(x+t) - f_k'(x)| \le 2x^2|t| + 10.46|x||t| + 12.16|t|. \qquad \square$$

CHAPTER III
BOUNDS OF MOMENTS FOR LOCALLY AND DISJOINT LOCALLY DEPENDENT CDO

In this work, we aim to provide bounds for approximating loss on a tranche of a locally dependent CDO by using the call function of the standard normal random variable. To do that, we need to determine moments of locally and disjoint locally dependent CDO which are provided in this chapter.

First, consider a standard CDO containing n assets. Assume that the i^{th} asset has a recovery rate R_i and a default time τ_i for $i = 1, 2, 3, \ldots, n$. Then, the total loss on a CDO at time T is

$$L(T) = \frac{1}{n}\sum_{i=1}^{n}(1 - R_i)\mathbb{I}\left(\tau_i \leq T\right).$$

In this work, we are attentive to the loss on a tranche of a CDO defined by

$$(L(T) - AP)^{+} - (L(T) - DP)^{+},$$

where AP and DP stand for attachment and detachment points of a tranche, respectively. Hence, the problem is restricted to approximating

$$E\left(L(T) - \tilde{k}\right)^{+},$$

where $\tilde{k}$ is a positive real number and $0 < \tilde{k} \leq 1$. Note that, when $\tilde{k} = 0$, $EL(T)^{+} = EL(T)$ is easily calculated.

Let Z be a standard normal random variable. To approximate $E\left(L(T) - \tilde{k}\right)^{+}$ by a call function of the standard normal random variable, we need to normalize

$L(T)$. Let

$$X_i = \frac{(1 - R_i)\left[\mathbb{I}(\tau_i \le T) - p_i\right]}{n\sqrt{\operatorname{Var} L(T)}},$$

where $p_i = P\left(\mathbb{I}(\tau_i \le T) = 1\right)$ and let $W = \sum_{i=1}^{n} X_i$. Then

$$W = \frac{L(T) - EL(T)}{\sqrt{\operatorname{Var} L(T)}}$$

with

$$EW = 0 \quad \text{and} \quad \operatorname{Var} W = 1.$$

Let $k = \dfrac{\tilde{k} - EL(T)}{\sqrt{\operatorname{Var} L(T)}}$. Then,

$$\begin{aligned} &\left| E\left(L(T) - \tilde{k}\right)^{+} - \sqrt{\operatorname{Var} L(T)}E\left(Z - k\right)^{+} \right| \\ &\quad = \sqrt{\operatorname{Var} L(T)}\left|E(W - k)^{+} - E(Z - k)^{+}\right|. \end{aligned}$$

Hence, the problem is transformed into finding a bound for $|E(W-k)^{+} - E(Z-k)^{+}|$.

Next, to obtain a bound for $|E(W-k)^{+} - E(Z-k)^{+}|$, we introduce the local dependence (LD) condition defined by Chen and Shao ([6]) in 2004.

Definition 3.1 (LD condition)**.** We say that random variables $X_1, X_2, X_3, \ldots, X_n$ satisfy the *local dependence* condition if for each $i = 1, 2, 3, \ldots, n$, there exist $A_i \subseteq B_i \subseteq C_i \subseteq \{1, 2, 3, \ldots, n\}$ such that X_i is independent of $X_{A_i^c}$, X_{A_i} is independent of $X_{B_i^c}$ and X_{B_i} is independent of $X_{C_i^c}$.

We can transform the LD condition into a chart. For example, consider random variables $X_1, X_2, X_3, \ldots, X_{10}$ with local dependence structure as shown in Figure 3.1.

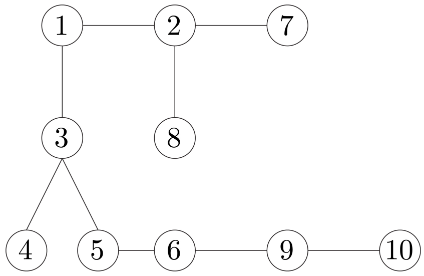


Figure 3.1: Example of locally dependent random variables

Note that, if two vertices are not adjacent, then they are independent. From Figure 3.1, we have

$$A_1 = \{1,2,3\}, \quad B_1 = \{1,2,3,4,5,7,8\}, \quad C_1 = \{1,2,3,4,5,6,7,8\},$$
$$A_2 = \{1,2,7,8\}, \quad B_2 = \{1,2,3,7,8\}, \quad C_2 = \{1,2,3,4,5,7,8\}, \text{etc.}$$

We have that X_1 is independent of $\{X_j, j \neq 1,2,3\}$, $\{X_1, X_2, X_3\}$ is independent of $\{X_6, X_9, X_{10}\}$ and $\{X_1, X_2, X_3, X_4, X_5, X_7, X_8\}$ is independent of $\{X_9, X_{10}\}$. On the one hand, we can rewrite Figure 3.1 into a diagram as shown in Figure 3.2 for each $i = 1, 2, 3, \ldots, 10$.

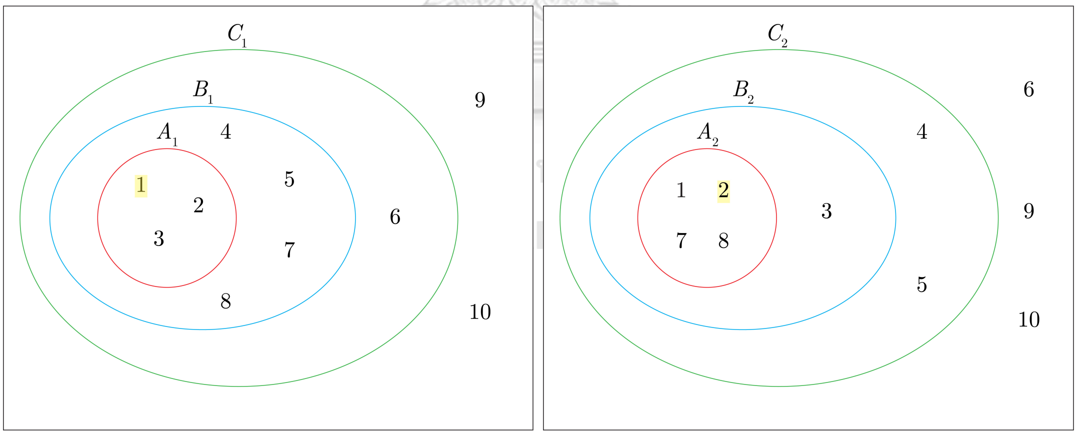


Figure 3.2: Example of A_1, B_1, C_1 and A_2, B_2, C_2

In a real world problem, assets are correlated when they have some common structure. For example, they are in the same organization or they use the same resource in their occupation. In addition, people in a family may be related, because they help each other when someone default. The next Figure 3.3 shows a real world example about a local dependence relation.

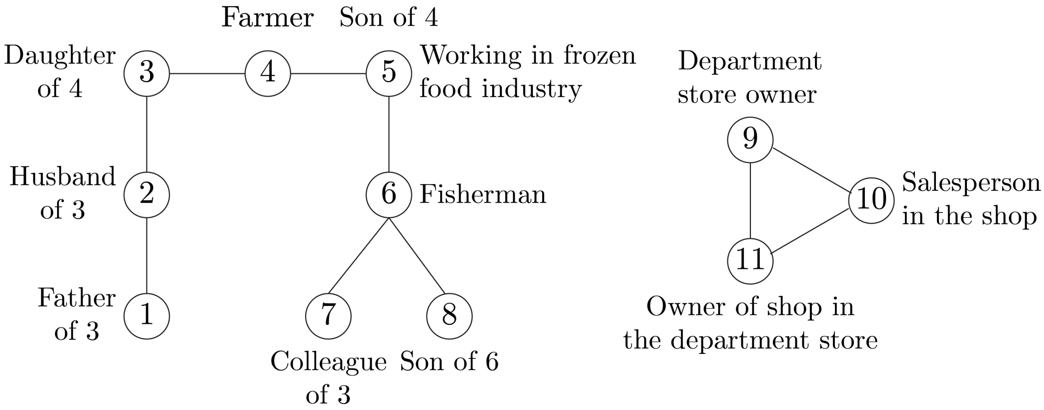


Figure 3.3: Example of local dependence relation

We can see that, when the daughter of 4 defaults, then her husband and her father may be affected because they are sharing money in their family. But other people are absolutely not disturbed. Consider the situation that farmer, his son and his daughter default. A default of 5 who works in a frozen food industry may be correlated with a fisherman due to the financial status of the industry or the lack of raw material from the oceans. A default of 3 may correlate with the financial status of her husband. But, other people are not interrupted. Conversely, a default of a department store owner does not affect other people except the owner of shop in the store and the salesperson in the shop.

From the definition of the LD condition and the given examples, we see that the structure of local dependence is quite complicated. Consequently, before proving the result about the LD assumption, we give a special and realistic case of the LD condition, called the disjoint local dependence (DLD) condition.

Definition 3.2 (DLD condition)**.** We say that random variables $X_1, X_2, X_3, \dots, X_n$ satisfy the *disjoint local dependence* condition if there exists a partition $\{A_i\}_{i=1}^{d}$ of $\{1, 2, 3, \dots, n\}$, where $d \le n$ such that for each $i = 1, 2, 3, \dots, d$, X_{A_i} is independent of $X_{A_i^c}$.

From the definition, we give an example of random variables $X_1, X_2, X_3, \dots, X_{10}$ that satisfy the DLD condition as follows.

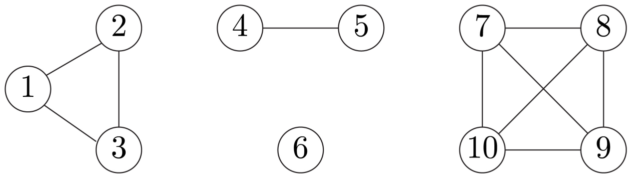


Figure 3.4: Example of the disjoint locally dependent random variables

From Figure 3.4, we have

$$A_1 = A_2 = A_3 = \{1, 2, 3\},$$

$$A_4 = A_5 = \{4, 5\},$$

$$A_6 = \{6\}$$

and

$$A_7 = A_8 = A_9 = A_{10} = \{7, 8, 9, 10\}.$$

It can be concluded that assets form different groups are independent. For example, X_1 is independent of X_4, and X_6 is independent of X_{10}.

In the real world situation, we can classify assets into groups due to their relation such as occupation, region or common resources. From Figure 3.5, we classify

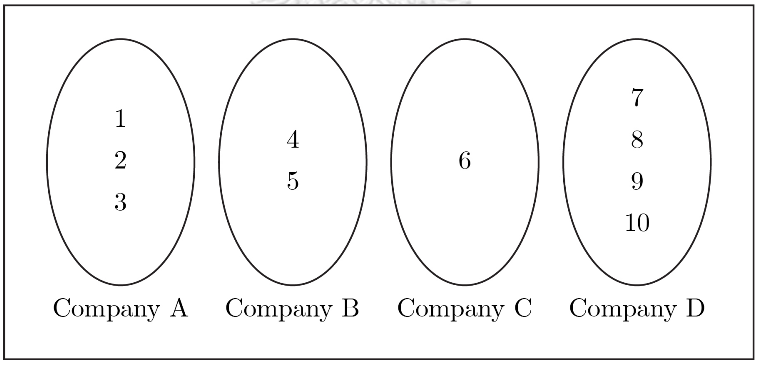


Figure 3.5: Example of disjoint local dependence relation

assets into groups due to their workplace. Notice that, each $i = 1, 2, 3, \ldots, 10$ represents a staff in each company that is indebted and is contained in a CDO. When a company encountered a problem, their staff may be affected but other companies are not disturbed.

Now, we propose bounds for moments of the LD CDO in Section 3.1 and moments of the DLD CDO in Section 3.2. They are useful facts for obtaining bounds for loss on a tranche of a CDO.

Throughout this work, we let

1. $Y_i = \sum_{j \in A_i} X_j$;
2. $p_i = P\left(\mathbb{I}(\tau_i \leq T) = 1\right), q_i = 1 - p_i$ and $p_{ij} = P(\mathbb{I}(\tau_i \leq T) = 1, \mathbb{I}(\tau_j \leq T) = 1)$;
3. $|A| = \max_{1 \leq i \leq n} |A_i|$ and $|B| = \max_{1 \leq i \leq n} |B_i|$;
4. $\kappa_1 = \max_{1 \leq i \leq n} \max\{|C_i|, |C_i^{-1}|\}$, where $C_i^{-1} = \{j \mid i \in C_j\}$.

From the above notations, we notice from Figure 3.1 that $C_1^{-1} = \{1, 2, 3, \ldots, 8\}$, $C_2^{-1} = \{1, 2, 3, 4, 5, 7, 8\}$, $C_3^{-1} = \{1, 2, 3, \ldots, 9\}$, etc. In addition, we have $\kappa_1 = 9$. From this example, we can see that κ_1 is closed to $n = 10$. But, in the real situation, n is mostly greater than 100. From this fact and by Remark 1.3, we assume in this work that κ_1 does not depend on n.

3.1 Bounds for Moments of Locally Dependent CDO

In this section, we provide a formula for $\operatorname{Var} L(T)$, upper bounds for the forth and the sixth moments of W under the local dependence condition.

Theorem 3.3. *Under the LD condition, we have*

1. $\operatorname{Var} L(T) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \in A_i} (1 - R_i)(1 - R_j)[p_{ij} - p_i p_j]$;
2. $EW^4 = 3 + \mathcal{O}\left(\frac{1}{n^3 \left(\operatorname{Var} L(T)\right)^2}\right)$;
3. $\left|\sum_{i=1}^{n} EX_i Y_i W^4\right| = 3 + \mathcal{O}\left(\frac{1}{n^2 \left(\operatorname{Var} L(T)\right)^{3/2}}\right)$;
4. $EW^6 = 15 + \mathcal{O}\left(\frac{1}{n^2 \left(\operatorname{Var} L(T)\right)^{3/2}}\right)$.

Furthermore, if $\operatorname{Var} L(T) = \mathcal{O}\left(\frac{1}{n}\right)$, *then*

1. $EW^4 = 3 + \mathcal{O}\left(\frac{1}{n}\right)$;

2. $EW^6 = 15 + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$.

Proof. 1. By the expression of $L(T)$, we have

$$\begin{aligned}\operatorname{Var} L(T) &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}(1-R_i)(1-R_j)\operatorname{Cov}\left(\mathbb{I}(\tau_i \le T, \tau_j \le T)\right)\\ &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j\in A_i}(1-R_i)(1-R_j)[p_{ij}-p_ip_j].\end{aligned}$$

2. By Lemma 3.1 in [6], we have

$$EW^4 \le 3 + 22\kappa_1^3\sum_{i=1}^{n} EX_i^4.$$

By the fact that

$$|X_i| = \left|\frac{(1-R_i)[\mathbb{I}(\tau_i \le T) - p_i]}{n\sqrt{\operatorname{Var} L(T)}}\right| \le \frac{1}{n\sqrt{\operatorname{Var} L(T)}}, \tag{3.1}$$

we have

$$EW^4 = 3 + \mathcal{O}\left(\frac{1}{n^3\left(\operatorname{Var} L(T)\right)^2}\right).$$

3. For $i = 1, 2, 3, \ldots, n$, let $Z_i = \sum_{j\in B_i} X_j$. By the fact that X_iY_i and $W - Z_i$ are independent, we have

$$\begin{aligned}&\sum_{i=1}^{n} EX_iY_iW^4\\ &= \sum_{i=1}^{n} E(X_iY_i)E(W-Z_i)^4 + \sum_{i=1}^{n} EX_iY_i\left(W^4 - (W-Z_i)^4\right)\\ &= \sum_{i=1}^{n} E(X_iY_i)E(W^4 - 4W^3Z_i + 6W^2Z_i^2 - 4WZ_i^3 + Z_i^4)\\ &\quad + \sum_{i=1}^{n} EX_iY_i\left(4W^3Z_i - 6W^2Z_i^2 + 4WZ_i^3 - Z_i^4\right)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} E(X_iY_i)E(W^4) - 4\sum_{i=1}^{n} E(X_iY_i)E(W^3Z_i) \\
&\quad + 6\sum_{i=1}^{n} E(X_iY_i)E(W^2Z_i^2) - 4\sum_{i=1}^{n} E(X_iY_i)E(WZ_i^3) \\
&\quad + \sum_{i=1}^{n} E(X_iY_i)E(Z_i^4) + 4\sum_{i=1}^{n} EX_iY_iZ_iW^3 - 6\sum_{i=1}^{n} EX_iY_iZ_i^2W^2 \\
&\quad + 4\sum_{i=1}^{n} EX_iY_iZ_i^3W - \sum_{i=1}^{n} EX_iY_iZ_i^4.
\end{aligned}
$$

By considering in the same manner with (3.1), we have

$$
|Y_i| \le \frac{|A_i|}{n\sqrt{\operatorname{Var} L(T)}} \le \frac{|A|}{n\sqrt{\operatorname{Var} L(T)}} \tag{3.2}
$$

and
$$
|Z_i| \le \frac{|B_i|}{n\sqrt{\operatorname{Var} L(T)}} \le \frac{|B|}{n\sqrt{\operatorname{Var} L(T)}}.
$$

These imply that

$$
\begin{aligned}
\left|\sum_{i=1}^{n} E(X_iY_i)E(W^3Z_i)\right| &\le \frac{|A||B|\left(EW^4\right)^{3/4}}{n^2\left(\operatorname{Var} L(T)\right)^{3/2}} = \mathcal{O}\left(\frac{1}{n^2\left(\operatorname{Var} L(T)\right)^{3/2}}\right), \\
\left|\sum_{i=1}^{n} E(X_iY_i)E(W^2Z_i^2)\right| &\le \frac{|A||B|^2 EW^2}{n^3\left(\operatorname{Var} L(T)\right)^{2}} = \mathcal{O}\left(\frac{1}{n^3\left(\operatorname{Var} L(T)\right)^{2}}\right), \\
\left|\sum_{i=1}^{n} E(X_iY_i)E(WZ_i^3)\right| &\le \frac{|A||B|^3\left(EW^2\right)^{1/2}}{n^4\left(\operatorname{Var} L(T)\right)^{5/2}} = \mathcal{O}\left(\frac{1}{n^4\left(\operatorname{Var} L(T)\right)^{5/2}}\right), \\
\left|\sum_{i=1}^{n} E(X_iY_i)E(Z_i^4)\right| &\le \frac{|A||B|^4}{n^5\left(\operatorname{Var} L(T)\right)^{3}} = \mathcal{O}\left(\frac{1}{n^5\left(\operatorname{Var} L(T)\right)^{3}}\right), \\
\left|\sum_{i=1}^{n} EX_iY_iZ_iW^3\right| &\le \frac{|A||B|\left(EW^4\right)^{3/4}}{n^2\left(\operatorname{Var} L(T)\right)^{3/2}} = \mathcal{O}\left(\frac{1}{n^2\left(\operatorname{Var} L(T)\right)^{3/2}}\right), \\
\left|\sum_{i=1}^{n} EX_iY_iZ_i^2W^2\right| &\le \frac{|A||B|^2 EW^2}{n^3\left(\operatorname{Var} L(T)\right)^{2}} = \mathcal{O}\left(\frac{1}{n^3\left(\operatorname{Var} L(T)\right)^{2}}\right), \\
\left|\sum_{i=1}^{n} EX_iY_iZ_i^3W\right| &\le \frac{|A||B|^3\left(EW^2\right)^{1/2}}{n^4\left(\operatorname{Var} L(T)\right)^{5/2}} = \mathcal{O}\left(\frac{1}{n^4\left(\operatorname{Var} L(T)\right)^{5/2}}\right)
\end{aligned}
$$

and
$$
\left|\sum_{i=1}^{n} EX_iY_iZ_i^4\right| \le \frac{|A||B|^4}{n^5\left(\operatorname{Var} L(T)\right)^{3}} = \mathcal{O}\left(\frac{1}{n^5\left(\operatorname{Var} L(T)\right)^{3}}\right).
$$

These imply that

$$\left|\sum_{i=1}^{n} EX_iY_iW^4\right| \leq \left|\sum_{i=1}^{n} E(X_iY_i)E(W^4)\right| + \mathcal{O}\left(\frac{1}{n^2\left(\mathrm{Var}\, L(T)\right)^{3/2}}\right). \qquad (3.3)$$

By the fact that X_i and $W - Y_i$ are independent and $EX_i = 0$, we have

$$\begin{aligned}\sum_{i=1}^{n} E(X_iY_i)E(W^4) &= E(W^4)\sum_{i=1}^{n}\sum_{j\in A_i} E(X_iX_j)\\ &= E(W^4)\sum_{i=1}^{n}\sum_{j=1}^{n} E(X_iX_j)\\ &= EW^4EW^2\\ &= EW^4.\end{aligned}$$

From this fact, (3.3) and Theorem 3.3(2), we have

$$\left|\sum_{i=1}^{n} EX_iY_iW^4\right| = 3 + \mathcal{O}\left(\frac{1}{n^2\left(\mathrm{Var}\, L(T)\right)^{3/2}}\right).$$

4. By the fact that X_i and $W - Y_i$ are independent and $EX_i = 0$, we have

$$\begin{aligned}EW^6 &= \sum_{i=1}^{n} EW^5X_i\\ &= \sum_{i=1}^{n} EX_i\left[W^5 - (W-Y_i)^5\right]\\ &= \sum_{i=1}^{n} EX_i\left(5W^4Y_i - 10W^3Y_i^2 + 10W^2Y_i^3 - 5WY_i^4 + Y_i^5\right)\\ &= 5\sum_{i=1}^{n} EX_iY_iW^4 - 10\sum_{i=1}^{n} EX_iY_i^2W^3 + 10\sum_{i=1}^{n} EX_iY_i^3W^2\\ &\quad - 5\sum_{i=1}^{n} EX_iY_i^4W + \sum_{i=1}^{n} EX_iY_i^5.\end{aligned}$$

By (3.1) and (3.2), we have

$$\left|\sum_{i=1}^{n} EX_iY_i^2W^3\right| \le \frac{|A|^2\left(EW^4\right)^{3/4}}{n^2\left(\operatorname{Var} L(T)\right)^{3/2}} = \mathcal{O}\left(\frac{1}{n^2\left(\operatorname{Var} L(T)\right)^{3/2}}\right),$$

$$\left|\sum_{i=1}^{n} EX_iY_i^3W^2\right| \le \frac{|A|^3EW^2}{n^3\left(\operatorname{Var} L(T)\right)^{2}} = \mathcal{O}\left(\frac{1}{n^3\left(\operatorname{Var} L(T)\right)^{2}}\right),$$

$$\left|\sum_{i=1}^{n} EX_iY_i^4W\right| \le \frac{|A|^4\left(EW^2\right)^{1/2}}{n^4\left(\operatorname{Var} L(T)\right)^{5/2}} = \mathcal{O}\left(\frac{1}{n^4\left(\operatorname{Var} L(T)\right)^{5/2}}\right)$$

and
$$\left|\sum_{i=1}^{n} EX_iY_i^5\right| \le \frac{|A|^5}{n^5\left(\operatorname{Var} L(T)\right)^{3}} = \mathcal{O}\left(\frac{1}{n^5\left(\operatorname{Var} L(T)\right)^{3}}\right).$$

From this fact and Theorem3.3(3), we have

$$\begin{aligned} EW^6 &= 5\left|\sum_{i=1}^{n} EX_iY_iW^4\right| + \mathcal{O}\left(\frac{1}{n^2\left(\operatorname{Var} L(T)\right)^{3/2}}\right) \\ &= 15 + \mathcal{O}\left(\frac{1}{n^2\left(\operatorname{Var} L(T)\right)^{3/2}}\right). \end{aligned}$$ □

3.2 Bounds for Moments of Disjoint Locally Dependent CDO

In this section, we present moments of W under the DLD condition. From the structure of DLD condition, we can group assets due to their relation. Hence, assume that the n assets can be classified into d groups and the i^{th} company has $m_i - m_{i-1}$ indebted personnel (for $i = 1, 2, \ldots, d$ when $m_0 = 0$ and $m_d = n$) as shown in Figure 3.6. Notice under the DLD condition that for $i = 1, 2, 3, \ldots, d$, $A_i = \{m_{i-1}+1, m_{i-1}+2, \ldots, m_i\}$.

Next, we use this classification to determine the moments of W.

Theorem 3.4. *Under the DLD condition, we have*

1. $\operatorname{Var} L(T) = \dfrac{1}{n^2}\displaystyle\sum_{i=1}^{d}\sum_{j\in A_i}\sum_{l\in A_i}(1-R_j)(1-R_l)[p_{jl} - p_jp_l];$

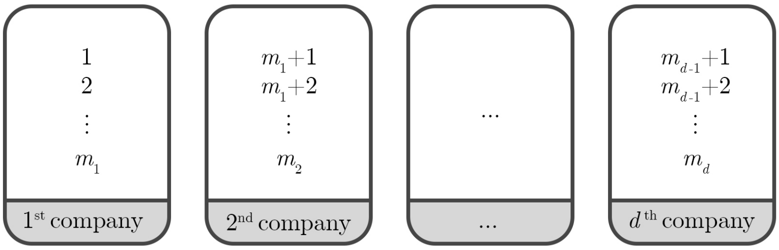


Figure 3.6: Classification of assets in a DLD CDO

2. $EW^4 \leq 3 + \sum_{i=1}^{d} EY_i^4$;

3. $EW^6 \leq 15 + \sum_{i=1}^{d} EY_i^6 + 15 \sum_{i=1}^{d} EY_i^4 + 10 \left(\sum_{i=1}^{d} EY_i^3 \right)^2$.

Proof. 1. By the expression of $\operatorname{Var} L(T)$ in Theorem 3.3(1) and the fact that $\{A_i\}_{i=1}^d$ are disjoint, we have

$$\operatorname{Var} L(T) = \frac{1}{n^2} \sum_{i=1}^{d} \sum_{j \in A_i} \sum_{l \in A_i} (1 - R_j)(1 - R_l)[p_{jl} - p_j p_l].$$

2. By the DLD condition, we have that Y_i and Y_j are independent for $i \neq j$. From this fact and $EY_i = 0$, we have

$$\sum_{i=1}^{d} EY_i^2 = \sum_{i=1}^{d} EY_i^2 + \sum_{i=1}^{d} \sum_{\substack{j=1 \\ j \neq i}}^{d} EY_i EY_j = E \left(\sum_{i=1}^{d} Y_i \right)^2 = EW^2 = 1. \quad (3.4)$$

Observe that $EY_{j_1}^3 Y_{j_2} = EY_{j_1}^2 Y_{j_2} Y_{j_3} = EY_{j_1} Y_{j_2} Y_{j_3} Y_{j_4} = 0$ for distinct index j_i. Hence,

$$\begin{aligned} EW^4 &= E \left(\sum_{i=1}^{d} Y_i \right)^4 \\ &= \sum_{i=1}^{d} EY_i^4 + 4 \sum_{j_1=1}^{d} \sum_{\substack{j_2=1 \\ j_2 \neq j_1}}^{d} EY_{j_1}^3 Y_{j_2} + 6 \sum_{j_1=1}^{d} \sum_{\substack{j_2=1 \\ j_2 < j_1}}^{d} EY_{j_1}^2 Y_{j_2}^2 \end{aligned}$$

$$+ 12\sum_{j_1=1}^{d}\sum_{\substack{j_2=1\\ j_2\neq j_1}}^{d}\sum_{\substack{j_3=1\\ j_3\neq j_1\\ j_3<j_2}}^{d} EY_{j_1}^2Y_{j_2}Y_{j_3} + 24\sum_{j_1=1}^{d}\sum_{\substack{j_2=1\\ j_2<j_1}}^{d}\sum_{\substack{j_3=1\\ j_3<j_2}}^{d}\sum_{\substack{j_4=1\\ j_4<j_3}}^{d} EY_{j_1}Y_{j_2}Y_{j_3}Y_{j_4}$$

$$\leq \sum_{i=1}^{d} EY_i^4 + 3\sum_{i=1}^{d}\sum_{\substack{j=1\\ j\neq i}}^{d} EY_i^2Y_j^2$$

$$\leq \sum_{i=1}^{d} EY_i^4 + 3\left(\sum_{i=1}^{d} EY_i^2\right)^2$$

$$\leq 3 + \sum_{i=1}^{d} EY_i^4. \tag{3.5}$$

3. By (3.4) and using the same argument as in (3.5), we have

$$EW^6 = \sum_{i=1}^{d} EY_i^6 + 15\sum_{i=1}^{d}\sum_{\substack{j=1\\ j\neq i}}^{d} EY_i^4EY_j^2 + 10\sum_{i=1}^{d}\sum_{\substack{j=1\\ j\neq i}}^{d} EY_i^3EY_j^3$$

$$+ 15\sum_{i=1}^{d}\sum_{\substack{j=1\\ j\neq i}}^{d}\sum_{\substack{l=1\\ l\neq i\\ l\neq j}}^{d} EY_i^2EY_j^2EY_l^2$$

$$\leq \sum_{i=1}^{d} EY_i^6 + 15\left(\sum_{i=1}^{d} EY_i^4\right)\left(\sum_{j=1}^{d} EY_j^2\right) + 10\left(\sum_{i=1}^{d} EY_i^3\right)^2$$

$$+ 15\left(\sum_{i=1}^{d} EY_i^2\right)^3$$

$$\leq \sum_{i=1}^{d} EY_i^6 + 15\sum_{i=1}^{d} EY_i^4 + 10\left(\sum_{i=1}^{d} EY_i^3\right)^2 + 15. \qquad \square$$

Corollary 3.5. *Under DLD condition, we have*

1. $EW^4 \leq 3 + \dfrac{d|A|^4}{n^4\left(\operatorname{Var} L(T)\right)^2}$;

2. $EW^6 \leq 15 + \dfrac{(1+10d)\,d|A|^6}{n^6\left(\operatorname{Var} L(T)\right)^3} + \dfrac{15d|A|^4}{n^4\left(\operatorname{Var} L(T)\right)^2}$.

Proof. By (3.2), we have

$$\sum_{i=1}^{d} EY_i^6 \leq \frac{d|A|^6}{n^6\left(\operatorname{Var} L(T)\right)^3},$$

$$\sum_{i=1}^{d} EY_i^4 \leq \frac{d|A|^4}{n^4 \left(\operatorname{Var} L(T)\right)^2}$$

and
$$\left(\sum_{i=1}^{d} EY_i^3\right)^2 \leq \frac{d^2|A|^6}{n^6 \left(\operatorname{Var} L(T)\right)^3}.$$

From these facts, Theorem 3.4(2) and Theorem 3.4(3), we have

$$EW^4 \leq 3 + \frac{d|A|^4}{n^4 \left(\operatorname{Var} L(T)\right)^2}$$

and
$$EW^6 \leq 15 + \frac{(1+10d)\, d|A|^6}{n^6 \left(\operatorname{Var} L(T)\right)^3} + \frac{15d|A|^4}{n^4 \left(\operatorname{Var} L(T)\right)^2}.$$ □

Next, we present two situations under DLD condition when we classify assets from their workplace and divide assets into d group as shown in Figure 3.6. The first situation deals with companies that tend to go bankrupt. While the second situation is a group of companies that may lay off some staff to maintain the financial liquidity of the companies. Under each situation, the explicit formula for moments of Y_i can be obtained.

Example 3.6 (Bankrupt assets)**.** Consider a CDO containing bankrupt assets. The assets correspond with loans of personnel from d companies. Each company tends to go bankrupt due to the global crisis. If a company goes bankrupt, then all personnel in the company are unemployed. Consequently, they default. In other words, when an asset defaults, then other assets in the same company also default. In addition, bankruptcy of a company does not affect other companies. Let p_{m_i} be the probability that the i^{th} company defaults. Then,

1. $\operatorname{Var} L(T) = \dfrac{1}{n^2} \displaystyle\sum_{i=1}^{d} p_{m_i} q_{m_i} \left(\sum_{j\in A_i} (1-R_j)\right)^2;$

2. $E|Y_i|^r = \dfrac{p_{m_i} q_{m_i} (p_{m_i}^{r-1} + q_{m_i}^{r-1})}{n^r (\operatorname{Var} L(T))^{r/2}} \left(\displaystyle\sum_{j\in A_i} (1-R_j)\right)^r$ for $i = 1, 2, 3, \ldots, d.$

Proof. 1. Consider the i^{th} company. Observe that, the probability of default for each asset in the i^{th} company is p_{m_i} and for the j^{th} and the l^{th} assets which are in the i^{th} company, we have the probability that the j^{th} and the l^{th} assets simultaneously default is

$$P(\mathbb{I}\left(\tau_j \leq T\right) = 1, \mathbb{I}\left(\tau_l \leq T = 1\right)) = p_{m_i}.$$

From these facts and Theorem 3.4(1.), we obtain that

$$\begin{aligned}
\operatorname{Var} L(T) &= \frac{1}{n^2} \sum_{i=1}^{d} \sum_{j \in A_i} \sum_{l \in A_i} (1 - R_j)(1 - R_l)[p_{jl} - p_j p_l] \\
&= \frac{1}{n^2} \sum_{i=1}^{d} (p_{m_i} - p_{m_i}^2) \sum_{j \in A_i} \sum_{l \in A_i} (1 - R_j)(1 - R_l) \\
&= \frac{1}{n^2} \sum_{i=1}^{d} p_{m_i} q_{m_i} \left(\sum_{j \in A_i} (1 - R_j) \right)^2.
\end{aligned}$$

2. From the situation, we know that if an asset defaults, then other assets in the same company default. Then, for the j^{th} and the l^{th} assets which are in the same company, we have

$$P(\mathbb{I}\left(\tau_j \leq T\right) = 1, \mathbb{I}\left(\tau_l \leq T\right) = 0) = 0,$$

and the probability that all assets in the i^{th} company simultaneously default is p_{m_i}. These imply that the chance that there is no default assets in the company is $1 - p_{m_i}$. Hence, we can conclude that, for each i, and $x_i \in \{0, 1\}$,

$$\begin{aligned}
&P\left(\mathbb{I}\left(\tau_{m_{i-1}+1} \leq T\right) = x_1, \mathbb{I}\left(\tau_{m_{i-1}+2} \leq T\right) = x_2, \ldots, \mathbb{I}\left(\tau_{m_i} \leq T\right) = x_{m_i - m_{i-1}}\right) \\
&= \begin{cases} p_{m_i} & \text{if } x_1 = x_2 = \cdots = x_{m_i - m_{i-1}} = 1; \\ q_{m_i} & \text{if } x_1 = x_2 = \cdots = x_{m_i - m_{i-1}} = 0; \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

This implies that, for $i = 1, 2, \ldots, d$ and $r \geq 1$,

$$E\left|\sum_{j\in A_i}(1-R_j)\left(\mathbb{I}(\tau_j \leq T) - p_j\right)\right|^r$$

$$= \sum_{x_1,x_2,\ldots,x_{m_i-m_{i-1}}} \left|\sum_{j\in A_i}(1-R_j)\left(x_j - p_j\right)\right|^r$$

$$\times P\left(\mathbb{I}\left(\tau_{m_{i-1}+1} \leq T\right) = x_1, \mathbb{I}\left(\tau_{m_{i-1}+2} \leq T\right) = x_2, \ldots, \mathbb{I}\left(\tau_{m_i} \leq T\right) = x_{m_i-m_{i-1}}\right) \tag{3.6}$$

$$= \left(\sum_{j\in A_i}(1-R_j)q_j\right)^r p_{m_i} + \left(\sum_{j\in A_i}(1-R_j)p_j\right)^r q_{m_i}$$

$$= p_{m_i}q_{m_i}^r\left(\sum_{j\in A_i}(1-R_j)\right)^r + p_{m_i}^r q_{m_i}\left(\sum_{j\in A_i}(1-R_j)\right)^r$$

$$= p_{m_i}q_{m_i}(p_{m_i}^{r-1} + q_{m_i}^{r-1})\left(\sum_{j\in A_i}(1-R_j)\right)^r.$$

This implies that

$$\begin{aligned}
E|Y_i|^r &= E\left|\sum_{j\in A_i} X_j\right|^r \\
&= \frac{1}{(\operatorname{Var} L(T))^{r/2}} E\left|\sum_{j\in A_i}\frac{1-R_j}{n}\left(\mathbb{I}(\tau_j \leq T) - p_j\right)\right|^r \\
&= \frac{p_{m_i}q_{m_i}(p_{m_i}^{r-1} + q_{m_i}^{r-1})}{n^r(\operatorname{Var} L(T))^{r/2}}\left(\sum_{j\in A_i}(1-R_j)\right)^r.
\end{aligned}$$ □

Under an economic contraction around the world, many companies must manage their financial status. One of many solutions to reduce the exceeding cost is a layoff. As a result, we consider a CDO containing laid-off assets in the next situation.

Example 3.7 (Laid–off assets)**.** The n assets in the CDO are split into a number of groups, and each group represents a company or a department. We suppose that each organization plans to lay off at most one employee. Hence, if our colleague is laid off, then we are still employed. On the other hand, the layoff of other companies does not affect our company. Moreover, it is possible that no coworker in the same company are laid off. Then,

1. $\operatorname{Var} L(T) = \dfrac{1}{n^2}\displaystyle\sum_{i=1}^{d}\left[\sum_{j\in A_i}(1-R_j)^2 p_j - \left(\sum_{j\in A_i}(1-R_j)p_j\right)^2\right]$;

2. $$E|Y_i|^r = \frac{1}{n^r(\operatorname{Var} L(T))^{r/2}}\sum_{j\in A_i} p_j\left|1-R_j-\sum_{l\in A_i}(1-R_l)p_l\right|^r + \frac{1}{n^r(\operatorname{Var} L(T))^{r/2}}(1-p_{A_i})\left(\sum_{j\in A_i}(1-R_j)p_j\right)^r,$$
   where $p_{A_i} = \sum_{j\in A_i} p_j$, for $i = 1, 2, 3, \ldots, d$.

Proof. 1. Let $i \neq j$ be assets in the same department. We know that each company can lay off at most one employee; as a result, there is no chance for layoff at least two employees. Hence

$$p_{ij} = P(\mathbb{I}(\tau_i \le T) = 1, \mathbb{I}(\tau_j \le T) = 1) = 0 \quad \text{for } i \neq j.$$

From this fact and Theorem 3.4(1), we have

$$\begin{aligned}
&\operatorname{Var} L(T)\\
&= \frac{1}{n^2}\sum_{i=1}^{d}\sum_{j\in A_i}\sum_{l\in A_i}(1-R_j)(1-R_l)[p_{jl}-p_jp_l]\\
&= \frac{1}{n^2}\sum_{i=1}^{d}\sum_{j\in A_i}(1-R_j)^2(p_j-p_j^2) - \frac{1}{n^2}\sum_{i=1}^{d}\sum_{j\in A_i}\sum_{l\in A_i-\{j\}}(1-R_j)(1-R_l)p_jp_l\\
&= \frac{1}{n^2}\sum_{i=1}^{d}\sum_{j\in A_i}(1-R_j)^2p_j - \frac{1}{n^2}\sum_{i=1}^{d}\sum_{j\in A_i}\sum_{l\in A_i}(1-R_j)(1-R_l)p_jp_l\\
&= \frac{1}{n^2}\sum_{i=1}^{d}\left[\sum_{j\in A_i}(1-R_j)^2p_j - \left(\sum_{j\in A_i}(1-R_j)p_j\right)^2\right].
\end{aligned}$$

2. Consider the i^{th} department for $i = 1, 2, \ldots, d$. Notice that it is impossible that at least two assets simultaneously default. If only one asset in this group

defaults, say i_0, then

$$P(\mathbb{I}(\tau_l \leq T) = 0 \text{ for all } l \in A_i - \{i_0\}, \mathbb{I}(\tau_{i_0} \leq T) = 1) = p_{i_0}.$$

Thus, the probability that only one asset in the i^{th} group defaults is $\sum_{j\in A_i} p_j$. Denote $p_{A_i} = \sum_{j\in A_i} p_j$. On the other hand, we obtain the chance that no assets in this group are laid off is $1 - p_{A_i}$. From these facts and (3.6), we have

$$\begin{aligned}
&E\left|\sum_{j\in A_i}(1-R_j)\left(\mathbb{I}(\tau_j \leq T) - p_j\right)\right|^r \\
&= \sum_{l\in A_i}\left|(1-R_l)(1-p_l) - \sum_{j\in A_i-\{l\}}(1-R_j)p_j\right|^r p_l \\
&\quad + \left(\sum_{j\in A_i}(1-R_j)p_j\right)^r (1-p_{A_i}) \\
&= \sum_{l\in A_i}\left|1-R_l-\sum_{j\in A_i}(1-R_j)p_j\right|^r p_l + \left(\sum_{j\in A_i}(1-R_j)p_j\right)^r (1-p_{A_i}).
\end{aligned}$$

Consequently,

$$\begin{aligned}
E|Y_i|^r &= \frac{1}{(\operatorname{Var} L(T))^{r/2}} E\left|\sum_{j\in A_i}\frac{1-R_j}{n}\left(\mathbb{I}(\tau_j \leq T) - p_j\right)\right|^r \\
&= \frac{1}{n^r(\operatorname{Var} L(T))^{r/2}}\left[\sum_{l\in A_i}\left|1-R_l-\sum_{j\in A_i}(1-R_j)p_j\right|^r p_l\right. \\
&\qquad\qquad \left. + \left(\sum_{j\in A_i}(1-R_j)p_j\right)^r (1-p_{A_i})\right].
\end{aligned}$$

□

CHAPTER IV
BOUNDS ON NORMAL APPROXIMATION FOR LOCALLY DEPENDENT CDO

In this chapter, we concentrate on a locally dependent CDO. We first give a uniform bound

$$\delta(n) := \sup_{k\in\mathbb{R}} \left| E(W-k)^+ - E(Z-k)^+ \right|$$

for LD random variables in Section 4.1. The non-uniform bound

$$\delta(n,k) := \left| E(W-k)^+ - E(Z-k)^+ \right|$$

which is a refinement of a uniform bound is provided in Section 4.2.

Continued from the previous chapter, we let

1. $\kappa_2 = \max_{1\le i\le n}\{|N(B_i)|\}$, where $N(B_i) = \{j \mid B_j \cap B_i \neq \varnothing\}$;
2. $\kappa_3 = \max_{1\le i\le n} \max\{|B_i|, |B_i^{-1}|\}$, where $B_i^{-1} = \{j \mid i \in B_j\}$;
3. $\kappa = \max\{\kappa_1, \kappa_2, \kappa_3\}$.

From Figure 3.1, we have $N(B_1) = \{1,2,3,\ldots,9\}, N(B_2) = \{1,2,3,\ldots,8\}$, $N(B_3) = \{1,2,3,\ldots,10\}, B_1^{-1} = \{1,2,3,4,5,7,8\}$ and $B_2^{-1} = \{1,2,3,7,8\}$. Additionally, we have $\kappa_2 = 10$ and $\kappa_3 = 7$. Hence, from these facts and by an example of κ_1 in Chapter 3, p.31, we have $\kappa = 10$. Under the same reason in Remark 1.3, we assume that κ does not depend on n.

4.1 Uniform Bound

In this section, we present a uniform bound on normal approximation for locally dependent random variables. The bound does not depend on k. The proof of this theorem is mainly motivated by the proof in Theorem 2.1 and Theorem 2.2 in [6]. Chen ([6]) introduced the LD condition for general random variables in 2004 and dealt with the Stein equation for $h(x) = \mathbb{I}(x \le k)$ for a fixed real number k. In this work, we consider a call function $h(x) = (x-k)^+$ which is used to determine loss on a tranche of a CDO. Hence, the modified proof in this work is slightly different from [6] because of the property of f_k in Proposition 2.3.

Theorem 4.1 (Uniform Bound)**.** *Under LD condition, we have*

$$\delta(n) = \mathcal{O}\left(\frac{1}{n^2 \left(\operatorname{Var} L(T)\right)^{3/2}}\right) + \mathcal{O}\left(\frac{1}{n^{3/2} \operatorname{Var} L(T)}\right).$$

Furthermore, if $\operatorname{Var} L(T) = \mathcal{O}\left(\frac{1}{n}\right)$, *then*

$$\delta(n) = \mathcal{O}\left(\frac{1}{\sqrt{n}}\right).$$

Proof. By modification of arguments in Theorem 2.1 and Theorem 2.2 in [6], p. 2009–2013, we have

$$\left|E(W-k)^+ - E(Z-k)^+\right| \le R_1 + R_2 + R_3, \tag{4.1}$$

where

$$R_1 = \|f_k'\|\gamma_{n,3} + \|f_k'\|(\kappa_2\gamma_{n,4})^{1/2}, \tag{4.2}$$

$$R_2 = \frac{2\|f_k'\|}{3}\gamma_{n,3}, \tag{4.3}$$

$$R_3 = \left|E\int_{|t|\le 1}\left[f_k'(W+t) - f_k'(W)\right]\widehat{K}(t)dt\right|,$$

$$\widehat{K}(t) = \sum_{i=1}^{n} X_i\left[\mathbb{I}\left(-Y_i \le t < 0\right) - \mathbb{I}\left(0 \le t \le -Y_i\right)\right]$$

and

$$\gamma_{n,r} = \sum_{i=1}^{n}\left(E|X_i|^r + E|Y_i|^r\right).$$

By (2.9), we have

$$R_1 \leq \sqrt{\frac{2}{\pi}}\gamma_{n,3} + \sqrt{\frac{2}{\pi}}(\kappa_2\gamma_{n,4})^{1/2} \tag{4.4}$$

and

$$R_2 \leq \frac{2}{3}\sqrt{\frac{2}{\pi}}\gamma_{n,3}. \tag{4.5}$$

Hence, it remains to determine R_3. By Proposition 2.3 and the fact that

$$\int_{|t|\leq 1} |t|\widehat{K}(t)dt \leq \frac{1}{2}\sum_{i=1}^{n} |X_i|\left(Y_i^2 \wedge 1\right), \tag{4.6}$$

where $a \wedge b = \min\{a, b\}$ for any real number a, b (see [6], p.2010), we obtain

$$\begin{aligned} R_3 &\leq 2E\int_{|t|\leq 1} W^2|t|\widehat{K}(t)dt + 10.46E\int_{|t|\leq 1} |W||t|\widehat{K}(t)dt + 12.16E\int_{|t|\leq 1}|t|\widehat{K}(t)dt \\ &\leq EW^2\sum_{i=1}^{n}|X_i|(Y_i^2 \wedge 1) + 5.23E|W|\sum_{i=1}^{n}|X_i|(Y_i^2 \wedge 1) + 6.08\sum_{i=1}^{n}E|X_i|(Y_i^2 \wedge 1). \end{aligned}$$

From this fact and the facts that

$$E|W|\sum_{i=1}^{n}|X_i|(Y_i^2 \wedge 1) \leq \left(\frac{2}{3} + \frac{4\kappa_3}{3}\right)\gamma_{n,3} \tag{4.7}$$

and

$$\sum_{i=1}^{n}E|X_i|(Y_i^2 \wedge 1) \leq \frac{2}{3}\gamma_{n,3} \tag{4.8}$$

where $\kappa_3 = \max_{1\leq i\leq n}\max\{|B_i|, |B_i^{-1}|\}$, (see [6], p.2012–2013), we have

$$R_3 \leq EW^2\sum_{i=1}^{n}|X_i|(Y_i^2 \wedge 1) + \left(7.54 + \frac{20.92}{3}\kappa_3\right)\gamma_{n,3}.$$

Hence, it remains to determine $EW^2\sum_{i=1}^{n}|X_i|(Y_i^2 \wedge 1)$. By the fact that

$$\left(\sum_{i=1}^{d} a_i\right)^k \leq d^{k-1}\sum_{i=1}^{d} a_i^k, \tag{4.9}$$

for $a_i > 0$ and $k, d \in \mathbb{N}$, we have

$$\begin{aligned} E\left(\sum_{i=1}^{n} |X_i|(Y_i^2 \wedge 1)\right)^2 &\leq E\left(\sum_{i=1}^{n} |X_i|Y_i^2\right)^2 \\ &\leq n\sum_{i=1}^{n} EX_i^2Y_i^4 \\ &\leq n\sum_{i=1}^{n} \left(EX_i^6\right)^{1/3}\left(EY_i^6\right)^{2/3} \\ &\leq \frac{n}{3}\sum_{i=1}^{n}(EX_i^6 + 2EY_i^6) \\ &\leq \frac{2n}{3}\gamma_{n,6}. \end{aligned} \tag{4.10}$$

From this fact and using the Hölder's inequality, we obtain

$$\begin{aligned} EW^2\sum_{i=1}^{n} |X_i|(Y_i^2 \wedge 1) &\leq \left(EW^4\right)^{1/2}\left[E\left(\sum_{i=1}^{n} |X_i|(Y_i^2 \wedge 1)\right)^2\right]^{1/2} \\ &\leq \left(\frac{2n}{3}EW^4\gamma_{n,6}\right)^{1/2}. \end{aligned}$$

Consequently, we conclude that

$$R_3 \leq \left(\frac{2n}{3}EW^4\gamma_{n,6}\right)^{1/2} + \left(7.54 + \frac{20.92\kappa_3}{3}\right)\gamma_{n,3}. \tag{4.11}$$

Combining (4.1), (4.4), (4.5) and (4.11), we obtain

$$\delta(n) \leq (8.88 + 6.98\kappa)\,\gamma_{n,3} + 0.8\,(\kappa\gamma_{n,4})^{1/2} + \left(\frac{2n}{3}EW^4\gamma_{n,6}\right)^{1/2},$$

where $\kappa = \max\{\kappa_2, \kappa_3\}$.

By (3.1) and (3.2), we have

$$\gamma_{n,r} \leq \frac{1 + |A|^r}{n^{r-1}\left(\operatorname{Var} L(T)\right)^{r/2}} = \mathcal{O}\left(\frac{1}{n^{r-1}\left(\operatorname{Var} L(T)\right)^{r/2}}\right). \tag{4.12}$$

From this fact and Theorem 3.3(2), we have

$$\delta(n) = \mathcal{O}\left(\frac{1}{n^2\left(\operatorname{Var} L(T)\right)^{3/2}}\right) + \mathcal{O}\left(\frac{1}{n^{3/2}\operatorname{Var} L(T)}\right). \qquad \square$$

4.2 Non-uniform Bound

In this section, we improve the uniform bound in Section 4.1 by proposing a non-uniform bound on normal approximation under local dependence. The non-uniform bound is sharper than the uniform bound when k is large enough.

Theorem 4.2 (Non-uniform Bound). *Under LD condition and for $k \geq 2$, we have*

$$\begin{aligned}\delta(n,k) = {} & C_1(k,\kappa)\mathcal{O}\left(\frac{1}{n^2\left(\operatorname{Var} L(T)\right)^{3/2}}\right) + C_2(k,\kappa)\mathcal{O}\left(\frac{1}{n^{3/2}\operatorname{Var} L(T)}\right)\\ & + \frac{1}{k}\mathcal{O}\left(\frac{1}{n^2\left(\operatorname{Var} L(T)\right)^{3/2}}\right),\end{aligned}$$

where $$C_1(k,\kappa) = \left(2 + \frac{2\kappa}{3}\right)\left(\frac{e^{-k^2/2}}{\sqrt{2\pi}k^2} + \frac{1}{k}\right) + \frac{1}{3k^2}$$

and $$C_2(k,\kappa) = \sqrt{\kappa}\left(\frac{e^{-k^2/2}}{\sqrt{2\pi}k^2} + \frac{1}{k}\right).$$

Furthermore, if $\operatorname{Var} L(T) = \mathcal{O}\left(\frac{1}{n}\right)$, *then*

$$\delta(n,k) = \left(C_1(k,\kappa) + C_2(k,\kappa) + \frac{1}{k}\right)\mathcal{O}\left(\frac{1}{\sqrt{n}}\right).$$

Proof. By Proposition 2.2, (4.2) and (4.3), we have

$$\begin{aligned}&|E(W-k)^+ - E(Z-k)^+|\\ &\leq \frac{5}{3}\left(\frac{e^{-k^2/2}}{\sqrt{2\pi}k^2} + \frac{1}{k}\right)\gamma_{n,3} + \left(\frac{e^{-k^2/2}}{\sqrt{2\pi}k^2} + \frac{1}{k}\right)\left(\kappa_2\gamma_{n,4}\right)^{1/2} + R_3.\end{aligned}$$

Thus, it remains to bound R_3. We use truncation technique to rewrite R_3 as shown:

$$R_3 \leq R_{3,1} + R_{3,2} + R_{3,3}, \tag{4.13}$$

where $\quad R_{3,1} = E\int_{|t|\leq 1} |f_k'(W+t) - f_k'(W)|\,\mathbb{I}(W > k)\widehat{K}(t)dt,$

$$R_{3,2} = E\int_{|t|\leq 1} |f_k'(W+t) - f_k'(W)|\,\mathbb{I}(W+t > k, W \leq k)\widehat{K}(t)dt$$

and $\quad R_{3,3} = E\int_{|t|\leq 1} |f_k'(W+t) - f_k'(W)|\,\mathbb{I}(W+t \leq k, W \leq k)\widehat{K}(t)dt.$

By Proposition 2.3 , we obtain that

$$R_{3,1} \leq E\left(2W^2 + 10.46|W| + 12.16\right)\mathbb{I}(W > k)\int_{|t|\leq 1} |t|\widehat{K}(t)dt$$

and $\quad R_{3,2} \leq E\left(2W^2 + 10.46|W| + 12.16\right)\mathbb{I}(W > k-1)\int_{|t|\leq 1} |t|\widehat{K}(t)dt.$

In each term of $R_{3,1}$, we use the Hölder's inequality, Markov's inequality, (4.6) and (4.10) to obtain that

$$\begin{aligned}
R_{3,1} &\leq \left\{2\left[EW^4\mathbb{I}(W > k)\right]^{1/2} + 10.46\left[EW^2\mathbb{I}(W > k)\right]^{1/2} + 12.16\left[P(W > k)\right]^{1/2}\right\} \\
&\qquad \times \left[E\left(\int_{|t|\leq 1} |t|\widehat{K}(t)dt\right)^2\right]^{1/2} \\
&\leq \frac{1}{k}\left(\sqrt{EW^6} + 5.23\sqrt{EW^4} + 6.08\right)\left[E\left(\sum_{i=1}^{n} |X_i|(Y_i^2 \wedge 1)\right)^2\right]^{1/2} \\
&\leq \frac{1}{k}\left(\sqrt{EW^6} + 5.23\sqrt{EW^4} + 6.08\right)\left(\frac{2n\gamma_{n,6}}{3}\right)^{1/2}.
\end{aligned} \tag{4.14}$$

Using the same argument of bounding $R_{3,1}$, we obtain

$$R_{3,2} \leq \frac{1}{k-1}\left(\sqrt{EW^6} + 5.23\sqrt{EW^4} + 6.08\right)\left(\frac{2n\gamma_{n,6}}{3}\right)^{1/2}$$
$$\leq \frac{2}{k}\left(\sqrt{EW^6} + 5.23\sqrt{EW^4} + 6.08\right)\left(\frac{2n\gamma_{n,6}}{3}\right)^{1/2}, \tag{4.15}$$

where we use the fact that $\frac{1}{k-1} \leq \frac{2}{k}$ for $k \geq 2$ in the last inequality.

To bound $R_{3,3}$, consider $|f_k'(x+t) - f_k'(x)|$ where $x + t \leq k$ and $x \leq k$. We use Stein equation (2.1), Proposition 2.1 and Proposition 2.2 to obtain

$$\begin{aligned}|f_k'(x+t) - f_k'(x)| &= |x(f_k(x+t) - f_k(x)) + tf_k(x+t)| \\ &\leq |x|\,\|f_k'\|\,|t| + |f_k(x+t)||t| \\ &\leq \left(\frac{e^{-k^2/2}}{\sqrt{2\pi}k^2} + \frac{1}{k}\right)|x||t| + \frac{|t|}{k^2},\end{aligned} \tag{4.16}$$

where we use the mean value theorem in the first inequality. From this fact and (4.6)–(4.8), we have

$$\begin{aligned}R_{3,3} &\leq E\left[\left(\frac{e^{-k^2/2}}{\sqrt{2\pi}k^2} + \frac{1}{k}\right)|W| + \frac{1}{k^2}\right]\int_{|t|\leq 1} |t|\widehat{K}(t)dt \\ &\leq \frac{1}{2}\left(\frac{e^{-k^2/2}}{\sqrt{2\pi}k^2} + \frac{1}{k}\right)\sum_{i=1}^n E|WX_i|(Y_i^2 \wedge 1) + \frac{1}{2k^2}\sum_{i=1}^n E|X_i|(Y_i^2 \wedge 1) \\ &\leq \left[\frac{1}{3}\left(\frac{e^{-k^2/2}}{\sqrt{2\pi}k^2} + \frac{1}{k} + \frac{1}{k^2}\right) + \frac{2\kappa_3}{3}\left(\frac{e^{-k^2/2}}{\sqrt{2\pi}k^2} + \frac{1}{k}\right)\right]\gamma_{n,3}.\end{aligned} \tag{4.17}$$

Combining (4.13)–(4.15) and (4.17), we obtain

$$\begin{aligned}\delta(n,k) \leq &\left[\left(2 + \frac{2\kappa}{3}\right)\left(\frac{e^{-k^2/2}}{\sqrt{2\pi}k^2} + \frac{1}{k}\right) + \frac{1}{3k^2}\right]\gamma_{n,3} + \sqrt{\kappa}\left(\frac{e^{-k^2/2}}{\sqrt{2\pi}k^2} + \frac{1}{k}\right)\gamma_{n,4}^{1/2} \\ &+ \frac{1}{k}\left(3\sqrt{EW^6} + 15.69\sqrt{EW^4} + 18.24\right)\left(\frac{2n\gamma_{n,6}}{3}\right)^{1/2}.\end{aligned}$$

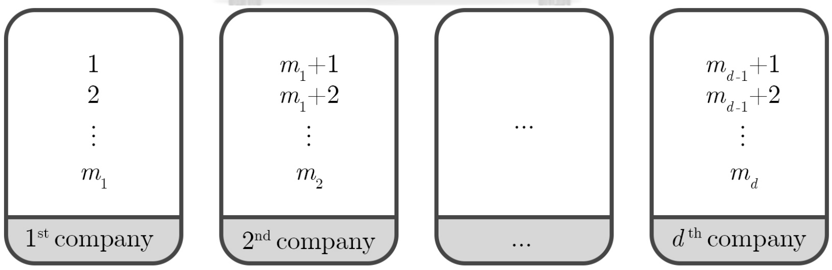
By (4.12), Theorem 3.3(2) and Theorem 3.3(4), we have

$$\delta(n,k) = C_1(k,\kappa)\mathcal{O}\left(\frac{1}{n^2\left(\operatorname{Var} L(T)\right)^{3/2}}\right) + C_2(k,\kappa)\mathcal{O}\left(\frac{1}{n^{3/2}\operatorname{Var} L(T)}\right) + \frac{1}{k}\mathcal{O}\left(\frac{1}{n^2\left(\operatorname{Var} L(T)\right)^{3/2}}\right). \qquad \square$$

CHAPTER V
BOUNDS ON NORMAL APPROXIMATION FOR DISJOINT LOCALLY DEPENDENT CDO

In this chapter, we consider a disjoint locally dependent CDO when each asset can be classified into disjoint group. Each group represents a company or a department. We use the Stein's method together with properties of the Stein solution f_k and its derivative f_k' presented in Chapter 2 to determine uniform bound in Section 5.1 and non-uniform bounds in Section 5.2 on normal approximation for disjoint locally dependent CDO. Moreover, we propose the bounds for bankrupt assets and the bounds for laid-off assets in Section 5.3.

Notice that, we use the notation appeared in Chapter 3 about the structure of DLD CDO throughout this chapter. Assume that there are d groups of disjoint assets from n assets, and the i^{th} group has $m_i - m_{i-1}$ indebted personnel (for $i = 1, 2, \ldots, d$ when $m_0 = 0$ and $m_d = n$).



Classification of assets in a DLD CDO.

5.1 Uniform Bound

Notice from the fact that $\{A_i\}_{i=1}^{d}$ are disjoint, so we can rewrite $W = \sum_{i=1}^{d} Y_i$. We next modify the argument in Theorem 4.1 to prove the following result.

Theorem 5.1 (Uniform Bound)**.** *Under the DLD condition, we have*

$$\delta(n) \le 24.97 \sum_{i=1}^{d} E|Y_i|^3 + 0.8 \left(\sum_{i=1}^{d} EY_i^4 \right)^{1/2} + \left(dEW^4 \sum_{i=1}^{d} EY_i^6 \right)^{1/2}.$$

Furthermore, if we use the fact that

$$|Y_i| \le \frac{|A_i|}{n\sqrt{\operatorname{Var} L(T)}} \quad \textit{and} \quad EW^4 \le 3 + \frac{d|A|^4}{n^4 \left(\operatorname{Var} L(T)\right)^2},$$

we have

$$\begin{aligned} \delta(n) \le\ & \frac{24.97 d|A|^3}{n^3 \left(\operatorname{Var} L(T)\right)^{3/2}} + \frac{0.8\sqrt{d}|A|^2}{n^2 \operatorname{Var} L(T)} \\ & + \left(3 + \frac{d|A|^4}{n^4 \left(\operatorname{Var} L(T)\right)^2} \right)^{1/2} \frac{d|A|^3}{n^3 \left(\operatorname{Var} L(T)\right)^{3/2}}. \end{aligned}$$

Proof. By the fact that X_{A_i} and $X_{A_i^c}$ are independent, we have that Y_i and $W - Y_i$ are independent. Hence,

$$\begin{aligned} EWf_k(W) &= \sum_{i=1}^{d} EY_i \left[f_k(W) - f_k(W - Y_i) \right] \\ &= \sum_{i=1}^{d} EY_i \int_{-Y_i}^{0} f_k'(W + t)dt \\ &= E \int_{-\infty}^{\infty} f_k'(W + t)\widetilde{K}(t)dt, \end{aligned} \tag{5.1}$$

where $\widetilde{K}(t) = \sum_{i=1}^{d} Y_i \left[\mathbb{I}\left(-Y_i \le t < 0\right) - \mathbb{I}\left(0 \le t \le -Y_i\right) \right]$. Note that

$$\begin{aligned} \int_{-\infty}^{\infty} \widetilde{K}(t)dt &= \int_{-\infty}^{\infty} \sum_{i=1}^{d} Y_i \left[\mathbb{I}\left(-Y_i \le t < 0\right) - \mathbb{I}\left(0 \le t \le -Y_i\right) \right] dt \\ &= \sum_{i=1}^{d} Y_i \left(\int_{-Y_i}^{0} \mathbb{I}(Y_i > 0)dt - \int_{0}^{-Y_i} \mathbb{I}(Y_i \le 0)dt \right) \\ &= \sum_{i=1}^{d} Y_i^2. \end{aligned} \tag{5.2}$$

Then,

$$E\int_{-\infty}^{\infty}\widetilde{K}(t)dt=\sum_{i=1}^{d}EY_i^2=EW^2=1.$$

From this fact, (5.1) and (2.2), we have

$$\begin{aligned}E(W-k)^+-E(Z-k)^+&=EWf_k(W)-Ef_k'(W)\\&=E\int_{-\infty}^{\infty}f_k'(W+t)\widetilde{K}(t)dt-E\int_{-\infty}^{\infty}f_k'(W)E\widetilde{K}(t)dt\\&=S_1+S_2+S_3,\end{aligned}\tag{5.3}$$

where
$$S_1=E\int_{-\infty}^{\infty}f_k'(W)\left[\widetilde{K}(t)-E\widetilde{K}(t)\right]dt,$$
$$S_2=E\int_{|t|>1}\left[f_k'(W+t)-f_k'(W)\right]\widetilde{K}(t)dt$$
and
$$S_3=E\int_{|t|\le 1}\left[f_k'(W+t)-f_k'(W)\right]\widetilde{K}(t)dt.\tag{5.4}$$

By (5.2), we obtain

$$|S_1|\le\|f_k'\|E\left|\sum_{i=1}^{d}\left(Y_i^2-EY_i^2\right)\right|.\tag{5.5}$$

To bound $E\left|\sum_{i=1}^{d}\left(Y_i^2-EY_i^2\right)\right|$, let $\overline{Y_i}=Y_i^2\mathbb{I}(|Y_i|\le 1)$ for $i=1,2,3,\ldots,d$. Then, we can follow the proof of Theorem 2.2 in [6], p.2013 to show that

$$\begin{aligned}&E\left|\sum_{i=1}^{d}\left(Y_i^2-EY_i^2\right)\right|\\&\le E\left|\sum_{i=1}^{d}\left(Y_i^2-EY_i^2\right)\mathbb{I}(|Y_i|\le 1)\right|+E\left|\sum_{i=1}^{d}\left(Y_i^2-EY_i^2\right)\mathbb{I}(|Y_i|>1)\right|\\&\le E\left|\sum_{i=1}^{d}(\overline{Y_i}-E\overline{Y_i})\right|+2\sum_{i=1}^{d}EY_i^2\mathbb{I}(|Y_i|>1)\end{aligned}$$

$$\leq \left(\operatorname{Var}\sum_{i=1}^{d}\overline{Y_i}\right)^{1/2} + 2\sum_{i=1}^{d}E|Y_i|^3. \tag{5.6}$$

Since Y_i's, $i = 1, 2, 3, \ldots, d$, are independent,

$$\operatorname{Var}\left(\sum_{i=1}^{d}\overline{Y_i}\right) = \sum_{i=1}^{d}\operatorname{Var}\overline{Y_i} \leq \sum_{i=1}^{d}E\overline{Y_i}^2 \leq \sum_{i=1}^{d}EY_i^4. \tag{5.7}$$

By (2.9) and (5.5)–(5.7), we obtain

$$|S_1| \leq \|f_k'\|\left(\sum_{i=1}^{d}EY_i^4\right)^{1/2} + 2\|f_k'\|\sum_{i=1}^{d}E|Y_i|^3 \tag{5.8}$$

$$\leq \sqrt{\frac{2}{\pi}}\left(\sum_{i=1}^{d}EY_i^4\right)^{1/2} + 2\sqrt{\frac{2}{\pi}}\sum_{i=1}^{d}E|Y_i|^3. \tag{5.9}$$

Consider S_2. By (2.9) and the fact that

$$\begin{aligned}
&\int_{|t|>1}\widetilde{K}(t)dt \\
&= \sum_{i=1}^{d}Y_i\int_{|t|>1}\left[\mathbb{I}\left(-Y_i \leq t < 0\right) - \mathbb{I}\left(0 \leq t \leq -Y_i\right)\right]dt \\
&= \sum_{i=1}^{d}Y_i\left[\int_{|t|>1}\mathbb{I}(Y_i > 0)\mathbb{I}(-Y_i \leq t < 0)dt - \int_{|t|>1}\mathbb{I}(Y_i \leq 0)\mathbb{I}(0 \leq t \leq -Y_i)dt\right] \\
&= \sum_{i=1}^{d}Y_i\left[\int_{-Y_i}^{-1}\mathbb{I}(Y_i > 1)dt - \int_{1}^{-Y_i}\mathbb{I}(Y_i < -1)dt\right] \\
&\leq \sum_{i=1}^{d}|Y_i|\left[\int_{-Y_i}^{0}\mathbb{I}(Y_i > 1)dt + \int_{0}^{-Y_i}\mathbb{I}(Y_i < -1)dt\right] \\
&= \sum_{i=1}^{d}|Y_i|\left(Y_i\mathbb{I}(Y_i > 1) - Y_i\mathbb{I}(Y_i < -1)\right) \\
&= \sum_{i=1}^{d}|Y_i|\left(|Y_i|\mathbb{I}(Y_i > 1) + |Y_i|\mathbb{I}(Y_i < -1)\right) \\
&\leq \sum_{i=1}^{d}Y_i^2\mathbb{I}(|Y_i| \geq 1)
\end{aligned}$$

$$\leq \sum_{i=1}^{d} |Y_i|^3,$$

we have

$$|S_2| \leq 2\|f_k'\| E \int_{|t|>1} \widetilde{K}(t)dt \leq 2\|f_k'\| \sum_{i=1}^{d} E|Y_i|^3 \leq 2\sqrt{\frac{2}{\pi}} \sum_{i=1}^{d} E|Y_i|^3. \tag{5.10}$$

For S_3, by Proposition 2.3 and the fact that

$$\begin{aligned}
&\int_{|t|\leq 1} |t|\widetilde{K}(t)dt \\
&= \sum_{i=1}^{d} Y_i \int_{|t|\leq 1} |t| \left[\mathbb{I}\left(-Y_i \leq t < 0\right) - \mathbb{I}\left(0 \leq t \leq -Y_i\right)\right] dt \\
&= \sum_{i=1}^{d} Y_i \left[\int_{|t|\leq 1} |t| \left(\mathbb{I}(Y_i > 0)\mathbb{I}\left(-Y_i \leq t < 0\right) - \mathbb{I}(Y_i \leq 0)\mathbb{I}\left(0 \leq t \leq -Y_i\right)\right) dt\right] \\
&= \sum_{i=1}^{d} Y_i \left(\int_{-Y_i \vee -1}^{0} |t|\mathbb{I}(Y_i > 0)dt - \int_{0}^{1\wedge -Y_i} |t|\mathbb{I}(Y_i \leq 0)dt\right) \\
&\leq \frac{1}{2} \sum_{i=1}^{d} |Y_i| \left(Y_i^2 \wedge 1\right), \qquad (5.11)
\end{aligned}$$

we have

$$\begin{aligned}
|S_3| &\leq 2E \int_{|t|\leq 1} W^2|t|\widetilde{K}(t)dt + 10.46E \int_{|t|\leq 1} |W||t|\widetilde{K}(t)dt + 12.16E \int_{|t|\leq 1} |t|\widetilde{K}(t)dt \\
&\leq EW^2 \sum_{i=1}^{d} |Y_i|(Y_i^2 \wedge 1) + 5.23E|W| \sum_{i=1}^{d} |Y_i|(Y_i^2 \wedge 1) + 6.08 \sum_{i=1}^{d} E|Y_i|(Y_i^2 \wedge 1) \\
&= S_{3,1} + S_{3,2} + S_{3,3}.
\end{aligned}$$

By (4.9), we have

$$E\left(\sum_{i=1}^{d} |Y_i| \left(Y_i^2 \wedge 1\right)\right)^2 \leq E\left(\sum_{i=1}^{d} |Y_i|^3\right)^2 \leq d\sum_{i=1}^{d} EY_i^6. \tag{5.12}$$

Hence,

$$S_{3,1} \le \left(EW^4\right)^{1/2} \left[E\left(\sum_{i=1}^{d} |Y_i|(Y_i^2 \wedge 1) \right)^2 \right]^{1/2} \le \left(dEW^4 \sum_{i=1}^{d} EY_i^6 \right)^{1/2}. \quad (5.13)$$

Modifying the idea in [6], p.2013, we obtain

$$\begin{aligned} E|WY_i|(Y_i^2 \wedge 1) &\le E|W - Y_i|E|Y_i|(Y_i^2 \wedge 1) + EY_i^2(Y_i^2 \wedge 1) \\ &\le (1 + E|Y_i|)E|Y_i|(Y_i^2 \wedge 1) + E|Y_i|^3 \\ &\le 2E|Y_i|^3 + E|Y_i|EY_i^2 \\ &\le 3E|Y_i|^3. \end{aligned} \quad (5.14)$$

This implies that

$$S_{3,2} \le 15.69 \sum_{i=1}^{d} E|Y_i|^3. \quad (5.15)$$

Moreover,

$$S_{3,3} = 6.08 \sum_{i=1}^{d} E|Y_i|(Y_i^2 \wedge 1) \le 6.08 \sum_{i=1}^{d} E|Y_i|^3. \quad (5.16)$$

We conclude from (5.13) and (5.15)–(5.16) that

$$|S_3| \le \left(dEW^4 \sum_{i=1}^{d} EY_i^6 \right)^{1/2} + 21.77 \sum_{i=1}^{d} E|Y_i|^3. \quad (5.17)$$

Combining (5.3), (5.9), (5.10) and (5.17), we obtain

$$\delta(n) \le 24.97 \sum_{i=1}^{d} E|Y_i|^3 + 0.8 \left(\sum_{i=1}^{d} EY_i^4 \right)^{1/2} + \left(dEW^4 \sum_{i=1}^{d} EY_i^6 \right)^{1/2}. \qquad \square$$

Next, we provide a non-uniform bound on normal approximation for disjoint locally dependent CDO. The bound is a refinement of the uniform bound in Theorem

5.2 when k is large enough.

5.2 Non-uniform Bound

Theorem 5.2 (Non-uniform Bound)**.** *Under the DLD condition with $k \geq 2$, we have*

$$\delta(n,k) \ \leq \ C_1(k)\sum_{i=1}^{d} E|Y_i|^3 \,+\, C_2(k)\left(\sum_{i=1}^{d} EY_i^4\right)^{1/2} \,+\, C_3(k)\left(d\sum_{i=1}^{d} EY_i^6\right)^{1/2},$$

where

$$C_1(k) = \frac{5.5e^{-k^2/2}}{\sqrt{2\pi}k^2} + \frac{5.5}{k} + \frac{1}{2k^2},$$

$$C_2(k) = \frac{e^{-k^2/2}}{\sqrt{2\pi}k^2} + \frac{1}{k}$$

and

$$C_3(k) = \frac{1}{k}\left(3\sqrt{EW^6} + 15.69\sqrt{EW^4} + 18.24\right).$$

Furthermore, if we use the fact that

$$|Y_i| \leq \frac{|A_i|}{n\sqrt{\operatorname{Var} L(T)}},$$

$$EW^4 \leq 3 + \frac{d|A|^4}{n^4\left(\operatorname{Var} L(T)\right)^2}$$

and

$$EW^6 \leq 15 + \frac{(1+10d)\,d|A|^6}{n^6\left(\operatorname{Var} L(T)\right)^3} + \frac{15d|A|^4}{n^4\left(\operatorname{Var} L(T)\right)^2},$$

we have

$$\delta(n,k) \leq \frac{C_1(k)d|A|^3}{n^3\left(\operatorname{Var} L(T)\right)^{3/2}} + \frac{C_2(k)\sqrt{d}|A|^2}{n^2 \operatorname{Var} L(T)} + \frac{\overline{C_3(k)}d|A|^3}{n^3(\operatorname{Var} L(T))^{3/2}},$$

where

$$\overline{C_3(k)} = \frac{1}{k}\left[3\left(15 + \frac{(1+10d)\,d|A|^6}{n^6\left(\operatorname{Var} L(T)\right)^3} + \frac{15d|A|^4}{n^4\left(\operatorname{Var} L(T)\right)^2}\right)^{1/2}\right.$$

$$+\, 15.69\left(3+\frac{d|A|^4}{n^4\left(\operatorname{Var}L(T)\right)^2}\right)^{1/2}+18.24\Bigg].$$

Proof. By (5.3), (5.4), (5.8) and (5.10), we have

$$\left|E(W-k)^+ - E(Z-k)^+\right| \le |S_1|+|S_2|+|S_3|, \tag{5.18}$$

where
$$|S_1| \le \|f_k'\|\left[\left(\sum_{i=1}^d EY_i^4\right)^{1/2} + 2\sum_{i=1}^d E|Y_i|^3\right],$$
$$|S_2| \le 2\|f_k'\|\sum_{i=1}^d E|Y_i|^3$$
and
$$|S_3| \le E\int_{|t|\le 1} |f_k'(W+t)-f_k'(W)|\,\widetilde{K}(t)dt.$$

From Proposition 2.2, we obtain

$$|S_1| \le \left(\frac{e^{-k^2/2}}{\sqrt{2\pi}k^2}+\frac{1}{k}\right)\left[\left(\sum_{i=1}^d EY_i^4\right)^{1/2} + 2\sum_{i=1}^d E|Y_i|^3\right] \tag{5.19}$$

and
$$|S_2| \le 2\left(\frac{e^{-k^2/2}}{\sqrt{2\pi}k^2}+\frac{1}{k}\right)\sum_{i=1}^d E|Y_i|^3. \tag{5.20}$$

Thus, it remains to consider S_3. By using the argument in (4.13), we have

$$|S_3| \le S_{3,1}+S_{3,2}+S_{3,3}, \tag{5.21}$$

where
$$S_{3,1} = E\int_{|t|\le 1} |f_k'(W+t)-f_k'(W)|\,\mathbb{I}(W>k)\widetilde{K}(t)dt,$$
$$S_{3,2} = E\int_{|t|\le 1} |f_k'(W+t)-f_k'(W)|\,\mathbb{I}(W+t>k, W\le k)\widetilde{K}(t)dt$$
and
$$S_{3,3} = E\int_{|t|\le 1} |f_k'(W+t)-f_k'(W)|\,\mathbb{I}(W+t\le k, W\le k)\widetilde{K}(t)dt.$$

By Proposition 2.3 , we obtain that

$$S_{3,1} \leq E\left(2W^2 + 10.46|W| + 12.16\right)\mathbb{I}(W > k)\int_{|t|\leq 1}|t|\widetilde{K}(t)dt$$

and
$$S_{3,2} \leq E\left(2W^2 + 10.46|W| + 12.16\right)\mathbb{I}(W > k-1)\int_{|t|\leq 1}|t|\widetilde{K}(t)dt.$$

By using the Hölder's inequality, Markov's inequality, (5.11) and (5.12), we obtain

$$\begin{aligned}
S_{3,1} &\leq \left\{2\left[EW^4\mathbb{I}(W > k)\right]^{1/2} + 10.46\left[EW^2\mathbb{I}(W > k)\right]^{1/2} + 12.16\left[P(W > k)\right]^{1/2}\right\} \\
&\qquad \times \left[E\left(\int_{|t|\leq 1}|t|\widetilde{K}(t)dt\right)^2\right]^{1/2} \\
&\leq \frac{1}{k}\left(\sqrt{EW^6} + 5.23\sqrt{EW^4} + 6.08\right)\left[E\left(\sum_{i=1}^{d}|Y_i|\left(Y_i^2 \wedge 1\right)\right)^2\right]^{1/2} \\
&\leq \frac{1}{k}\left(\sqrt{EW^6} + 5.23\sqrt{EW^4} + 6.08\right)\left(d\sum_{i=1}^{d}EY_i^6\right)^{1/2}. \qquad (5.22)
\end{aligned}$$

Using the same argument of bounding $S_{3,1}$, we obtain

$$\begin{aligned}
S_{3,2} &\leq \frac{1}{k-1}\left(\sqrt{EW^6} + 5.23\sqrt{EW^4} + 6.08\right)\left(d\sum_{i=1}^{d}EY_i^6\right)^{1/2} \\
&\leq \frac{2}{k}\left(\sqrt{EW^6} + 5.23\sqrt{EW^4} + 6.08\right)\left(d\sum_{i=1}^{d}EY_i^6\right)^{1/2}, \qquad (5.23)
\end{aligned}$$

where we use the fact that $\frac{1}{k-1} \leq \frac{2}{k}$ for $k \geq 2$ in the last inequality.

To bound $S_{3,3}$, we use (4.16), (5.11) and (5.14) to obtain

$$\begin{aligned}
S_{3,3} &\leq E\left[\left(\frac{e^{-k^2/2}}{\sqrt{2\pi}k^2} + \frac{1}{k}\right)|W| + \frac{1}{k^2}\right]\int_{|t|\leq 1}|t|\widetilde{K}(t)dt \\
&\leq \frac{1}{2}\left(\frac{e^{-k^2/2}}{\sqrt{2\pi}k^2} + \frac{1}{k}\right)\sum_{i=1}^{n}E|WY_i|(Y_i^2 \wedge 1) + \frac{1}{2k^2}\sum_{i=1}^{n}E|Y_i|^3 \\
&\leq \left(\frac{3e^{-k^2/2}}{2\sqrt{2\pi}k^2} + \frac{3}{2k} + \frac{1}{2k^2}\right)\sum_{i=1}^{d}E|Y_i|^3. \qquad (5.24)
\end{aligned}$$

Combining (5.18)–(5.24), we obtain

$$\delta(n,k) \le \left(\frac{5.5e^{-k^2/2}}{\sqrt{2\pi}k^2} + \frac{5.5}{k} + \frac{1}{2k^2}\right)\sum_{i=1}^{d} E|Y_i|^3 + \left(\frac{e^{-k^2/2}}{\sqrt{2\pi}k^2} + \frac{1}{k}\right)\left(\sum_{i=1}^{d} EY_i^4\right)^{1/2}$$
$$+ \frac{1}{k}\left(3\sqrt{EW^6} + 15.69\sqrt{EW^4} + 18.24\right)\left(d\sum_{i=1}^{d} EY_i^6\right)^{1/2}. \qquad \square$$

Next, we provide two situations under DLD condition. In each situation, we compute $\operatorname{Var} L(T)$ and find the exact value of $\sum_{i=1}^{d} E|Y_i|^r$ for $r \ge 1$.

5.3 Examples of Disjoint Locally Dependent CDO

In this section, we provide uniform and non-uniform bounds for loss on a tranche of the DLD CDO under two situations. Additionally, we set specific parameters to compare the bounds.

Example 5.3. Under the situation in Example 3.6, we have

1. the uniform bound for loss on a tranche of CDO containing bankrupt assets is

$$\delta(n) \le 24.97\gamma_{d,3} + 0.8\gamma_{d,4}^{1/2} + \left[d\gamma_{d,6}\left(3+\gamma_{d,4}\right)\right]^{1/2};$$

2. for $k \ge 2$, the non-uniform bound for loss on a tranche of CDO containing bankrupt assets is

$$\delta(n,k) \le C_1(k)\gamma_{d,3} + C_2(k)\gamma_{d,4}^{1/2} + C_3(k)\left(d\gamma_{d,6}\right)^{1/2},$$

where $$\gamma_{d,r} = \frac{1}{n^r(\operatorname{Var} L(T))^{r/2}}\sum_{i=1}^{d} p_{m_i}q_{m_i}(p_{m_i}^{r-1} + q_{m_i}^{r-1})\left(\sum_{j\in A_i}(1-R_j)\right)^r,$$

$$EW^4 \le 3 + \gamma_{d,4},$$

$$EW^6 \le 15 + \gamma_{d,6} + 15\gamma_{d,4} + 10\gamma_{d,3}^2$$

and $$\operatorname{Var} L(T) = \frac{1}{n^2}\sum_{i=1}^{d} p_{m_i} q_{m_i}\left(\sum_{j\in A_i}(1-R_j)\right)^2.$$

The proof of Theoerm 5.3 is completed by applying Example 3.6, Theorem 3.4, Theorem 5.1 and Theorem 5.2.

To compare the uniform and non-uniform bounds, we consider Example 5.3 with the following parameters: $d = n/2, p = p_i, R = R_i$ and $m_i - m_{i-1} = 2$. Therefore, the bounds for loss on a tranche of the CDO with bankrupt assets for $k \geq 2$ are

$$\delta(n) \leq \frac{24.97\sqrt{2}(p^2+q^2)}{\sqrt{npq}} + \frac{0.8\sqrt{2(p^3+q^3)}}{\sqrt{npq}} + \sqrt{\frac{2(p^5+q^5)}{np^2q^2}\left(3+\frac{2(p^3+q^3)}{npq}\right)}$$

and

$$\delta(n,k) \leq \frac{\sqrt{2}(p^2+q^2)C_1(k)}{\sqrt{npq}} + \frac{\sqrt{2(p^3+q^3)}C_2(k)}{\sqrt{npq}} + \sqrt{\frac{2(p^5+q^5)}{np^2q^2}}C_3(k),$$

where $C_1(k), C_2(k)$ and $C_3(k)$ are presented in Theorem 5.2,

$$EW^4 \leq 3 + \frac{2(p^3+q^3)}{npq}$$

and $$EW^6 \leq 15 + \frac{4(p^5+q^5)}{(npq)^2} + \frac{30(p^3+q^3)+20(p^2+q^2)^2}{npq}.$$

The numerical results of uniform and non-uniform bounds for loss on a tranche of the CDO containing bankrupt assets ($\sqrt{\operatorname{Var} L(T)}\delta(n)$ and $\sqrt{\operatorname{Var} L(T)}\delta(n,k)$) are presented with parameters $R = 0.7$ and $p = 0.5$ as follows.

n	Uniform	Non-uniform			
		$\tilde{k}=0.3$	$\tilde{k}=0.5$	$\tilde{k}=0.7$	$\tilde{k}=0.9$
50	0.16509	0.07704	0.03299	0.02099	0.01539
100	0.08253	0.02711	0.01161	0.00739	0.00542
150	0.05502	0.01474	0.00631	0.00402	0.00295
200	0.04126	0.00956	0.00410	0.00261	0.00192

Table 5.1: Uniform and non-uniform bounds for loss on a tranche of CDO with bankrupt assets

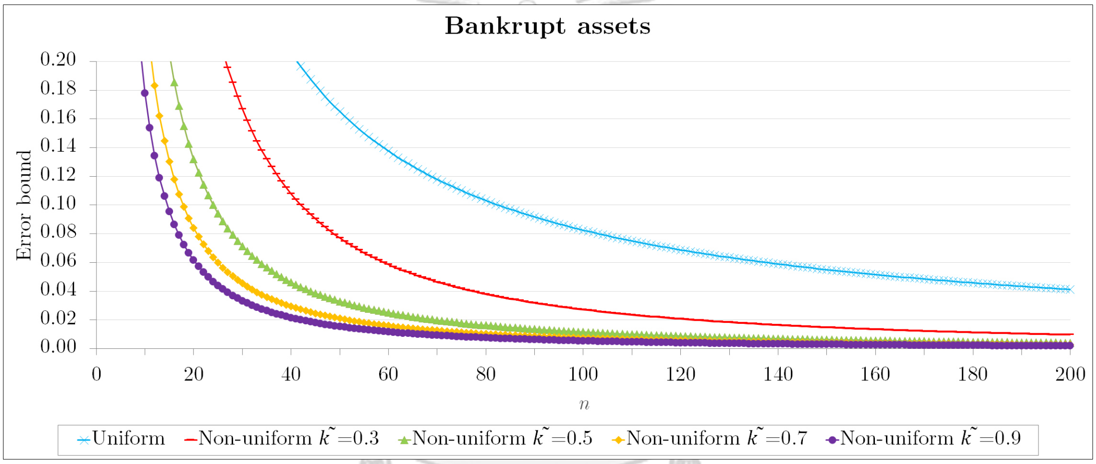


Figure 5.1: Uniform and non-uniform bounds for loss on a tranche of CDO with bankrupt assets

From Table 5.1 and Figure 5.1, observe that a uniform bound steadily declines when n grows but non-uniform bounds have diminished dramatically, especially, when $\tilde{k}$ tends to 1. Moreover, when n is fixed, non-uniform bounds are significantly smaller than the uniform bound.

In the next example, we make use of the situation in Example 3.7. We approximate an average of loss on a tranche of the CDO containing laid-off assets and propose the uniform and non-uniform bounds from the approximation.

Example 5.4. Under the situation in Example 3.7, we have

1. the uniform bound for loss on a tranche of CDO containing laid-off assets is

$$\delta(n) \leq 24.97\beta_{d,3} + 0.8\beta_{d,4}^{1/2} + \left[d\beta_{d,6}\left(3+\beta_{d,4}\right)\right]^{1/2};$$

2. for $k \geq 2$, the non-uniform bound for loss on a tranche of CDO containing laid-off assets is

$$\delta(n,k) \leq C_1(k)\beta_{d,3} + C_2(k)\beta_{d,4}^{1/2} + C_3(k)\left(d\beta_{d,6}\right)^{1/2},$$

where $\quad EW^4 \leq 3 + \beta_{d,4},$

$$EW^6 \leq 15 + \beta_{d,6} + 15\beta_{d,4} + 10\beta_{d,3}^2,$$

$$\beta_{d,r} = \frac{1}{n^r(\operatorname{Var} L(T))^{r/2}}\sum_{i=1}^{d}\left[\sum_{j\in A_i} p_j\left|1-R_j-\sum_{l\in A_i}(1-R_l)p_l\right|^r + (1-p_{A_i})\left(\sum_{j\in A_i}(1-R_j)p_j\right)^r\right]$$

and

$$\operatorname{Var} L(T) = \frac{1}{n^2}\sum_{i=1}^{d}\left[\sum_{j\in A_i}(1-R_j)^2 p_j - \left(\sum_{j\in A_i}(1-R_j)p_j\right)^2\right].$$

The proof of Theoerm 5.4 is completed by applying Example 3.7, Theorem 3.4, Theorem 5.1 and Theorem 5.2.

Next, we compare the bounds by applying Example 5.4 with the following parameters: $d = n/2, p = p_i$ and $m_i - m_{i-1} = 2$. We obtain that

$$\delta(n) \leq \frac{24.97\sqrt{2}(\bar{p}^2+\bar{q}^2)}{\sqrt{n\bar{p}\bar{q}}} + \frac{0.8\sqrt{2(\bar{p}^3+\bar{q}^3)}}{\sqrt{n\bar{p}\bar{q}}} + \sqrt{\frac{2(\bar{p}^5+\bar{q}^5)}{n\bar{p}^2\bar{q}^2}\left(3+\frac{2(\bar{p}^3+\bar{q}^3)}{n\bar{p}\bar{q}}\right)}$$

and

$$\delta(n,k) \leq \frac{\sqrt{2}(\bar{p}^2+\bar{q}^2)C_1(k)}{\sqrt{n\bar{p}\bar{q}}} + \frac{\sqrt{2(\bar{p}^3+\bar{q}^3)}C_2(k)}{\sqrt{n\bar{p}\bar{q}}} + \sqrt{\frac{2(\bar{p}^5+\bar{q}^5)}{n\bar{p}^2\bar{q}^2}}C_3(k),$$

where $C_1(k), C_2(k)$ and $C_3(k)$ are presented in Theorem 5.2,

$$\bar{p} = 2p,$$
$$EW^4 \leq 3 + \frac{2(\bar{p}^3 + \bar{q}^3)}{n\bar{p}\bar{q}}$$
and
$$EW^6 \leq 15 + \frac{4(\bar{p}^5 + \bar{q}^5)}{(n\bar{p}\bar{q})^2} + \frac{30(\bar{p}^3 + \bar{q}^3) + 20(\bar{p}^2 + \bar{q}^2)^2}{n\bar{p}\bar{q}}.$$

Notice that uniform and non-uniform bounds for loss on a tranche of the CDO containing laid-off assets are $\sqrt{\operatorname{Var} L(T)}\delta(n)$ and $\sqrt{\operatorname{Var} L(T)}\delta(n,k)$, respectively. By setting additional parameters $R = 0.7$ and $p = 0.4$, we obtain uniform and non-uniform bounds as follows.

n	Uniform bound	Non-uniform bounds			
		$\tilde{k} = 0.3$	$\tilde{k} = 0.5$	$\tilde{k} = 0.7$	$\tilde{k} = 0.9$
50	0.11622	0.02590	0.01227	0.00804	0.00598
100	0.05648	0.00658	0.00312	0.00204	0.00152
150	0.03718	0.00298	0.00141	0.00093	0.00069
200	0.02768	0.00170	0.00081	0.00053	0.00040

Table 5.2: Uniform and non-uniform bounds for loss on a tranche of the CDO with laid-off assets

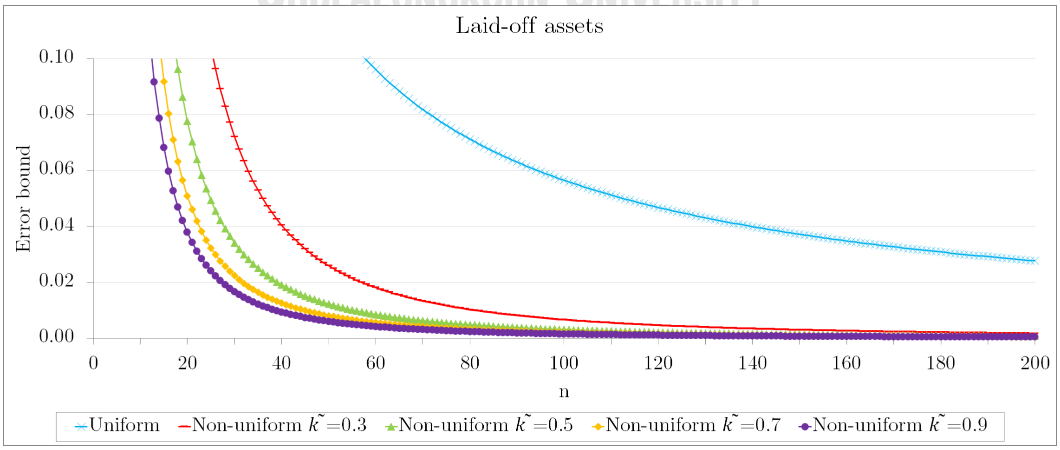


Figure 5.2: Uniform and non-uniform bounds for loss on a tranche of the CDO with laid-off assets

From Table 5.2 and Figure 5.2, we see that when n grows, both uniform and non-uniform bounds are actually declined. For each n, the non-uniform bounds decrease when $\tilde{k}$ goes up and the non-uniform bound is the sharpest bound when $\tilde{k} = 1$. In addition, the non-uniform bound is exactly smaller than the uniform bound when $\tilde{k}$ is only 0.3.

CHAPTER VI
FURTHER RESEARCH

In this dissertation, we concentrate on loss on a tranche of a CDO under some dependent structure. We approximate an average loss on a tranche of a CDO by an average of call function for a standard normal random variable. In addition, uniform and non–uniform bounds for the approximation are proposed. While proving the non–uniform bounds, the sixth moments of W is appeared. Moreover, the rate of convergence of the uniform bound is $\frac{1}{\sqrt{n}}$, while the rate of convergence of the non–uniform bound is $\frac{1}{k\sqrt{n}}$, where k is an attachment or a detachment point for the tranche of a CDO.

Additionally, we present two situations under the disjoint local dependence condition. The first example is a CDO containing bankrupt assets and the second one is a CDO containing laid–off assets. In the CDO containing laid–off assets, we assume that each company can lay off at most one employee.

Therefore, some interesting questions arise for a future research as follows.

1. Although the rate of convergence of the non–uniform bound is $\frac{1}{k\sqrt{n}}$, many terms in the bound have an exponential rate in terms of k. Moreover, each random variable X_i in the scenario of CDO is bounded. Therefore, the question is that "can we refine a non–uniform bound from a polynomial rate, $\frac{1}{k}$, to an exponential rate?"

2. In [7], they focused on a CDO containing independent assets and proposed a correction term that makes the rate of convergence of the bound to be $\frac{1}{n}$. Thus, an interesting question is that "can we improve the rate of convergence by proposing some correction terms?"

3. Can we reduce the sixth moments of W appeared by using the Hölder's

inequality in the non–uniform bound?

4. Can we generalize the condition in the CDO containing laid–off assets to be laying off at most c_i employees in each i^{th} company for positive integer c_i?

REFERENCES

[1] Burtschell, X., Gregory, J.: Beyond the Gaussian copula: Stochastic and local correlation, *J. Credit Risk* **3**, 31–62 (2007).

[2] Burtschell, X., Gregory, J., Laurent, J.P.: A comparative analysis of CDO pricing models under the factor copula framework, *J. Derivatives* **16**, 9–37 (2009).

[3] Buzková, P., Teplý, P.: Collateralized debt obligations valuation using the one factor Gaussian copula model, *Prague Econ. Pap.* **2012**, 30–49 (2012).

[4] Cao, L., Jingqing, Z., Lim, K., Zhao, Z.: An empirical study of pricing and hedging collateralized debt obligation (CDO), *Adv. Econom.* **22**, 15–54 (2008).

[5] Chen, L.H.Y., Goldstein, L., Shao, Q.M.: *Normal Approximation by Stein's Method*, Springer Science & Business Media, New York, 2010.

[6] Chen, L.H.Y., Shao, Q.M.: Normal approximation under local dependence, *Ann. Probab.* **32**, 1985–2028 (2004).

[7] El Karoui, N., Jiao, Y.: Stein's method and zero bias transformation for CDO tranche pricing, *Finance Stoch.* **13**, 151–180 (2009).

[8] El Karoui, N., Jiao, Y., Kurtz, D.: Gaussian and Poisson approximation: application to CDO tranche pricing. *J. Comput. Finance* **2**, 1–28 (2008).

[9] Glasserman, P., Suchintabandid, S., Correlation expansion for CDO pricing, *J. Bank. Finance* **31**, 1375–1398 (2007).

[10] Gregory, J., Laurent, J.P.: Basket default swaps, CDOs and factor copulas, *J. Risk* **7**, 103–122 (2005).

[11] Hull, J., White, A.: Valuation of a CDO and an n^{th} to default CDS without Monte Carlo simulation, *J. Derivatives* **12**, 1–40 (2004).

[12] Jongpreechaharn, S., Neammanee, K.: A constant of approximation of the expectation of call function by the expectation of normal distributed random variable, *Proc. for* 11^{th} *Conf. Sci. Technol. for Youth Year 2016* 1: 112–121 (2006).

[13] Jongpreechaharn, S., Neammanee, K.: Normal approximation for call function via Stein's method, *Commun. Stat. Theory Methods* **48**, 3498–3517 (2019).

[14] Kalemanova, A., Schmid, B., Werner, R.: The normal inverse Gaussian distribution for synthetic CDO pricing, *J. Derivatives* **14**, 80–94 (2007).

[15] Pagés, G., Wilbertz, B.: Dual quantization for random walks with application to credit derivatives, *J. Comput. Finance* **16**, 33–60 (2009).

[16] Stein, C.: A bound for the error in the normal approximation to the distribution of a sum of dependent random variables, *Proc. Sixth Berkeley Symp. Math. Statist. Probab.* **2**, 586–602 (1972).

[17] Stein, C.: *Approximate Computational of Expectations*, Institute of Mathematical Statistics, Hayward, 1986.

[18] Totouom, D., Armstrong, M.: Credit risk dependence modeling with dynamic copula: An application to CDO tranches. *Econom. Risk Manage.* **22**, 85–102 (2015).

[19] Yang, J., Hurd, T., Zhang, X.: Saddlepoint approximation method for pricing CDOs, *Comput. Finance* **10**, 1–24 (2006).

VITA

Miss Suporn Jongpreechaharn was born on May 9th, 1991 in Songkhla, Thailand. She graduated with high school degree from Sadao Khanchai Kamphalanon Anusorn School, Songkhla. She received a Bachelor of Science Program in Mathematics, Department of Mathematics and Statistics, Faculty of Science, Prince of Songkla University in 2013. She received a Master of Science Program in Mathematics, Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University in 2016. She has received scholarship from Human Resource Development in Science Project (Science Achievement Scholarship of Thailand, SAST) since she was studying in Bachelor degree. She further received a Doctor of Philosophy Program in Mathematics in academic year 2019.