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SOME PROPERTIES OF EXTENDING HYPERMODULES,
C_{11}-HYPERMODULES AND t-EXTENDING HYPERRINGS

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ในงานวิจัยนี้ เรานิยามและศึกษาเอกซ์เทนดิงไฮเพอร์มอดูล ซีหนึ่งหนึ่ง-ไฮเพอร์มอดูลและที-เอกซ์เทนดิงไฮเพอร์ริง เราอธิบายลักษณะเอกซ์เทนดิงไฮเพอร์มอดูล ซีหนึ่งหนึ่ง-ไฮเพอร์มอดูลและที-เอกซ์เทนดิงไฮเพอร์ริงในหลายแนวทางภายใต้เงื่อนไขบางประการของไฮเพอร์มอดูลและไฮเพอร์ริง ยิ่งไปกว่านั้น เราศึกษาสมบัติบางประการที่เกี่ยวข้องกับไฮเพอร์มอดูลย่อยของเอกซ์เทนดิงไฮเพอร์มอดูลและซีหนึ่งหนึ่ง-ไฮเพอร์มอดูล โดยเฉพาะอย่างยิ่ง เราศึกษาซีหนึ่งหนึ่ง-ไฮเพอร์มอดูลในกรณีที่ซีหนึ่งหนึ่ง-ไฮเพอร์มอดูลสามารถแยกออกเป็นผลบวกตรงของสองไฮเพอร์มอดูลย่อยได้

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In this research, extending hypermodules, C_{11}-hypermodules and t-extending hyperrings are defined and studied. We characterize extending hypermodules, C_{11}-hypermodules and t-extending hyperrings in many ways under some conditions of hypermodules and hyperrings. Moreover, some properties concerning subhypermodules of extending hypermodules and C_{11}-hypermodules are investigated. Especially, we study C_{11}-hypermodules in the case that they can be decomposed as a direct sum of two subhypermodules.

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CONTENTS

NOTATION

$\mathcal{P}(H)$	the power set of a set H
$\mathcal{P}^*(H)$	the power set of a set H excluding the empty set
$\|H\|$	the cardinality of a set H
$\mathbb{N}$	the set of natural numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{N}_0$	the set of natural numbers including 0
$\|x\|$	the absolute value of a real number x
$\operatorname{ann}(x)$	the set of annihilators of x in an R-hypermodule M
$\operatorname{End}_0(M)$	the set of hypermodule homomorphisms $f : M \to M$ with $f(0) = 0$
$M \cong M'$	R-hypermodules M and M' are isomorphic
$N \leq M$	N is a subhypermodule of an R-hypermodule M
$N \leq_{\oplus} M$	N is a direct summand of an R-hypermodule M
$N \leq_p M$	N is a projection invariant subhypermodule of an R-hypermodule M
$N \leq_{ess} M$	N is an essential subhypermodule of an R-hypermodule M
$N \leq_{cl} M$	N is a closed subhypermodule of an R-hypermodule M
$N \leq_{tess} M$	N is a t-essential subhypermodule of an R-hypermodule M
$N \leq_{tcl} M$	N is a t-closed subhypermodule of an R-hypermodule M
$Z(M)$	the singular subhypermodule of an R-hypermodule M
$Z_2(M)$	the second singular subhypermodule of an R-hypermodule M

CHAPTER I
INTRODUCTION

Extending modules (also known as CS-modules) are an interesting topic in module theory which has been studied for several years. Let R be a ring with identity. According to Tercan and Yücel [14], an R-module M is called an *extending module* if every submodule of M is essential in a direct summand of M. There are many generalizations of extending modules which have been studied by many authors; for examples, Smith et al. [5, 12, 13], Birkenmeier et al. [3, 4] and Asgari et al. [1, 2]. One of generalizations of extending modules is C_{11}-modules which have been investigated by Smith and Tercan [12], Birkenmeier and Tercan [4]. An R-module M is called a C_{11}-*module* if every submodule of M has a complement in M which is a direct summand of M; moreover, a ring R is called a C_{11}-*ring* if R is a C_{11}-module (R is viewed as an R-module). In 2011, Asgari and Haghany [1] provided the concept of t-extending modules which is also another generalization of extending modules. According to Asgari and Haghany [1], an R-module M is called a t-*extending module* if every t-closed submodule of M is a direct sumand of M. In the same way as C_{11}-rings, a ring R is called a t-*extending ring* if R is a t-extending module. Ones can observe that there are many results concerning extending modules, C_{11}-modules, C_{11}-rings and t-extending rings. However, there are few works concerning the concepts of extending modules, C_{11}-modules, C_{11}-rings and t-extending rings by using the structures of hypermodules and hyperrings. In this research, we extend the notions of extending modules, C_{11}-modules, C_{11}-rings and t-extending rings to extending hypermodules, C_{11}-hypermodules, C_{11}-hyperrings and t-extending hyperrings, respectively, which is the main purpose of this research. It is well-known that there are different notions of hyperrings and hypermodules (see [7, 8, 11]). In this research, we focus on hyperrings and hypermodules investigated by Siraworakun [11] in 2012. Furthermore, we would like to mention that although most of properties in this work are similar to those in modules, the important point is to develop tools in hypermodules and hyperrings to prove those properties.

In Chapter I, we introduce canonical hypergroups, hyperrings and hypermodules in Section 1.1, Section 1.2 and Section 1.3, respectively. In addition, their examples are presented in this chapter.

In Chapter II, we first give the concept of direct sums and then introduce hypermodule homomorphisms used in order to define projection invariant subhypermodules. Moreover, some results concerning direct sums, hypermodule homomorphisms and projection invariant subhypermodules are presented in this chapter. Especially, isomorphism theorems for hypermodules are given in Section 2.2.

In Chapter III, the notions of essential subhypermodules, complements, closed subhypermodules, the singular subhypermodule, the second singular subhypermodule, t-essential subhypermodules and t-closed subhypermodules of an R-hypermodule are given. These subhypermodules play important roles in order to define extending hypermodules, C_{11}-hypermodules and t-extending hyperrings in Chapter IV. In addition, some characterizations of closed subhypermodules, the singular subhypermodule and the second singular subhypermodule are provided.

In Chapter IV, we present characterizations of extending hypermodules, C_{11}-hypermodules and t-extending hyperrings in Section 4.1, Section 4.2 and Section 4.3, respectively. Moreover, we give some results concerning C_{11}-hypermodules in the case that they can be decomposed as a direct sum of two subhypermodules in Section 4.2 and also provide some properties of C_{11}-hyperrings in Section 4.3.

1.1 Canonical Hypergroups

In this section, we present the notion of canonical hypergroups (see [6]) introduced by Mittas in 1970 which play an important role in order to define hyperrings and hypermodules in Section 1.2 and Section 1.3, respectively.

For a nonempty set H, let $\mathcal{P}(H)$ denote the power set of H, $\mathcal{P}^*(H) = \mathcal{P}(H)\backslash\{\varnothing\}$ and $|H|$ the cardinality of H.

Definition 1.1.1. [6] A *hyperoperation* on a nonempty set H is a function from $H \times H$ into $\mathcal{P}^*(H)$. A *hypergroupoid* is a pair $(H, \circ)$ of a nonempty set H and a hyperoperation $\circ$ on H.

Let $(H, \circ)$ be a hypergoupoid. For nonempty subsets X and Y of H and $a \in H$, let

$$X \circ Y = \bigcup_{x\in X, y\in Y} x \circ y, \quad X \circ a = X \circ \{a\} \quad \text{and} \quad a \circ X = \{a\} \circ X.$$

A hypergroupoid $(H, \circ)$ is said to be *commutative* if

$$x \circ y = y \circ x \quad \text{for all } x, y \in H.$$

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if

$$x \circ (y \circ z) = (x \circ y) \circ z \quad \text{for all } x, y, z \in H.$$

A semihypergroup $(H, \circ)$ is called a *hypergroup* if

$$H \circ x = x \circ H = H \quad \text{for all } x \in H.$$

Example 1.1.2. [6] Let H be a nonempty set. Define a hyperoperation on H by

$$x \circ y = H \quad \text{for all } x, y \in H.$$

Then $(H, \circ)$ is a hypergroup. This hypergroup is called the *total hypergroup*.

Example 1.1.3. [6] Let G be a group. For $x, y \in G$, define a hyperoperation $\circ$ on G by

$$x \circ y = \langle x, y\rangle, \quad \text{the subgroup of } G \text{ generated by } x \text{ and } y.$$

Then $(G, \circ)$ is a hypergroup.

Example 1.1.4. [6] Let N be a normal subgroup of a group G. Define a hyperoperation $\circ$ on G by

$$x \circ y = xyN \quad \text{for all } x, y \in G.$$

Then $(G, \circ)$ is a hypergroup.

Definition 1.1.5. [6] Let $(H, \circ)$ be a hypergroupoid. An element a in H is called an *identity* of H if $x \in (x \circ a) \bigcap (a \circ x)$ for all $x \in H$. Moreover, an element e in H is called a *scalar identity* of H if $x \circ e = e \circ x = \{x\}$ for all $x \in H$.

In general, an identity of a hypergroupoid may not be unique. For a total hypergroup $(H, \circ)$ with $|H| \geq 2$, it can be seen that every element in H is an identity of H. However, a scalar identity of a hypergroupoid is unique. In fact, if e and e^* are scalar identities of a hypergroupoid $(H, \circ)$, then $\{e\} = e \circ e^* = \{e^*\}$, so $e = e^*$.

Definition 1.1.6. [6] Let $(H, \circ)$ be a hypergroup endowed with at least one identity. An element $x' \in H$ is called an *inverse* of $x \in H$ if there exists an identity a of H such that $a \in (x \circ x') \bigcap (x' \circ x)$.

For a total hypergroup $(H, \circ)$ with $|H| \geq 2$ and $x \in H$, we see that all elements in H are inverses of x. This concludes that an inverse of each element in a hypergroup may not be unique.

Definition 1.1.7. [6] Let $(H, \circ)$ be a hypergroup endowed with at least one identity. Then $(H, \circ)$ is said to be *reversible* if for any $x, y, z \in H$ with $x \in y \circ z$, there exist inverses y' of y and z' of z such that $y \in x \circ z'$ and $z \in y' \circ x$.

Next, we provide the definition of canonical hypergroups which generalize abelian groups. The role of canonical hypergroups in hyperrings and hypermodules is similar to abelian groups in rings and modules, respectively.

Definition 1.1.8. [6] Let $(H, \circ)$ be a hypergroup. Then $(H, \circ)$ is called a *canonical hypergroup* if it satisfies the following properties:

(i) $(H, \circ)$ is commutative;

(ii) $(H, \circ)$ has the scalar identity;

(iii) each element $x \in H$ has a unique inverse, denoted by x^{-1}; and

(iv) $(H, \circ)$ is reversible.

Definition 1.1.9. [6] Let $(H, \circ)$ be a canonical hypergroup. For a nonempty subset X of H, let

$$X^{-1} = \{x^{-1} : x \in X\}.$$

Proposition 1.1.10. [10] *Let $(H, \circ)$ be a canonical hypergroup. Then $(x^{-1})^{-1} = x$ and $(x \circ y)^{-1} = x^{-1} \circ y^{-1}$ for all $x, y \in H$.*

Proposition 1.1.11. [11] *Let $(H, \circ)$ be a canonical hypergroup with the scalar identity 0. Then for all nonempty subsets A, B and C of H,*

(i) $A \circ B = B \circ A$;

(ii) $A \circ \{0\} = A$;

(iii) $(A \circ B) \circ C = A \circ (B \circ C)$; *and*

(iv) $(A \circ B)^{-1} = A^{-1} \circ B^{-1}$.

Next, we provide some examples of canonical hypergroups used in order to establish some examples of hyperrings and hypermodules in Section 1.2 and Section 1.3, respectively.

Example 1.1.12. [11] Let H be a nonempty set with $|H| \geq 2$. Choose an element in H and denote it by 0. Define a hyperoperation $\circ$ on H by, for any $a, b \in H$,

$$a \circ b = \begin{cases} \{a\}, & \text{if } b = 0, \\ \{b\}, & \text{if } a = 0, \\ H, & \text{if } a = b \neq 0, \\ \{a, b\}, & \text{if } a \neq b, a \neq 0 \text{ and } b \neq 0. \end{cases}$$

Then $(H, \circ)$ is a canonical hypergroup with 0 as the scalar identity, and the inverse of each element in H is itself.

Example 1.1.13. [10] Let $t \in \mathbb{R}$ be such that $0 < t \leq 1$ and $M = [0, t]$ or $M = [0, t)$. Define a hyperoperation $\oplus$ on M by, for any $x, y \in M$,

$$x \oplus y = \begin{cases} \{\max\{x, y\}\}, & \text{if } x \neq y, \\ [0, x], & \text{if } x = y. \end{cases}$$

Then $(M, \oplus)$ is a canonical hypergroup with 0 as the scalar identity, and the inverse of each element in M is itself.

Proposition 1.1.14. *Let $(M, \oplus)$ be the canonical hypergroup defined in Example 1.1.13 and $\alpha, \beta \in M$ with $\alpha \leq \beta$. Then $[0, \alpha] \oplus [0, \beta] = [0, \beta]$.*

Proof. By Proposition 1.1.11(i) and (ii), we obtain that $[0, \beta] = \{0\} \oplus [0, \beta] \subseteq [0, \alpha] \oplus [0, \beta]$. To show that $[0, \alpha] \oplus [0, \beta] \subseteq [0, \beta]$, let $\gamma \in [0, \alpha] \oplus [0, \beta]$. Then there exist $\alpha_1 \in [0, \alpha]$ and $\beta_1 \in [0, \beta]$ such that $\gamma \in \alpha_1 \oplus \beta_1$. If $\alpha_1 = \beta_1$, then $\alpha_1 \oplus \beta_1 = [0, \beta_1]$, so $\gamma \in [0, \beta_1] \subseteq [0, \beta]$. Suppose that $\alpha_1 \neq \beta_1$. If $\alpha_1 > \beta_1$, then $\alpha_1 \oplus \beta_1 = \{\max\{\alpha_1, \beta_1\}\} = \{\alpha_1\}$, so $\gamma = \alpha_1 \in [0, \alpha] \subseteq [0, \beta]$ since $\alpha \leq \beta$. Moreover, if $\alpha_1 < \beta_1$, then $\alpha_1 \oplus \beta_1 = \{\max\{\alpha_1, \beta_1\}\} = \{\beta_1\}$ which implies that $\gamma = \beta_1 \in [0, \beta]$. This shows that $[0, \alpha] \oplus [0, \beta] \subseteq [0, \beta]$. Hence, $[0, \alpha] \oplus [0, \beta] = [0, \beta]$. $\square$

Example 1.1.15. [10] Let $a \in \mathbb{R}$ be such that $a \geq 1$ and $R = [a, \infty) \cup \{0\}$ or $R = (a, \infty) \cup \{0\}$. Define a hyperoperation $\oplus$ on R by, for any $x, y \in R$,

$$x \oplus y = \begin{cases} \{y\}, & \text{if } x = 0, \\ \{x\}, & \text{if } y = 0, \\ [x, \infty) \cup \{0\}, & \text{if } x = y \neq 0, \\ \{\min\{x, y\}\}, & \text{if } x \neq y, x \neq 0 \text{ and } y \neq 0. \end{cases}$$

Then $(R, \oplus)$ is a canonical hypergroup with 0 as the scalar identity, and the inverse of each element in R is itself.

Example 1.1.16. [10] Let $a \in \mathbb{R}$ be such that $0 < a \leq 1$ and $R = [-a, a]$ or $R = (-a, a)$. Define a hyperoperation $\oplus$ on R by, for any $x, y \in R$,

$$x \oplus y = \begin{cases} \{x\}, & \text{if } y = x, \\ [-|x|, |x|], & \text{if } y = -x, \\ \{x\}, & \text{if } |y| < |x|, \\ \{y\}, & \text{if } |y| > |x|, \end{cases}$$

where $|z|$ denotes the absolute value of a real number z. Then $(R, \oplus)$ is a canonical hypergroup with 0 as the scalar identity of R, and $-x$ is the inverse of $x \in R$.

1.2 Hyperrings

In general, there are different concepts of hyperrings (see [7]). However, we are mainly interested in hyperrings investigated by Siraworakun [11] in this research. In this section, we provide the concept of hyperrings and their examples. Finally, the concept of quotient hyperrings is introduced. From now on, for a canonical hypergroup $(H, +)$, let 0 and $-a$ denote the scalar identity of H and the inverse of $a \in H$, respectively; moreover, for any $x_1, x_2, \ldots, x_k \in H$ with $k \in \mathbb{N}$, let $\sum_{i=1}^{k} x_i$ denote $x_1 + x_2 + \cdots + x_k$ and for the case $k = 1$, "$z \in \sum_{i=1}^{k} x_i$" represents "$z = x_1$" for all $z \in H$.

Definition 1.2.1. [11] A *hyperring* is a structure $(R, +, \bullet)$ where $+$ and $\bullet$ are hyperoperations on R satisfying the following properties:

(i) $(R, +)$ is a canonical hypergroup;

(ii) $(R, \bullet)$ is a semihypergroup;

(iii) $a \bullet (b + c) \subseteq (a \bullet b) + (a \bullet c)$ and $(b + c) \bullet a \subseteq (b \bullet a) + (c \bullet a)$ for all $a, b, c \in R$; and

(iv) $a \bullet (-b) = (-a) \bullet b = -(a \bullet b)$ for all $a, b \in R$.

If equalities hold in (iii), then the hyperring R is said to be *strongly distributive.*

A hyperring $(R, +, \bullet)$ is said to be *commutative* if $a \bullet b = b \bullet a$ for all $a, b \in R$. For convenience, we sometimes abbreviate a hyperring $(R, +, \bullet)$ by a hyperring R and $a \bullet b$ by ab for all $a, b \in R$; moreover, we abbreviate $A \bullet B$ by AB for all $\varnothing \neq A, B \subseteq R$.

Example 1.2.2. [11] Let R be an abelian group with $|R| \geq 2$. Define a hyperoperation $+$ on R by $a + b = \{ab\}$ for all $a, b \in R$. Moreover, define a hyperoperation

$\bullet$ on R by $a \bullet b = \langle a, b \rangle$, the subgroup of R generated by the set $\{a, b\}$, for all $a, b \in R$. Then $(R, +, \bullet)$ is a hyperring.

Example 1.2.3. [10] Let $(R, +)$ be the canonical hypergroup defined in Example 1.1.15. Then $(R, +, \bullet)$ is a strongly distributive hyperring where $\bullet$ is the hyperoperation on R defined by $a \bullet b = \{a \cdot b\}$ for all $a, b \in R$ ($\cdot$ is the usual multiplication on $\mathbb{R}$).

Proposition 1.2.4. *Let $(R, +)$ be the canonical hypergroup defined in Example 1.1.12 where $R = \mathbb{N} \cup \{0\} := \mathbb{N}_0$. Then $(\mathbb{N}_0, +, \bullet)$ is a hyperring where $\bullet$ is the hyperoperation on $\mathbb{N}_0$ defined by $a \bullet b = \{a \cdot b\}$ for all $a, b \in \mathbb{N}_0$ ($\cdot$ is the usual multiplication on $\mathbb{N}_0$).*

Proof. It is easy to see that $(\mathbb{N}_0, \bullet)$ is a semihypergroup since $(\mathbb{N}_0, \cdot)$ is a semigroup. Note that $-r = r$ for all $r \in \mathbb{N}_0$. This implies that $a \bullet (-b) = (-a) \bullet b = -(a \bullet b)$ for all $a, b \in \mathbb{N}_0$. It remains to show that $a \bullet (b + c) \subseteq (a \bullet b) + (a \bullet c)$ and $(b + c) \bullet a \subseteq (b \bullet a) + (c \bullet a)$ for all $a, b, c \in \mathbb{N}_0$. Let $a, b, c \in \mathbb{N}_0$. First, we show that $a \bullet (b + c) \subseteq (a \bullet b) + (a \bullet c)$.

Case 1: $b = 0$ or $c = 0$. Without loss of generality, assume that $b = 0$. Then $a \bullet (b + c) = a \bullet \{c\} = \{a \cdot c\}$ and $(a \bullet b) + (a \bullet c) = \{0\} + (a \bullet c) = \{a \cdot c\}$.

Case 2: $b \neq 0$ and $c \neq 0$.

Subcase 2.1: $b = c$. Then $b + c = \mathbb{N}_0$. If $a = 0$, then

$$a \bullet (b + c) = 0 \bullet \mathbb{N}_0 = \bigcup_{d \in \mathbb{N}_0} (0 \bullet d) = \{0\} = \{0\} + \{0\} = (a \bullet b) + (a \bullet c).$$

Suppose that $a \neq 0$. Then $a \cdot b = a \cdot c \neq 0$, so $(a \bullet b) + (a \bullet c) = \{a \cdot b\} + \{a \cdot c\} = \mathbb{N}_0$. Therefore,

$$a \bullet (b + c) = a \bullet \mathbb{N}_0 = \bigcup_{d \in \mathbb{N}_0} (a \bullet d) = \bigcup_{d \in \mathbb{N}_0} \{a \cdot d\} = \{a \cdot d : d \in \mathbb{N}_0\} \subseteq \mathbb{N}_0 = (a \bullet b) + (a \bullet c).$$

Subcase 2.2: $b \neq c$. Then $b + c = \{b, c\}$. If $a = 0$, then $a \bullet (b + c) = 0 \bullet \{b, c\} = \{0\} = (a \bullet b) + (a \bullet c)$. In the case $a \neq 0$, we get $a \cdot b, a \cdot c \neq 0$ and

$a \cdot b \neq a \cdot c$. Hence,

$$a \bullet (b+c) = a \bullet \{b,c\} = \{a \cdot b, a \cdot c\} = \{a \cdot b\} + \{a \cdot c\} = (a \bullet b) + (a \bullet c).$$

This shows that $a \bullet (b+c) \subseteq (a \bullet b) + (a \bullet c)$. Since $(\mathbb{N}_0, \bullet)$ is commutative, $(b+c) \bullet a \subseteq (b \bullet a) + (c \bullet a)$. Therefore, $(\mathbb{N}_0, +, \bullet)$ is a hyperring. □

In the hyperring $(\mathbb{N}_0, +, \bullet)$ given in Proposition 1.2.4, one can see that

$$2 \bullet (2+2) = 2 \bullet \mathbb{N}_0 = \bigcup_{n \in \mathbb{N}_0} (2 \bullet n) = \bigcup_{n \in \mathbb{N}_0} \{2 \cdot n\} = 2 \cdot \mathbb{N}_0,$$

but

$$(2 \bullet 2) + (2 \bullet 2) = \{4\} + \{4\} = \mathbb{N}_0$$

which means that $2 \bullet (2+2) \neq (2 \bullet 2) + (2 \bullet 2)$. Hence, the hyperring $(\mathbb{N}_0, +, \bullet)$ is not strongly distributive.

Proposition 1.2.5. [11] *Let A, B and C be nonempty subsets of a hyperring R. The following statements hold.*

(i) $(-A)B = A(-B) = -(AB)$.

(ii) $A(B+C) \subseteq AB + AC$.

(iii) $(A+B)C \subseteq AC + BC$.

Definition 1.2.6. [11] Let R be a hyperring. A nonempty subset I of R is called a *subhyperring* of R if I is a hyperring under the same hyperoperations on R.

Definition 1.2.7. [11] Let I be a subhyperring of a hyperring R. We say that I is a *left* (*right*) *hyperideal* of R if $ra \subseteq I$ $(ar \subseteq I)$ for all $a \in I$ and $r \in R$. Moreover, I is called a *hyperideal* of R if I is both a left and a right hyperideal of R.

Proposition 1.2.8. [11] *Let I be a nonempty subset of a hyperring R. Then I is a left (right) hyperideal of R if and only if $a - b \subseteq I$ and $ra \subseteq I$ $(ar \subseteq I)$ for all $a, b \in I$ and $r \in R$.*

Next, we give some examples of hyperideals of a hyperring.

Proposition 1.2.9. *Let* $(R, +, \bullet)$ *be the hyperring defined in Example 1.2.3 where* $R = [1, \infty) \cup \{0\}$. *Then* $[\alpha, \infty) \cup \{0\}$ *is a hyperideal of* R *for all* $\alpha \geq 1$.

Proof. Let $\alpha \geq 1$. Suppose that $a, b \in [\alpha, \infty) \cup \{0\}$ and $r \in [1, \infty) \cup \{0\}$. Note that $-x = x$ for all $x \in [1, \infty) \cup \{0\}$. Then $a - b = a + b$.

Case 1: $a = 0$ or $b = 0$. Without loss of generality, let $a = 0$. Then $a + b = \{b\} \subseteq [\alpha, \infty) \cup \{0\}$.

Case 2: $a \neq 0$ and $b \neq 0$. If $a = b$, then $a + b = [a, \infty) \cup \{0\} \subseteq [\alpha, \infty) \cup \{0\}$. Assume that $a \neq b$. Then $a + b = \{\min\{a, b\}\} \subseteq [\alpha, \infty) \cup \{0\}$. This shows that $a - b \subseteq [\alpha, \infty) \cup \{0\}$.

Next, we show that $r \bullet a \subseteq [\alpha, \infty) \cup \{0\}$. If $r = 0$ or $a = 0$, then $r \bullet a = \{ra\} = \{0\} \subseteq [\alpha, \infty) \cup \{0\}$. If $r \geq 1$ and $a \neq 0$, then $ra \geq a$, so that $r \bullet a = \{ra\} \subseteq [ra, \infty) \cup \{0\} \subseteq [a, \infty) \cup \{0\} \subseteq [\alpha, \infty) \cup \{0\}$. Since R is commutative, $r \bullet a = a \bullet r$. By Proposition 1.2.8, we conclude that $[\alpha, \infty) \cup \{0\}$ is a hyperideal of R. $\square$

Proposition 1.2.10. *Let* $(\mathbb{N}_0, +, \bullet)$ *be the hyperring defined in Proposition 1.2.4. Then only* $\{0\}$ *and* $\mathbb{N}_0$ *are hyperideals of* $\mathbb{N}_0$.

Proof. It is clear that $0 - 0 = \{0\}$ and $r \bullet 0 = \{0\} = 0 \bullet r$ for all $r \in \mathbb{N}_0$. Hence, $\{0\}$ is a hyperideal of $\mathbb{N}_0$ by Proposition 1.2.8. Moreover, assume that I is a nonzero hyperideal of $\mathbb{N}_0$. Let $0 \neq a \in I$. Thus, $\mathbb{N}_0 = a + a = a - a \subseteq I \subseteq \mathbb{N}_0$. This implies that $I = \mathbb{N}_0$. $\square$

Proposition 1.2.11. [11] *Let* I *and* J *be left (right) hyperideals of a hyperring* R. *Then* $I + J$ *and* $I \cap J$ *are also left (right) hyperideals of* R.

Corollary 1.2.12. *Let* I *and* J *be hyperideals of a hyperring* R. *Then* $I + J$ *and* $I \cap J$ *are also hyperideals of* R.

Next, we provide the concept of quotient hyperrings (see [11]) established by Siraworakun in 2012. Let P be a hyperideal of a hyperring R. Then the relation ρ on R defined as follows:

$$a\rho b \text{ if and only if } a + P = b + P \quad \text{for all } a, b \in R$$

is an equivalence relation. The set of equivalence classes of elements in R is denoted by R/P, i.e., $R/P = \big\{[a]_\rho : a \in R\big\}$ where $[a]_\rho$ is the equivalence class of $a \in R$. According to Siraworakun [11], it can be shown that $R/P = \{a + P : a \in R\}$ and $a \in b + P$ if and only if $a + P = b + P$ for all $a, b \in R$; moreover, he verified that $(R/P, \oplus, \otimes)$ is a hyperring where $\oplus$ and $\otimes$ are hyperoperations on R/P defined by, for all $a, b \in R$,

$$(a + P) \oplus (b + P) = \{x + P : x \in a + b\} \text{ and } (a + P) \otimes (b + P) = \{y + P : y \in ab\}.$$

In addition, P is the scalar identity of $(R/P, \oplus)$, and $(-r) + P$ is the inverse of $r + P \in R/P$. The hyperring $(R/P, \oplus, \otimes)$ is called the *quotient hyperring.*

1.3 Hypermodules

It is similar to hyperrings that there are several types of hypermodules (see [8]). However, we only focus on hypermodules investigated by Siraworakun [11] in 2012. In this section, we introduce the definition of hypermodules and their examples; moreover, some preliminary properties involving hypermodules are provided. Finally, the concept of quotient hypermodules are presented.

Definition 1.3.1. [11] Let $(R, +, \bullet)$ be a hyperring. An *R-hypermodule* is a structure $(M, \oplus, \diamond)$ such that $(M, \oplus)$ is a canonical hypergroup, and $\diamond$ is a *multivalued scalar operation*, i.e., a function from $R \times M$ into $\mathcal{P}^*(M)$, such that for all $a, b \in R$ and $x, y \in M$:

(i) $a \diamond (x \oplus y) \subseteq (a \diamond x) \oplus (a \diamond y)$;

(ii) $(a + b) \diamond x \subseteq (a \diamond x) \oplus (b \diamond x)$;

(iii) $(a \bullet b) \diamond x = a \diamond (b \diamond x)$; and

(iv) $a \diamond (-x) = (-a) \diamond x = -(a \diamond x)$.

If equalities hold in both (i) and (ii), then the R-hypermodule is said to be *strongly distributive.*

For convenience, we sometimes abbreviate an R-hypermodule $(M, \oplus, \diamond)$ by an R-hypermodule M, denoted by ${}_RM$, and $a \diamond x$ by ax for all $a \in R$ and $x \in M$. Form now on, for an R-hypermodule M, we use the symbol $+$ for the canonical hypergroups $(R, +)$ and $(M, +)$ in the hyperring R and the R-hypermodule M, respectively; however, they are not the same unless we specify. Moreover, it is clear that a hyperring $(R, +, \bullet)$ can be viewed as an R-hypermodule by considering the hyperoperation $\bullet$ as the multivalued scalar operation. Let M be an R-hypermodule. For $\varnothing \neq A \subseteq R$, $\varnothing \neq X \subseteq M, r \in R$ and $y \in M$, let

$$AX = \bigcup_{a \in A, x \in X} ax, \quad Ay = A\{y\} \quad \text{and} \quad rX = \{r\}X.$$

Proposition 1.3.2. [11] *Let M be an R-hypermodule. Then for any nonempty subsets A and B of R and nonempty subsets X and Y of M:*

(i) $A(X + Y) \subseteq AX + AY$;

(ii) $(A + B)X \subseteq AX + BX$;

(iii) $(AB)X = A(BX)$; *and*

(iv) $A(-X) = (-A)X = -(AX)$.

Definition 1.3.3. [11] A nonempty subset N of an R-hypermodule M is called a *subhypermodule* of M, denoted by $N \leq M$, if N is an R-hypermodule under the same hyperoperation on M and the multivalued scalar operation.

Proposition 1.3.4. [11] *Let N be a nonempty subset of an R-hypermodule M. Then N is a subhypermodule of M if and only if $x - y \subseteq N$ and $rx \subseteq N$ for all $x, y \in N$ and $r \in R$.*

For a hyperring R, if we view R as an R-hypermodule, then subhypermodules of R and left hyperideals of R are identical; moreover, in the case that the hyperring R is commutative, we obtain that subhypermodules of R and hyperideals of R coincide.

Example 1.3.5. Let $(R, +, \bullet)$ be the hyperring defined in Example 1.2.2. If $(R, +, \bullet)$ is viewed as an R-hypermodule by considering $\bullet$ as the multivalued scalar operation, then only R is the subhypermodule of itself.

As the previous example, only R is the subhypermodule of ${}_RR$ and $|R| > 1$, so $\{0\}$ is not a subhypermodule of ${}_RR$. This means that $\{0\}$ may not be a subhypermodule in general. However, we focus on hypermodules which $\{0\}$ must be a subhypermodule throughout this research. Such an R-hypermodule exists as the following proposition.

Proposition 1.3.6. *Let $(R, \oplus, \bullet)$ be the hyperring defined in Example 1.2.3 where $R = [s, \infty) \cup \{0\}$ with $s \geq 1$ and let $(M, +)$ be the canonical hypergroup defined in Example 1.1.13 where $M = [0, t]$ with $0 < t \leq 1$. Define a multivalued scalar operation $\diamond$ by, for any $a \in R$ and $x \in M$,*

$$a \diamond x = \begin{cases} \{0\}, & \text{if } a = 0, \\ [0, \frac{x}{a}], & \text{if } a \neq 0. \end{cases}$$

Then $(M, +, \diamond)$ is a strongly distributive R-hypermodule and $\{0\}$ is a subhypermodule of M.

Proof. Let $a, b \in R$ and $x, y \in M$. First, we show that $a \diamond (x+y) = (a \diamond x) + (a \diamond y)$. If $a = 0$, then $a \diamond x = \{0\} = a \diamond y$, so

$$a \diamond (x + y) = \bigcup_{z \in x+y} (0 \diamond z) = \{0\} = \{0\} + \{0\} = (a \diamond x) + (a \diamond y).$$

Suppose that $a \geq s$.

Case 1: $x \neq y$. Without loss of generality, assume that $x < y$. Then

$$a \diamond (x + y) = a \diamond \{\max\{x, y\}\} = a \diamond \{y\} = [0, \frac{y}{a}].$$

Note that $\frac{x}{a} < \frac{y}{a}$. By Proposition 1.1.14,

$$(a \diamond x) + (a \diamond y) = [0, \frac{x}{a}] + [0, \frac{y}{a}] = [0, \frac{y}{a}].$$

Case 2: $x = y$. Then $x + y = [0, x]$. Thus,

$$a \diamond (x + y) = a \diamond [0, x] = \bigcup_{z \in [0,x]} (a \diamond z) = \bigcup_{z \in [0,x]} [0, \frac{z}{a}] = [0, \frac{x}{a}].$$

By Proposition 1.1.14, $(a \diamond x) + (a \diamond y) = [0, \frac{x}{a}] + [0, \frac{x}{a}] = [0, \frac{x}{a}]$. This shows that $a \diamond (x + y) = (a \diamond x) + (a \diamond y)$.

Next, we show that $(a \oplus b) \diamond x = (a \diamond x) + (b \diamond x)$.

Case 1: $a = 0$ and $b = 0$. Hence,

$$(a \oplus b) \diamond x = \{0\} = \{0\} + \{0\} = (a \diamond x) + (b \diamond x).$$

Case 2: $a = 0$ and $b \geq s$. In this case, we obtain

$$(a \oplus b) \diamond x = \{b\} \diamond x = [0, \frac{x}{b}] = \{0\} + [0, \frac{x}{b}] = (a \diamond x) + (b \diamond x).$$

Case 3: $a \geq s$ and $b = 0$. This case is similar to Case 2.

Case 4: $a \geq s$ and $b \geq s$.

Subcase 4.1: $a = b$. In this case, we get $a \oplus b = [a, \infty) \cup \{0\}$. Hence,

$$(a \oplus b) \diamond x = \big(\{0\} \cup [a, \infty)\big) \diamond x = \{0\} \cup \Big(\bigcup_{d \in [a,\infty)} (d \diamond x)\Big)$$

$$= \cup\Big(\bigcup_{d \in [a,\infty)} [0, \frac{x}{d}]\Big) = [0, \frac{x}{a}].$$

By Proposition 1.1.14,

$$(a \diamond x) + (b \diamond x) = (a \diamond x) + (a \diamond x) = [0, \frac{x}{a}] + [0, \frac{x}{a}] = [0, \frac{x}{a}].$$

Subcase 4.2: $a \neq b$. Without loss of generality, assume that $a < b$. Then $a \oplus b = \{\min\{a, b\}\} = \{a\}$. Thus, $(a \oplus b) \diamond x = \{a\} \diamond x = [0, \frac{x}{a}]$. Note that $\frac{x}{b} \leq \frac{x}{a}$. By Proposition 1.1.14, $(a \diamond x) + (b \diamond x) = [0, \frac{x}{a}] + [0, \frac{x}{b}] = [0, \frac{x}{a}]$. This shows that $(a \oplus b) \diamond x = (a \diamond x) + (b \diamond x)$.

Next, we show that $(a \bullet b) \diamond x = a \diamond (b \diamond x)$. If $a = 0$ or $b = 0$, then $(a \bullet b) \diamond x = \{0\} = a \diamond (b \diamond x)$. Suppose that $a \geq s$ and $b \geq s$. Then $ab \neq 0$ since $s \geq 1$. Therefore,

$(a \bullet b) \diamond x = \{ab\} \diamond x = [0, \dfrac{x}{ab}]$ and

$$a \diamond (b \diamond x) = a \diamond [0, \frac{x}{b}] = \bigcup_{z \in [0, \frac{x}{b}]} (a \diamond z) = \bigcup_{z \in [0, \frac{x}{b}]} [0, \frac{z}{a}] = [0, \frac{x}{ab}].$$

This shows that $(a \bullet b) \diamond x = a \diamond (b \diamond x)$.

Finally, we show that $a \diamond (-x) = (-a) \diamond x = -(a \diamond x)$. Note that $-c = c$ for all $c \in R$ and $-z = z$ for all $z \in M$. Hence, $a \diamond (-x) = a \diamond x = (-a) \diamond x$. Moreover, $-(a \diamond x) = \{-z : z \in a \diamond x\} = \{z : z \in a \diamond x\} = a \diamond x$. This implies that $a \diamond (-x) = (-a) \diamond x = -(a \diamond x)$.

We conclude that $(M, +, \diamond)$ is a strongly distributive R-hypermodule. Note that $0 - 0 = \{0\}$. Moreover, for any $r \in R$, if $r = 0$, then $r \diamond 0 = \{0\}$. Note that $r \diamond 0 = [0, \dfrac{0}{r}] = \{0\}$ in the case $r \neq 0$. Therefore, $\{0\}$ is a subhypermodule of M by Proposition 1.3.4. □

From now on, only R-hypermodules such that $\{0\}$ is a subhypermodule are considered.

Proposition 1.3.7. *All subhypermodules of the R-hypermodule $M = [0, t]$ $(0 < t \leq 1)$ defined in Proposition 1.3.6 are $\{0\}, [0, x]$ and $[0, x)$ for some $x \in (0, t]$.*

Proof. Recall that $\{0\}$ is a subhypermodule of M by Proposition 1.3.6. Let N be a nonzero subhypermodule of M. Then N is nonempty and bounded above. This implies that sup$N$ exists, say x. It follows that $x \in (0, t]$ because $\{0\} \neq N \subseteq [0, t]$.

Case 1: $x \in N$. In this case, we claim that $N = [0, x]$. It is obvious that $N \subseteq [0, x]$. Moreover, $[0, x] = x + x \subseteq N$. Hence, $N = [0, x]$.

Case 2: $x \notin N$. Claim that $N = [0, x)$. Clearly, $N \subseteq [0, x)$ since $x =$ sup$N$ and $x \notin N$. To show that $[0, x) \subseteq N$, let $\alpha \in [0, x)$. Then $\alpha < x$. Thus, there exists $y \in N$ such that $\alpha < y < x$. Therefore, $\alpha \in [0, y] = y + y \subseteq N$ since $N \leq M$. This shows that $N = [0, x)$.

Conversely, recall that $R = [s, \infty) \cup \{0\}$ $(s \geq 1)$. Let $z \in (0, t]$. To show that $[0, z]$ is a subhypermodule of M, let $a, b \in [0, z]$ and $r \in R = [s, \infty) \cup \{0\}$. Without loss of generality, assume that $a \leq b$. By Proposition 1.1.14, we obtain that $[0, a] + [0, b] = [0, b]$. Hence,

$$a - b = a + (-b) = a + b \subseteq [0, a] + [0, b] = [0, b] \subseteq [0, z].$$

If $r = 0$, then $ra = \{0\} \subseteq [0, z]$. Moreover, $ra = [0, \frac{a}{r}] \subseteq [0, a] \subseteq [0, z]$ in the case $r \geq s \geq 1$. Therefore, $[0, z]$ is a subhypermodule of M by Proposition 1.3.4. Similarly, $[0, z)$ is a subhypermodule of M. □

Example 1.3.8. Let $(\mathbb{N}_0, +, \bullet)$ be the hyperring defined in Proposition 1.2.4. Consider $\mathbb{N}_0$ as an $\mathbb{N}_0$-hypermodule whose multivalued scalar operation is the hyperoperation $\bullet$. By Proposition 1.2.10, only $\{0\}$ and $\mathbb{N}_0$ are hyperideals of $\mathbb{N}_0$. Therefore, there are only two subhypermodules of $\mathbb{N}_0$, namely $\{0\}$ and $\mathbb{N}_0$.

By considering $\mathbb{N}_0$ as an $\mathbb{N}_0$-hypermodule in the previous example, we observe that $\mathbb{N}_0 2 = \bigcup_{n \in \mathbb{N}_0} n \bullet 2 = \bigcup_{n \in \mathbb{N}_0} \{n \cdot 2\} = \{n \cdot 2 : n \in \mathbb{N}_0\}$ is not a subhypermodule of $\mathbb{N}_0$ since only $\{0\}$ and $\mathbb{N}_0$ are subhypermodules of $\mathbb{N}_0$. Hence, for an R-hypermodule M, $m \in M$ and a left hyperideal I of R, $Im = \{y \in am : a \in I\}$ may not be a subhypermodule of M; however, Im is always a subhypermodule of M provided that M is strongly distributive as shown in the following proposition.

Proposition 1.3.9. *Let M be a strongly distributive R-hypermodule, I a left hyperideal of R and $m \in M$. Then Im is a subhypermodule of M.*

Proof. Note that $\varnothing \neq 0m \subseteq Im$, so $Im \neq \varnothing$. To show that $Im \leq M$, let $x, y \in Im$ and $r \in R$. Then there exist $a_1, a_2 \in I$ such that $x \in a_1 m$ and $y \in a_2 m$. Since M is strongly distributive, $a_1 m + (-a_2)m = (a_1 + (-a_2))m$. Therefore,

$$a_1 m - a_2 m = a_1 m + (-(a_2 m)) = a_1 m + (-a_2)m = (a_1 + (-a_2))m = (a_1 - a_2)m.$$

Note that $a_1 - a_2 \subseteq I$ since I is a left hyperideal of R. Hence,

$$x - y \subseteq a_1 m - a_2 m = (a_1 - a_2)m \subseteq Im.$$

Moreover, $rx \subseteq r(a_1 m) = (ra_1)m \subseteq Im$. We conclude that $Im \leq M$ by Proposition 1.3.4. □

In order to define the singular subhypermodule in Section 3.3, the following definition is needed.

Definition 1.3.10. Let M be an R-hypermodule and $x \in M$. An element r in R is called an *annihilator* of x if $rx = \{0\}$. The set of all annihilators of x is denoted by $\operatorname{ann}(x)$, i.e.,

$$\operatorname{ann}(x) = \big\{r \in R : rx = \{0\}\big\}.$$

Example 1.3.11. Let $(R, +, \bullet)$ be the hyperring defined in Example 1.2.2. Consider R as an R-hypermodule, and let $0 \neq x \in R$. Then $r \bullet x = \langle r, x\rangle \neq \{0\}$ for all $r \in R$. Hence, $\operatorname{ann}(x) = \varnothing$.

In genaral, for an R-hypermodule M and $x \in M$, we see that $\operatorname{ann}(x)$ may be the empty set as in the previous example; however, if $\operatorname{ann}(x)$ is nonempty, then it forms a left hyperideal of R. Note that $\{0\}$ is a subhypermodule. Hence, for an R-hypermodule M and $r \in R$, we obtain $r\{0\} \subseteq \{0\}$ from Proposition 1.3.4 but $r\{0\} \neq \varnothing$, so $r\{0\} = \{0\}$.

Proposition 1.3.12. *Let M be an R-hypermodule and $x \in M$. If* $\operatorname{ann}(x)$ *is nonempty, then* $\operatorname{ann}(x)$ *is a left hyperideal of R.*

Proof. Assume that $\operatorname{ann}(x) \neq \varnothing$. Let $a, b \in \operatorname{ann}(x)$ and $r \in R$. Then $ax = \{0\} = bx$. To show that $a - b \subseteq \operatorname{ann}(x)$, let $c \in a - b$. Thus, $cx \subseteq (a - b)x \subseteq ax - bx = \{0\} - \{0\} = \{0\}$. This implies that $cx = \{0\}$, i.e., $c \in \operatorname{ann}(x)$. Hence, $a - b \subseteq \operatorname{ann}(x)$. Next, let $d \in ra$. Then $dx \subseteq (ra)x = r(ax) = r\{0\} = \{0\}$. This forces that $dx = \{0\}$. Thus, $d \in \operatorname{ann}(x)$. This shows that $ra \subseteq \operatorname{ann}(x)$. By Proposition 1.2.8, we conclude that $\operatorname{ann}(x)$ is a left hyperideal of R. □

Proposition 1.3.13. [11] *Let K and N be subhypermodules of an R-hypermodule M. Then $K + N$ and $K \cap N$ are subhypermodules of M.*

Recall that for subhypermodules K and N of an R-hypermodule M,

$$K + N = \bigcup_{x \in K, y \in N} x + y = \big\{z \in M : \exists x \in K \exists y \in N, z \in x + y\big\}.$$

Therefore, for any subhypermodules $N_1, N_2, \ldots, N_k$ of an R-hypermodule M with $k \in \mathbb{N}$, we can define $\sum_{i=1}^{k} N_i$ to be $N_1 + N_2 + \cdots + N_k$, i.e.,

$$\sum_{i=1}^{k} N_i = N_1+N_2+\cdots+N_k = \{x \in M : \exists n_1 \in N_1 \exists n_2 \in N_2 \ldots \exists n_k \in N_k, x \in \sum_{i=1}^{k} n_i\}.$$

Corollary 1.3.14. *Let $N_1, N_2, \ldots, N_k$ be subhypermodules of an R-hypermodule M where $k \in \mathbb{N}$. Then $\sum_{i=1}^{k} N_i$ and $\bigcap_{i=1}^{k} N_i$ are subhypermodules of M.*

Next, we give a proposition which is similar to the modularity condition in module theory. The proof of this proposition is straightforward, but it is quite important to our works because we can transform between the sum of subhypermodules and the intersection of subhypermodules.

Proposition 1.3.15. (Modularity Condition) *Let H, K and L be subhypermodules of an R-hypermodule M such that $K \subseteq H$. Then $H \cap (K + L) = K + (H \cap L)$.*

Proof. Let $x \in H \cap (K + L)$. Then $x \in H$ and there exist $k \in K$ and $l \in L$ such that $x \in k+l$. Thus, $l \in x-k \subseteq H$. This means that $l \in H \cap L$, so $x \in K+(H \cap L)$. Hence, $H \cap (K + L) \subseteq K + (H \cap L)$. Next, $K + (H \cap L) \subseteq H + (H \cap L) \subseteq H$ because $K \subseteq H$. Clearly, $K+(H \cap L) \subseteq K+L$. Then $K+(H \cap L) \subseteq H \cap (K+L)$. Therefore, $H \cap (K + L) = K + (H \cap L)$. □

Next, the concept of quotient hypermodules investigated by Siraworakun [11] is presented.

Let N be a subhypermodule of an R-hypermodule M. Siraworakun defined M/N to be the set $\{x + N : x \in M\}$ and proved that $x \in y + N$ if and only if $x + N = y + N$ for all $x, y \in M$. Especially, $x + N = N$ if and only if $x \in N$; moreover, he proved that $(M/N, \boxplus, \boxdot)$ is an R-hypermodule where $\boxplus$ is the hyperoperation on M/N and $\boxdot$ is the multivalued scalar operation defined by

$$(x + N) \boxplus (y + N) = \{t + N : t \in x + y\} \text{ and } r \boxdot (x + N) = \{t + N : t \in rx\}$$

for all $x, y \in M$ and $r \in R$. The scalar identity of $(M/N, \boxplus)$ is N (we sometimes use the symbol $\bar{0}$ instead), and $(-x) + N$ is the inverse of $x + N \in M/N$, i.e.,

$-(x+N) = (-x)+N$ for all $x \in M$. The R-hypermodule $(M/N, \boxplus, \boxdot)$ is called the *quotient R-hypermodule.*

Furthermore, Siraworakun provided the form of subhypermodules of quotient hypermodules as follows.

Proposition 1.3.16. [11] *Let N be a subhypermodule of an R-hypermodule M. Then every subhypermodule of M/N is in the form K/N, where K is a subhypermodule of M containing N.*

Next, we give a result concerning the uniqueness of subhypermodules of quotient hypermodules in the previous proposition.

Proposition 1.3.17. *Let N be a subhypermodule of an R-hypermodule M and $\tilde{K}$ a subhypermodule of M/N. Then there exists uniquely $K \leq M$ containing N such that $\tilde{K} = K/N$.*

Proof. By Proposition 1.3.16, there exists $K \leq M$ containing N such that $\tilde{K} = K/N$. Let K' be a subhypermodule of M containing N such that $K/N = K'/N$. Suppose that $k \in K$. Then $k+N \in K/N = K'/N$. Thus, there exists $k' \in K'$ such that $k+N = k'+N$. Then $k \in k'+N \subseteq k'+K' = K'$. Hence, $K \subseteq K'$. Similarly, $K' \subseteq K$. Therefore, $K = K'$. □

Let $(M/N, \boxplus, \boxdot)$ be the quotient hypermodule and let N' be a subhypermodule of M containing N. We can define a hyperoperation $\uplus$ on $(M/N)/(N'/N)$ and a multivalued scalar operation $\circledast$ by

$$[(x+N)\boxplus(N'/N)]\uplus[(y+N)\boxplus(N'/N)] = \{(t+N)\boxplus(N'/N) : t+N \in (x+N)\boxplus(y+N)\},$$

$$r \circledast ((x+N) \boxplus (N'/N)) = \{(t+N) \boxplus (N'/N) : t+N \in r \boxdot (x+N)\}$$

for all $x, y \in M$ and $r \in R$. Then $((M/N)/(N'/N), \uplus, \circledast)$ is also the quotient R-hypermodule.

In this thesis, for any subhypermodules N and N' of an R-hypermodule M with $N' \leq N$, we use the symbols $\boxplus$ and $\boxdot$ for the quotient R-hypermodule M/N

throughout this work; moreover, the symbols $\uplus$ and $\circledast$ are used for the quotient R-hypermodule $(M/N)/(N'/N)$.

We end this section with some results regarding annihilators of an element in quotient hypermodules.

Proposition 1.3.18. *Let $(M/N, \boxplus, \boxdot)$ be the quotient R-hypermodule and $x \in M$, $\varnothing \neq I \subseteq R$. Then $I \boxdot (x + N) = \{N\}$ if and only if $Ix \subseteq N$. In particular,*

$$\operatorname{ann}(x+N) \;=\; \{r \in R : r \boxdot (x+N) = \{N\}\} = \{r \in R : rx \subseteq N\}.$$

Proof. By the definition of quotient hypermodules, we obtain that

$$\begin{aligned} I \boxdot (x+N) = \bigcup_{r \in I} r \boxdot (x+N) = \bigcup_{r \in I} \{t + N : t \in rx\} \\ = \{t + N : t \in \bigcup_{r \in I} rx\} = \{t + N : t \in Ix\}. \end{aligned}$$

It is straightforward that $\{t + N : t \in Ix\} = \{N\}$ if and only if $Ix \subseteq N$.

□

Proposition 1.3.19. *Let K and N be subhypermodules of an R-hypermodule M such that $K \cap N = \{0\}$. Then* $\operatorname{ann}(k+N) = \operatorname{ann}(k)$, *i.e.,* $\{r \in R : rk \subseteq N\} = \{r \in R : rk = \{0\}\}$ *for all $k \in K$.*

Proof. Let $k \in K$. If $r \in \operatorname{ann}(k)$, then $rk = \{0\} \subseteq N$, so $r \in \operatorname{ann}(k+N)$. This implies that $\operatorname{ann}(k) \subseteq \operatorname{ann}(k+N)$. Next, let $s \in \operatorname{ann}(k+N)$. Then $sk \subseteq N$. Since $K \leq M$, we obtain $sk \subseteq K$. Thus, $sk \subseteq K \cap N = \{0\}$. This forces that $sk = \{0\}$. This means that $s \in \operatorname{ann}(k)$. Hence, $\operatorname{ann}(k+N) \subseteq \operatorname{ann}(k)$. We conclude that $\operatorname{ann}(k+N) = \operatorname{ann}(k)$. □

CHAPTER II
SUBHYPERMODULES AND HOMOMORPHISMS

2.1 Direct Sums and Projection Invariant Subhypermodules

In this section, we first explore the concepts of direct sums of subhypermodules and direct summands of hypermodules which lead us to define extending hypermodules, C_{11}-hypermodules and t-extending hyperrings in Chapter IV.

Definition 2.1.1. Let $N_1, N_2, \ldots, N_k$ be subhypermodules of an R-hypermodule M where $k \in \mathbb{N}$ with $k \geq 2$. Then M is called the *direct sum* of $N_1, N_2, \ldots, N_k$, denoted by $M = \bigoplus_{i=1}^{k} N_i$ or $M = N_1 \oplus N_2 \oplus \cdots \oplus N_k$, if $M = \sum_{i=1}^{k} N_i$ and $N_j \cap (\sum_{\substack{i=1 \\ i\neq j}}^{k} N_i) = \{0\}$ for all $j \in \{1, 2, \ldots, k\}$.

Definition 2.1.2. Let N be a subhypermodule of an R-hypermodule M. We say that N is a *direct summand* of M, denoted by $N \leq_{\oplus} M$, if there exists $N' \leq M$ such that $M = N \oplus N'$.

In module theory, if a module M is the direct sum of submodules $N_1, N_2, \ldots,$ N_k, then every element in M can be written uniquely as a sum of elements in $N_1, N_2, \ldots, N_k$. In hypermodules, the uniqueness concerning elements in direct sums is presented as follows.

Proposition 2.1.3. *Let $N_1, N_2, \ldots, N_k$ be subhypermodules of an R-hypermodule M such that $M = \sum_{i=1}^{k} N_i$ where $k \in \mathbb{N}$ with $k \geq 2$. The following statements are equivalent:*

(i) $N_j \cap (\sum_{\substack{i=1 \\ i\neq j}}^{k} N_i) = \{0\}$ *for each* $j \in \{1, 2, \ldots, k\}$;

(ii) *for each* $x \in M$, *there exist uniquely* $n_1 \in N_1, n_2 \in N_2, \dots, n_k \in N_k$ *such that* $x \in \sum_{i=1}^{k} n_i$;

(iii) *for any* $n_1 \in N_1, n_2 \in N_2, \dots, n_k \in N_k$, *if* $0 \in \sum_{i=1}^{k} n_i$, *then* $n_i = 0$ *for all* $i \in \{1, 2, \dots, k\}$.

Proof. (i)⇒(ii) Assume that (i) holds. Let $x \in M$. Since $M = \sum_{i=1}^{k} N_i$, there exist $n_1 \in N_1, n_2 \in N_2, \dots, n_k \in N_k$ such that $x \in \sum_{i=1}^{k} n_i$. Assume that there exist $m_1 \in N_1, m_2 \in N_2, \dots, m_k \in N_k$ such that $x \in \sum_{i=1}^{k} m_i$. Let $j \in \{1, 2, \dots, k\}$. Since $x \in \sum_{i=1}^{k} n_i$ and $x \in \sum_{i=1}^{k} m_i$, we can write $x \in n_j + y$ and $x \in m_j + z$ for some $y \in \sum_{\substack{i=1 \\ i \neq j}}^{k} n_i$ and $z \in \sum_{\substack{i=1 \\ i \neq j}}^{k} m_i$, respectively. Thus,

$$n_j \in x - y \subseteq (m_j + z) - y = m_j + (z - y) \subseteq m_j + (\textstyle\sum_{\substack{i=1 \\ i \neq j}}^{k} N_i).$$

This means that $n_j \in m_j + a$ for some $a \in \sum_{\substack{i=1 \\ i \neq j}}^{k} N_i$. Then $a \in n_j - m_j \subseteq N_j$. By the assumption, $a \in N_j \cap (\sum_{\substack{i=1 \\ i \neq j}}^{k} N_i) = \{0\}$, so $a = 0$. Hence, $n_j \in m_j + 0 = \{m_j\}$, i.e., $n_j = m_j$.

(ii)⇒(iii) Assume that (ii) holds. Let $n_1 \in N_1, n_2 \in N_2, \dots, n_k \in N_k$ be such that $0 \in \sum_{i=1}^{k} n_i$. Moreover, note that $0 \in 0 + \cdots + 0$. By the assumption, $n_i = 0$ for all $i \in \{1, 2, \dots, k\}$.

(iii)⇒(i) Assume that (iii) holds. Let $j \in \{1, 2, \dots, k\}$ and $x \in N_j \cap (\sum_{\substack{i=1 \\ i \neq j}}^{k} N_i)$. Then $x \in N_j$ and there exist $n_1 \in N_1, \dots, n_{j-1} \in N_{j-1}, n_{j+1} \in N_{j+1}, \dots, n_k \in N_k$ such that $x \in \sum_{\substack{i=1 \\ i \neq j}}^{k} n_i$. Because $N_j \leq M$, it follows that $-x \in N_j$. Hence,

$$0 \in x - x \subseteq n_1 + \cdots + n_{j-1} + (-x) + n_{j+1} + \cdots + n_k \in \textstyle\sum_{i=1}^{k} N_i.$$

By the assumption, $-x = 0$, so $x = 0$. Therefore, $N_j \cap (\sum_{\substack{i=1 \\ i \neq j}}^{k} N_i) = \{0\}$. □

Next, some elementary properties of direct sums are provided.

Proposition 2.1.4. *Let* K_1, K_2, N_1 *and* N_2 *be subhypermodules of an* R*-hypermodule* M *such that* $K_1 \leq N_1$ *and* $K_2 \leq N_2$. *If* $N_1 \cap N_2 = \{0\}$, *then* $K_1 \oplus K_2 = (K_1 \oplus N_2) \cap (N_1 \oplus K_2)$.

Proof. Assume that $N_1 \cap N_2 = \{0\}$. Then $K_1 \cap K_2 = \{0\}, K_1 \cap N_2 = \{0\}$ and $N_1 \cap K_2 = \{0\}$. Clearly, $K_1 \oplus K_2 \subseteq (K_1 \oplus N_2) \cap (N_1 \oplus K_2)$. It remains to show that $(K_1 \oplus N_2) \cap (N_1 \oplus K_2) \subseteq K_1 \oplus K_2$. Let $z \in (K_1 \oplus N_2) \cap (N_1 \oplus K_2)$. Then there exist $k_1 \in K_1, k_2 \in K_2, n_1 \in N_1$ and $n_2 \in N_2$ such that $z \in k_1 + n_2$ and $z \in n_1 + k_2$. Then $n_2 \in z - k_1 \subseteq (n_1 + k_2) - k_1 = (n_1 - k_1) + k_2$. Thus, $n_2 \in n_1' + k_2$ for some $n_1' \in n_1 - k_1 \subseteq N_1$. Then $n_1' \in n_2 - k_2 \subseteq N_2$, so $n_1' \in N_1 \cap N_2 = \{0\}$, i.e., $n_1' = 0$. Hence, $n_2 \in 0 + k_2 = \{k_2\} \subseteq K_2$. Then $z \in K_1 \oplus K_2$. This shows that $(K_1 \oplus N_2) \cap (N_1 \oplus K_2) \subseteq K_1 \oplus K_2$. Therefore, $K_1 \oplus K_2 = (K_1 \oplus N_2) \cap (N_1 \oplus K_2)$. □

Proposition 2.1.5. *Let K_1, K_2 and N_1 be subhypermodules of an R-hypermodule M such that $K_1 \cap K_2 = K_2 \cap N_1 = \{0\}$ and let $K = K_1 \oplus K_2$ and $N = K_2 \oplus N_1$. If $K_1 \cap N = \{0\}$, then $K + N = (K_1 \oplus K_2) \oplus N_1$.*

Proof. Assume that $K_1 \cap N = \{0\}$. First, we show that $(K_1 \oplus K_2) \cap N_1 = \{0\}$. Let $x \in (K_1 \oplus K_2) \cap N_1$. Then $x \in N_1$ and $x \in k_1 + k_2$ for some $k_1 \in K_1$ and $k_2 \in K_2$. Thus, $k_1 \in x - k_2 \subseteq N$, so $k_1 \in K_1 \cap N = \{0\}$, i.e., $k_1 = 0$. Then $x \in 0 + k_2 = \{k_2\} \subseteq K_2$. Hence, $x \in K_2 \cap N_1 = \{0\}$, i.e., $x = 0$. Thus, $(K_1 \oplus K_2) \cap N_1 = \{0\}$. Note also that $(K_1 \oplus K_2) \oplus N_1 = K \oplus N_1 \subseteq K + N$.

It remains to show that $K + N \subseteq (K_1 \oplus K_2) \oplus N_1$. Let $y \in K + N$. Then there exist $k \in K$ and $n \in N$ such that $y \in k + n$. Since $K = K_1 \oplus K_2$, we obtain $k \in k_1 + k_2$ for some $k_1 \in K_1$ and $k_2 \in K_2$. Similarly, since $N = K_2 \oplus N_1$, there exist $k_2' \in K_2$ and $n_1 \in N_1$ such that $n \in k_2' + n_1$. Hence,

$$y \in k + n \subseteq (k_1 + k_2) + (k_2' + n_1) = [k_1 + (k_2 + k_2')] + n_1 \subseteq (K_1 \oplus K_2) \oplus N_1$$

This shows that $K + N \subseteq (K_1 \oplus K_2) \oplus N_1$. Therefore, $K + N = (K_1 \oplus K_2) \oplus N_1$. □

From the previous proposition, note that $K_2 \subseteq K_2 \oplus N_1 = N$, so

$$K \cap N = (K_1 \oplus K_2) \cap N = K_2 + (N \cap K_1) = K_2 + \{0\} = K_2$$

by the Modularity Condition. This concludes that $K \cap N$ may not be $\{0\}$, so the sum $K + N$ may not be direct.

Next, we provide the definition of homomorphisms for hypermodules. Moreover, some preliminary properties of homomorphisms are given.

Definition 2.1.6. [11] Let M and M' be R-hypermodules. A function $f : M \to M'$ is called a *(hypermodule) homomorphism* if

$$f(x+y) = f(x) + f(y) \quad \text{and} \quad f(rx) = rf(x)$$

for all $x, y \in M$ and $r \in R$.

For a hypermodule homomorphism $f : M \to M'$, let $\ker(f)$ denote the *kernel* of f defined by

$$\ker(f) = \{x \in M : f(x) = 0\}.$$

Next, we provide a result of homomorphisms concerning subsets of hyperrings and hypermodules.

Proposition 2.1.7. *Let M and M' be R-hypermodules, and let $f : M \to M'$ be a homomorphism. Then $f(X+Y) = f(X) + f(Y)$ and $f(AX) = Af(X)$ for all $\varnothing \neq X, Y \subseteq M$ and $\varnothing \neq A \subseteq R$.*

Proof. Let $\varnothing \neq X, Y \subseteq M$ and $\varnothing \neq A \subseteq R$. First, we show that $f(X+Y) = f(X) + f(Y)$. Let $z \in X + Y$. Then there exist $x \in X$ and $y \in Y$ such that $z \in x + y$. Thus, $f(z) \in f(x+y) = f(x) + f(y) \subseteq f(X) + f(Y)$. Therefore, $f(X+Y) \subseteq f(X)+f(Y)$. Next, let $z' \in f(X)+f(Y)$. Then there exist $x' \in X$ and $y' \in Y$ such that $z' \in f(x')+f(y')$. Thus, $z' \in f(x')+f(y') = f(x'+y') \subseteq f(X+Y)$. This means that $f(X) + f(Y) \subseteq f(X+Y)$. Hence, $f(X+Y) = f(X) + f(Y)$.

Finally, we show that $f(AX) = Af(X)$. Let $t \in AX$. Then there exist $a \in A$ and $x \in X$ such that $t \in ax$. Thus, $f(t) \in f(ax) = af(x) \subseteq Af(X)$. This shows that $f(AX) \subseteq Af(X)$. Next, let $t' \in Af(X)$. Then there exist $a' \in A$ and $x' \in X$ such that $t' \in a'f(x')$. Thus, $t' \in a'f(x') = f(a'x') \subseteq f(AX)$, so $Af(X) \subseteq f(AX)$. Therefore, $f(AX) = Af(X)$. □

According to [11], Siraworakun provided some elementary properties of homomorphisms sending 0 to 0 which concern inverses and subhypermodules.

Proposition 2.1.8. [11] *Let $f : M \to M'$ be a hypermodule homomorphism such that $f(0) = 0$. The following statements hold.*

(i) $f(-x) = -f(x)$ *for all* $x \in M$.

(ii) *If* $N \leq M$, *then* $f(N) \leq M'$.

(iii) *If* $N' \leq M'$, *then* $f^{-1}(N') \leq M$.

As the previous proposition, $\ker(f) = f^{-1}(\{0\}) \leq M$ and $f(M) \leq M'$ for any hypermodule homomorphism $f : M \to M'$ with $f(0) = 0$.

Proposition 2.1.9. *Let* $f : M \to M'$ *be a hypermodule homomorphism such that* $f(0) = 0$. *Then* f *is a monomorphism if and only if* $\ker(f) = \{0\}$.

Proof. ($\Rightarrow$) This is obvious.
($\Leftarrow$) Assume that $\ker(f) = \{0\}$. Let $x, y \in M$ be such that $f(x) = f(y)$. By Proposition 2.1.8(i),

$$0 \in f(x) - f(y) = f(x) + (-f(y)) = f(x) + f(-y) = f(x + (-y)) = f(x - y).$$

Then $f(z) = 0$ for some $z \in x - y$. Hence, $z \in \ker(f) = \{0\}$, i.e., $z = 0$. This means that $x \in z + y = 0 + y = \{y\}$, i.e., $x = y$. Therefore, f is injective. □

In hypermodules, there is no conclusion to insist that homomorphisms send 0 to 0; however, we give a necessary and sufficient condition that makes a homomorphism sending 0 to 0 as follows.

Proposition 2.1.10. *Let* $f : M \to M'$ *be a hypermodule homomorphism. Then* $f(0) = 0$ *if and only if* $0 \in f(M)$.

Proof. ($\Rightarrow$) This is obvious.
($\Leftarrow$) Assume that $0 \in f(M)$. Then $f(x) = 0$ for some $x \in M$. Note that $\{x\} = 0 + x$, so $f(x) \in f(0 + x) = f(0) + f(x)$. Hence, $f(0) \in f(x) - f(x) = 0 - 0 = \{0\}$, i.e., $f(0) = 0$. □

Proposition 2.1.11. *Let* N *be a subhypermodule of an* R*-hypermodule* M. *Define* $g : M \to M/N$ *by* $g(m) = m + N$ *for all* $m \in M$. *Then* g *is a surjective homomorphism.*

Proof. It is clear that g is surjective. To show that g is a homomorphism, let $x, y \in M$ and $r \in R$. Let $z \in x+y$. Then $g(z) = z+N \in (x+N) \boxplus (y+N) = g(x) \boxplus g(y)$. Thus, $g(x+y) \subseteq g(x) \boxplus g(y)$. Let $z' + N \in g(x) \boxplus g(y) = (x+N) \boxplus (y+N)$. Then there exists $z'' \in x + y$ such that $z' + N = z'' + N$. Thus,

$$z' + N = z'' + N = g(z'') \in g(x + y).$$

This means that $g(x) \boxplus g(y) \subseteq g(x + y)$. Therefore, $g(x + y) = g(x) \boxplus g(y)$. Next, let $a \in rx$. Then $a + N \in r \boxdot (x + N) = r \boxdot g(x)$. Thus, $g(rx) \subseteq r \boxdot g(x)$. Let $a' + N \in r \boxdot g(x) = r \boxdot (x+N)$. Then there exists $a'' \in rx$ such that $a' + N = a'' + N$. Thus, $a' + N = a'' + N = g(a'') \in g(rx)$. This shows that $r \boxdot g(x) \subseteq g(rx)$. Therefore, $g(rx) = r \boxdot g(x)$. We conclude that g is a homomorphism. □

The map g in Proposition 2.1.11 is called the *canonical map.* Moreover, this map always sends 0 to $\bar{0}$.

The following proposition is similar to the fact in module theory. However, we require the condition that hypermodule homomorphisms map 0 to itself.

In this research, for an R-hypermodule M, let

$$\mathrm{End}_0(M) = \{f : M \to M : f \text{ is a hypermodule homomorphism and } f(0) = 0\}.$$

Proposition 2.1.12. *Let M be an R-hypermodule and $f \in \mathrm{End}_0(M)$. If $f^2 = f$, then $M = f(M) \oplus \ker(f)$.*

Proof. Assume that $f^2 = f$. First, we show that $M = f(M) + \ker(f)$. It suffices to show that $M \subseteq f(M) + \ker(f)$. Let $m \in M$. Hence,

$$\begin{aligned} f(m - f(m)) &= f(m + (-f(m))) \\ &= f(m) + f(-f(m)) \\ &= f(m) - f^2(m) \quad \text{(by Proposition 2.1.8(i))} \\ &= f(m) - f(m). \end{aligned}$$

This implies that $0 \in f(m - f(m))$. Then there exists $k \in m - f(m)$ such that $f(k) = 0$, i.e., $k \in \ker(f)$. Since $k \in m - f(m)$, we obtain $m \in f(m) + k$. This means that $m \in f(M)+\ker(f)$. Hence, $M \subseteq f(M)+\ker(f)$. Next, if $x \in f(M)\cap\ker(f)$, then $f(x)=0$ and $f(y)=x$ for some $y \in M$, so $x=f(y)=f^2(y)=f(x)=0$. This means that $f(M)\cap\ker(f)=\{0\}$. Therefore, $M=f(M)\oplus\ker(f)$. □

Proposition 2.1.13. *Let N be a subhypermodule of an R-hypermodule M and $f \in \mathrm{End}_0(M)$ with $f^2=f$. If $N \leq \ker(f)$, then $(f(M)\oplus N)/N$ is a direct summand of M/N.*

Proof. Assume that $N \leq \ker(f)$. Define $F : M/N \to M/N$ by $F(x+N)=f(x)+N$ for all $x \in M$. Let $x, y \in M$ be such that $x+N=y+N$. Then $x \in y+N$. Thus, there exists $k \in N \leq \ker(f)$ such that $x \in y+k$. Hence,

$$f(x) \in f(y+k)=f(y)+f(k)=f(y)+0 \subseteq f(y)+N.$$

This means that $f(x)+N=f(y)+N$. Therefore, F is well-defined. Since f is a homomorphism, $f(0)=0$ and $f^2=f$, we obtain that F is a homomorphism, $F(\bar{0})=\bar{0}$ and $F^2=F$, respectively. By Proposition 2.1.12, we obtain that $M/N = F(M/N)\oplus\ker(F)$. Note that $f(M)\cap N \leq f(M)\cap\ker(f)=\{0\}$, so $f(M)\cap N=\{0\}$. Claim that $F(M/N)=(f(M)\oplus N)/N$. If $x \in M$, then $F(x+N)=f(x)+N \in f(M)/N \subseteq (f(M)\oplus N)/N$, so $F(M/N)\subseteq (f(M)\oplus N)/N$. Finally, let $y+N \in (f(M)\oplus N)/N$ where $y \in f(M)\oplus N$. Then there exist $m \in M$ and $n \in N$ such that $y \in f(m)+n$. Thus,

$$y+N \in (f(m)+N)\boxplus(n+N)=(f(m)+N)\boxplus N=\{f(m)+N\}.$$

This means that $y+N=f(m)+N=F(m+N)$. Hence, $(f(M)\oplus N)/N \subseteq F(M/N)$. Therefore, $(f(M)\oplus N)/N=F(M/N)$ is a direct summand of M/N. □

Proposition 2.1.14. *Let $N_1, N_2, \dots, N_k$ be subhypermodules of an R-hypermodule M such that $M=\bigoplus_{i=1}^{k} N_i$ where $k \in \mathbb{N}$ with $k \geq 2$. Let $j \in \{1,2,\dots,k\}$. Define $\pi_j : M \to N_j$ by*

$$\pi_j(x)=n_j \quad \textit{for all} \quad x \in \textstyle\sum_{i=1}^{k} n_i.$$

Then π_j is a surjective homomorphism and $\pi_j^2=\pi_j$.

Proof. By Proposition 2.1.3(ii), π_j is well-defined. First, we show that π_j is a homomorphism. Let $x, y \in M$ and $r \in R$. Then there exist $n_1, n_1' \in N_1, \ldots, n_k, n_k' \in N_k$ such that $x \in \sum_{i=1}^{k} n_i$ and $y \in \sum_{i=1}^{k} n_i'$. Thus, $\pi_j(x) = n_j$ and $\pi_j(y) = n_j'$. Then

$$x + y \subseteq \sum_{i=1}^{k} n_i + \sum_{i=1}^{k} n_i' = \sum_{i=1}^{k} (n_i + n_i').$$

To show that $\pi_j(x+y) \subseteq \pi_j(x) + \pi_j(y)$, let $a \in x + y$. Then there exist $n_1'' \in n_1 + n_1' \subseteq N_1, \ldots, n_k'' \in n_k + n_k' \subseteq N_k$ such that $a \in \sum_{i=1}^{k} n_i''$. Then $\pi_j(a) = n_j'' \in n_j + n_j' = \pi_j(x) + \pi_j(y)$. Hence, $\pi_j(x+y) \subseteq \pi_j(x) + \pi_j(y)$. Next, let $b \in \pi_j(x) + \pi_j(y)$. Since $x \in \sum_{i=1}^{k} n_i$ and $y \in \sum_{i=1}^{k} n_i'$, we can write $x \in n_j + l$ and $y \in n_j' + l'$ for some $l, l' \in \sum_{\substack{i=1 \\ i \neq j}}^{k} N_i$. Then $n_j \in x - l$ and $n_j' \in y - l'$. Thus,

$$b \in \pi_j(x) + \pi_j(y) = n_j + n_j' \subseteq (x - l) + (y - l') = (x + y) - (l + l').$$

Then there exist $z \in x + y$ and $z' \in l + l' \subseteq \sum_{\substack{i=1 \\ i \neq j}}^{k} N_i$ such that $b \in z - z'$. Since $z' \in \sum_{\substack{i=1 \\ i \neq j}}^{k} N_i$, there exist $m_1 \in N_1, \ldots, m_{j-1} \in N_{j-1}, m_{j+1} \in N_{j+1}, \ldots, m_k \in N_k$ such that $z' \in \sum_{\substack{i=1 \\ i \neq j}}^{k} m_i$. Note that $b \in \pi_j(x) + \pi_j(y) \subseteq N_j$. Hence,

$$z \in z' + b \subseteq m_1 + \cdots + m_{j-1} + b + m_{j+1} + \cdots + m_k.$$

This means that $b = \pi_j(z) \in \pi_j(x+y)$. Therefore, $\pi_j(x) + \pi_j(y) \subseteq \pi_j(x+y)$. This shows that $\pi_j(x+y) = \pi_j(x) + \pi_j(y)$. To show that $\pi_j(rx) \subseteq r\pi_j(x)$, let $c \in rx$. Note that $rx \subseteq r(\sum_{i=1}^{k} n_i) \subseteq \sum_{i=1}^{k} rn_i$. Then there exist $t_1 \in rn_1 \subseteq N_1, \ldots, t_k \in rn_k \subseteq N_k$ such that $c \in \sum_{i=1}^{k} t_i$. This implies that $\pi_j(c) = t_j \in rn_j = r\pi_j(x)$. Hence, $\pi_j(rx) \subseteq r\pi_j(x)$. Next, let $d \in r\pi_j(x)$. Recall that $n_j \in x - l$ for some $l \in \sum_{\substack{i=1 \\ i \neq j}}^{k} N_i$. Thus,

$$r\pi_j(x) = rn_j \subseteq r(x - l) \subseteq rx - rl.$$

Then there exist $p \in rx$ and $q \in rl$ such that $d \in p - q$. Since $\sum_{\substack{i=1 \\ i \neq j}}^{k} N_i \leq M$, we obtain $q \in rl \subseteq \sum_{\substack{i=1 \\ i \neq j}}^{k} N_i$. Then there exist $q_1 \in N_1, \ldots, q_{j-1} \in N_{j-1}, q_{j+1} \in N_{j+1}, \ldots, q_k \in N_k$ such that $q \in \sum_{\substack{i=1 \\ i \neq j}}^{k} q_i$. Note that $d \in r\pi_j(x) \subseteq N_j$. Therefore,

$$p \in d + q \subseteq q_1 + \cdots + q_{j-1} + d + q_{j+1} + \cdots + q_k.$$

This means that $d = \pi_j(p) \in \pi_j(rx)$. Thus, $r\pi_j(x) \subseteq \pi_j(rx)$. This shows that $r\pi_j(x) = \pi_j(rx)$. Therefore, π_j is a homomorphism.

Note that $\pi_j(x_j) = x_j$ for all $x_j \in N_j$. Thus, π_j is surjective. Let $m \in M$. Then there exist $x_1 \in N_1, \dots, x_k \in N_k$ such that $m \in \sum_{i=1}^{k} x_i$. Then $\pi_j(m) = x_j$. Thus, $\pi_j^2(m) = \pi_j(x_j) = x_j = \pi_j(m)$. We conclude that $\pi_j^2 = \pi_j$. □

The map π_j in Proposition 2.1.14 is called the *projection map* on N_j. It is clear that projection maps always send 0 to itself.

Corollary 2.1.15. *Let N be a subhypermodule of an R-hypermodule M. If $N \leq_{\oplus} M$, then there exists $g \in \mathrm{End}_0(M)$ such that $g^2 = g, g(M) = N$ and $M = N \oplus \ker(g)$.*

Proof. This follows from Proposition 2.1.12 and Proposition 2.1.14 by choosing g to be the projection map on N. □

Proposition 2.1.16. *Let H, L and N be subhypermodules of an R-hypermodule M such that $M = H \oplus L$ and let $\pi_L : M \to L$ be the projection map on L. If $H \cap N = \{0\}$, then $H \oplus N = H \oplus \pi_L(N)$.*

Proof. Note that $H \cap \pi_L(N) \leq H \cap L = \{0\}$ because $\pi_L(N) \leq L$. Then $H \cap \pi_L(N) = \{0\}$. To show that $H \oplus N \subseteq H \oplus \pi_L(N)$, let $x \in H \oplus N$. Then $x \in h_1 + n_1$ for some $h_1 \in H$ and $n_1 \in N$. Since $M = H \oplus L$, there exist $h_2 \in H$ and $l \in L$ such that $n_1 \in h_2 + l$. Then $\pi_L(n_1) = l$. Hence,

$$x \in h_1 + n_1 \subseteq h_1 + (h_2 + l) = (h_1 + h_2) + \pi_L(n_1) \subseteq H \oplus \pi_L(N).$$

This shows that $H \oplus N \subseteq H \oplus \pi_L(N)$. Next, let $y \in H \oplus \pi_L(N)$. Then there exist $h' \in H$ and $n' \in N$ such that $y \in h' + \pi_L(n')$. Since $M = H \oplus L$, there exist $h'' \in H$ and $l' \in L$ such that $n' \in h'' + l'$. Then $\pi_L(n') = l' \in n' - h''$. Thus,

$$y \in h' + \pi_L(n') \subseteq h' + (n' - h'') = (h' - h'') + n' \subseteq H \oplus N.$$

This shows that $H \oplus \pi_L(N) \subseteq H \oplus N$. Therefore, $H \oplus N = H \oplus \pi_L(N)$. □

Next, we give the concept of projection invariant subhypermodules whose properties concern homomorphisms investigated in Section 4.2. In addition, some basic properties of projection invariant subhypermodules are given.

Definition 2.1.17. Let N be a subhypermodule of an R-hypermodule M. We say that N is a *projection invariant subhypermodule* of M, denoted by $N \leq_p M$, if $f(N) \subseteq N$ for all $f \in \text{End}_0(M)$ with $f^2 = f$.

Definition 2.1.18. Let I be a hyperideal of a hyperring R. We say that I is a *projection invariant hyperideal* of R if I is a projection invariant subhypermodule of ${}_RR$.

Proposition 2.1.19. *Let K and N be projection invariant subhypermodules of an R-hypermodule M. Then $K+N$ and $K \cap N$ are projection invariant subhypermodules of M.*

Proof. The proof is straightforward. □

Proposition 2.1.20. *Let K, N and P be subhypermodules of an R-hypermodule M such that $M = K \oplus N$. If $P \leq_p M$, then $P = (P \cap K) \oplus (P \cap N)$.*

Proof. Assume that $P \leq_p M$. It is clear that $(P \cap K) \cap (P \cap N) = \{0\}$ and $(P \cap K) \oplus (P \cap N) \subseteq P$. Hence, it suffices to show that $P \subseteq (P \cap K) \oplus (P \cap N)$. Let $p \in P$. Then $p \in k + n$ for some $k \in K$ and $n \in N$. Recall that $\pi_K^2 = \pi_K$ and $\pi_N^2 = \pi_N \in \text{End}_0(M)$. Then $\pi_K(P) \subseteq P$ since $P \leq_p M$. Hence, $k = \pi_K(p) \in P$. Similarly, $n \in P$. This means that $p \in (P \cap K) \oplus (P \cap N)$. Thus, $P \subseteq (P \cap K) \oplus (P \cap N)$. This concludes that $P = (P \cap K) \oplus (P \cap N)$. □

Corollary 2.1.21. *Let K, N and P be subhypermodules of an R-hypermodule M such that $M = K \oplus N$. If $P \leq_p M$ and $P \cap K = \{0\}$, then $P \leq N$.*

Proof. By Proposition 2.1.20, we obtain that $P = \{0\} \oplus (P \cap N) = P \cap N \leq N$. □

2.2 Isomorphism Theorems

In this section, we give the concept of hypermodule isomorphisms and then present isomorphism theorems of hypermodules.

Definition 2.2.1. Let M and M' be R-hypermodules. We say that M is *isomorphic* to M', denoted by $M \cong M'$, if there exists a bijective hypermodule homomorphism between M and M'.

From the previous defintion, we observe that isomorphisms automatically send 0 to 0 by the surjectivity and Proposition 2.1.10.

Next, we provide isomorphism theorems of hypermodules. The proofs are straightforward and similar to isomorphism theorems of modules.

Proposition 2.2.2. (First Isomorphism Theorem) *Let $f : M \to M'$ be a hypermodule homomorphism such that $f(0) = 0$. Then $M/\ker(f) \cong f(M)$.*

Proof. For convenience, let $K = \ker(f)$. Define $\bar{f} : M/K \to f(M)$ by

$$\bar{f}(x + K) = f(x) \quad \text{for all } x \in M.$$

First, we show that $\bar{f}$ is well-defined. To see this, let $x, y \in M$ be such that $x + K = y + K$. Then $x \in y + K$. Thus, there exists $k \in K$ such that $x \in y + k$. This implies that $f(x) \in f(y + k) = f(y) + f(k) = f(y) + 0 = \{f(y)\}$, i.e., $f(x) = f(y)$. Hence, $\bar{f}$ is well-defined.

Next, we show that $\bar{f}$ is a homomorphism. Let $x, y \in M$ and $r \in R$. To show that $\bar{f}[(x+K) \boxplus (y+K)] \subseteq \bar{f}(x+K) + \bar{f}(y+K)$, let $z + K \in (x+K) \boxplus (y+K)$. Then there exists $z_1 \in x + y$ such that $z + K = z_1 + K$. Thus,

$$\bar{f}(z_1 + K) = f(z_1) \in f(x + y) = f(x) + f(y) = \bar{f}(x + K) + \bar{f}(y + K),$$

but then $\bar{f}(z + K) = \bar{f}(z_1 + K)$, so that $\bar{f}(z + K) \in \bar{f}(x + K) + \bar{f}(y + K)$. Hence, $\bar{f}[(x+K) \boxplus (y+K)] \subseteq \bar{f}(x+K) + \bar{f}(y+K)$. Next, let $z' \in \bar{f}(x+K) + \bar{f}(y+K)$. Note that $\bar{f}(x + K) + \bar{f}(y + K) = f(x) + f(y) = f(x + y)$. Then there exists $z_1' \in x + y$ such that $f(z_1') = z'$. This implies that $z_1' + K \in (x + K) \boxplus (y + K)$ and $\bar{f}(z_1' + K) = f(z_1') = z'$. This means that $z' \in \bar{f}[(x + K) \boxplus (y + K)]$. Thus, $\bar{f}(x+K)+\bar{f}(y+K) \subseteq \bar{f}[(x+K)\boxplus(y+K)]$. This shows that $\bar{f}[(x+K)\boxplus(y+K)] = \bar{f}(x+K)+\bar{f}(y+K)$. Moreover, let $a+K \in r \boxdot (x+K)$. Then there exists $a_1 \in rx$ such that $a + K = a_1 + K$. Hence,

$$\bar{f}(a+K) = \bar{f}(a_1+K) = f(a_1) \in f(rx) = rf(x) = r\bar{f}(x+K).$$

Therefore, $\bar{f}(r \boxdot (x+K)) \subseteq r\bar{f}(x+K)$. Next, let $b \in r\bar{f}(x+K)$. Note that $r\bar{f}(x+K) = rf(x) = f(rx)$. Then there exists $b_1 \in rx$ such that $f(b_1) = b$. Hence, $b_1+K \in r\boxdot(x+K)$ and $\bar{f}(b_1+K) = f(b_1) = b$. This means that $b \in \bar{f}(r\boxdot(x+K))$. Thus, $r\bar{f}(x+K) \subseteq \bar{f}(r\boxdot(x+K))$. This shows that $\bar{f}(r\boxdot(x+K)) = r\bar{f}(x+K)$. We conclude that $\bar{f}$ is a homomorphism. It is obvious that $\bar{f}$ is surjective, so $\bar{f}(\bar{0}) = 0$. Finally, let $a+K \in \ker(\bar{f})$. Then $f(a) = \bar{f}(a+K) = 0$. This means that $a \in K$, so $a+K = K$. This shows that $\ker(\bar{f}) = \{\bar{0}\}$. By Proposition 2.1.9, $\bar{f}$ is injective. Therefore, $M/\ker(f) \cong f(M)$. □

Proposition 2.2.3. (Second Isomorphism Theorem) *Let K and N be subhypermodules of an R-hypermodule M. Then $N/(N \cap K) \cong (K+N)/K$.*

Proof. Define $f : N \to (K+N)/K$ by $f(x) = x+K$ for all $x \in N$. To show that f is a homomorphism, let $x, y \in N$ and $r \in R$. First, we show that $f(x+y) = f(x) \boxplus f(y)$. Let $z \in x+y$. Then $z+K \in (x+K) \boxplus (y+K)$, and

$$f(z) = z+K \in (x+K) \boxplus (y+K) = f(x) \boxplus f(y).$$

Thus, $f(x+y) \subseteq f(x) \boxplus f(y)$. Let $z'+K \in f(x) \boxplus f(y)$. Note that $f(x) \boxplus f(y) = (x+K) \boxplus (y+K)$. Then there exists $z_1' \in x+y$ such that $z'+K = z_1'+K$. Hence, $z'+K = z_1'+K = f(z_1') \in f(x+y)$. This means that $f(x) \boxplus f(y) \subseteq f(x+y)$. Hence, $f(x+y) = f(x) \boxplus f(y)$. Next, we show that $f(rx) = r \boxdot f(x)$. Let $a \in rx$. Note that $a+K \in r\boxdot(x+K)$. Thus, $f(a) = a+K \in r\boxdot(x+K) = r\boxdot f(x)$. This shows that $f(rx) \subseteq r\boxdot f(x)$. Next, let $b+K \in r\boxdot f(x)$. Note that $r\boxdot f(x) = r\boxdot(x+K)$. Then there exists $b_1 \in rx$ such that $b_1+K = b+K$. Thus, $b+K = b_1+K = f(b_1) \in f(rx)$, so $r \boxdot f(x) \subseteq f(rx)$. Hence, $f(rx) = r \boxdot f(x)$. This shows that f is a homomorphism. To show that f is surjective, let $c \in K+N$. Then there exists $k \in K$ and $n \in N$ such that $c \in k+n$. Thus, $n \in c+(-k)$. This implies that $n+K \in (c+K) \boxplus (-k+K) = (c+K) \boxplus K = \{c+K\}$, i.e., $n+K = c+K$. Thus,

$f(n) = n + K = c + K$. Therefore, f is surjective and then $f(0) = \bar{0}$. Finally,

$$\begin{aligned}\ker(f) = \{x \in N : f(x) = \bar{0}\} &= \{x \in N : x + K = K\} \\ &= \{x \in N : x \in K\} = N \cap K.\end{aligned}$$

By the First Isomorphism Theorem, we conclude that

$$N/(N \cap K) \cong (K + N)/K.$$

□

Proposition 2.2.4. (Third Isomorphism Theorem) *Let K and N be subhypermodules of an R-hypermodule M such that $K \leq N$. Then*

$$(M/K)/(N/K) \cong M/N.$$

Proof. Define $f : M/K \to M/N$ by $f(x+K) = x+N$ for all $x \in M$. To show that f is well-defined, let $x, y \in M$ be such that $x+K = y+K$. Then $x \in y+K \subseteq y+N$ since $K \leq N$. Thus, $x + N = y + N$. Hence, f is well-defined. Next, we show that f is a homomorphism. Let $x, y \in M$ and $r \in R$. First, we show that

$$f[(x + K) \boxplus_K (y + K)] = f(x + K) \boxplus_N f(y + K).$$

Let $z+K \in (x+K) \boxplus_K (y+K)$. Then there exists $z_1 \in x+y$ such that $z+K = z_1+K$. Hence, $z_1 + N \in (x + N) \boxplus_N (y + N)$. Moreover,

$$f(z + K) = f(z_1 + K) = z_1 + N \in (x + N) \boxplus_N (y + N) = f(x + K) \boxplus_N f(y + K).$$

Hence, $f[(x+K) \boxplus_K (y+K)] \subseteq f(x+K) \boxplus_N f(y+K)$. To show that $f(x+K) \boxplus_N f(y+K) \subseteq f[(x+K) \boxplus_K (y+K)]$, let $z' + N \in f(x+K) \boxplus_N f(y+K)$. Note that $f(x + K) \boxplus_N f(y + K) = (x + N) \boxplus_N (y + N)$. Then there exists $z_1' \in x + y$ such that $z_1' + N = z' + N$. Thus, we have $z_1' + K \in (x + K) \boxplus_K (y + K)$. Therefore,

$$z' + N = z_1' + N = f(z_1' + K) \in f[(x + K) \boxplus_K (y + K)].$$

This shows that $f(x + K) \boxplus_N f(y + K) \subseteq f[(x + K) \boxplus_K (y + K)]$. Next, we show that $f(r \boxdot_K (x + K)) = r \boxdot_N f(x + K)$. Let $a + K \in r \boxdot_K (x + K)$. Then there exists $a_1 \in rx$ such that $a + K = a_1 + K$. Thus, we get $a_1 + N \in r \boxdot_N (x + N)$. Moreover,

$$f(a + K) = f(a_1 + K) = a_1 + N \in r \boxdot_N (x + N) = r \boxdot_N f(x + K).$$

Hence, $f(r \boxdot_K (x + K)) \subseteq r \boxdot_N f(x + K)$. Let $a' + N \in r \boxdot_N f(x + K)$. Note that $r \boxdot_N f(x+K) = r \boxdot_N (x+N)$. Then there exists $a'_1 \in rx$ such that $a' + N = a'_1 + N$. Now, we have $a'_1 + K \in r \boxdot_K (x + K)$. Therefore,

$$a' + N = a'_1 + N = f(a'_1 + K) \in f(r \boxdot_K (x + K)).$$

This shows that $r \boxdot_N f(x + K) \subseteq f(r \boxdot_K (x + K))$. Hence, f is a homomorphism. It is obvious that f is surjective and then $f(\bar{0}) = \bar{0}$. In addition,

$$\begin{aligned} \ker(f) &= \{x + K \in M/K : f(x + K) = N\} \\ &= \{x + K \in M/K : x + N = N\} \\ &= \{x + K \in M/K : x \in N\} \\ &= N/K. \end{aligned}$$

By the First Isomorphism Theorem, we conclude that

$$(M/K)/(N/K) \cong M/N.$$

□

CHAPTER III
SOME SPECIAL SUBHYPERMODULES

In module theory, essential submodules, complements, closed submodules, the singular submodules and the second singular submodules are special submodules which have been studied in many directions for several years. In 2011, Asgari and Haghany [1] provided the concepts of t-essential submodules and t-closed submodules in order to define t-extending modules; moreover, they gave characterizations of t-extending modules. In this research, we extend the concepts of these submodules to subhypermodules consisting of essential subhypermodules, complements, closed subhypermodules, the singular subhypermodule, the second singular subhypermodule, t-essential subhypermodules and t-closed subhypermodules. Moreover, some properties of these subhypermodules are given. Especially, we present characterizations of closed subhypermodules, the singular subhypermodule and the second singular subhypermodule.

3.1 Essential Subhypermodules

We begin this chapter with the concept of essential subhypermodules.

Definition 3.1.1. A subhypermodule N of an R-hypermodule M is called an *essential subhypermodule* of M (or *essential* in M), denoted by $N \leq_{ess} M$, if $L = \{0\}$ for any $L \leq M$ with $N \cap L = \{0\}$.

Remark 3.1.2. Let N be a subhypermodule of an R-hypermodule M. Then $N \leq_{ess} M$ if and only if $N \cap L \neq \{0\}$ for all $\{0\} \neq L \leq M$.

Remark 3.1.3. Let M be an R-hypermodule. Then every subhypermodule of M is always essential in itself. In particular, $\{0\} \leq_{ess} M$ if and only if $M = \{0\}$.

Example 3.1.4. Let $M = [0, t]$ where $0 < t \leq 1$ be the R-hypermodule defined in Proposition 1.3.6. It follows from Proposition 1.3.7 that each nonzero subhypermodule of M is of the form $[0, x]$ or $[0, x)$ for some $x \in (0, t]$. Therefore every nonzero subhypermodule of M is essential in M.

Next, we give some properties of essential subhypermodules.

Proposition 3.1.5. *Let M and M' be R-hypermodules. The following statements hold.*

(i) Let $K \leq N \leq M$. Then $K \leq_{ess} M$ if and only if $K \leq_{ess} N$ and $N \leq_{ess} M$.

(ii) Let $f : M \to M'$ be a homomorphism such that $f(0) = 0$ and $N' \leq M'$. If $N' \leq_{ess} M'$, then $f^{-1}(N') \leq_{ess} M$.

(iii) Let $\{K_i\}_{i=1}^k$ and $\{N_i\}_{i=1}^k$ be families of subhypermodules of M where $k \in \mathbb{N}$. If $K_i \leq_{ess} N_i$ for all $i \in \{1, 2, \ldots, k\}$, then $\cap_{i=1}^k K_i \leq_{ess} \cap_{i=1}^k N_i$.

(iv) Let $K_1,\ K_2,\ N_1,\ N_2 \leq M$ be such that $K_1 \cap K_2 = \{0\}$. If $K_1 \leq_{ess} N_1$ and $K_2 \leq_{ess} N_2$, then $K_1 \oplus K_2 \leq_{ess} N_1 \oplus N_2$.

Proof. (i) Assume that $K \leq_{ess} M$. Clearly, $K \leq_{ess} N$. If $\{0\} \neq L \leq M$, then $\{0\} \neq K \cap L \leq N \cap L$ since $K \leq_{ess} M$. Hence, $N \leq_{ess} M$.

Conversely, assume $K \leq_{ess} N$ and $N \leq_{ess} M$. Let $L \leq M$ be such that $K \cap L = \{0\}$. Then $K \cap (N \cap L) = \{0\}$. Now, $N \cap L = \{0\}$ because $K \leq_{ess} N$. Since $N \leq_{ess} M$, we obtain $L = \{0\}$. This shows that $K \leq_{ess} M$.

(ii) Assume that $N' \leq_{ess} M'$. By Proposition 2.1.8(iii), we obtain $f^{-1}(N') \leq M$. Let $\{0\} \neq L \leq M$. If $f(L) = \{0\}$, then $L \leq \ker(f) \leq f^{-1}(N')$, so $\{0\} \neq L = f^{-1}(N') \cap L$. Suppose that $f(L) \neq \{0\}$. Note that $\{0\} \neq f(L) \leq M'$ by Proposition 2.1.8(ii). Since $N' \leq_{ess} M'$, we know that $N' \cap f(L) \neq \{0\}$. Then there exists $0 \neq l \in L$ such that $0 \neq f(l) \in N'$. This means that $0 \neq l \in f^{-1}(N') \cap L$. Then $f^{-1}(N') \cap L \neq \{0\}$. Therefore, $f^{-1}(N') \leq_{ess} M$.

(iii) Assume that $K_i \leq_{ess} N_i$ for all $i \in \{1, 2, \ldots, k\}$. Let $\{0\} \neq L \leq \cap_{i=1}^k N_i$. Since $K_1 \leq_{ess} N_1$ and $\{0\} \neq L \leq \cap_{i=1}^k N_i \leq N_1$, we get $K_1 \cap L \neq \{0\}$. Similarly, since $K_2 \leq_{ess} N_2$ and $\{0\} \neq K_1 \cap L \leq N_2$, we also obtain $(K_1 \cap K_2) \cap L = K_2 \cap (K_1 \cap L) \neq \{0\}$. By repeating this process k times, $(\cap_{i=1}^{k} K_i) \cap L \neq \{0\}$. This shows that $\cap_{i=1}^{k} K_i \leq_{ess} \cap_{i=1}^{k} N_i$.

(iv) Assume $K_1 \leq_{ess} N_1$ and $K_2 \leq_{ess} N_2$. By (iii), we get $\{0\} = K_1 \cap K_2 \leq_{ess} N_1 \cap N_2$. This forces that $N_1 \cap N_2 = \{0\}$ from Remark 3.1.3. For each $i \in \{1, 2\}$, let $\pi_i : N_1 \oplus N_2 \to N_i$ be the projection map on N_i. By (ii),

$$\pi_1^{-1}(K_1) \leq_{ess} N_1 \oplus N_2 \quad \text{and} \quad \pi_2^{-1}(K_2) \leq_{ess} N_1 \oplus N_2.$$

By (iii), we get $\pi_1^{-1}(K_1) \cap \pi_2^{-1}(K_2) \leq_{ess} N_1 \oplus N_2$. Next, we show that

$$\pi_1^{-1}(K_1) = K_1 \oplus N_2 \quad \text{and} \quad \pi_2^{-1}(K_2) = N_1 \oplus K_2.$$

Let $x \in \pi_1^{-1}(K_1)$. Because $x \in N_1 \oplus N_2$, it follows that $x \in n_1 + n_2$ for some $n_1 \in N_1$ and $n_2 \in N_2$. Thus, $n_1 = \pi_1(x) \in K_1$. This implies that $x \in K_1 \oplus N_2$. Let $y \in K_1 \oplus N_2$. Then there exist $k_1 \in K_1 \leq N_1$ and $n_2 \in N_2$ such that $y \in k_1 + n_2$. Thus, $\pi_1(y) = k_1 \in K_1$, i.e., $y \in \pi_1^{-1}(K_1)$. Hence, $\pi_1^{-1}(K_1) = K_1 \oplus N_2$. Similarly, $\pi_2^{-1}(K_2) = N_1 \oplus K_2$. By Proposition 2.1.4,

$$K_1 \oplus K_2 = (K_1 \oplus N_2) \cap (N_1 \oplus K_2) = \pi_1^{-1}(K_1) \cap \pi_2^{-1}(K_2) \leq_{ess} N_1 \oplus N_2.$$

This completes the proof. □

In module theory, any arbitrary intersections of essential submodules of an R-module may not be essential in that R-module. From this conclusion, we also conclude that arbitrary intersection of essential subhypermodules of an R-hypermodule may not be essential in that R-hypermodule in general.

Next, we provide a characterization of essential subhypermodules of an R-hypermodule under certain conditions.

Proposition 3.1.6. *Let M be a strongly distributive R-hypermodule such that $m \in Rm$ for all $m \in M$ and $N \leq M$. Then $N \leq_{ess} M$ if and only if $N \cap Rx \neq \{0\}$ for all $0 \neq x \in M$.*

Proof. Assume that $N \leq_{ess} M$ and $0 \neq x \in M$. Then $Rx \leq M$ by Proposition 1.3.9. Now, $0 \neq x \in Rx$. By the essentiality of N in M, we conclude that $N \cap Rx \neq \{0\}$.

Conversely, assume $N \cap Rx \neq \{0\}$ for all $0 \neq x \in M$. To show that $N \leq_{ess} M$, let $\{0\} \neq L \leq M$. Then there exists $0 \neq y \in L$. By the assumption, we obtain that $N \cap Ry \neq \{0\}$. Since $L \leq M$, we get $Ry \subseteq L$. Thus, $\{0\} \neq N \cap Ry \subseteq N \cap L$. This shows that $N \leq_{ess} M$. □

In general, R-hypermodules M might not satisfy the condition that $m \in Rm$ for all $m \in M$ such as the R'-hypermodule M' in Proposition 1.3.6 where $M' = [0, \frac{1}{2}]$ and $R' = \{0\} \cup [2, \infty)$ because $R'\frac{1}{2} = [0, \frac{1}{4}]$, so $\frac{1}{2} \notin R'\frac{1}{2}$; however, if we let $R'' = \{0\} \cup [1, \infty)$, then

$$R''m = \bigcup_{r \in \{0\} \cup [1,\infty)} rm = \{0\} \cup \Big(\bigcup_{r \in [1,\infty)} [0, \frac{m}{r}] \Big) = [0, m]$$

for all $m \in M'$, so the R''-hypermodule M' satisfies the condition that $m \in R''m$ for all $m \in M'$.

3.2 Complements and Closed Subhypermodules

In this section, we first give the concepts of complements and closed subhypermodules and then provide some properties of these subhypermodules involving the essentiality of subhypermodules.

Definition 3.2.1. Let N be a subhypermodule of an R-hypermodule M. A subhypermodule K of M is called a *complement* of N in M if it is maximal under inclusion in the set $\{L \leq M : L \cap N = \{0\}\}$, i.e., $K \cap N = \{0\}$ and $K = K'$ for any $K \leq K' \leq M$ with $K' \cap N = \{0\}$.

We observe that for a subhypermodule N of an R-hypermodule M, by applying Zorn's Lemma to the set $\{L \leq M : L \cap N = \{0\}\}$, it has a maximal element under inclusion, i.e., a complement K of N in M exists. This concludes that every subhypermodule of an R-hypermodule always has a complement.

Moreover, let H and N be subhypermodules of an R-hypermodule M such that $H \cap N = \{0\}$, there exists a complement K of N in M such that $H \leq K$ (consider the set $\{L \leq M : L \cap N = \{0\} \text{ and } H \leq L\}$).

Proposition 3.2.2. *Let K and N be subhypermodules of an R-hypermodule M such that $M = K \oplus N$. Then K and N are complements of each other in M.*

Proof. To show that K is a complement of N in M, let $K' \leq M$ be such that $K \subseteq K'$ and $K' \cap N = \{0\}$. Let $k' \in K'$. Since $M = K \oplus N$, there exist $k \in K$ and $n \in N$ such that $k' \in k + n$. Then $n \in k' - k \subseteq K'$, so $n \in K' \cap N = \{0\}$, i.e., $n = 0$. This means that $k' \in k + 0 = \{k\} \subseteq K$. Thus, $K' \subseteq K$. This shows that $K = K'$. Hence, K is a complement of N in M. Similarly, N is a complement of K in M. □

Proposition 3.2.3. *Let K and N be subhypermodules of an R-hypermodule M. If K is a complement of N in M, then $N \oplus K \leq_{ess} M$.*

Proof. Assume that K is a complement of N in M. Let $L \leq M$ be such that $(N \oplus K) \cap L = \{0\}$. Claim that $N \cap (K + L) = \{0\}$. To see this, let $x \in N \cap (K + L)$. Then $x \in N$ and $x \in k + l$ for some $k \in K$ and $l \in L$. Thus $l \in x - k \subseteq N \oplus K$. This means that $l \in (N \oplus K) \cap L = \{0\}$, i.e., $l = 0$. Then $x \in k + 0 = \{k\} \subseteq K$, so $x \in N \cap K = \{0\}$, i.e., $x = 0$. Therefore, $N \cap (K + L) = \{0\}$. Since K is a complement of N in M, we get $K = K + L$. Then $L \leq K + L = K \leq N \oplus K$. This means that $L = (N \oplus K) \cap L = \{0\}$. We conclude that $N \oplus K \leq_{ess} M$. □

In module theory, consider $\mathbb{Z}_{30}$ as a $\mathbb{Z}$-module, it can be seen that $\langle \overline{15} \rangle \oplus \langle \overline{6} \rangle = \langle \overline{3} \rangle \leq_{ess} \mathbb{Z}_{30}$, but $\langle \overline{6} \rangle$ is not a complement of $\langle \overline{15} \rangle$ in $\mathbb{Z}_{30}$ because $\langle \overline{6} \rangle \leq \langle \overline{2} \rangle$ and $\langle \overline{2} \rangle \cap \langle \overline{15} \rangle = \{\overline{0}\}$. Similarly, $\langle \overline{15} \rangle$ is not a complement of $\langle \overline{6} \rangle$ in $\mathbb{Z}_{30}$. This example asserts that the converse of the above proposition does not hold in general because hypermodules generalize modules. However, we give a sufficient condition such that the converse of the above proposition holds.

Proposition 3.2.4. *Let K and N be subhypermodules of an R-hypermodule M. If $K \leq_{\oplus} M$ and $N \oplus K \leq_{ess} M$, then K is a complement of N in M.*

Proof. Assume that $K \leq_{\oplus} M$ and $N \oplus K \leq_{ess} M$. Then there exists $K' \leq M$ such that $M = K \oplus K'$. Let $L \leq M$ be such that $K \subseteq L$ and $L \cap N = \{0\}$. By the Modularity Condition,

$$K \oplus (L \cap K') = L \cap (K \oplus K') = L \cap M = L.$$

We claim that $L \cap K' = \{0\}$. We first show that $(N \oplus K) \cap (L \cap K') = \{0\}$. To see this, let $x \in (N \oplus K) \cap (L \cap K')$. Then $x \in L, x \in K'$ and $x \in n + k$ for some $n \in N$ and $k \in K$. Thus, $n \in x - k \subseteq L$. This means that $n \in L \cap N = \{0\}$, i.e., $n = 0$. Then $x \in 0 + k = \{k\} \subseteq K$, so $x \in K \cap K' = \{0\}$, i.e., $x = 0$. Hence, $(N \oplus K) \cap (L \cap K') = \{0\}$. Therefore, $L \cap K' = \{0\}$ since $N \oplus K \leq_{ess} M$. This imples that $K = L$. We conclude that K is a complement of N in M. □

Proposition 3.2.5. *Let H, L and N be subhypermodules of an R-hypermodule M such that $H \leq_{ess} L$. Then N is a complement of L in M if and only if N is a complement of H in M.*

Proof. Assume that N is a complement of L in M. Clearly, $N \cap H = \{0\}$. Let $N' \leq M$ be such that $N \subseteq N'$ and $N' \cap H = \{0\}$. Then $H \cap (L \cap N') = \{0\}$. Since $H \leq_{ess} L$, we get $L \cap N' = \{0\}$. This implies that $N = N'$ since N is a complement of L in M. Hence, N is a complement of H in M.

Conversely, assume that N is a complement of H in M. Note that $H \cap (N \cap L) = \{0\}$. Since $H \leq_{ess} L$, we have $N \cap L = \{0\}$. If $N'' \leq M$ with $N \subseteq N''$ and $N'' \cap L = \{0\}$, then $N'' \cap H = \{0\}$, which implies that $N = N''$ since N is a complement of H in M. Therefore, N is a complement of L in M. □

Proposition 3.2.6. *Let H, K and N be subhypermodules of an R-hypermodule M such that $K \leq N$. If H is a complement of K in M and $H \cap N = \{0\}$, then $K \leq_{ess} N$.*

Proof. Assume that H is a complement of K in M and $H \cap N = \{0\}$. Let $L \leq N$ be such that $K \cap L = \{0\}$. We observe that $H \cap L \leq H \cap N = \{0\}$. Claim that

$K \cap (L \oplus H) = \{0\}$. Let $x \in K \cap (L \oplus H)$. Then $x \in K$ and $x \in l + h$ for some $l \in L$ and $h \in H$. Thus, $h \in x - l \subseteq N$. This means that $h \in H \cap N = \{0\}$, i.e., $h = 0$. It follows that $x \in l + 0 = \{l\} \subseteq L$, so $x \in K \cap L = \{0\}$, i.e., $x = 0$. Thus, $K \cap (L \oplus H) = \{0\}$. Since H is a complement of K in M, we obtain $L \oplus H = H$. Then $L \leq L \oplus H = H$. Thus, $L = H \cap L = \{0\}$. This concludes that $K \leq_{ess} N$. □

Next, we give the concept of closed subhypermodules which is similar to the concept of closed submodules in module theory. Moreover, we also give an equivalent condition for closed subhypermodules concerning the essentiality of subhypermodules.

Definition 3.2.7. A subhypermodule K of an R-hypermodule M is called a *closed subhypermodule* of M (or *closed* in M), denoted by $K \leq_{cl} M$, if there exists $K' \leq M$ such that K is a complement of K' in M.

By Proposition 3.2.2, we immediately obtain that every direct summand of an R-hypermodule M is a closed subhypermodule of M.

Proposition 3.2.8. *Let C be a subhypermodule of an R-hypermodule M. Then $C \leq_{cl} M$ if and only if $C = N$ for any $N \leq M$ with $C \leq_{ess} N$.*

Proof. Assume that $C \leq_{cl} M$. Then there exists $C' \leq M$ such that C is a complement of C' in M. Let $N \leq M$ be such that $C \leq_{ess} N$. We see that $C \cap (N \cap C') \leq C \cap C' = \{0\}$. Now, $N \cap C' = \{0\}$ since $C \leq_{ess} N$. Because C is a complement of C' in M, we conclude that $C = N$.

Conversely, assume that $C = N$ for any $N \leq M$ with $C \leq_{ess} N$. Let C' be a complement of C in M. To show that C is also a complement of C' in M, let $K \leq M$ be such that $C \subseteq K$ and $K \cap C' = \{0\}$. Thus, $C \leq_{ess} K$ by Proposition 3.2.6. By the assumption, $C = K$. This shows that C is a complement of C' in M. Hence, $C \leq_{cl} M$. □

Corollary 3.2.9. *Let C be a subhypermodule of an R-hypermodule M. If $C \leq_{cl} M$, then $C \leq_{cl} N$ for any $C \leq N \leq M$.*

Proof. This follows form Proposition 3.2.8. $\square$

Proposition 3.2.10. *Let K and N be subhypermodules of an R-hypermodule M with $K \leq N$. If $K \leq_{cl} N$ and $N \leq_{cl} M$, then $K \leq_{cl} M$.*

Proof. Assume that $K \leq_{cl} N$ and $N \leq_{cl} M$. We show that $K \leq_{cl} M$ by applying Proposition 3.2.8. Let $L \leq M$ be such that $K \leq_{ess} L$. We show that $K = L$. Since $K \leq_{cl} N$ and $N \leq_{cl} M$, there exist $K' \leq N$ and $N' \leq M$ such that K is a complement of K' in N and N is a complement of N' in M, respectively. We divide the details of the proof into three steps as follows.

(i) First, we show that $L \cap (K' + N') = \{0\}$.

Claim that $K \cap (K' + N') = \{0\}$. Let $x \in K \cap (K' + N')$. Then $x \in K$ and $x \in k' + n'$ for some $k' \in K'$ and $n' \in N'$. Thus $n' \in x - k' \subseteq N$. This means that $n' \in N \cap N' = \{0\}$, i.e., $n' = 0$. Then $x \in k' + 0 = \{k'\} \subseteq K'$, so $x \in K \cap K' = \{0\}$, i.e., $x = 0$. This shows that $K \cap (K' + N') = \{0\}$. Hence,

$$K \cap [L \cap (K' + N')] \leq K \cap (K' + N') = \{0\}.$$

Since $K \leq_{ess} L$, we obtain $L \cap (K' + N') = \{0\}$.

(ii) We next show that $K = N \cap (L + N')$.

First, we claim that $K' \cap (L + N') = \{0\}$. Let $y \in K' \cap (L + N')$. Then $y \in K'$ and there exist $l \in L$ and $n' \in N'$ such that $y \in l + n'$. Then $l \in y - n' \subseteq K' + N'$. This implies that $l \in L \cap (K' + N') = \{0\}$ by (i). Thus, $l = 0$. Then $y \in 0 + n' = \{n'\} \subseteq N'$, so $y \in K' \cap N' \leq N \cap N' = \{0\}$, i.e., $y = 0$. Hence, $K' \cap (L + N') = \{0\}$. It follows that

$$K' \cap [N \cap (L + N')] = K' \cap (L + N') = \{0\}.$$

We observe that $K \leq N \cap (L + N')$. Since K is a complement of K' in N, this forces that $K = N \cap (L + N')$.

(iii) Finally, we show that $L \leq N$.

Claim that $N' \cap (N + L) \leq L$. To see this, let $z \in N' \cap (N + L)$. Then $z \in N'$ and there exist $n \in N$ and $l \in L$ such that $z \in n + l$. Thus, $n \in z - l \subseteq N' + L$. Then $n \in N \cap (L + N') = K \leq L$ by (ii). Hence, $z \in n + l \subseteq L$. This shows that

$N' \cap (N+L) \leq L$. We observe that

$$K \cap [N' \cap (N+L)] \leq N \cap N' = \{0\}.$$

Thus, $N' \cap (N+L) = \{0\}$ because $K \leq_{ess} L$. Since N is a complement of N' in M, this forces that $N = N+L$. Therefore, $L \leq N+L = N$.

Now, $L \leq N$ by (iii) and recall that $K \leq_{cl} N$ and $K \leq_{ess} L$. By Proposition 3.2.8, $K = L$. This concludes that $K \leq_{cl} M$ by Proposition 3.2.8 again. □

Proposition 3.2.11. *Let M be a strongly distributive R-hypermodule such that $m \in Rm$ for all $m \in M$, and let K and L be closed subhypermodules of M. Then K is a complement of L in M if and only if L is a complement of K in M.*

Proof. Assume that K is a complement of L in M. Then $K \cap L = \{0\}$. To show that L is a complement of K in M, let $L' \leq M$ be such that $L \leq L'$ and $K \cap L' = \{0\}$. Claim that $L \leq_{ess} L'$. To see this, assume that $0 \neq x \in L'$. We show that $L \cap Rx \neq \{0\}$. It follows from Proposition 3.2.3 that $K \oplus L \leq_{ess} M$. This gives $(K \oplus L) \cap Rx \neq \{0\}$ by Proposition 3.1.6. Let $0 \neq y \in (K \oplus L) \cap Rx$. Then there exist $k \in K, l \in L$ and $r \in R$ such that $y \in k + l$ and $y \in rx$. Thus, $k \in y - l \subseteq rx - l \subseteq L'$. This implies that $k \in K \cap L' = \{0\}$, i.e., $k = 0$. Then $y \in 0 + l = \{l\} \subseteq L$. Hence, $0 \neq y \in L \cap Rx$. We obtain $L \leq_{ess} L'$ by Proposition 3.1.6, but then $L \leq_{cl} M$, so that $L = L'$ by Proposition 3.2.8. Therefore, L is a complement of K in M. The converse can be proved in the similar way as above. □

We know from Remark 3.1.3 that each subhypermodule of an R-hypermodule is always essential in itself. Moreover, we can show that for a subhypermodule of an R-hypermodule, there exists a closed subhypermodule such that the subhypermodule is essential in that closed subhypermodule.

Proposition 3.2.12. *Let K be a subhypermodule of an R-hypermodule M. Then there exists $N \leq M$ such that $K \leq_{ess} N$ and $N \leq_{cl} M$.*

Proof. Let H be a complement of K in M. Then $H \cap K = \{0\}$. By applying Zorn's Lemma to the set $\{L \leq M : H \cap L = \{0\}$ and $K \leq L\}$, there exists a complement

N of H in M such that $K \leq N$. Then $N \leq_{cl} M$. Note that $H \cap N = \{0\}$. By Proposition 3.2.6, we conclude that $K \leq_{ess} N$. □

Finally, we give some properties concerning essential subhypermodules and closed subhypermodules of quotient R-hypermodules

Proposition 3.2.13. *Let K and N be subhypermodules of an R-hypermodule M such that $K \leq N$. The following statements hold.*

(i) If $N/K \leq_{ess} M/K$, then $N \leq_{ess} M$.

(ii) If $N \leq_{ess} M$ and $K \leq_{cl} M$, then $N/K \leq_{ess} M/K$.

Proof. (i) Assume that $N/K \leq_{ess} M/K$. To show that $N \leq_{ess} M$, let $L \leq M$ be such that $N \cap L = \{0\}$. By the Modularity Condition,

$$N \cap (L + K) = K + (N \cap L) = K + \{0\} = K,$$

so $(L + K)/K \cap N/K = \{K\}$. Because $N/K \leq_{ess} M/K$, it follows that

$$(L + K)/K = \{K\}, \text{ i.e., } L + K = K.$$

Thus, $L \leq L+K = K \leq N$. This means that $L = L \cap N = \{0\}$. Hence, $N \leq_{ess} M$.

(ii) Assume that $N \leq_{ess} M$ and $K \leq_{cl} M$. Let L' be a subhypermodule of M containing K such that $N/K \cap L'/K = \{K\}$. Then $N \cap L' = K$. Since $K \leq_{cl} M$, there exists $K' \leq M$ such that K is a complement of K' in M. We see that $N \cap (L' \cap K') = K \cap K' = \{0\}$. Next, $L' \cap K' = \{0\}$ because $N \leq_{ess} M$. Since K is a complement of K' in M, we obtain $L' = K$. Therefore, $N/K \leq_{ess} M/K$. □

The results regarding closed subhypermodules reverse the results of essential subhypermodules as above.

Proposition 3.2.14. *Let K and N be subhypermodules of an R-hypermodule M such that $K \leq N$. The following statements hold.*

(i) If $N \leq_{cl} M$, then $N/K \leq_{cl} M/K$.

(ii) If $N/K \leq_{cl} M/K$ and $K \leq_{cl} M$, then $N \leq_{cl} M$.

Proof. (i) Assume that $N \leq_{cl} M$. Let L be a subhypermodule of M containing K such that $N/K \leq_{ess} L/K$. Then we get $N \leq_{ess} L$ by Proposition 3.2.13(i), but $N \leq_{cl} M$, so $N = L$ by Proposition 3.2.8. This gives $N/K = L/K$. By Proposition 3.2.8, we conclude that $N/K \leq_{cl} M/K$.

(ii) Assume that $N/K \leq_{cl} M/K$ and $K \leq_{cl} M$. Let $L \leq M$ be such that $N \leq_{ess} L$. Since $K \leq_{cl} M$ and $K \leq N$, by Corollary 3.2.9, we get $K \leq_{cl} L$. By Proposition 3.2.13(ii), we obtain $N/K \leq_{ess} L/K$. Because $N/K \leq_{cl} M/K$, it follows that $N/K = L/K$, so $N = L$. We conclude that $N \leq_{cl} M$ by Proposition 3.2.8. □

3.3 The Singular Subhypermodule and The Second Singular Subhypermodule

Throughout this section, all hyperrings are required to be commutative. In this section, we define the singular subhypermodule and the second singular subhypermodule which are similar to the concepts of the singular submodule and the second singular submodule, respectively. Some properties of these subhypermodules used in Chapter IV are given.

For an R-hypermodule M and $m \in M$, recall that

$$\operatorname{ann}(m) = \{r \in R : rm = \{0\}\};$$

moreover, in the case that $\operatorname{ann}(m) \neq \varnothing$, Proposition 1.3.12 yields that $\operatorname{ann}(m)$ forms a left hyperideal of R, so it is a subhypermodule of ${}_RR$.

Proposition 3.3.1. *Let M be an R-hypermodule where R is commutative. Define*

$$Z(M) = \{m \in M : \operatorname{ann}(m) \leq_{ess} {}_RR\}.$$

Then $Z(M)$ is a subhypermodule of M.

Proof. Note that $R0 = \{0\}$, so $\operatorname{ann}(0) = R \leq_{ess} {}_RR$. This means that $0 \in Z(M)$. Hence, $Z(M) \neq \varnothing$. Let $x, y \in Z(M)$ and $r \in R$. Then $\operatorname{ann}(x), \operatorname{ann}(y) \leq_{ess} {}_RR$. Thus, $\operatorname{ann}(x) \cap \operatorname{ann}(y) \leq_{ess} {}_RR$ by Proposition 3.1.5(iii). Let $z \in x - y$. To show that $z \in Z(M)$, we claim that $\operatorname{ann}(x) \cap \operatorname{ann}(y) \leq \operatorname{ann}(z)$. Let $a \in \operatorname{ann}(x) \cap \operatorname{ann}(y)$. Then $ax = \{0\} = ay$. Thus,

$$az \subseteq a(x-y) \subseteq ax - ay = \{0\} - \{0\} = \{0\},$$

so $az = \{0\}$. This means that $a \in \operatorname{ann}(z)$. Hence, $\operatorname{ann}(x) \cap \operatorname{ann}(y) \leq \operatorname{ann}(z)$. By Proposition 3.1.5(i), we obtain $\operatorname{ann}(z) \leq_{ess\ R} R$. This shows that $z \in Z(M)$. Hence, $x - y \subseteq Z(M)$.

Let $z' \in rx$. Claim that $\operatorname{ann}(x) \leq \operatorname{ann}(z')$. Let $b \in \operatorname{ann}(x)$. Then $bx = \{0\}$. By the commutivity of R,

$$bz' \subseteq b(rx) = (br)x = (rb)x = r(bx) = r\{0\} = \{0\}.$$

This implies that $bz' = \{0\}$, so that $b \in \operatorname{ann}(z')$. Thus, $\operatorname{ann}(x) \leq \operatorname{ann}(z')$. Then $\operatorname{ann}(z') \leq_{ess\ R} R$ by Proposition 3.1.5(i), so $z' \in Z(M)$. This shows that $rx \subseteq Z(M)$. This concludes that $Z(M) \leq M$ from Proposition 1.3.4. $\square$

For a module M' over a ring R' (commutativity is not assumed), the set

$$\left\{x \in M' : \{s \in R' : sx = 0\} \text{ is essential in } R'\right\}$$

can be verified that it is a submodule of M' and it is called the *singular submodule* of M'.

However, for an R-hypermodule M, the condition that R is commutative is important in order to illustrate that

$$Z(M) = \{m \in M : \operatorname{ann}(m) \leq_{ess\ R} R\}$$

is a subhypermodule of M in Proposition 3.3.1. Hence, the commutivity of hyerrings in this section is necessary.

Definition 3.3.2. Let R be a commutative hyperring. The subhypermodule $Z(M)$ of an R-hypermodule M defined in Proposition 3.3.1 is called the *singular subhypermodule*.

Definition 3.3.3. Let M be an R-hypermodule where R is commutative. We say that M is a *singular hypermodule* if $Z(M) = M$, and M is a *nonsingular hypermodule* if $Z(M) = \{0\}$.

From Proposition 1.3.17, for a subhypermodule N of an R-hypermodule M, we know that for each $\tilde{K} \leq M/N$, there exists a unique subhypermodule K of M containing N such that $K/N = \tilde{K}$. Form this conclusion, we can define the second singular subhypermodule as below.

Definition 3.3.4. Let M be an R-hypermodule where R is commutative. The *second singular subhypermodule* of M, denoted by $Z_2(M)$, is the subhypermodule of M containing $Z(M)$ such that

$$Z_2(M)/Z(M) = Z(M/Z(M)).$$

Definition 3.3.5. Let R be a commutative hyperring. An R-hypermodule M is said to be *Z_2-torsion* if $Z_2(M) = M$.

Remark 3.3.6. Every singular hypermodule is always Z_2-torsion.

Recall that for any subhypermodules N and N' of an R-hypermodule M with $N' \leq N$, we use the symbols $\boxplus$ and $\boxdot$ for the quotient R-hypermodule $(M/N, \boxplus, \boxdot)$ throughout this thesis; moreover, the symbols $\uplus$ and $\circledast$ are used for the quotient R-hypermodule $\big((M/N)/(N'/N), \uplus, \circledast\big)$.

Proposition 3.3.7. *Let M be an R-hypermodule where R is commutative. The following statements hold.*

(i) $Z(M) = \{x \in M : Ix = \{0\}$ for some $I \leq_{ess} {}_RR\}$.

(ii) $Z_2(M) = \{x \in M : Ix \subseteq Z(M)$ for some $I \leq_{ess} {}_RR\}$.

(iii) $Z(M) \leq_{ess} Z_2(M)$.

Proof. For convenience, let $K = \{x \in M : Ix = \{0\}$ for some $I \leq_{ess} {}_RR\}$ and $L = \{x \in M : Ix \subseteq Z(M)$ for some $I \leq_{ess} {}_RR\}$.

(i) Let $x \in Z(M)$. Then $\operatorname{ann}(x) \leq_{ess} {}_RR$. We see that $\operatorname{ann}(x)x = \{0\}$. This means that $x \in K$. Hence, $Z(M) \subseteq K$. Next, let $y \in K$. Then $Iy = \{0\}$ for some $I \leq_{ess} {}_RR$. This implies that $I \leq \operatorname{ann}(y)$. By Proposition 3.1.5(i), $\operatorname{ann}(y) \leq_{ess} {}_RR$, so $y \in Z(M)$. This shows that $K \subseteq Z(M)$. Therefore, $Z(M) = K$.

(ii) Let $x' \in Z_2(M)$. Then

$$x' + Z(M) \in Z_2(M)/Z(M) = Z(M/Z(M)).$$

By (i), $J \boxdot (x' + Z(M)) = \{Z(M)\}$ for some $J \leq_{ess\ R} R$. Thus, $Jx' \subseteq Z(M)$. This means that $x' \in L$. Hence, $Z_2(M) \subseteq L$. Next, let $y' \in L$. Then $J'y' \subseteq Z(M)$ for some $J' \leq_{ess\ R} R$. Thus, $J' \boxdot (y' + Z(M)) = \{Z(M)\}$. By (i),

$$y' + Z(M) \in Z(M/Z(M)) = Z_2(M)/Z(M).$$

This implies that $y' \in Z_2(M)$, so $L \subseteq Z_2(M)$. Therefore, $Z_2(M) = L$.

(iii) Let $N \leq Z_2(M)$ with $N \cap Z(M) = \{0\}$. To show that $N = \{0\}$, let $x \in N$. By (ii), $J''x \subseteq Z(M)$ for some $J'' \leq_{ess\ R} R$. Since $x \in N$ and $N \leq M$, we get $J''x \subseteq N$. Then $J''x \subseteq N \cap Z(M) = \{0\}$, i.e., $J''x = \{0\}$. By (i), $x \in Z(M)$, so $N \leq Z(M)$. Hence, $N = N \cap Z(M) = \{0\}$. This shows that $Z(M) \leq_{ess} Z_2(M)$. □

Corollary 3.3.8. *Let M be an R-hypermodule where R is commutative. Then $Z(M) = \{0\}$ if and only if $Z_2(M) = \{0\}$.*

Proof. This follows from Proposition 3.3.7(i) and (ii). □

Corollary 3.3.9. *Let N be a subhypermodule of an R-hypermodule M where R is commutative. The following statements hold.*

(i) $Z(N) = Z(M) \cap N$.

(ii) $Z_2(N) = Z_2(M) \cap N$.

(iii) N is a singular hypermodule if and only if $N \leq Z(M)$. In particular, $Z(M)$ is always a singular hypermodule.

(iv) N is Z_2-torsion if and only if $N \leq Z_2(M)$. In particular, $Z_2(M)$ is always Z_2-torsion.

Proof. (i) This follows from Proposition 3.3.7(i).

(ii) We obtain from Proposition 3.3.7(ii) that

$$\begin{aligned}
Z_2(M) \cap N &= \{x \in M : Ix \subseteq Z(M) \text{ for some } I \leq_{ess} {}_RR\} \cap N \\
&= \{x \in N : Ix \subseteq Z(M) \text{ for some } I \leq_{ess} {}_RR\} \\
&= \{x \in N : Ix \subseteq Z(M) \cap N \text{ for some } I \leq_{ess} {}_RR\} \quad (\text{since } N \leq M) \\
&= \{x \in N : Ix \subseteq Z(N) \text{ for some } I \leq_{ess} {}_RR\} \qquad (\text{by(i)}) \\
&= Z_2(N).
\end{aligned}$$

(iii) and (iv) are directly obtained from (i) and (ii), respectively. □

Proposition 3.3.10. *Let M and M' be R-hypermodules where R is commutative. Let $f : M \to M'$ be a homomorphism such that $f(0) = 0$. Then $f(Z(M)) \leq Z(M')$ and $f(Z_2(M)) \leq Z_2(M')$.*

Proof. It suffices to show that $f(Z(M)) \subseteq Z(M')$ and $f(Z_2(M)) \subseteq Z_2(M')$. First, let $x \in Z(M)$. By Proposition 3.3.7(i), $Ix = \{0\}$ for some $I \leq_{ess} {}_RR$. Hence,

$$If(x) = f(Ix) = f(\{0\}) = \{0\}.$$

Then $f(x) \in Z(M')$ by Proposition 3.3.7(i). We conclude that $f(Z(M)) \subseteq Z(M')$.

Finally, let $y \in f(Z_2(M))$. By Proposition 3.3.7(ii), we get $Jy \subseteq Z(M)$ for some $J \leq_{ess} {}_RR$. Therefore,

$$Jf(y) = f(Jy) \subseteq f(Z(M)) \subseteq Z(M').$$

Now, $f(y) \in Z_2(M')$ by Proposition 3.3.7(ii). Hence, $f(Z_2(M)) \subseteq Z_2(M')$. □

Corollary 3.3.11. *Let M and M' be R-hypermodules where R is commutative. The following statements hold.*

(i) Let $f : M \to M'$ be a surjective homomorphism. If M is a singular hypermodule (Z_2-torsion), then M' is also a singular hypermodule (Z_2-torsion).

(ii) Let $g : M' \to M$ be an injective homomorphism such that $g(0) = 0$. If M is a nonsingular hypermodule, then M' is also a nonsingular hypermodule.

Proof. (1) Assume that $Z(M) = M$. By the surjectivity of f and Proposition 2.1.10, we have $f(0) = 0$, By Proposition 3.3.10,

$$Z(M') \leq M' = f(M) = f(Z(M)) \leq Z(M').$$

Hence, $Z(M') = M'$.

(2) Assume that $Z(M) = \{0\}$. By Proposition 2.1.9, $\ker(g) = \{0\}$. Now, $g(Z(M')) \leq Z(M) = \{0\}$ by Proposition 3.3.10. This means that $Z(M') \leq \ker(g) = \{0\}$. Therefore, $Z(M') = \{0\}$. □

Corollary 3.3.12. *Let N be a subhypermodule of a Z_2-torsion R-hypermodule M where R is commutative. Then N and M/N are Z_2-torsion R-hypermodules.*

Proof. By Corollary 3.3.9(iv), N is Z_2-torsion. Moreover, we conclude that M/N is Z_2-torsion which follows from Corollary 3.3.11(i) by choosing $g : M \to M/N$ to be the canonical map. □

Proposition 3.3.13. *Let R be a commutative hyperring. Let K and N be subhypermodules of an R-hypermodule M such that $K \leq N$ and $N/K = Z(M/K)$. If M/K is Z_2-torsion, then M/N is a singular hypermodule.*

Proof. Assume that M/K is Z_2-torsion. Then $Z_2(M/K) = M/K$. Hence,

$$Z[(M/K)/Z(M/K)] = Z_2(M/K)/Z(M/K) = (M/K)/Z(M/K).$$

This means that $(M/K)/Z(M/K)$ is a singular hypermodule, but then

$$(M/K)/(N/K) = (M/K)/Z(M/K),$$

so $(M/K)/(N/K)$ is a singular hypermodule. By the Third Isomorphism Theorem, M/N is a singular hypermodule. □

Proposition 3.3.14. *Let R be a commutative hyperring. Let K and N be subhypermodules of an R-hypermodule M such that $M = N \oplus K$. Then $Z(M) = Z(N) \oplus Z(K)$.*

Proof. By Proposition 3.3.10, we know that $Z(M) \leq_p M$. We conclude that $Z(M) = Z(N) \oplus Z(K)$ by Proposition 2.1.20 and Corollary 3.3.9(i). □

Next, we give some results involving the singularity and the nonsingularity of strongly distributive commutative hyperrings.

Proposition 3.3.15. *Let I and J be hyperideals of a strongly distributive commutative hyperring R such that $I \leq J$. If $I \leq_{ess} J$, then ${}_R(J/I)$ is singular.*

Proof. Assume that $I \leq_{ess} J$. We show that $Z(J/I) = J/I$. It remains to show that $J/I \subseteq Z(J/I)$. Note that $Z(J/I) = Z(R/I) \cap J/I$ by Corollary 3.3.9(i). So, it suffices to show that $J/I \subseteq Z(R/I)$. To see this, let $a + I \in J/I$ where $a \in J$. Recall that

$$Z(R/I) = \{r + I : \text{ann}(r + I) \leq_{ess} {}_R R\} \text{ and } \text{ann}(a + I) = \{r \in R : ra \subseteq I\}.$$

Claim that $\text{ann}(a + I) \leq_{ess} {}_R R$. To see this, let K be a hyperideal of R such that $\text{ann}(a + I) \cap K = \{0\}$. We show that $K = \{0\}$. Since I is a hyperideal of R, we obtain $Ia \subseteq I$. This implies that $I \subseteq \text{ann}(a + I)$. Now, $Ka \subseteq K$ because K is a hyperideal of R. Then, $I \cap Ka \subseteq \text{ann}(a + I) \cap K = \{0\}$, so $I \cap Ka = \{0\}$. Since $a \in J$ and J is a hyperideal of R, we have $Ka \subseteq J$. Moreover, $Ka \leq J$, this follows from Proposition 1.3.9. Because $I \leq_{ess} J$, $Ka \leq J$ and $I \cap Ka = \{0\}$, it follows that $Ka = \{0\} \subseteq I$. This means that $K \subseteq \text{ann}(a + I)$. Therefore,

$$K = \text{ann}(a + I) \cap K = \{0\}.$$

This shows that $\text{ann}(a + I) \leq_{ess} {}_R R$. Then $a + I \in Z(R/I)$. This concludes that $J/I \subseteq Z(R/I)$. □

Corollary 3.3.16. *Let I, J and K be hyperideals of a strongly distributive commutative hyperring R such that $I \leq J \leq K$. If $J/I \leq_{ess} K/I$, then ${}_R[(K/I)/(J/I)]$ is singular.*

Proof. Assume that $J/I \leq_{ess} K/I$. Then $J \leq_{ess} K$ by Proposition 3.2.13(i). By Proposition 3.3.15, ${}_R(K/J)$ is singular. We conclude that ${}_R[(K/I)/(J/I)]$ is singular by the Third Isomorphism Theorem. □

Proposition 3.3.17. *Let R be a strongly distributive commutative hyperring. The following statements hold.*

(i) $Z_2(R) \leq_{cl} {}_RR$.

(ii) ${}_R(R/Z_2(R))$ *is nonsingular.*

Proof. (i) Let J be a hyperideal of R such that $Z_2(R) \leq_{ess} J$. Then $Z(R) \leq_{ess} J$ by Proposition 3.3.7(iii) and Proposition 3.1.5(i). Moreover, we know that ${}_R(J/Z(R))$ is singular by Proposition 3.3.15. We show that $Z_2(R) = J$ which suffices to show that $J \subseteq Z_2(R)$. Let $a \in J$. Then $a + Z(R) \in J/Z(R) = Z(J/Z(R))$. By Proposition 3.3.7(i),

$$K \boxdot (a + Z(R)) = \{Z(R)\} \quad \text{for some } K \leq_{ess} {}_RR.$$

This implies that $Ka \subseteq Z(R)$ which gives $a \in Z_2(R)$ by Proposition 3.3.7(ii). This shows that $J \subseteq Z_2(R)$. Hence, $Z_2(R) = J$. We conclude that $Z_2(R) \leq_{cl} {}_RR$ by Proposition 3.2.8.

(ii) By Proposition 1.3.16, there exists a hyperideal I of R containing $Z_2(R)$ such that $I/Z_2(R) = Z(R/Z_2(R))$. We show that $Z_2(R) = I$. First, we claim that $Z_2(R) \leq_{ess} I$. To see this, let $I' \leq I$ be such that $Z_2(R) \cap I' = \{0\}$. To show that $I' = \{0\}$, let $c \in I'$. Because $I' \subseteq I$, it follows that

$$c + Z_2(R) \in I/Z_2(R) = Z(R/Z_2(R)).$$

By Proposition 3.3.7(i),

$$L \boxdot (c + Z_2(R)) = \{Z_2(R)\} \quad \text{for some } L \leq_{ess} {}_RR.$$

This implies that $Lc \subseteq Z_2(R)$. Since I' is a hyperideal of R and $c \in I'$, we obtain $Lc \subseteq I'$. Then $Lc \subseteq Z_2(R) \cap I' = \{0\}$, so $Lc = \{0\} \subseteq Z(R)$. Now, we get $c \in Z_2(R)$ by Proposition 3.3.7(ii). Thus, $c \in Z_2(R) \cap I' = \{0\}$, i.e., $c = 0$. This means that $I' = \{0\}$. Hence, $Z_2(R) \leq_{ess} I$. By Proposition 3.2.8 and (i), we obtain $Z_2(R) = I$. Therefore, ${}_R(R/Z_2(R))$ is nonsingular. □

Corollary 3.3.18. *Let I be a hyperideal of a strongly distributive commutative hyperring R. The following statements hold.*

(i) $Z_2(R/I) \leq_{cl} {}_R(R/I)$.

(ii) ${}_R[(R/I)/(Z_2(R/I))]$ *is nonsingular.*

Proof. The proofs are similar to the proofs of Proposition 3.3.17. □

Proposition 3.3.19. *Let I and J be hyperideals of a strongly distributive commutative hyperring R such that $I \leq J$. If ${}_R(J/I)$ and ${}_R[(R/I)/(J/I)]$ are Z_2-torsion, then ${}_R(R/I)$ is Z_2-torsion.*

Proof. Assume ${}_R(J/I)$ and ${}_R[(R/I)/(J/I)]$ are Z_2-torsion. Then

$$Z_2(J/I) = J/I \text{ and } Z_2[(R/I)/(J/I)] = (R/I)/(J/I).$$

We show that $Z_2(R/I) = R/I$. It suffices to show that $R/I \subseteq Z_2(R/I)$. To see this, let $r + I \in R/I$ where $r \in R$. Then

$$(r + I) \boxplus (J/I) \in (R/I)/(J/I) = Z_2[(R/I)/(J/I)].$$

By Proposition 3.3.7(ii),

$$K \circledast [(r + I) \boxplus (J/I)] \subseteq Z[(R/I)/(J/I)] \quad \text{for some } K \leq_{ess} {}_R R.$$

Claim that $K \boxdot (r + I) \subseteq Z_2(R/I)$. Let $a + I \in K \boxdot (r + I)$. Then

$$(a+I) \boxplus (J/I) \in (K \boxdot (r+I)) \boxplus (J/I) = K \circledast [(r+I) \boxplus (J/I)] \subseteq Z[(R/I)/(J/I)].$$

By Proposition 3.3.7(i),

$$L \circledast [(a + I) \boxplus (J/I)] = \{J/I\} \quad \text{for some } L \leq_{ess} {}_R R.$$

Thus, $L \boxdot (a+I) \subseteq J/I$. Since ${}_R(J/I)$ is Z_2-torsion, by Corollary 3.3.9(iv), we get $J/I \subseteq Z_2(R/I)$. Then $L \boxdot (a + I) \subseteq Z_2(R/I)$. Therefore,

$$L \circledast [(a + I) \boxplus Z_2(R/I)] = \{Z_2(R/I)\}.$$

By Proposition 3.3.7(i) and Corollary 3.3.18(ii),

$$(a + I) \boxplus Z_2(R/I) \in Z[(R/I)/(Z_2(R/I))] = \{Z_2(R/I)\}.$$

This means that $a + I \in Z_2(R/I)$. This shows that $K \boxdot (r + I) \subseteq Z_2(R/I)$. It follows from $K \boxdot (r + I) \subseteq Z_2(R/I)$ that

$$K \circledast [(r + I) \boxplus Z_2(R/I)] = \{Z_2(R/I)\}.$$

By Proposition 3.3.7(i) and Corollary 3.3.18(ii),

$$(r+I)\boxplus Z_2(R/I) \in Z[(R/I)/(Z_2(R/I))] = \{Z_2(R/I)\}.$$

Thus, $r+I \in Z_2(R/I)$. This shows that $R/I \subseteq Z_2(R/I)$. Hence, $Z_2(R/I) = R/I$. We conclude that ${}_R(R/I)$ is Z_2-torsion. □

3.4 t-Essential Subhypermodules and t-Closed Subhypermodules

Throughout this section, all hyperrings are required to be commutative. In this section, we use the second singular subhypermodule to define t-essential subhypermodules, and then the concept of t-closed subhypermodules is given by using t-essential subhypermodules. In addition, we present characterizations of t-essential subhypermodules and t-closed subhypermodules; however, we only focus on strongly distributive hyperrings by considering them as hypermodules over itself.

Definition 3.4.1. Let R be a commutative hyperring. A subhypermodule N of an R-hypermodule M is called a *t-essential subhypermodule* of M (or *t-essential* in M), denoted by $N \leq_{tess} M$, if $L \leq Z_2(M)$ for any $L \leq M$ with $N \cap L \leq Z_2(M)$.

According to Corollary 3.3.8, for an R-hypermodule M where R is commutative, essential subhypermodules of M and t-essential subhypermodules of M coincide when the R-hypermodule M is a nonsingular hypermodule, i.e., $Z(M) = \{0\}$.

Definition 3.4.2. A hyperideal I of a commutative hyperring R is called a *t-essential hyperideal* of R if I is a t-essential subhypermodule of ${}_RR$.

Remark 3.4.3. Let N be a subhypermodule of an R-hypermodule M where R is commutative. If N is Z_2-torsion, then $K \leq_{tess} N$ for any $K \leq N$.

Proposition 3.4.4. *Let I be a hyperideal of a strongly distributive commutative hyperring R. The following statements are equivalent:*

(i) $I \leq_{tess\ R}R$;

(ii) $(I + Z_2(R))/Z_2(R) \leq_{ess\ R}(R/Z_2(R))$;

(iii) $I + Z_2(R) \leq_{ess\ R}R$;

(iv) $_R(R/I)$ *is* Z_2*-torsion.*

Proof. (i)⇒(ii) Assume that $I \leq_{tess\ R}R$. Claim that $I + Z_2(R) \leq_{ess\ R}R$. Let J be a complement of I in R. Then $I \cap J = \{0\} \leq Z_2(R)$. Since $I \leq_{tess\ R}R$, we get $J \leq Z_2(R)$, so $I \oplus J \leq I + Z_2(R)$. Then $I \oplus J \leq_{ess\ R}R$ by Proposition 3.2.3. Thus, $I + Z_2(R) \leq_{ess\ R}R$ by Proposition 3.1.5(i). By Proposition 3.2.13(ii) and Proposition 3.3.17(i), we conclude that $(I + Z_2(R))/Z_2(R) \leq_{ess\ R}(R/Z_2(R))$.

(ii)⇒(iii) This follows from Proposition 3.2.13(i).

(iii)⇒(iv) Assume that $I + Z_2(R) \leq_{ess\ R}R$. Then $_R(R/(I + Z_2(R)))$ is singular by Proposition 3.3.15; moreover, it is Z_2-torsion by Remark 3.3.6. By the Third Isomorphism Theorem, $_R[(R/I)/((I + Z_2(R))/I)]$ is also Z_2-torsion. Recall that $Z_2(R)$ is Z_2-torsion. By Corollary 3.3.12, $Z_2(R)/(I \cap Z_2(R))$ is also Z_2-torsion. Therefore, $(I+Z_2(R))/I$ is Z_2-torsion by the Second Isomorphism Theorem. Now,

$$_R[(R/I)/((I + Z_2(R))/I)] \text{ and } (I + Z_2(R))/I$$

are Z_2-torsion. We conclude that $_R(R/I)$ is Z_2-torsion by Proposition 3.3.19.

(iv)⇒(i) Assume that $_R(R/I)$ is Z_2-torsion. Let I' be a hyperideal of R containing I such that $I'/I = Z(R/I)$. Thus, $_R(R/I')$ is singular by Proposition 3.3.13. To show that $I \leq_{tess\ R}R$, let J be a hyperideal of R such that $I \cap J \leq Z_2(R)$. We show that $J \leq Z_2(R)$. Let $a \in J$. Then $a + I' \in R/I' = Z(R/I')$ since $_R(R/I')$ is singular. By Proposition 3.3.7(i),

$$L \boxdot (a + I') = \{I'\} \quad \text{for some } L \leq_{ess\ R}R.$$

Then $La \subseteq I'$. Claim that $La \subseteq Z_2(R)$. Let $t \in La$. Then $t + I \in I'/I = Z(R/I)$. By Proposition 3.3.7(i),

$$K \boxdot (t + I) = \{I\} \quad \text{for some } K \leq_{ess\ R}R.$$

Hence, $Kt \subseteq I$. Because J is a hyperideal of R and $a \in J$, it follows that

$$Kt \subseteq K(La) = (KL)a \subseteq J.$$

This implies that $Kt \subseteq I \cap J \subseteq Z_2(R)$. Thus, $K \boxdot (t + Z_2(R)) = \{Z_2(R)\}$ By Proposition 3.3.7(i) and Proposition 3.3.17(ii),

$$t + Z_2(R) \in Z(R/Z_2(R)) = \{Z_2(R)\}.$$

Thus, $t \in Z_2(R)$. This shows that $La \subseteq Z_2(R)$. Hence, $L \boxdot (a + Z_2(R)) = \{Z_2(R)\}$. By Proposition 3.3.7(i) and Proposition 3.3.17(ii),

$$a + Z_2(R) \in Z(R/Z_2(R)) = \{Z_2(R)\}.$$

Thus, $a \in Z_2(R)$. This shows that $J \leq Z_2(R)$. Therefore, $I \leq_{tess} {}_RR$. □

For an essential hyperideal I of a strongly distributive commutative hyperring R, ${}_R(R/I)$ is singular from Proposition 3.3.15, and it is Z_2-torsion from Remark 3.3.6 which implies that I is a t-essential hyperideal of R by Proposition 3.4.4. This conculdes that every essential hyperideal of a strongly distributive commutative hyperring R is a t-essential hyperideal of R, and they coincide when ${}_RR$ is nonsingular; moreover, by Proposition 3.2.3 and Proposition 3.4.4, we can conclude that every complement of $Z_2(R)$ in R is a t-essential hyperideal of R.

Definition 3.4.5. Let R be a commutative hyperring. A subhypermodule K of an R-hypermodule M is called a *t-closed subhypermodule* of M (or *t-closed* in M), denoted by $K \leq_{tcl} M$, if $K = K'$ for any $K' \leq M$ with $K \leq_{tess} K'$.

Definition 3.4.6. A hyperideal I of a commutative hyperring R is called a *t-closed hyperideal* of R if I is a t-closed subhypermodule of ${}_RR$.

For a strongly distributive commutative hyperring R, every essential hyperideal of R is a t-essential hyperideal of R by Proposition 3.4.4, and from this reason, we can conclude that every t-closed hyperideal of R is a closed hyperideal of R; moreover, they are identical when ${}_RR$ is nonsingular.

Proposition 3.4.7. *Let J be a hyperideal of a strongly distributive commutative hyperring R. The following statements are equivalent:*

(i) $J \leq_{tcl} {}_RR$*;*

(ii) J *contains* $Z_2(R)$ *and* $J/Z_2(R) \leq_{cl} {}_R(R/Z_2(R))$*;*

(iii) J *contains* $Z_2(R)$ *and* $J \leq_{cl} {}_RR$*;*

(iv) ${}_R(R/J)$ *is nonsingular.*

Proof. (i)⇒(ii) Assume that $J \leq_{tcl} {}_RR$. By Proposition 3.3.9(iv), we know that $Z_2(R)$ is Z_2-torsion. Then $J \cap Z_2(R)$ and $Z_2(R)/(J \cap Z_2(R))$ are Z_2-torsion by Corollary 3.3.12. Hence, $(J + Z_2(R))/J$ is Z_2-torsion by the Second Isomorphism Theorem. In addition, by Proposition 3.4.4, we obtain $J \leq_{tess} J + Z_2(R)$, but then $J \leq_{tcl} {}_RR$, so $J + Z_2(R) = J$. It follows that $Z_2(R) \leq J + Z_2(R) = J$.

Next, we show that $J/Z_2(R) \leq_{cl} {}_R(R/Z_2(R))$. Let J' be a hyperideal of R containing $Z_2(R)$ such that $J/Z_2(R) \leq_{ess} J'/Z_2(R)$. Note that $J + Z_2(R) = J$ since $Z_2(R) \leq J$. Thus, we can view

$$(J + Z_2(R))/Z_2(R) \leq_{ess} J'/Z_2(R).$$

By Proposition 3.4.4, $J \leq_{tess} J'$. Because $J \leq_{tcl} {}_RR$, it follows that $J = J'$, so $J/Z_2(R) = J'/Z_2(R)$. Therefore, $J/Z_2(R) \leq_{cl} {}_R(R/Z_2(R))$ by Proposition 3.2.8.

(ii)⇒(iii) This follows from Proposition 3.2.14(ii) and Proposition 3.3.17(i).

(iii)⇒(iv) Assume that J contains $Z_2(R)$ and $J \leq_{cl} {}_RR$. Let J' be a hyperideal of R containing J such that $Z(R/J) = J'/J$. We show that $J' = J$. Since $J \leq_{cl} {}_RR$, by Proposition 3.2.8, it suffices to show that $J \leq_{ess} J'$. Let $I' \leq J'$ be such that $I' \cap J = \{0\}$. To show that $I' = \{0\}$, let $a \in I'$. Then $a + J \in J'/J = Z(R/J)$. By Proposition 3.3.7(i),

$$K \boxdot (a + J) = \{J\} \quad \text{for some } K \leq_{ess} {}_RR.$$

Thus, $Ka \subseteq J$. Since I' is a hyperideal of R and $a \in I'$, we get $Ka \subseteq I'$. This implies that $Ka \subseteq I' \cap J = \{0\}$, i.e., $Ka = \{0\}$. By Proposition 3.3.7(i), we have $a \in Z(R) \leq Z_2(R) \leq J$. This implies that $a \in I' \cap J = \{0\}$, i.e., $a = 0$. Thus, $I' = \{0\}$. This shows that $J \leq_{ess} J'$. Then $J' = J$. Hence, $Z(R/J) = \{J\}$. We conclude that ${}_R(R/J)$ is nonsingular.

(iv)⇒(i) Assume that ${}_R(R/J)$ is nonsingular. Then $Z(R/J) = \{J\}$. Thus, $Z_2(R/J) = \{J\}$ by Corollary 3.3.8. To show that $J \leq_{tcl} {}_R R$, let J' be a hyperideal of R such that $J \leq_{tess} J'$. By Proposition 3.4.4, we obtain J'/J is Z_2-torsion. Hence,

$$J'/J = Z_2(J'/J) \leq Z_2(R/J) = \{J\}.$$

This forces that $J'/J = \{J\}$, i.e., $J' = J$. Therefore, $J \leq_{tcl} {}_R R$. □

CHAPTER IV
EXTENDING HYPERMODULES, C_{11}-HYPERMODULES AND t-EXTENDING HYPERRINGS

In this chapter, we give the concepts of extending hypermodules, C_{11}-hypermodules and t-extending hypermodules which generalize extending modules, C_{11}-modules and t-extending modules, respectively. The main purpose of this chapter is to present characterizations of extending hypermodules, C_{11}-hypermodules and t-extending hyperrings. Moreover, decompositions of C_{11}-hypermodules are investigated.

4.1 Extending Hypermodules

Let us start with the concept of extending hypermodules which concerns direct summands and essential subhypermodules.

Definition 4.1.1. An R-hypermodule M is called an *extending hypermodule* if for each $N \leq M$, there exists $D \leq_{\oplus} M$ such that $N \leq_{ess} D$.

Definition 4.1.2. A hyperring R is called an *extending hyperring* if ${}_RR$ is an extending hypermodule.

First, characterizations of extending hypermodules involving closed subhypermodules and essentiality of direct sums of two subhypermodules are given. In addition, we characterize strongly distributive extending hypermodules M satisfying the condition that $m \in Rm$ for all $m \in M$ by using the lifting of homomorphisms from some subhypermodules of M into M itself.

Theorem 4.1.3. *Let M be an R-hypermodule. The following statements are equivalent:*

(i) M is an extending hypermodule;

(ii) every closed subhypermodule of M is a direct summand of M;

(iii) for any $K, L \leq M$ with $K \cap L = \{0\}$, there exists a direct summand D of M such that $L \leq D$ and $D \oplus K \leq_{ess} M$.

Proof. (i)⇒(ii) Assume that M is an extending hypermodule. Let $C \leq_{cl} M$. Then there exists $C' \leq_{\oplus} M$ such that $C \leq_{ess} C'$. By Proposition 3.2.8, $C = C'$.

(ii)⇒(iii) Assume (ii) holds. Let $K, L \leq M$ be such that $K \cap L = \{0\}$. By applying Zorn's lemma to the set $\{H \leq M : K \cap H = \{0\}$ and $L \leq H\}$, there exists $D \leq M$ such that D is a complement of K in M and $L \leq D$. Then $D \leq_{cl} M$. By the assumption and Proposition 3.2.3, we obtain $D \leq_{\oplus} M$ and $D \oplus K \leq_{ess} M$, respectively.

(iii)⇒(i) Assume (iii) holds. Suppose that $L \leq M$. Let K be a complement of L in M. Then $K \cap L = \{0\}$. By the assumption, there exists $D \leq_{\oplus} M$ such that $L \leq D$ and $D \oplus K \leq_{ess} M$. By Proposition 3.2.6., we conclude that $L \leq_{ess} D$. Therefore, M is an extending hypermodule. □

We already know that every direct summand is a closed subhypermodule, but the converse does not hold in general. However, by Theorem 4.1.3, we can summarize that direct summands and closed subhypermodules of an R-hypermodule are identical provided that the R-hypermodule is an extending hypermodule.

Proposition 4.1.4. *Let M be a strongly distributive R-hypermodule such that $m \in Rm$ for all $m \in M$. Then M is an extending hypermodule if and only if for every closed subhypermodule K of M there exists a complement L of K in M such that every homomorphism $f : K \oplus L \to M$ with $f(0) = 0$ can be extended to a homomorphism $\bar{f} : M \to M$.*

Proof. Assume that M is an extending hypermodule. Let $K \leq_{cl} M$. Then $K \leq_{\oplus} M$ by Theorem 4.1.3, so $M = K \oplus L$ for some $L \leq M$. Thus, L is a complement of K

in M by Proposition 3.2.2 and the result regarding homomorphisms is clear.

Conversely, let $C \leq_{cl} M$. By the assumption, there exists a complement D of C in M such that every homomorphism $f : C \oplus D \to M$ with $f(0) = 0$ can be lifted to a homomorphism $\bar{f} : M \to M$. We show that $C \leq_{\oplus} M$. Let $\pi_C : C \oplus D \to M$ be the projection map on C. Then π_C is a homomorphism with $\pi_C(0) = 0$. Thus there exists a homomorphism $\bar{\pi}_C : M \to M$ such that $\bar{\pi}_C(a) = \pi_C(a)$ for all $a \in C \oplus D$. Especially, $\bar{\pi}_C(c) = c$ for all $c \in C$ and $\bar{\pi}_C(d) = 0$ for all $d \in D$. Note that $C \leq \bar{\pi}_C(M)$ and $D \leq \ker(\bar{\pi}_C)$. Claim that $C \leq_{ess} \bar{\pi}_C(M)$. To see this, assume $0 \neq z \in \bar{\pi}_C(M)$. Then $\bar{\pi}_C(y) = z$ for some $y \in M$. We observe that $y \notin \ker(\bar{\pi}_C)$ since $z \neq 0$. Thus, $y \notin D$. By the assumption, $y \in Ry \leq D + Ry$. This implies that $D \subsetneq D + Ry$. Thus, $C \cap (D + Ry) \neq \{0\}$ since D is a complement of C in M. Let $0 \neq z_0 \in C \cap (D + Ry)$. Then there exist $d \in D$ and $y_0 \in Ry$ such that $z_0 \in d + y_0$. Since $y_0 \in Ry$, there exists $r \in R$ such that $y_0 \in ry$. Therefore,

$$z_0 = \bar{\pi}_C(z_0) \in \bar{\pi}_C(d + y_0) = \bar{\pi}_C(d) + \bar{\pi}_C(y_0) \subseteq 0 + \bar{\pi}_C(ry)$$
$$= \bar{\pi}_C(ry) = r\bar{\pi}_C(y) = rz.$$

This means that $z_0 \in Rz$. Hence, $0 \neq z_0 \in C \cap Rz$. This implies that $C \leq_{ess} \bar{\pi}_C(M)$ by Proposition 3.1.6, but $C \leq_{cl} M$, so $C = \bar{\pi}_C(M)$ by Proposition 3.2.8. Moreover, if $m \in M$, then $\bar{\pi}_C(m) \in C$, so $\bar{\pi}_C^2(m) = \bar{\pi}_C(m)$. This means that $\bar{\pi}_C^2 = \bar{\pi}_C$. By Proposition 2.1.12, we can write $M = \bar{\pi}_C(M) \oplus \ker(\bar{\pi}_C)$. This shows that $C \leq_{\oplus} M$. Therefore, M is an extending hypermodule by Theorem 4.1.3. □

In modules, a submodule of an extending module may not be extending. It follows that a subhypermodule of an extending hypermodule may not be an extending hypermodule in general. However, it can be shown that every closed subhypermodule of an extending hypermodule is also an extending hypermodule which is similar to the result in modules.

Proposition 4.1.5. *Every closed subhypermodule of an extending hypermodule is also an extending hypermodule.*

Proof. Let C be a closed subhypermodule of an extending hypermodule M. Let $K \leq_{cl} C$. Then $K \leq_{cl} M$ by Proposition 3.2.10. This implies that $K \leq_{\oplus} M$ by Theorem 4.1.3. Then $M = K \oplus K'$ for some $K' \leq M$. Note that we can write $C = K \oplus (K' \cap C)$ because $C \leq K$. This means that $K \leq_{\oplus} C$. By Theorem 4.1.3, we conclude that C is an extending hypermodule. □

4.2 C_{11}-Hypermodules

According to [4, 14], C_{11}-modules can be characterized in many way, and there are several results of C_{11}-modules concerning their submodules. In this section, we characterize C_{11}-hypermodules. In addition, projection invariant subhypermodules of C_{11}-hypermodules are investigated.

Definition 4.2.1. An R-hypermodule M is called a C_{11}*-hypermodule* if for each $N \leq M$, there exists a complement K of N in M such that $K \leq_{\oplus} M$.

Definition 4.2.2. A hyperring R is called a C_{11}*-hyperring* if ${}_RR$ is a C_{11}-hypermodule.

Note that every subhypermodule of an R-hypermodule always has a complement which is also a closed subhypermodule. By Theorem 4.1.3, we can conclude that every extending hypermodule is always a C_{11}-hypermodule, but the converse does not hold.

Next, we give characterizations of C_{11}-hypermodules regarding closed subhypermodules and essentiality of direct sums of subhypermodules.

Theorem 4.2.3. *Let M be an R-hypermodule. The following statements are equivalent:*

(i) M is a C_{11}-hypermodule;

(ii) for every closed subhypermodule C of M, there exists a direct summand D of M such that D is a complement of C in M;

(iii) for every closed subhypermodule C of M, there exists a direct summand D of M such that $D \oplus C \leq_{ess} M$;

(iv) for every subhypermodule N of M, there exists a direct summand D of M such that $D \oplus N \leq_{ess} M$.

Proof. (i)⇒(ii) This follows directly from the definition of C_{11}-hypermodules.

(ii)⇒(iii) This is obtained from Proposition 3.2.3.

(iii)⇒(iv) Assume that (iii) holds. Let $N \leq M$. Then there exists $C \leq M$ such that $N \leq_{ess} C$ and $C \leq_{cl} M$ by Proposition 3.2.12. By the assumption, there exists $D \leq_{\oplus} M$ such that $D \oplus C \leq_{ess} M$. Thus, D is a complement of C in M by Proposition 3.2.4. Moreover, D is a complement of N in M by Proposition 3.2.5. By Proposition 3.2.3, we conclude that $D \oplus N \leq_{ess} M$.

(iv)⇒(i) This follows from Proposition 3.2.4. □

Next, we show that any direct sums of two C_{11}-hypermodules must be a C_{11}-hypermodule.

Proposition 4.2.4. *Let K_1 and K_2 be subhypermodules of an R-hypermodule M such that $M = K_1 \oplus K_2$. If K_1 and K_2 are C_{11}-hypermodules, then M is a C_{11}-hypermodule.*

Proof. Assume that K_1 and K_2 are C_{11}-hypermodules. Let $N \leq M$. Since K_1 is a C_{11}-hypermodule, by Theorem 4.2.3, there exists $D_1 \leq_{\oplus} K_1$ such that $D_1 \oplus (N \cap K_1) \leq_{ess} K_1$. By the Modularity Condition,

$$K_1 \cap (D_1 \oplus N) = D_1 \oplus (N \cap K_1) \leq_{ess} K_1.$$

Since K_2 is a C_{11}-hypermodule and $(D_1 \oplus N) \cap K_2 \leq K_2$, by Theorem 4.2.3 again, there exists $D_2 \leq_{\oplus} K_2$ such that $D_2 \oplus [(D_1 \oplus N) \cap K_2] \leq_{ess} K_2$. By the Modularity Condition,

$$K_2 \cap [D_2 \oplus (D_1 \oplus N)] = D_2 \oplus [(D_1 \oplus N) \cap K_2] \leq_{ess} K_2.$$

Let $D = D_2 \oplus D_1$. Since $M = K_1 \oplus K_2, D_1 \leq_{\oplus} K_1$ and $D_2 \leq_{\oplus} K_2$, it follows that $D \leq_{\oplus} M$. In addition, $K_2 \cap (D \oplus N) \leq_{ess} K_2$. Note that $K_1 \cap (D_1 \oplus N) \leq K_1 \cap (D \oplus N)$, but then $K_1 \cap (D_1 \oplus N) \leq_{ess} K_1$, so $K_1 \cap (D \oplus N) \leq_{ess} K_1$ by Proposition 3.1.5(i). Hence,

$$[K_1 \cap (D \oplus N)] \oplus [K_2 \cap (D \oplus N)] \leq_{ess} K_1 \oplus K_2 = M$$

by Proposition 3.1.5(iv). Moreover,

$$[K_1 \cap (D \oplus N)] \oplus [K_2 \cap (D \oplus N)] \leq (K_1 \oplus K_2) \cap (D \oplus N) = M \cap (D \oplus N) = D \oplus N.$$

Thus, $D \oplus N \leq_{ess} M$ by Proposition 3.1.5(i). By Theorem 4.2.3, we conclude that M is a C_{11}-hypermodule. □

According to Smith and Tercan [13], a direct summand of a C_{11}-module may not be a C_{11}-module. This implies that a direct summand of a C_{11}-hypermodule may not be a C_{11}-hypermodule. In this research, we show that if a C_{11}-hypermodule can be decomposed as a direct sum of two subhypermodules, then the subhypermodules are also C_{11}-hypermodules when at least one of them is a projection invariant subhypermodule. To illustrate this statement, the next proposition is needed.

Proposition 4.2.5. *Every projection invariant subhypermodule of a C_{11}-hypermodule is also a C_{11}-hypermodule.*

Proof. Let P be a projection invariant subhypermodule of a C_{11}-hypermodule M. To show that P is a C_{11}-hypermodule, let $N \leq P$. Since M is a C_{11}-hypermodule, there exists $D \leq_{\oplus} M$ such that D is a complement of N in M. Then $M = D \oplus D'$ for some $D' \leq M$. Thus, $P = (P \cap D) \oplus (P \cap D')$ by Proposition 2.1.20, so $P \cap D \leq_{\oplus} P$. Moreover, $N \oplus D \leq_{ess} M$ by Proposition 3.2.3. By the Modularity Condition,

$$N \oplus (P \cap D) = P \cap (N \oplus D) \leq_{ess} P.$$

By Theorem 4.2.3, we conclude that P is a C_{11}-hypermodule. □

Proposition 4.2.6. *Let K_1 and K_2 be subhypermodules of a C_{11}-hypermodule M such that $M = K_1 \oplus K_2$. If $K_1 \leq_p M$, then both K_1 and K_2 are C_{11}-hypermodules.*

Proof. Assume that $K_1 \leq_p M$. By Proposition 4.2.5, we obtain that K_1 is a C_{11}-hypermodule. It remains to show that K_2 is a C_{11}-hypermodule. To see this, let $N_2 \leq K_2$. Since M is a C_{11}-hypermodule, by Theorem 4.2.3, there exists $D \leq_{\oplus} M$ such that $D \oplus (K_1 \oplus N_2) \leq_{ess} M$. Then $M = D \oplus D'$ for some $D' \leq M$. Thus,

$K_1 = (K_1 \cap D) \oplus (K_1 \cap D')$ by Proposition 2.1.20. Since $K_1 \cap D = \{0\}$, we obtain $K_1 = K_1 \cap D'$, so $K_1 \leq D'$. Thus, we can write $D' = K_1 \oplus (K_2 \cap D')$. Then $M = D \oplus K_1 \oplus (K_2 \cap D')$. Let $\pi_2 : K_1 \oplus K_2 \to K_2$ be the projection map on K_2. Note that $K_1 \cap D = \{0\}$. By Proposition 2.1.16, $K_1 \oplus D = K_1 \oplus \pi_2(D)$. Hence,

$$M = D \oplus K_1 \oplus (K_2 \cap D') = K_1 \oplus \pi_2(D) \oplus (K_2 \cap D').$$

Since $\pi_2(D) \leq_{\oplus} M$ and $\pi_2(D) \leq K_2$, we obtain $\pi_2(D) \leq_{\oplus} K_2$. Note that

$$K_1 \oplus \pi_2(D) \oplus N_2 = K_1 \oplus D \oplus N_2 \leq_{ess} M.$$

Then $K_2 \cap (K_1 \oplus \pi_2(D) \oplus N_2) \leq_{ess} K_2$. By the Modularity Condition,

$$K_2 \cap (K_1 \oplus \pi_2(D) \oplus N_2) = (\pi_2(D) \oplus N_2) \oplus (K_2 \cap K_1) = \pi_2(D) \oplus N_2.$$

Thus, $\pi_2(D) \oplus N_2 \leq_{ess} K_2$. By Theorem 4.2.3, K_2 is a C_{11}-hypermodule. □

Proposition 4.2.5 yields that projection invariant subhypermodules of a C_{11}-hypermodule M are also C_{11}-hypermodules; moreover, if they are also closed subhypermodules of M, then they are direct summands of M.

Proposition 4.2.7. *Let C be a subhypermodule of a C_{11}-hypermodule M. If $C \leq_p M$ and $C \leq_{cl} M$, then $C \leq_{\oplus} M$.*

Proof. Since M is a C_{11}-hypermodule, there exists $D \leq_{\oplus} M$ such that D is a complement of C in M. Then $M = D \oplus D'$ for some $D' \leq M$. By Proposition 3.2.3, $C \oplus D \leq_{ess} M$. This implies that $D' \cap (C \oplus D) \leq_{ess} D'$. Thus, $C \leq D'$ by Corollary 2.1.21. By the Modularity Condition,

$$C = C \oplus (D \cap D') = D' \cap (C \oplus D) \leq_{ess} D',$$

but $C \leq_{cl} M$ which concludes that $C = D' \leq_{\oplus} M$ by Proposition 3.2.8. □

Proposition 4.2.8. *Let M be a C_{11}-hypermodule. Then for each $X \leq_p M$, there exist $K_1, K_2 \leq M$ such that $X \leq_{ess} K_2$ and $M = K_1 \oplus K_2$.*

Proof. Let $X \leq_p M$. Since M is a C_{11}-hypermodule, there exists $K_1 \leq_{\oplus} M$ such that K_1 is a complement of X in M. Then $M = K_1 \oplus K_2$ for some $K_2 \leq M$. Let $\pi_1 : K_1 \oplus K_2 \to K_1$ be the projection map on K_1. Then $\pi_1^2 = \pi_1 \in \mathrm{End}_0(M)$,

$\pi_1(M) = K_1$ and $\ker(\pi_1) = K_2$. Claim that $X \leq K_2$. To see this, let $x \in X$. Since $X \leq_p M$, we obtain $\pi_1(x) \in X$. Thus, $\pi_1(x) \in K_1 \cap X = \{0\}$. This implies that $x \in \ker(\pi_1) = K_2$. Hence, $X \leq K_2$. By Proposition 3.2.6, we obtain $X \leq_{ess} K_2$. □

4.3 t-Extending Hyperrings and C_{11}-Hyperrings

Throughout this section, all hyperrings are required to be commutative. First, we give the concept of t-extending hypermodules defined from t-closed subhypermodules given in Section 3.4. Unfortunately, there is a property concerning the essentiality and the singularity (Proposition 3.3.15) which cannot be proved on any hypermodules. However, the problem can be solved on any strongly distributive hyperrings by considering them as hypermodules over itself. Hence, we only focus on t-extending hyperrings throughout this work. In this section, we give characterizations of t-extending hyperrings; moreover, we are interested in C_{11}-hyperrings. Finally, some properties of C_{11}-hyperrings R involving the second singular subhypermodule of R are investigated.

Definition 4.3.1. Let R be a commutative hyperring. An R-hypermodule M is called a *t-extending hypermodule* if every t-closed subhypermodule of M is a direct summand of M.

Definition 4.3.2. A commutative hyperring R is called a *t-extending hyperring* if $_RR$ is a t-extending hypermodule.

Theorem 4.3.3. *Let R be a strongly distributive commutative hyperring. The following statements are equivalent:*

(i) R is a t-extending hyperring;

(ii) there exists a hyperideal I of R such that $R = Z_2(R) \oplus I$ and I is an extending hyperring;

(iii) every hyperideal of R containing $Z_2(R)$ is essential in a direct summand of R;

(iv) every hyperideal of R is t-essential in a direct summand of R.

Proof. (i)⇒(ii) Assume that R is a t-extending hyperring. By Proposition 3.3.17(i) and Proposition 3.4.7, we obtain $Z_2(R) \leq_{tcl} {}_RR$. Thus, $Z_2(R) \leq_{\oplus} R$ since R is t-extending. Then there exists a hyperideal I of R such that $R = Z_2(R) \oplus I$.

Next, we show that I is extending. Let $J \leq_{cl} I$. We show that $J \leq_{\oplus} I$. First, claim that I/J is nonsingular. By Proposition 3.3.14, we can write $Z(R) = Z(Z_2(R)) \oplus Z(I)$. Note that $Z(Z_2(R)) = Z(R)$ by Proposition 3.3.9(i). This implies that $Z(I) = \{0\}$, so $J \leq_{tcl} I$ since $J \leq_{cl} I$. Therefore, I/J is nonsingular from Proposition 3.4.7. The claim is proved. Next, we show that $Z_2(R) \oplus J \leq_{tcl} {}_RR$. By proposition 3.4.7, it suffices to show that ${}_R[R/(Z_2(R) \oplus J)]$ is nonsingular, i.e.,

$$Z[R/(Z_2(R) \oplus J)] = \{Z_2(R) \oplus J\}.$$

Recall that

$$Z[R/(Z_2(R) \oplus J)] = \left\{x + (Z_2(R) \oplus J) : \operatorname{ann}(x + (Z_2(R) \oplus J)) \leq_{ess} {}_RR\right\} \text{ and}$$

for each $x \in R$,

$$\operatorname{ann}(x + (Z_2(R) \oplus J)) = \{z \in R : zx \subseteq Z_2(R) \oplus J\}.$$

Let $r + (Z_2(R) \oplus J) \in Z[R/(Z_2(R) \oplus J)]$. Then $\operatorname{ann}(r + (Z_2(R) \oplus J)) \leq_{ess} {}_RR$. Since $R = Z_2(R) \oplus I$, there exist $z \in Z_2(R)$ and $a \in I$ such that $r \in z + a$. We show that $\operatorname{ann}(a + J) = \operatorname{ann}(r + (Z_2(R) \oplus J))$. Let $s \in \operatorname{ann}(a + J)$. Then $sa \subseteq J$. Thus, $sr \subseteq s(z + a) \subseteq sz + sa \subseteq Z_2(R) \oplus J$. This means that $s \in \operatorname{ann}(r + (Z_2(R) \oplus J))$. Therefore, $\operatorname{ann}(a + J) \subseteq \operatorname{ann}(r + (Z_2(R) \oplus J))$. Let $t \in \operatorname{ann}(r + (Z_2(R) \oplus J))$. Then $tr \subseteq Z_2(R) \oplus J$. Since $r \in z + a$, we can write $a \in r - z$. Hence,

$$ta \subseteq t(r - z) \subseteq tr - tz \subseteq Z_2(R) \oplus J.$$

Since $a \in I$ and I is a hyperideal of R, it follows that $ta \subseteq I$. By the Modularity Condition,

$$ta \subseteq I \cap (Z_2(R) \oplus J) = J \oplus (I \cap Z_2(R)) = J.$$

This means that $t \in \operatorname{ann}(a + J)$, so $\operatorname{ann}(r + (Z_2(R) \oplus J)) \subseteq \operatorname{ann}(a + J)$. Thus, $\operatorname{ann}(a + J) = \operatorname{ann}(r + (Z_2(R) \oplus J)) \leq_{ess} {}_RR$, so $a + J \in Z(I/J) = \{J\}$ since I/J is nonsingular by the claim. This means that $a \in J$, so $r \in z + a \subseteq Z_2(R) \oplus J$.

Therefore, ${}_R(R/(Z_2(R) \oplus J))$ is nonsingular. Thus, $Z_2(R) \oplus J \leq_{tcl} {}_RR$ by proposition 3.4.7. Since ${}_RR$ is t-extending, $Z_2(R) \oplus J \leq_\oplus R$. Then there exists a hyperideal J' of R such that $R = (Z_2(R) \oplus J) \oplus J'$. Since $J \leq I$, we can write $I = [(Z_2(R) \oplus J') \cap I] \oplus J$. This shows that $J \leq_\oplus I$. By Proposition 4.1.3, we conclude that I is an extending hyperring.

(ii)⇒(iii) Assume that there exists a hyperideal I of R such that I is an extending hyperring and $R = Z_2(R) \oplus I$. Let J be a hyperideal of R such that $Z_2(R) \leq J$. Then $J = Z_2(R) \oplus (I \cap J)$. Since I is extending, there exists a hyperideal I' of R such that $I \cap J \leq_{ess} I'$ and $I' \leq_\oplus I$. By Proposition 3.1.5(iv),

$$J = Z_2(R) \oplus (I \cap J) \leq_{ess} Z_2(R) \oplus I'.$$

Claim that $I' \leq_\oplus R$. Since $I' \leq_\oplus I$, there exists $I'' \leq I$ such that $I = I' \oplus I''$. Hence,

$$R = Z_2(R) \oplus I = (Z_2(R) \oplus I') \oplus I''.$$

This implies that $Z_2(R) \oplus I' \leq_\oplus R$.

(iii)⇒(iv) Assume that (iii) holds. Let I be a hyperideal of R. Note that $I + Z_2(R)$ contains $Z_2(R)$. By the assumption, there exists a hyperideal J of R such that $I + Z_2(R) \leq_{ess} J$ and $J \leq_\oplus R$. By Proposition 3.4.4, we conclude that $I \leq_{tess} J$.

(iv)⇒(i) Assume that (iv) holds. Let I be a t-closed hyperideal of R. By the assumption, there exists a hyperideal J of R such that $I \leq_{tess} J$ and $J \leq_\oplus R$. Since $I \leq_{tcl} {}_RR$, we get $I = J$. Hence, $I \leq_\oplus R$. We conclude that R is a t-extending hyperring. □

Proposition 4.3.4. *Let I be a projection invariant hyperideal of a commutative C_{11}-hyperring R with $Z[{}_R(R/I)] = \{I\}$. Then ${}_R(R/I)$ is also a C_{11}-hypermodule.*

Proof. Let $\tilde{J} \leq R/I$. Then there exists a hyperideal J of R containing I such that $\tilde{J} = J/I$. Since R is a C_{11}-hyperring, there exists $J' \leq_\oplus R$ such that J' is a complement of J in R. By Proposition 2.1.15, there exists $f^2 = f \in \mathrm{End}_0(R)$ such that $f(R) = J'$. We divide the details of the proof into two steps as follows.

(1) We show that $(J' \oplus I)/I \leq_{\oplus} R/I$.

Note that $f(R) \cap I = J' \cap I \leq J' \cap J = \{0\}$, so $f(R) \cap I = \{0\}$. Then $f(I) \subseteq I$ since I is a projection invariant hyperideal of R. Hence, $f(I) \subseteq f(R) \cap I = \{0\}$. This means that $f(I) = \{0\}$, i.e., $I \subseteq \ker(f)$. Thus, $(J' \oplus I)/I \leq_{\oplus} R/I$ by Proposition 2.1.13.

(2) Claim that $[(J' \oplus I)/I] \oplus [J/I] \leq_{ess\ R}(R/I)$.

In this step, we divide the details of the proof into two steps.

(2.1) First, we claim that $(J' \oplus J)/I \leq_{ess\ R}(R/I)$.

To show that $(J' \oplus J)/I \leq_{ess} R/I$, let $\tilde{K} \leq R/I$ be such that $(J' \oplus J)/I) \cap \tilde{K} = \{I\}$. Then there exists a hyperideal K of R containing I such that $\tilde{K} = K/I$. We show that $K = I$. It suffices to show that $K \subseteq I$. To see this, let $k \in K$. Since $J' \oplus J$ is a hyperideal of R, we get $(J' \oplus J)k \subseteq J' \oplus J$. Similarly, $(J' \oplus J)k \subseteq K$ since K is a hyperideal of R and $k \in K$. Moreover, if $a \in (J' \oplus J)k$, then $a + I \in (J' \oplus J)/I \cap K/I = \{I\}$, so $a \in I$. This implies that $(J' \oplus J)k \subseteq I$. Recall that $\text{ann}(k+I) = \{r \in R : rk \subseteq I\}$. Hence, $J' \oplus J \subseteq \text{ann}(k+I)$. Since J' is a complement of J in R, by Proposition 3.2.3, $J' \oplus J \leq_{ess\ R} R$. Thus, $\text{ann}(k + I) \leq_{ess\ R} R$ by Proposition 3.1.5(i). This implies that $k + I \in Z[_R(R/I)] = \{I\}$, so $k \in I$. Therefore, $K \subseteq I$. This shows that $(J' \oplus J)/I \leq_{ess\ R}(R/I)$.

(2.2) Finally, $[(J' \oplus I)/I] \oplus [J/I] = (J' \oplus J)/I$.

The proof of this step is straightforward.

From step (1) and step (2), by Proposition 3.2.4, we obtain that $(J' \oplus I)/I$ is a complement of $\tilde{J} = J/I$ in R/I. Hence, $_R(R/I)$ is a C_{11}-hypermodule. □

Proposition 4.3.5. *Let R be a strongly distributive commutative C_{11}-hyperring. Then $Z_2(R)$ is also a C_{11}-hyperring and it is a direct summand of R.*

Proof. By Proposition 3.3.10 and Proposition 3.3.17, we obtain $Z_2(R) \leq_p R$ and $Z_2(R) \leq_{cl} R$, respectively. Hence, $Z_2(R)$ is a C_{11}-hyperring by Proposition 4.2.5. Moreover, $Z_2(R)$ is a direct summand of R by Proposition 4.2.7. □

Corollary 4.3.6. *Let R be a strongly distributive commutative C_{11}-hyperring. Then there exists a hyperideal J of R such that $R = Z_2(R) \oplus J$ and J is a C_{11}-*

hyperring.

Proof. By Proposition 4.3.5, there exists a hyperideal J of R with $R = Z_2(R) \oplus J$. Note that $Z_2(R) \leq_p R$ by Proposition 3.3.10. Hence, J is a C_{11}-hyperring by Proposition 4.2.6. □

Lemma 4.3.7. *Let R be a strongly distributive commutative hyperring and $f^2 = f \in \mathrm{End}_0(R)$. If there exists a hyperideal K of R such that $K \leq_p R$ and $K \leq_{ess} f(R)$, then $f(R) + Z(R) \leq_p R$.*

Proof. Assume that K is a hyperideal of R satisfying $K \leq_p R$ and $K \leq_{ess} f(R)$. By Proposition 3.3.15, ${}_R(f(R)/K)$ is singular. Moreover, $R = f(R) \oplus \ker(f)$ by Proposition 2.1.12. Let $\pi_{\ker(f)}$ be the projection map on $\ker(f)$.

First, claim that $(\pi_{\ker(f)}gf)(R) \subseteq Z(R)$ for all $g^2 = g \in \mathrm{End}_0(R)$. Let $r \in R$ and $g^2 = g \in \mathrm{End}_0(R)$. By the singularity of $f(R)/K$,

$$f(r) + K \in f(R)/K = Z[{}_R(f(R)/K)].$$

By Proposition 3.3.7(i),

$$H \boxdot (f(r) + K) = \{K\} \quad \text{for some } H \leq_{ess} {}_RR.$$

Then, $Hf(r) \subseteq K$. This implies that $f(Hr) \subseteq K$. Recall that $K \leq_p R$. Hence,

$$H[(\pi_{\ker(f)}gf)(r)] = (\pi_{\ker(f)}gf)(Hr) \subseteq (\pi_{\ker(f)}g)(K) \subseteq \pi_{\ker(f)}(K) \subseteq K \subseteq f(R).$$

Note that $H[(\pi_{\ker(f)}gf)(r)] = (\pi_{\ker(f)}gf)(Hr) \subseteq \ker(f)$. This means that

$$H[(\pi_{\ker(f)}gf)(r)] \subseteq f(R) \cap \ker(f) = \{0\}.$$

Thus, $H[(\pi_{\ker(f)}gf)(r)] = \{0\}$. By Proposition 3.3.7(i), $(\pi_{\ker(f)}gf)(r) \in Z(R)$. Hence, the claim is proved.

Finally, we show that $f(R) + Z(R) \leq_p R$. Suppose that $h^2 = h \in \mathrm{End}_0(R)$ and $x \in f(R) + Z(R)$. Then there exist $a \in R$ and $z \in Z(R)$ such that $x \in f(a) + z$. Since $R = f(R) \oplus \ker(f)$, there exist $b \in R$ and $k \in \ker(f)$ such that $h(f(a)) \in f(b) + k$. Thus,

$$(\pi_{\ker(f)}hf)(a) \in \pi_{\ker(f)}(f(b) + k) = \pi_{\ker(f)}(f(b)) + \pi_{\ker(f)}(k) = 0 + k = \{k\}.$$

Thus, $k = (\pi_{\ker(f)} hf)(a) \in Z(R)$ by the claim. Note that $Z(R) \leq_p R$ by Proposition 3.3.10. Therefore,

$$h(x) \in h(f(a)+z) = h(f(a)) + h(z) \subseteq f(b) + k + h(z) \subseteq f(R) + Z(R).$$

This shows that $h(f(R)+Z(R)) \subseteq f(R)+Z(R)$. Hence, $f(R)+Z(R) \leq_p R$. □

Theorem 4.3.8. *Let I be a projection invariant hyperideal of a strongly distributive commutative C_{11}-hyperring R. Then there exist hyperideals J_1 and J_2 of R such that $I \leq_{ess} J_2$ and $R = J_1 \oplus J_2$. Moreover,*

(i) if $Z_2(J_1)$ is a C_{11}-hyperring, then J_1 is also a C_{11}-hyperring;

(ii) if $Z_2(J_2)$ is a C_{11}-hyperring, then J_2 is also a C_{11}-hyperring.

Proof. By Proposition 4.2.8, there exist hyperideals J_1 and J_2 of R such that $I \leq_{ess} J_2$ and $R = J_1 \oplus J_2$. In case $I = J_2$, by Proposition 4.2.6, we immediately obtain that J_1 and J_2 are C_{11}-hyperrings. There is nothing to prove. Hence, we are interested in the case that $I \neq J_2$. Since $Z_2(R) \leq_p R$, by Proposition 2.1.20, we can write

$$Z_2(R) = (Z_2(R) \cap J_1) \oplus (Z_2(R) \cap J_2),$$

but then $Z_2(R) \cap J_1 = Z_2(J_1)$ and $Z_2(R) \cap J_2 = Z_2(J_2)$ by Corollary 3.3.9. Therefore, $Z_2(R) = Z_2(J_1) \oplus Z_2(J_2)$. By Corollary 4.3.6, there exists a hyperideal J of R such that $R = Z_2(R) \oplus J$ and J is a C_{11}-hyperring. Thus, we can write $R = Z_2(J_1) \oplus Z_2(J_2) \oplus J$. Since $Z_2(J_1) \leq J_1$ and $Z_2(J_2) \leq J_2$, we obtain

$$J_1 = Z_2(J_1) \oplus [(Z_2(J_2) \oplus J) \cap J_1] \quad \text{and} \quad J_2 = Z_2(J_2) \oplus [(Z_2(J_1) \oplus J) \cap J_2],$$

respectively. For convenience, let

$$J_1' = (Z_2(J_2) \oplus J) \cap J_1 \quad \text{and} \quad J_2' = (Z_2(J_1) \oplus J) \cap J_2.$$

Hence,

$$J_1 = Z_2(J_1) \oplus J_1' \quad \text{and} \quad J_2 = Z_2(J_2) \oplus J_2'.$$

Recall that $Z_2(R) = Z_2(J_1) \oplus Z_2(J_2)$. We see that $Z_2(J_1) \cap J_2 \leq J_1 \cap J_2 = \{0\}$, so $Z_2(J_1) \cap J_2 = \{0\}$. By Proposition 2.1.5,

$$Z_2(R) + J_2 = Z_2(J_1) \oplus Z_2(J_2) \oplus J_2'.$$

This implies that

$$R = J_1 \oplus J_2 = Z_2(J_1) \oplus Z_2(J_2) \oplus J_1' \oplus J_2' = (Z_2(R) + J_2) \oplus J_1'.$$

Let $f : (Z_2(R) + J_2) \oplus J_1' \to Z_2(R) + J_2$ be the projection map on $Z_2(R) + J_2$. Then $f^2 = f \in \mathrm{End}_0(R)$ and $f(R) = Z_2(R) + J_2$. Since $I \leq_{ess} J_2 = Z_2(J_2) \oplus J_2'$, by Proposition 3.1.5(iv),

$$Z_2(J_1) \oplus I \leq_{ess} Z_2(J_1) \oplus Z_2(J_2) \oplus J_2' = Z_2(R) + J_2.$$

Note that $Z_2(J_1) \oplus I \leq Z_2(R) + I$. This implies that $Z_2(R) + I \leq_{ess} Z_2(R) + J_2 = f(R)$. Since $Z_2(R) \leq_p R$ and $I \leq_p R$, we obtain $Z_2(R) + I \leq_p R$. By Lemma 4.3.7, $f(R) + Z(R) \leq_p R$. Moreover, $Z(R) \leq Z_2(R) \leq Z_2(R) + J_2 = f(R)$. Hence,

$$Z_2(R) + J_2 = f(R) = f(R) + Z(R) \leq_p R.$$

By applying Proposition 4.2.6 to $R = (Z_2(R) + J_2) \oplus J_1'$, we obtain that $Z_2(R) + J_2$ and J_1' are C_{11}-hyperrings. Recall that $J_1 = Z_2(J_1) \oplus J_1'$. If $Z_2(J_1)$ is a C_{11}-hyperring, then J_1 is a C_{11}-hyperring by Proposition 4.2.4. Thus, the proof of (i) is complete. Now, we know that $Z_2(R) + J_2$ is a C_{11}-hyperring. Note that

$$Z_2(R) + J_2 = Z_2(J_1) \oplus Z_2(J_2) \oplus J_2' = Z_2(R) \oplus J_2'.$$

This means that $Z_2(R) \oplus J_2'$ is a C_{11}-hyperring. We see that

$$Z_2(R) = Z_2(R) \cap (Z_2(R) \oplus J_2') = Z_2(Z_2(R) \oplus J_2') \leq_p Z_2(R) \oplus J_2'.$$

By applying Proposition 4.2.6 again to $Z_2(R) \oplus J_2'$, we conclude that J_2' is a C_{11}-hyperring. Recall that $J_2 = Z_2(J_2) \oplus J_2'$. The proof of (ii) follows from Proposition 4.2.4 which is similar to (i). □

From theorem 4.3.8, for a strongly distributive commutative C_{11}-hyperring R and $I \leq_p R$, there exist two hyperideals J_1 and J_2 of R such that $I \leq_{ess} J_2$ and $R = J_1 \oplus J_2$, but there is no conclusion to assert that J_1 and J_2 are C_{11}-hyperrings in the case $I \neq J_2$, although R is a C_{11}-hyperring; however, the condition that

$Z_2(J_1)$ is a C_{11}-hyperring guarantees that J_1 is a C_{11}-hyperring, and the case of J_2 is similar to J_1.

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