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HOLOGRAPHIC RENORMALIZATION GROUP FLOWS FROM N=2 4D GAUGED SUPERGRAVITY

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A Thesis Submitted in Partial Fulfillment of the Requirements

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ในทฤษฎีสนามควอนตัมการศึกษาอันตรกิริยาระหว่างอนุภาคมูลฐานทำได้โดยใช้แผนภาพไฟน์แมน (Feynman diagram) ในการคำนวณในระดับที่อนุภาคมีอันตรกิริยาแบบรุนแรงนั้นลูปของแผนภาพไฟน์แมนจะให้ผลที่เป็นค่าอนันต์ ปัญหานี้สามารถแก้ไขได้ด้วยวิธีที่เรียกว่ารีนอร์มัลไลเซชัน (renormalization) ซึ่งสามารถกำจัดค่าอนันต์ที่เกินมาได้โดยพิจารณาการเปลี่ยนแปลงของพารามิเตอร์ที่บ่งบอกการเกิดอันตรกิริยาที่ขึ้นกับการเปลี่ยนระดับของพลังงาน การเปลี่ยนแปลงของระบบทางกายภาพภายใต้สเกลที่แตกต่างกันสร้างกรุปที่เรียกว่ากรุปรีนอร์มัลไลเซชัน (renormalization group) การโฟลว์ของกรุปรีนอร์มัลไลเซชัน (RG flows) อธิบายการเปลี่ยนรูป (deformation) จากทฤษฎีสนามคอนฟอร์มอลที่ระดับพลังงานสูง (UV) ไปเป็นทฤษฎีสนามที่พลังงานต่ำ (IR) ซึ่งอาจจะมีสมมาตรคอนฟอร์มอลหลงเหลือหรือไม่มีสมาตรนี้อยู่เลยก็ได้ วิทยานิพนธ์เล่มนี้ศึกษาการโฟลว์กรุปรีนอร์มัลไลเซชันจากเกจซูเปอร์กราวิตี $N = 2$ ซึ่งเกจกรุปที่พิจารณาคือกรุป $SO(2) \times SO(6)$ คำตอบที่ได้อธิบายการโฟลว์จากทฤษฎีสนามคอนฟอร์มอลที่มีสมมาตรคอนฟอร์มอล $N = 2$ ในสามมิติไปเป็นทฤษฎีสนามคอนฟอร์มอลที่สูญเสียสมมาตรคอนฟอร์มอลโดยใช้ทฤษฎีที่เรียกว่าความสอดคล้อง AdS/CFT (AdS/CFT correspondence) หรือ โฮโลกราฟฟิก AdS/CFT (AdS/CFT holography).

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In quantum field theories, interactions between particles can be studied by using Feynman diagrams. Calculating loop Feynman diagrams results in infinite values. This problem can be solved by using a process of renormalization that is the method of removing an infinity. Changing a physical system under different scales forms a group called renormalization group. Some renormalization group flows (RG flows) describe deformations of a conformal field theory (CFT) to another conformal or non-conformal theories, resulting in the deformations of a UV conformal fixed point to another fixed point or a non-conformal phase in the IR. In this work, we study holographic RG flows from $N = 2$ gauged supergravity with $SO(2) \times SO(6)$ gauge group. The solutions describe RG flows from the $N = 2$ CFT to non-conformal field theory in three dimensions driven by mass deformations according to the so-called AdS/CFT correspondence or AdS/CFT holography.

Department:	Physics	Student's Signature	....
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Chapter I

INTRODUCTION

1.1 String Theory Overview

James Clerk Maxwell's contributions to electromagnetism indeed played a crucial role in shaping our understanding of light as an electromagnetic wave, which propagates at a constant velocity in a vacuum. This posed a conflict with Newtonian mechanics, which assumed that velocity was relative to the observer's frame of reference.

In 1905, Albert Einstein published his theory of special relativity based on two postulates: the constancy of the speed of light in all inertial reference frames and the equivalence of all inertial reference frames with the same laws of physics. This theory provided a profound understanding of space and time, introducing concepts such as time dilation and length contraction resulting from the relative nature of simultaneity.

To generalize this theory, Einstein extends his ideas to include gravity, leading to the development of general relativity, which was published between 1907 and 1915. General relativity introduced the revolutionary concept that gravity is not a force acting at a distance, as described by Newtonian gravity, but rather a consequence of the curvature of spacetime caused by the presence of mass and energy. The equivalence principle posited that no experiment can distinguish between the effects of gravity and those of an accelerating reference frame.

General relativity successfully explained various gravitational phenomena, including the anomalous perihelion shifts of planets and the bending of light around massive objects. Moreover, the theory predicted the existence of black holes and gravitational waves, both of which have been observed in subsequent years, providing robust confirmation of Einstein's revolutionary ideas.

On the other hand, in the early 1900s, physicists couldn't use classical physics to explain the radiation of metals at high temperatures until Max Planck proposed the idea that was the beginning of modern physics. Max Planck suggests that matter can absorb and release only discrete energy, which is carried on to the photoelectric effect by Albert Einstein, published in 1905. In the photoelectric effect, light consists of tiny packets of energy known as photons or light quanta. The idea of a quantum implies that light or electromagnetic wave can behave as a particle, which was used to construct the spectrum of the hydrogen model by Niels Bohr in 1913. After that, Louis de Broglie studies special relativity together with the idea of quantum, leading to a complementary idea, which says that matter can also behave as a wave called a matter wave. The wave-particle duality means that we can't specify the wave state or particle of the physical system at the level of subatomic particles.

In 1926, Erwin Schrodinger made an effort to construct an equation for describing the de Broglie wave in a situation where an electron has high speed. By combining special relativity and quantum theory, the equation is created with the form

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \nabla^2\psi + \frac{m^2c^2}{\hbar^2}\psi = 0. \tag{1.1}$$

In the above equation, ψ corresponds to the wave function, and m represents the mass of a particle. The equation, also known as the Klein-Gordon equation, predicts the fine structure of the hydrogen atom incorrectly because the spin of the electron is not taken into account. Although this equation is unsuccessful in describing an electron in the hydrogen atom, Schrodinger found that the non-relativistic limit of this equation predicts the correct spectrum of hydrogen atom correctly. This limit provides the Schrodinger equation that is used at present, written as

$$i\hbar\frac{\partial\psi}{\partial t}(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right]\psi(x,t) = H\psi(x,t), \tag{1.2}$$

where $V(x,t)$ is a function of potential and H represents Hamiltonian's system. After the proposing of Schrodinger equation, Max Born interprets the wave function as a probability amplitude, proposing that the probability of finding a particle at each position depends on $|\psi(x,t)|^2$.

In the same period (1925–1926), Werner Heisenberg, Max Born, and Pascual Jordan developed the concept of matrix mechanics, which is a formulation of quantum mechanics called the Heisenberg picture. This formalism is in contrast to the wave formalism of Schrodinger, called the Schrodinger picture, which implies that states of a particle or physical system are described by time-dependent wave functions, but the physical operators are invariant in time. In the case of the Heisenberg picture, the states of a system are invariant in time, but the physical operators depend on time.

In 1927, Paul Dirac introduced the transformation theory, which provided a powerful and elegant framework for understanding quantum mechanics. The key idea was to combine two different representations of physical states, and show their equivalence. In Dirac's formalism, a physical state in quantum mechanics is represented by a vector in a mathematical space called a Hilbert space. The ket notation, represented by the symbol $|\Psi\rangle$, represents the state vector in this space. For example, if $|\Psi\rangle$ represents a particle's position, then the value of $|\Psi\rangle$ at a particular point would give the probability amplitude of finding the particle at that position. On the other hand, the bra notation, represented by the symbol $\langle\Psi|$, in dual space of the ket space, representing the adjoint vector. It corresponds to the complex conjugate transpose of the ket vector. If $|\Psi\rangle$ represents the state vector, then $\langle\Psi|$ represents the corresponding dual vector, or "bra" vector. The inner product of a ket and a bra vector is represented as $\langle\Psi|\phi\rangle$, where $|\phi\rangle$ is another state vector. The inner product is a complex number that represents the probability amplitude for transitioning from state $|\phi\rangle$ to state $|\Psi\rangle$. Dirac's transformation theory demonstrated the equivalence between Heisenberg and Schrödinger representations. This equivalence allowed for the formulation of quantum mechanics in a more compact and elegant manner.

It facilitated the development of various mathematical techniques in quantum mechanics, including the development of quantum operators, wave functions, and the Schrödinger equation.

In 1928, Paul Dirac indeed made a significant contribution to theoretical physics with the formulation of the Dirac equation:

$$\gamma^{\mu}\partial_{\mu}\Psi - m\Psi = 0, \tag{1.3}$$

where γ^{μ} is 4×4 gamma matrix that satisfy the property:

$$\{\gamma^{\mu}, \gamma^{v}\} = \gamma^{\mu}\gamma^{v} + \gamma^{v}\gamma^{\mu} = 2\eta^{\mu v}\mathbb{I}_{4\times 4}, \tag{1.4}$$

where $\eta^{\mu\nu}$ is Minkowski's metric. The Dirac equation is a relativistic wave equation that describes the behavior of fermions, such as electrons, in a relativistic quantum mechanical framework. It was developed as a relativistic extension of the Schrödinger equation, which was inadequate for describing particles moving at speeds close to the speed of light. The Dirac equation predicted the existence of antiparticles, which are particles with the same mass but opposite charge to their corresponding particles. In the case of the electron, the Dirac equation predicted the existence of the positron, which was later discovered in 1932 by Carl D. Anderson, confirming Dirac's prediction. The discovery of antiparticles and the success of the Dirac equation in describing their behavior provided strong basis for the concept of quantum field theory. In quantum field theory, particles are not considered as separate entities but rather as excitations of underlying fields that permeate spacetime. These fields are described by mathematical functions that depend on both time and 3D space coordinates. Particles are then interpreted as quantized excitations or disturbances in these fields.

Quantum field theory has become a fundamental framework for understanding particle physics and quantum interactions in a relativistic context. It has been successfully applied to describe the behavior of elementary particles and their interactions, providing a consistent and powerful theory for understanding the subatomic world. The Standard Model of particle physics, which describes the electromagnetic, weak, and strong nuclear forces and the particles that mediate these forces, is based on quantum field theory and has been extensively tested and confirmed through experiments.

However, when it comes to the effect of gravity, the standard model fails to explain gravity as it does with other forces. Gravity remains outside the scope of the standard model, and the theory of gravity lacks of normalization, leading to infinite and uncontrollable results. This issue arises from the fundamental mismatch between the smooth spacetime concept of general relativity and the discrete nature of quantum concepts. In other words, the smooth continuum of spacetime in general relativity does not align with the discrete nature of quantum mechanics, causing a fundamental incompatibility between the two theories. This discrepancy has been a central challenge in theoretical physics and is one of the reasons why developing a complete theory of quantum gravity remains an open and active area of research.

One of the popular theories of quantum gravity is string theory, which was originally developed to explain the spectrum of interactions among hadronic particles. It all began when Gabriele Veneziano made a significant discovery in 1968. He found that the scattering of hadrons could be explained using the beta function of Euler. Subsequently, in 1969-1970, Holger Bech Nielsen and Leonard Susskind proposed that the spectrum of hadronic scattering is not due to point particles, but rather arises from one-dimensional objects known as strings. In this model, all hadron particles are considered to be the same type of string, but their properties, such as mass, spin, and charge, differ based on the frequency of vibration of the string. However, the string theory faced some challenges. One such problem was the requirement for 26 dimensions of spacetime to preserve Lorentz symmetry. Additionally, the spectrum of the theory predicted the existence of a particle known as the tachyon, which has an imaginary mass, posing difficulties for the theory's consistency. Furthermore, another issue arose with the appearance of massless particles in the spectrum that have a spin of 2. Such particles were not found in the list of known hadrons, which presented a discrepancy between the theory and experimental observations.

The early version of string theory was referred to as the Bosonic string theory since it only incorporated bosonic particles (particles with integer spin). Subsequent developments, such as superstring theory and its various versions, aimed to address some of these issues by introducing fermions (particles with half-integer spin) and additional dimensions of spacetime.

Superstring theory is a type of string theory that goes beyond the original Bosonic string theory. It includes fermionic particles in its spectrum and addresses some of the issues faced by the Bosonic version. One of the problems solved in superstring theory is the tachyon, which is removed through the implementation of supersymmetry. Supersymmetry introduces a symmetry between bosons and fermions, which helps stabilize the theory. In the context of superstring theory, the spacetime dimensions that preserve Lorentz symmetry are reduced to ten dimensions, which is a significant development from the initial requirement of 26 dimensions in Bosonic string theory. Moreover, in superstring theory, the massless particle with spin 2 is believed to be the graviton, which is the hypothetical force carrier particle of gravity. This graviton is responsible for mediating gravitational interactions between particles. Superstring theory is considered a promising candidate for a unified theory of elementary particles and their interactions. However, it can be formulated in multiple ways, resulting in five different versions: type I, type IIA, type IIB, and two versions of heterotic string theory.

Interestingly, the five versions of superstring theory are considered special limiting cases of a more comprehensive theory known as M-theory. M-theory was proposed by Edward Witten and exists in eleven dimensions. At low energy levels, M-theory effectively describes a theory called supergravity in eleven dimensions. This supergravity theory involves the existence of extended objects known as branes, specifically the $M2$ and $M5$ branes, which play crucial roles in understanding certain

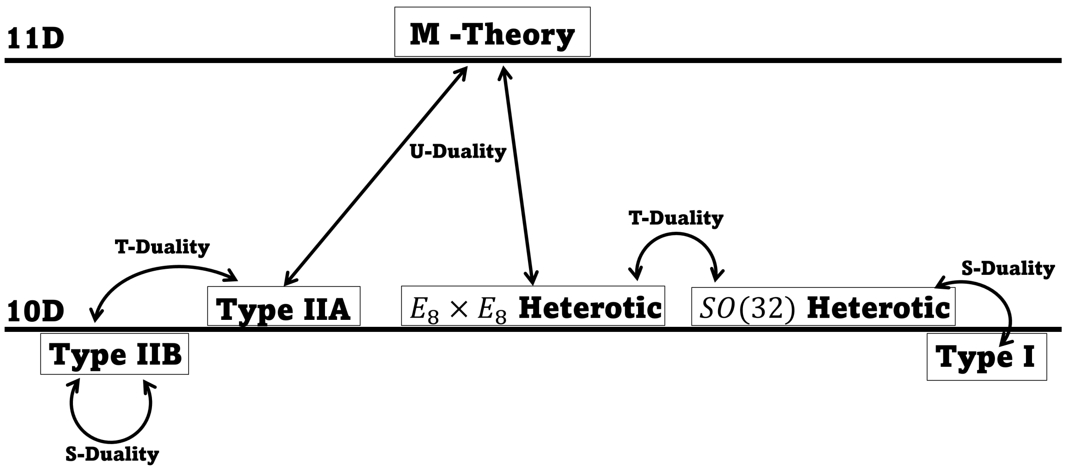

Figure 1.1: The relationship between five-string version and M-theory

aspects of M-theory and its connections to other physical phenomena.

1.2 Problem Statement

Quantum field theory (QFT) is a framework that combines quantum mechanics and special relativity to describe the behavior of elementary particles and their interactions. It is one of the fundamental theories in modern physics and provides a mathematical description of fields and particles at the quantum level. The calculation in QFT base on perturbation in terms Feynman diagrams. However, when performing these calculations, divergent quantities may appear, leading to infinite results.

To remove these infinite results, physicists use the essential concept called renormalization, which provides a systematic approach to address these infinities and obtain meaningful and finite predictions. When performing renormalization, the parameters of the theory are adjusted to absorb infinities and make the calculations finite. These parameters, such as masses m and coupling constants g, depend on the energy scale μ at which they are measured. The dependence of these parameters is described by RG equations of the form

$$\beta(g) = \mu\frac{\partial g}{\partial \mu}, \quad \text{and} \quad \gamma(m) = \frac{\mu}{m}\frac{\partial m}{\partial \mu}, \tag{1.5}$$

which govern the change of these parameters as the energy scale changes. The RG equations describe how the values of the parameters ”flow” as we move from a high-energy scale (UV) to a low-energy scale (IR). In some situations, the coupling constants are invariant under scale transformation ($\beta = \mu\frac{\partial g}{\partial \mu} = 0$), which is known as the conformal fixed point. The quantum field theory (QFT) that is invariant under scale transformation is a conformal field theory or CFT (a quantum field theory that possesses conformal symmetry). Studying renormalization group flows is crucial for understanding the universality of physical theories, as well as their critical behavior and phase transitions. However, in the case of strong coupling ($g \gg 1$), calculating RG flows in QFT becomes challenging as it involves a non-perturbative computation. To address this difficulty, a powerful tool called the AdS/CFT correspondence, or holography, can be employed.

The AdS/CFT correspondence, also known as the gauge/gravity duality or holographic duality, was first proposed by Juan Maldacena in a landmark paper (Maldacena, 1999). It is a powerful theoretical framework that establishes an equivalence between certain gravitational theories in Anti-de Sitter (AdS) spacetimes and quantum field theories (QFTs) with conformal symmetry living on the boundary of that spacetime. The central idea of the AdS/CFT correspondence is that the gravitational theory in the bulk, often described by supergravity, is dual to the QFT on the boundary. This duality implies that the two theories provide different descriptions of the same physics and are equivalent in a profound sense. It allows us to translate computations and phenomena in one theory to the other, opening up new avenues for studying strongly coupled field theory systems in terms of weakly coupled gravitational theories. For example, quantities of interest in the QFT, such as correlation functions or the study of renormalization group (RG) flows, can be mapped to computations in the gravitational theory. This mapping provides valuable insights into the behavior of strongly coupled systems that are difficult to access using traditional

QFT methods. The AdS/CFT correspondence has found applications in various areas of theoretical physics, including quantum gravity, string theory, and condensed matter physics. By employing consistent truncations, lower-dimensional gauged supergravity solutions can be uplifted to higher-dimensional theories, in ten or eleven dimensions, within the framework of string/M-theory. This process leads to the establishment of complete AdS/CFT dualities, providing a deeper understanding of the connections between gravity and quantum field theory.

In the context of the AdS/CFT correspondence, holographic RG flows refer to the description of renormalization group flows in the strong coupling limit of a quantum field theory (QFT) using the dual gravitational theory in Anti-de Sitter (AdS) space. The scalar potential $V(\phi)$ of the gauged supergravity in AdS space contains important information about the conformal fixed points and dynamics of the corresponding QFT at strong coupling. The solution of gauged supergravity that corresponds to an RG flow from one conformal fixed point to another conformal fixed point, or to a non-conformal fixed point, is given by the domain wall solution in the form of $AAdS_4$ (asymptotically anti-de Sitter space) space-time. The concept of AAdS implies that QFT exhibit a conformal fixed point precisely when AAdS is in the AdS form, with the theory being non-conformal at other positions. The domain wall solution provides a geometric description of the RG flow in the dual QFT, where the radial direction corresponds to the energy scale of the theory. By studying the properties of the domain wall solutions and the behavior of the scalar potential in the gauged supergravity, one can gain insights into the dynamics and phase structure of the corresponding QFT, particularly in the strong coupling regime.

1.3 Thesis Objective

1. To understand gauged supergravity.
2. To find solution of domain wall asymptotic to 4-dimension anti-de sitter space.

3. To study holographic RG flows in 3-dimensional conformal field theory.

1.4 Scope of Work

- Focus on the study of holographic RG flows originating from $N = 2$ gauged supergravity with an $SO(2) \times SO(6)$ gauge group.
- Explore the utilization of $SO(2) \times SO(4)$ and $U(3)$ scalar singlets within the context of holographic RG flows.
- Explore the behavior, implications, and mechanisms of RG flows that drive transitions from a $N = 2$ Conformal Field Theory (CFT) to non-conformal field theories in three dimensions.

1.5 Research Implication

In recent times, a number of physicists are beginning to posit that M theory could potentially serve as the elusive ”theory of everything.” The pursuit of understanding M theory holds the promise of providing insights into the fundamental nature of the universe.

The AdS/CFT correspondence, often referred to as the holographic principle, has emerged as an invaluable tool in comprehending M theory. This correspondence links certain gravitational theories (in Anti-de Sitter space) with certain conformal field theories (CFTs), thus allowing us to extract insights from one domain to understand the other. Holographic RG flow solutions, which describe the changes in a theory as it's examined at different energy scales, offer a particularly enlightening perspective on strong coupling phenomena in QFT. This is advantageous because it's often more feasible to calculate RG flow in these solutions than in traditional quantum field theory settings.

Notably, this holographic approach not only aids in the understanding of strong coupling within QFT but also extends its benefits to the realm of condensed matter physics. This is essential for accurately describing and modeling complex physical systems in condensed matter physics.

Nowadays, physicist believe that M-theory is a theoretical framework that has been proposed as a candidate for a unified theory encompassing all four fundamental forces of the universe, including gravity. It extends the ideas of string theory and introduces various extended objects, such as M2-branes and M5-branes. M2-branes are two-dimensional objects, and M5-branes are five-dimensional objects. These branes play a crucial role in the dynamics of M-theory. The low-energy dynamics of M2-branes are effectively described by a three-dimensional supersymmetric quantum field theory known as ABJM theory. The holographic duality, expressed as $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathrm{CFT}_3$, establishes a correspondence between eleven-dimensional supergravity in an $\mathrm{AdS}_4 \times \mathrm{S}^7$ spacetime and a conformal field theory (CFT) in three dimensions. This duality provides a powerful tool for studying the strongly coupled regime of M-theory using the techniques of a dual quantum field theory. In this thesis, the direction appears to focus on delving into the dynamics of M2-branes using four-dimensional supergravity and holographic tools. This could involve exploring the properties and behavior of M2-branes in the context of the holographic dual ABJM theory. Moreover, Conformal field theory in three dimensions, which arises in the holographic dual, has applications beyond fundamental physics. It is also used to understand concepts such as phase transitions and mass deformations in the context of condensed matter physics.

Chapter II

SUPERSYMMETRY

The symmetry that connected all the different types of particles in our universe, from electrons to photons and everything in between, was discovered by physicists in the 1970s. This relationship, known as supersymmetry, depends on the strange quantum feature of spin and may hold the key to a fresh perspective on physics.

2.1 Symmetry

Symmetry is a transformation that preserves the physics properties of a system. The symmetry transformation comprises both continuous transformations, including translation and rotation, as well as discrete transformations, such as time reversal and parity. These symmetries are known as discrete and continuous symmetry, respectively.

Two general classes of symmetry can be classified. A global symmetry is one that preserves a feature for a transformation that is equally applied at all points in spacetime. A local symmetry is one that preserves a property's invariance when a potentially different symmetry transformation is applied at each point in spacetime. In contrast, a global symmetry is not parameterized by the spacetime coordinates. Since it serves as the foundation for gauge theories, local symmetries have a significant impact on physics.

Noether's theorem

Consider a set of fields such as the $\phi^i(x)$, $i = 1, ..., n$ that belong in any representation (scalars, vector, and spinor). The equation of motion can be constructed from the action

$$S[\phi^i] = \int d^4x \mathscr{L}(\phi^i(x), \partial_\mu \phi^i(x)). \tag{2.1}$$

A mapping of the configuration space, $\phi^i(x) \to \phi^{i'}(x')$, that contains the property that if the original field configuration, $\phi(x)$, is a solution of the equations of motion, then the transformed configuration, $\phi'(x')$, is likewise a solution. This mapping represents a general symmetry that satisfies

$$S[\phi] = S[\phi']. \tag{2.2}$$

The variation of the action (2.1), with respect to ϕ^i, becomes

$$\delta_\phi S = \int d^4x \left[\frac{\partial \mathscr{L}}{\partial \phi^i} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial \partial_\mu \phi^i}\right)\right] \delta\phi^i + \int d^4x \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial \partial_\mu \phi^i} \delta\phi^i\right). \tag{2.3}$$

In the above equation, it shows that the Lagrangian density $\mathscr{L}(\phi^i, \partial_\mu \phi^i)$ transforms as a total derivative, which has no effect on the equation of motion. In equation (2.3), the surface term, $\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial \partial_\mu \phi^i} \delta\phi^i\right)$, can be neclected to obtain the equation of motion, written as

$$\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial \partial_\mu \phi^i}\right) - \frac{\partial \mathscr{L}}{\partial \phi^i} = 0. \tag{2.4}$$

The Euler-Lagrange equations of motion are named for this equation, which is generated by the condition, $\delta_{\phi^i} S = 0$, for all configurations $\delta\phi^i$.

We now consider a general field variation with infinitesimal symmetry by

$$\delta\phi^i(x) = \phi^{i'}(x) - \phi^i(x) = \epsilon^a \Delta_a \phi^i(x), \tag{2.5}$$

in which ϵ^a are constant on spacetime, $\partial_\mu \epsilon^a = 0$. As special cases, this property includes the various internal and spacetime symmetries discussed in the following section. Where Δ_a are the symmetry generator associated with the Lie algebra

$$[\Delta_a, \Delta_b] = f_{ab}{}^c \Delta_c, \tag{2.6}$$

leaving the action invariant, and ${f_{ab}}^c$ are the structure constant of a symmetry group.

The action is invariant when the transformation is a symmetry, and the variation in the Lagrangian density is expressed as total derivative K_a^μ, denoted by

$$\begin{aligned}\delta\mathscr{L} = \epsilon^a \partial_\mu K_a^\mu =& \epsilon^a \left[\frac{\partial\mathscr{L}}{\partial\partial_\mu\phi^i}\partial_\mu\Delta_a\phi^i + \frac{\partial\mathscr{L}}{\partial\phi^i}\Delta_a\phi^i\right] \\ =& \epsilon^a \left[\partial_\mu\left(\frac{\partial\mathscr{L}}{\partial\partial_\mu\phi^i}\Delta_a\phi^i\right) - \left[\partial_\mu\left(\frac{\partial\mathscr{L}}{\partial\partial_\mu\phi^i}\right) - \frac{\partial\mathscr{L}}{\partial\phi^i}\right]\Delta_a\phi^i\right], \end{aligned} \tag{2.7}$$

We can rewrite the above equation as

$$\partial_\mu\left[K_a^\mu - \frac{\partial\mathscr{L}}{\partial\partial_\mu\phi^i}\Delta_a\phi^i\right] = \left[\partial_\mu\left(\frac{\partial\mathscr{L}}{\partial\partial_\mu\phi^i}\right) - \frac{\partial\mathscr{L}}{\partial\phi^i}\right]\Delta_a\phi^i. \tag{2.8}$$

Using equation (2.4) we can rewrite (2.7) in the form $\partial_\mu J_a^\mu = 0$, where J_a^μ is the Noether current

$$J_a^\mu = K_a^\mu - \frac{\partial\mathscr{L}}{\partial\partial_\mu\phi^i}\Delta_a\phi^i. \tag{2.9}$$

This current is conserved, which means that $\partial_\mu J_a^\mu = 0$ for all solutions of the system's equations of motion.

For each conserved current one can define an integrated Noether charge, which is a constant of the motion, i.e. independent of time

$$Q_a = \int_{\Sigma_t} d^3x n_\mu J_a^\mu. \tag{2.10}$$

Every point on a space-like surface Σ_t has a time-like normal vector n_μ. The conservation of Q_a can be shown directly by choosing Σ_t at constant time, $n^\mu = \partial_t$, which provides

$$Q_a = \int d^3x J_a^0, \tag{2.11}$$

and

$$\frac{dQ_a}{dt} = \int d^3x \partial_0 J_a^0 = -\int d^3x \partial_i J_a^i = -\int d^2x \tilde{n}_i J_a^i \tag{2.12}$$

To obtain the above equation, we use $\partial_\mu J_a^\mu = 0$ and Stoke's theorem to change volume integration into surface integration that contains the orthogonal vector $\tilde{n}_i$. The boundary condition imposed that the fields at boundary vanish, $J_a^i \to 0$, so that

$$\frac{dQ_a}{dt} = 0. \tag{2.13}$$

We can determine the Noether current in an efficient way if we assume that the symmetry parameters depend on spacetime, $\partial_\mu \epsilon^a(x) \neq 0$. The variation of the action becomes

$$\begin{aligned}
\delta S &= \int d^4x \left[\frac{\partial \mathscr{L}}{\partial \partial_\mu \phi^i} \partial_\mu \delta \phi^i + \frac{\partial \mathscr{L}}{\partial \phi^i} \delta \phi^i \right] \\
&= \int d^4x \left[\frac{\partial \mathscr{L}}{\partial \partial_\mu \phi^i} \partial_\mu (\epsilon^a \Delta_a \phi^i) + \frac{\partial \mathscr{L}}{\partial \phi^i} \epsilon^a \Delta_a \phi^i \right] \\
&= \int d^4x \left[\partial_\mu \epsilon^a \frac{\partial \mathscr{L}}{\partial \partial_\mu \phi^i} \Delta_a \phi^i + \epsilon^a \partial_\mu K_a^\mu \right] \\
&= -\int d^4x \partial_\mu \epsilon^a \left[K_a^\mu - \frac{\partial \mathscr{L}}{\partial \partial_\mu \phi^i} \partial_\mu \Delta_a \phi^i \right] + \int d^4x \partial_\mu (\epsilon^a K_a^\mu)
\end{aligned} \tag{2.14}$$

The second term in the previous equation can be neglected by applying the boundary condition. The first term can be written in the form of (2.9), so that (2.14) becomes:

$$\delta S = -\int d^4x \partial_\mu \epsilon^a J_a^\mu, \tag{2.15}$$

which the Noether current is a negative of the coefficient of $\partial_\mu \epsilon^a$.

Internal symmetry

An internal symmetry is a transformation that only affects the fields, not the spacetime, and preserves the action or lagrangian invariant. The form of symmetry transformation is

$$\phi^i(x) \to \phi^{i'}(x) = U^i{}_j \phi^j(x), \tag{2.16}$$

which spacetime coordinates remain unchanged $x^{\mu'} = x^\mu$. The matrix $U^i{}_j$ is a representation of the group element G, which explains the symmetry in theory. For the internal symmetry that is continuous, the matrix $U^i{}_j$ can be written in the form of an exponential as

$$U(\theta) = e^{\theta^a t_a}, \tag{2.17}$$

in which θ^a are constant parameters, and ${(t_a)^i}_j$ are generators that correspond to Lie algebra

$$[t_a, t_b] = {f_{ab}}^c t_c. \tag{2.18}$$

From equation (2.5), we can write an infinitesimal transformation ($|\theta^a| << 1$) for an internal symmetry transformation as

$$\delta\phi^i(x) = \theta^a {(t_a)^i}_j \phi^j(x) \tag{2.19}$$

In the case of an invariant Largrangian density and spacetime-independence transformation of ${U^i}_j$, yielding $K_a^\mu = 0$, and the Neother current of this symmetry shows as

$$J_a^\mu = -\frac{\partial \mathscr{L}}{\partial\partial_\mu\phi^i} {(t_a)^i}_j \phi^j(x). \tag{2.20}$$

To give some example, we focus on systems of scalar fields with typical kinetic terms. The action is

$$S = -\frac{1}{2}\int d^4x \left[\partial_\mu\phi_i\partial^\mu\phi^i + m^2\phi_i\phi^i\right]. \tag{2.21}$$

So that, the Neother current is written as

$$J_a^\mu = \partial^\mu\phi_i(t_a)^i_j\phi^j. \tag{2.22}$$

Spacetime symmetries

The Poincare group, which consists of rotational transformation and translation in four-dimensional spacetime, is the symmetry group that describes spacetime symmetry . In accordance with this symmetry, the coordinates transform as

$$x^{\mu'} = \Lambda^\mu{}_\nu x^\nu + a^\mu, \tag{2.23}$$

where Λ^μ_ν is a Lorentz group element with no parity and time reversal, and a^μ is the translation parameters. The elements of the Lorentz group must correspond to conditions

$$\det\{\Lambda\} = +1 \tag{2.24}$$

and

$$\Lambda^{\mu}{}_{\rho}\eta_{\mu\nu}\Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}. \tag{2.25}$$

We now introduce the Lie algebra of the Lorentz group, expanding $\Lambda^{\mu}{}_{\nu}$ to infinitesimal transformation. It follows that

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu}. \tag{2.26}$$

$\omega^{\mu}{}_{\nu}$ is parameter of Lorentz transformation, and it must be an anti-symmetric matrix in order to satisfy equation (2.25), which is as follows:

$$\begin{aligned}
(\delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu})\eta_{\mu\sigma}(\delta^{\sigma}_{\rho} + \omega^{\sigma}{}_{\rho}) &= \eta_{\nu\rho} \\
(\delta^{\mu}_{\nu}\delta^{\sigma}_{\rho} + \delta^{\mu}_{\nu}\omega^{\sigma}{}_{\rho} + \delta^{\sigma}_{\rho}\omega^{\mu}{}_{\nu})\eta_{\mu\sigma} &= \eta_{\nu\rho} \\
\eta_{\nu\rho} + \omega_{\nu\rho} + \omega_{\rho\nu} &= \eta_{\nu\rho} \\
\omega_{\nu\rho} &= -\omega_{\rho\nu}.
\end{aligned} \tag{2.27}$$

Only the first order in $\omega^{\mu}{}_{\nu}$ is taken into consideration in the second line, and ω with two lower indices is given by $\omega_{\mu\nu} = \eta_{\mu\rho}\omega^{\rho}{}_{\nu}$. It has six independent parameters and can be conveniently parametrized in the from $\omega_{\mu\nu}\Sigma^{\mu\nu}$ by using six anti-symmetric matrices as

$$\Sigma_{[\rho\sigma]}{}^{\mu}{}_{\nu} = \delta^{\mu}_{\rho}\eta_{\nu\sigma} - \delta^{\mu}_{\sigma}\eta_{\rho\nu} = -\Sigma_{[\sigma\rho]}{}^{\mu}{}_{\nu}. \tag{2.28}$$

The matrices Σ are the generators of the Lorentz group $SO(1,3)$, which satisfy Lie algebra

$$[\Sigma_{[\mu\nu]}, \Sigma_{[\rho\sigma]}] = f_{[\mu\nu][\rho\sigma]}{}^{[\alpha\beta]}\Sigma_{[\alpha\beta]} \tag{2.29}$$

with structure constant following as

$$f_{[\mu\nu][\rho\sigma]}{}^{[\alpha\beta]} = 8\eta_{[\rho[\nu}\delta^{[\alpha}_{\nu]}\delta^{\beta]}_{\sigma]}. \tag{2.30}$$

Depending on the kind of field $\Phi(x)$, there are several ways to represent generators $\Sigma_{\mu\nu}$. The generators for scalar, vector, and spinor can be represented as

$$\text{scalar}: \quad \Sigma_{\mu\nu} = 0 \tag{2.31}$$

$$\text{vector}: \quad \Sigma_{[\rho\sigma]}{}^{\mu}{}_{\nu} = \delta^{\mu}_{\rho}\eta_{\nu\sigma} - \delta^{\mu}_{\sigma}\eta_{\rho\nu} \tag{2.32}$$

$$\text{spinor}: \quad \Sigma_{\mu\nu} = \frac{1}{4}[\gamma^{\mu}, \gamma^{\nu}] \tag{2.33}$$

Consider the field that transforms under the Poincare group in the form of

$$\Phi'(x') = \hat{U}(\Lambda)\Phi(x), \tag{2.34}$$

where $\hat{U}(\Lambda)$ are elements of the Lorentz group in the representation of $\Phi(x)$. For infinitesimal transformations $x^{\mu'} = \omega^{\mu}{}_{\nu}x^{\nu} + \epsilon^{\mu}$, we can expand the transformations in the above equation as follows:

$$\hat{U}(\mathbf{I} + \omega) = \mathbf{I} + \frac{1}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}. \tag{2.35}$$

Consider the left-hand side of equation (2.34) and expand the field $\Phi'(x')$ around $x^{\mu'} = x^{\mu}$, which shows as

$$\begin{aligned}\Phi'(x') &= \Phi'(x^{\mu} + \omega^{\mu}{}_{\nu}x^{\nu} + \epsilon^{\mu}) \\ &= \Phi'(x) + (\omega^{\mu}{}_{\nu}x^{\nu} + \epsilon^{\mu})\partial_{\mu}\Phi'(x) \\ &= \Phi'(x) + \omega^{\mu\nu}x_{\nu}\partial_{\mu}\Phi(x) + \epsilon^{\mu}\partial_{\mu}\Phi(x).\end{aligned} \tag{2.36}$$

Because the coefficient of $\partial_{\mu}\Phi'(x)$ corresponds to the first order of parameters $(\omega^{\mu}{}_{\nu}, \epsilon^{\mu})$, we substituted $\Phi'(x)$ by $\Phi(x)$ in the second term in the previous procedure.

The transformation on the right-hand side of equation (2.34) can be written as

$$\hat{U}(\mathbf{I} + \omega)\Phi(x) = \Phi(x) + \frac{1}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\Phi(x). \tag{2.37}$$

Equations (2.36) and (2.37) enable us to rewrite equation (2.34) as

$$\Phi'(x) + \omega^{\mu\nu}x_{\nu}\partial_{\mu}\Phi(x) + \epsilon^{\mu}\partial_{\mu}\Phi(x) = \Phi(x) + \frac{1}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\Phi(x). \tag{2.38}$$

Consequently, the fields' variation becomes

$$\delta\Phi(x) = -\epsilon^{\mu}\partial_{\mu}\Phi(x) + \frac{1}{2}\omega^{\mu\nu}(\Sigma_{\mu\nu} + x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})\Phi(x) \tag{2.39}$$

The Lagrangian density is invariant under Lorentz group and translation in the form $\mathscr{L}'(x + a) = \mathscr{L}'(x) - \epsilon^{\mu}\partial_{\mu}\mathscr{L}'(x) = \mathscr{L}(x)$, so that the variation becomes

$$\delta\mathscr{L}(x) = -\epsilon^{\mu}\partial_{\mu}\mathscr{L}(x) = -\epsilon^{\mu}P_{\mu}\mathscr{L}(x), \tag{2.40}$$

where P_μ is a generator of translation whose meaning is four-momentum.

Noether current of Poincare symmetry can be considered separately under translation and Lorentz transfromation. For $\omega_{\mu\nu} = 0$ equation (2.39) becomes

$$\delta\Phi(x) = -\epsilon^\mu \partial_\mu \Phi(x) \longrightarrow \Delta_\mu \Phi(x) = \partial_\mu \Phi(x), \tag{2.41}$$

and equation (2.40) provides $K^\nu_\mu = \delta^\nu_\mu \mathscr{L}$. It is important to note that the symmetry parameters are $-\epsilon^\mu$, and the indices a in equation (2.5) are changed into spacetime indices μ.

By using the definition of the Noether current in equation (2.9), we have

$$T^\mu{}_\nu = -\frac{\partial \mathscr{L}}{\partial \partial_\mu \Phi} \partial_\nu \Phi + \delta^\mu_\nu \mathscr{L}. \tag{2.42}$$

This conserved current is called the energy-momentum tensor, which corresponds to the conservation equation:

$$\partial_\mu T^\mu{}_\nu = 0 \tag{2.43}$$

In the case of Lorentz symmetry, where $\epsilon^\mu = 0$, utilizing equation (2.5) results in ϵ^a being represented as $\omega^{\mu\nu}$. This leads to the following expression:

$$\delta\Phi(x) = \frac{1}{2}\omega^{\mu}\Delta_{\mu\nu}\Phi(x). \tag{2.44}$$

By using equation (2.39), we have

$$\Delta_{\mu\nu}\Phi(x) = \frac{1}{2}\omega^{\mu\nu}\Sigma_{\mu\nu}\Phi(x) + x_\mu \partial_\nu \Phi(x) - x_\nu \partial_\mu \Phi(x). \tag{2.45}$$

Lagrangian density is invariant under transformation in the form of $x^{\mu'} = x^\mu + \omega^\mu{}_\nu x^\nu$, which provides

$$\begin{aligned}
\delta \mathscr{L} &= -\omega^{\mu\nu} x_\nu \partial_\mu \mathscr{L} \\
&= -\frac{1}{2}\omega^{\mu\nu}(x_\nu \partial_\mu - x_\mu \partial_\nu)\mathscr{L} \\
&= -\frac{1}{2}\omega^{\mu\nu}[\partial_\mu(x_\nu \mathscr{L}) - \partial_\nu(x_\mu \mathscr{L})] \\
&= -\frac{1}{2}\omega^{\mu\nu}\partial_\rho(\delta^\rho_\nu x_\mu \mathscr{L} - \delta^\rho_\mu x_\nu \mathscr{L}).
\end{aligned} \tag{2.46}$$

From the above equation, we can define K_a^μ as

$$K^\rho_{\mu\nu} = \delta^\rho_\nu x_\mu \mathscr{L} - \delta^\rho_\mu x_\nu \mathscr{L}. \tag{2.47}$$

So that the Noether current can be written as

$$M^\mu_{\rho\sigma} = -\frac{\partial \mathscr{L}}{\partial \partial_\mu \Phi}(\Sigma_{\sigma\rho}\Phi + x_\rho \partial_\sigma \Phi - x_\sigma \partial_\rho \Phi) + \delta^\mu_\sigma x_\rho \mathscr{L} - \delta^\mu_\rho x_\sigma \mathscr{L}, \tag{2.48}$$

and using the energy-momentum tensor in equitation (2.42) the above equation becomes

$$M^\mu_{\rho\sigma} = -\frac{\partial \mathscr{L}}{\partial \partial_\mu \Phi}\Sigma_{\sigma\rho}\Phi + x_\rho T^\mu{}_\sigma - x_\sigma T^\mu{}_\rho. \tag{2.49}$$

For example, in the case of scalar fields $(\Sigma_{\mu\nu} = 0)$, the above equation becomes

$$M^\mu_{\rho\sigma} = x_\rho T^\mu{}_\sigma - x_\sigma T^\mu{}_\rho, \tag{2.50}$$

and under conservation conditions, we have

$$\partial_\mu M^\mu_{\rho\sigma} = \eta_{\mu\rho} T^\mu{}_\sigma - \eta_{\mu\sigma} T^\mu{}_\rho = 0. \tag{2.51}$$

This condition imposes that the energy-momentum tensor be a symmetric tensor $(T_{\rho\sigma} = T_{\sigma\rho})$.

In the event that $\Sigma_{\mu\nu} \neq 0$, the conservation equation $\partial_\mu M^\mu_{\rho\sigma}$ becomes

$$\partial_\rho \left[\frac{\partial \mathscr{L}}{\partial \partial_\rho \Phi}\Sigma_{\mu\nu}\Phi\right] = T_{\mu\nu} - T_{\nu\mu}, \tag{2.52}$$

which means that the anti-symmetric part of energy-momentum tensor $(T_{\mu\nu})$ depends on $\Sigma_{\mu\nu}$.

Consider equation $\partial_\mu J^\mu_a = 0$: we can introduce new alternative currents that satisfy the conservation law, as shown below

$$\tilde{J}^\mu = J^\mu + \partial_\nu A^{\mu\nu}, \tag{2.53}$$

which satisfy $\partial_\mu \tilde{J}^\mu = 0$ for anti-symmetric $A^{\mu\nu}$. As a result of this, the new current corresponds to the conservation law and provides the same charge because

$$\int d^3x \partial_i A^{0i} = \int d^2x \tilde{n}_i A^{0i} = 0. \tag{2.54}$$

We apply the constraint that all fields vanish at the boundary in the last step.

If we define the new energy-momentum tensor as

$$\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\rho A^{\rho\mu,\nu}, \tag{2.55}$$

$A^{\rho\mu,\nu} = -A^{\mu\rho,\nu}$, then $T^{\mu\nu}$ and $\Theta^{\mu\nu}$ correspond to conservation law and also provide the same Noether charge. We can infer that both tensors are physically equivalent because of $\partial_\mu \Theta^{\mu\nu} = 0$, and

$$\int d^3x \Theta^{0\nu} = \int d^3x T^{0\nu} + \int d^3x \partial_i A^{i0,\nu} = \int d^3x T^{0\nu}. \tag{2.56}$$

From the above result, we can construct the symmetric tensor $\Theta^{\mu\nu}$ by replacing $\Theta^{\mu\nu}$ in equation (2.52) and using relation $\Theta^{\mu\nu} = \Theta^{\nu\mu}$. We have

$$-\partial_\rho \left[\frac{\partial \mathscr{L}}{\partial \partial_\rho \Phi} \Sigma_{\mu\nu} \Phi \right] = \partial_\rho (A^{\rho\mu,\nu} - A^{\rho\nu,\mu}). \tag{2.57}$$

If we neglect the constant of integration, the above equation becomes

$$A^{\rho\mu,\nu} - A^{\rho\nu,\mu} = -F^{\rho,\mu\nu}, \tag{2.58}$$

where

$$F^{\rho,\mu\nu} = \frac{\partial \mathscr{L}}{\partial \partial_\rho \Phi} \Sigma_{\mu\nu} \Phi. \tag{2.59}$$

In terms of $F^{\rho,\mu\nu}$, the equation (2.58) can be written as

$$A^{\rho\mu,\nu} = -\frac{1}{2}(F^{\rho,\mu\nu} + F^{\mu,\nu\rho} - F^{\nu,\rho\mu}). \tag{2.60}$$

When we substitute the previous equation in (2.55) , the energy-momentum tensor becomes

$$\Theta^{\mu\nu} = T^{\mu\nu} - \frac{1}{2}\partial_\rho \left[\frac{\partial \mathscr{L}}{\partial \partial_\rho \Phi} \Sigma^{\mu\nu} \Phi - \frac{\partial \mathscr{L}}{\partial \partial_\mu \Phi} \Sigma^{\rho\nu} \Phi - \frac{\partial \mathscr{L}}{\partial \partial_\nu \Phi} \Sigma^{\rho\mu} \Phi \right]. \tag{2.61}$$

The tensor $\Theta^{\mu\nu}$ is also known as the Belinfante-Rosenfeld tensor. Generally, when we consider the energy-momentum tensor, we also refer to it as $\Theta^{\mu\nu}$.

Additionally, we should note that a scalar field, which is explained by a general action of the form

$$S = \int d^4x \left[-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \right], \tag{2.62}$$

provides energy-momentum tensor in the form

$$\begin{aligned} T^\mu{}_\nu &= \frac{\partial \mathscr{L}}{\partial\partial_\mu\phi}\partial_\nu\phi + \delta^\mu_\nu \mathscr{L} \\ &= \partial^\mu\phi\partial_\nu\phi - \delta^\mu_\nu \left(\frac{1}{2}\partial_\rho\phi\partial^\rho\phi + V(\phi) \right). \end{aligned} \tag{2.63}$$

The charges for $T^\mu{}_\nu$ and $M^\mu_{\rho\sigma}$ can be calculated as

$$P_\mu = \int d^3x T^0{}_\mu \qquad \text{and} \qquad \mathscr{M}_{\mu\nu} = \int d^3x M^0_{\mu\nu}, \tag{2.64}$$

which are four-momentum and Lorentz generators, respectively. The conservation of M_{ij} means that the angular momentum is preserved, and the conservation of M_{0i} means that the centers of relativistic mass of the systems have constant velocity.

2.2 Supersymmetry

The symmetry that was discussed in the previous section was compiled under a no-go theorem provided by Coleman and Mandula. This theorem indicates that, in addition to CPT (charge conjugation, parity transformation, and time reversal), Poincare symmetries and an internal symmetry group G are the only possible symmetries of the S-matrix. In other words, the most general symmetry in field theory, is a direct product of Poincare symmetries and an internal symmetry group, that is $ISO(1,3) \times G$.

The supersymmetry algebra

The Coleman-Mandula theorem is crucial to the symmetry that corresponds to Lie algebra. To extend the boundary of Lie algebra, Haag, Lopuszanski, and

Sohnius generalized the notion of Lie algebra to include anti-commutators by imposing graded Lie algebra

$$[T_a, T_b\} = T_a T_b - (-1)^{\eta_a \eta_b} T_b T_a = f_{ab}{}^c T_c. \tag{2.65}$$

Bosonic ($T_a = B$) and fermionic ($T_a = F$) generators are the two categories that form the Lie algebra in supersymmetry. The number of graded $\eta_a = 0$ imposes the bosonic generators, while the graded $\eta_a = 1$ imposes the fermionic generators. The graded algebra, equation (2.65) states that the general structure of supersymmetry algebra becomes

$$[B, B^{'}] = B^{''}, \quad [B, F] = F^{'}, \quad \{F, F^{'}\} = B. \tag{2.66}$$

Consider the product of F^i, $i = 1, ..., N$ in representation $(j, j + \frac{1}{2})$ and $F^{i\dagger}$ in conjugate representation $(j + \frac{1}{2}, j)$. The anti-commutator between F^i and $F^{i\dagger}$ provides maximal spin representation as

$$\begin{aligned} \{F^i, F^{j\dagger}\} : &\qquad (2j + \frac{1}{2}, 2j + \frac{1}{2}) \\ \{F^i, F^j\} : &\qquad (2j, 2j + 1) \\ \{F^{i\dagger}, F^{j\dagger}\} : &\qquad (2j + 1, 2j). \end{aligned} \tag{2.67}$$

According to anti-commutator in (2.66), the above result must be a bosonic generator in Poincare symmetries or an internal symmetry $(J^{\mu\nu}, P^\mu, t_a)$ that belongs in representation $(1, 0) \oplus (0, 1), (\frac{1}{2}, \frac{1}{2})$ and $(0, 0)$, respectively. If $j > 0$, then $2j + \frac{1}{2} > \frac{1}{2}$, $2j + 1 > 1$, and $2j > 0$, which mean that

$$\{F^i, F^{j\dagger}\} = \{F^i, F^j\} = \{F^{i\dagger}, F^{j\dagger}\} = 0. \tag{2.68}$$

The above equations imply that the fermion generator $(j, j + \frac{1}{2}) \oplus (j + \frac{1}{2}, j)$ that contains $j > 0$ vanishes in supersymmetry algebra. As a result, all possible fermion generators must be spinor belonging to representation $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ of Lorentz group.

As generators in representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, define Q_{ai}, $a, b = 1, 2$ and $\overline{Q}^i_{\dot{a}}$, $\dot{a}, \dot{b} = 1, 2$, respectively, which are also known as supercharges. As we will see in the following section, supersymmetry is generated by the supercharge and used to identify the spin number of the particles. The supercharge number is represented by the indices $i = 1, 2, 3, ..., N$, so that the number of all supercharge is $4N$.

The supercharges are spinors in the Lorentz group, so that the commutator between Q_{ai}, $\overline{Q}^i_{\dot{a}}$ and $J^{\mu\nu}$ can be written as

$$[Q_{ai}, J_{\mu\nu}] = i(\sigma_{\mu\nu})_a{}^b Q_{bi} \qquad \text{and} \qquad [\overline{Q}^i_{\dot{a}}, J_{\mu\nu}] = i(\overline{\sigma}_{\mu\nu})_{\dot{a}}{}^{\dot{b}}\overline{Q}^i_{\dot{b}}, \tag{2.69}$$

where $(\sigma_{\mu\nu})_a{}^b$ and $(\bar{\sigma}_{\mu\nu})_{\dot{a}}{}^{\dot{b}}$ are Lorentz generator in $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representation, respectively. The other commutator can be found by using the super-Jacobian identity that is defined

$$(-1)^{\eta_a\eta_b}[[T_a, T_b\}, T_c\} + (-1)^{\eta_a\eta_b}[[T_b, T_c\}, T_a\} + (-1)^{\eta_a\eta_b}[[T_c, T_a\}, T_b\} = 0. \tag{2.70}$$

Consider the anti-commutator $\{Q_{ai}, \overline{Q}^j_{\dot{a}}\}$ that the product $[(\frac{1}{2}, 0) \otimes (0, \frac{1}{2})]_s = (\frac{1}{2}, \frac{1}{2})$ only gives the bosonic generator, P_μ as a possible generator. As a result, this anti-commutator can be written as

$$\{Q_{ai}, \overline{Q}^j_{\dot{a}}\} = -C^j{}_i \sigma_{\mu a\dot{a}} P^\mu, \tag{2.71}$$

where the minus sign corresponds to the convention of $\sigma_\mu = (-\mathbf{I}, \sigma^i)$.

The matrix $C^j{}_i$ can be found by considering the property of the commutator. Since the anti-commutator is hermitian, $C^j{}_i$ must also be hermitian and can be turned into a diagonal matrix. In terms of eigenvalues, matrix $C^j{}_i$ can be defined as $C^j{}_i = c_i \delta^j_i$. and the anti-commutator can be rewritten follow

$$\{Q_{ai}, \overline{Q}^j_{\dot{a}}\} = -\frac{1}{2}\delta^j_i \sigma_{\mu a\dot{a}} P^\mu, \tag{2.72}$$

where $C^i{}_i$ are eigenvalues and absorbed in the supercharge.

Take into account the commutator between Q_{ai} and P^μ, where the result of an anti-symmetric product provides $[(\frac{1}{2}, 0) \otimes (\frac{1}{2}, \frac{1}{2})]_A = (0, \frac{1}{2})$. This result implies that the generator on the right-hand side of the commutator must be $\overline{Q}^j_{\dot{a}}$. Therefore, the commutator can be written as

$$[Q_{ai}, P^\mu] = -\frac{1}{2} c_{ij} \sigma^\mu_{a\dot{a}} \overline{Q}^{\dot{a}j}, \tag{2.73}$$

and the conjugate hermitian of this commutator becomes

$$[Q^{\dot{a}i}, P^\mu] = -(c_{ij})^\dagger \overline{\sigma}^{\mu \dot{a} a} Q_{aj}. \tag{2.74}$$

Using the super-Jacobian identity, the algebra becomes

$$[P^\mu, [P^\nu, Q_{ai}]] + [P^\nu, [Q_{ai}, P^\mu]] + [Q_{ai}, [P^\mu, P^\nu]] = 0. \tag{2.75}$$

Using $[P^\mu, P^\nu] = 0$, the previous equation provides $cc^\dagger = 0$ because $\sigma^\mu \overline{\sigma}^\nu \neq 0$. Equation (2.75) becomes

$$[P^\mu, Q_{ai}] = [P^\mu, \overline{Q}^i_{\dot{a}}] = 0. \tag{2.76}$$

The anti-commutator $\{Q_{ai}, Q_{bj}\}$ can be found by considering the representation $(\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0) = (0,0) \oplus (1,0)$, which indicates that the right-hand side of the anti-commutator must be a scalar Z^{ij} and anti-symmetric tensor $Y^{ij}_{\mu\nu} = Y^{ji}_{\mu\nu} = -Y^{ij}_{\nu\mu}$. The form of the anti-commutator can be written as

$$\{Q_{ai}, Q_{bj}\} = -\frac{1}{2}\epsilon_{ab} Z^{ij} + \frac{1}{2}\sigma^{\mu\nu}_{ab} Y^{ij}_{\mu\nu}, \tag{2.77}$$

where the form of the tensor $Y^{ij}_{\mu\nu}$ is considered by using equation (2.71), $[P^\mu, Q_{ai}] = [P^\mu, P^\nu] = 0$, and super-Jacobian identity

$$[P^\mu, \{Q_{ai}, Q_{bj}\}] - \{Q_{bj}, [P^\mu, Q_{ai}]\} + \{Q_{ai}, [Q_{bj}, P^\mu]\} = 0. \tag{2.78}$$

The above equation provides $[P^\mu, \{Q_{ai}, Q_{bj}\}] = 0$, which implies that $Y^{ij}_{\mu\nu} = 0$ because of $[P^\mu, \sigma^{\nu\rho}] \neq 0$.

The generator Z_{ij} is a scalar in the Lorentz group, which means that Z_{ij} can be written in the form of an internal symmetry generator T_A, denoted by

$$Z_{ij} = a^A_{ij} T_A. \tag{2.79}$$

Where T_A corresponds to Lie algebra

$$[T_A, T_B] = f_{AB}{}^C T_C. \tag{2.80}$$

Let Q_{ai} and $\overline{Q}^{i}_{\dot{a}}$ transform under internal symmetry, denoted by

$$[Q_{ai}, T_A] = (S_A)_i{}^j Q_{aj} \quad \text{and} \quad [T_A, \overline{Q}^{i}_{\dot{a}}] = (S^{*A})^i{}_j \overline{Q}^{j}_{\dot{a}}, \tag{2.81}$$

where $(S_A)_i{}^j$ is generator T_A in representation of supercharge.

Using the equation (2.71) and $[P^\mu, T_A] = 0$ the super-Jacobian identity becomes

$$[T_A, \{Q_{Ai}, \overline{Q}^{j}_{\dot{a}}\}] + \{Q_{ai}, [\overline{Q}^{j}_{\dot{a}}, T_A]\} - \{\overline{Q}^{j}_{\dot{a}}, [T_A, Q_{ai}]\} = 0. \tag{2.82}$$

As a result, $S^{A\dagger} = S^A$, and the super-Jacobian identity is written as

$$[T_A, \{Q_{ai}, Q_{bj}\}] + \{Q_{ai}, [Q_{bj}, T_A]\} - \{Q_{bj}, [T_A, Q_{ai}]\} = 0, \tag{2.83}$$

which provides the commutator between T_A and Z_{ij}, denoted by

$$[T_A, Z_{ij}] = (S_A)_i{}^j Z_{jk} - (S_A)_j{}^k Z_{ik}. \tag{2.84}$$

The equation (2.84) implies that Z_{ij} is an invariant subalgebra. When using the equation (2.72) and (2.76) in super-Jacobian identity

$$[Q_{ai}, \{Q_{bj}, \overline{Q}^{k}_{\dot{c}}\}] + [Q_{bj}, \{\overline{Q}^{k}_{\dot{c}}, Q_{ai}\}] + [\overline{Q}^{k}_{\dot{c}}, \{Q_{ai}, Q_{bj}\}] = 0, \tag{2.85}$$

the result provides $[\overline{Q}^{k}_{\dot{a}}, Z_{ij}] = 0$, and $[Q_{ak}, Z_{ij}] = 0$. Additionally, combining super-Jacobian identity with a generator $(Q, \overline{Q}, Z)$ and $[P^\mu, Z_{ij}] = 0$ can provide the solution as

$$[Z_{ij}, Z_{kl}] = \epsilon^{ab}[\{Q_{ai}, Q_{bj}\}, Z_{kl}] = 0. \tag{2.86}$$

In other words, the above equation can be written as $a^A_{kl}[Z_{ij}, T_A] = 0$, which implies that $[Q_{ak}, Z_{ij}] = 0$ for $a^A_{kl} \neq 0$. As a consequence, the generator Z_{ij} commutes with all generators, which are called "central charges."

All supersymmetry algebra can be summarized as follows:

$$
\begin{aligned}
&[P^{\rho}, J^{\mu\nu}] = i(\eta^{\mu\rho}P^{\nu} - \eta^{\nu\rho}P^{\mu}), \qquad [P^{\mu}, P^{\nu}] = 0 \\
&[J^{\mu\nu}, J^{\rho\sigma}] = -i(J^{\mu\sigma}\eta^{\nu\rho} - J^{\nu\sigma}J\eta^{\mu\rho} + J^{\nu\rho}\eta^{\mu\sigma} - J^{\mu\rho}\eta^{\nu\sigma}) \\
&[P^{\mu}, Q_{ai}] = [P^{\mu}, \overline{Q}^{i}_{\dot{a}}] = 0, \qquad \{Q_{ai}, \overline{Q}^{j}_{\dot{a}}\} = -\frac{1}{2}\delta^{j}_{i}\sigma_{\mu a\dot{a}}P^{\mu} \\
&[Q_{ai}, J_{\mu\nu}] = i(\sigma_{\mu\nu})_{a}{}^{b}Q_{bi}, \qquad [\overline{Q}^{\dot{a}i}, J_{\mu\nu}] = i(\overline{\sigma}_{\mu\nu})^{\dot{a}}{}_{\dot{b}}\overline{Q}^{\dot{b}i} \\
&\{Q_{ai}, Q_{bj}\} = -\frac{1}{2}\epsilon_{ab}Z_{ij}, \qquad \{\overline{Q}^{i}_{\dot{a}}, \overline{Q}^{j}_{\dot{b}}\} = -\frac{1}{2}\epsilon_{\dot{a}\dot{b}}Z^{ij} \\
&[Q_{ai}, T_{A}] = (S_{A})_{i}{}^{j}Q_{aj}, \qquad [\overline{Q}^{i}_{\dot{a}}, T_{A}] = (S^{*}_{A})^{i}{}_{j}\overline{Q}^{j}_{\dot{a}} \\
&Z_{ij} = a^{A}_{ij}T_{A}, \qquad [T_{A}, T_{B}] = f_{AB}{}^{C}T_{C},
\end{aligned}
\tag{2.87}
$$

and $Z^{ij} = (Z_{ij})^{\dagger}$, which commute with all generator.

R symmetry

R symmetries, which are arranged as an automorphism group of supersymmetry, are symmetries that do not commute with supercharge Q_{ai}. Let T_A represent the generators of R symmetry, which corresponds to Lie algebra

$$[T_A, T_B] = f_{AB}{}^{C}T_C. \tag{2.88}$$

The commutator between T_A and Q_{ai} can be written as

$$[T_A, Q_{ai}] = (U_A)_i{}^{j}Q_{aj} \quad \text{and} \quad [T_A, Q^{i}_{\dot{a}}] = (U_A)^{i}{}_{j}Q^{j}_{\dot{a}}, \tag{2.89}$$

where $(U_A)^{i}{}_{j}$ is complex conjugate of $(U_A)_i{}^{j}$.

The super-Jacobian identity follows as

$$[[T_A, T_B], Q_{ai}] + [[T_B, Q_{ai}], T_A] + [[Q_{ai}, T_A], T_B] = 0, \tag{2.90}$$

which provides the commutator

$$[U_A, U_B] = -f_{AB}{}^{C}U_C. \tag{2.91}$$

The previous equation implies that the matrix $-(U_a)_i{}^j$ is a representation of R symmetry. Using the super-Jacobian identity

$$\{[T_A, Q_{ai}], \overline{Q}^j_{\dot{b}}\} + \{[T_A, \overline{Q}^j_{\dot{b}}], Q_{ai}\} + [\{Q_{ai}, \overline{Q}^j_{\dot{b}}\}, T_A] = 0, \tag{2.92}$$

and $[P^\mu, T_A] = 0$, which provide

$$-\frac{1}{2}\sigma^\mu_{a\dot{b}} P_\mu[(U_A)_i{}^k \delta^j_k + (U_A)^j{}_k \delta^k_i] = 0 \tag{2.93}$$

or

$$(U_A)_i{}^j = -(U_A)^j{}_i = -((U_A)_j{}^i)^*. \tag{2.94}$$

As a result, the matrix $(U_A)_i{}^j$ is anti-hermitian, and the R symmetry of supersymmetry that consists of $4N$ supercharge is the $U(N)$ group. As we will see in the following section, R symmetries are crucial in gauged supergravity and superconformal theory.

2.3 Supersymmetry representation

In the event that supersymmetry is a symmetry of spacetime, particles must belong in the supersymmetry representation. Because of $[P^\mu, Q_{ai}] = 0$, Casimir's operator $P^2 = P^\mu P_\mu$, suggests that all particles in any supersymmetry representation have the same mass. However, in the case of $[Q_{ia}, J^{\mu\nu}] \neq 0$, the operator W^2 (W^μ is Pauli–Lubanski vector) is not a Casimir operator of supersymmetry, yielding differing spin numbers for the particles.

Consider the anti-commutator

$$\{Q_{ai}, \overline{Q}^j_{\dot{b}}\} = -\frac{1}{2}\delta^j_i \sigma_{\mu a\dot{b}} P^\mu, \tag{2.95}$$

and find the trace of indices a and $\dot{b}$ that give

$$\mathrm{Tr}(Q_i\overline{Q}^j + \overline{Q}^j Q_i) = \delta^j_i P^0 \tag{2.96}$$

for Tr$(\sigma_0) = -2$, and Tr$(\sigma_i) = 0$. If there are any physical states $|\psi\rangle$ in Hilbert space where all states have a positive norm, then

$$\langle\psi| (Q\overline{Q} + \overline{Q}Q) |\psi\rangle = \langle\psi| QQ^\dagger |\psi\rangle + \langle\psi| Q^\dagger Q |\psi\rangle > 0. \tag{2.97}$$

Because of $\langle\psi| QQ^\dagger |\psi\rangle = ||Q^\dagger\psi||^2 > 0$ and $\langle\psi| Q^\dagger Q |\psi\rangle = ||Q\psi||^2 > 0$, the equation (2.96) provides $\langle\psi| P^0 |\psi\rangle > 0$, which implies that the energy ($P^0 = E$) in supersymmetry theory is always positive.

Assuming that ψ_B and ψ_F are the respective boson and fermion states, the supersymmetry generator will take ψ_B and ψ_F into

$$Q_{ai} |\psi_B\rangle \sim |\psi_F\rangle \quad \text{and} \quad Q_{ai} |\psi_F\rangle \sim |\psi_B\rangle \,. \tag{2.98}$$

These states are classified by defining the fermion number operator, which has two properties:

$$(-)^F |\psi_B\rangle = |\psi_B\rangle \quad , \quad (-)^F |\psi_F\rangle = -|\psi_F\rangle \,. \tag{2.99}$$

The anti-commutator between $(-)^F$ and Q_{ai} can be derived by following

$$\begin{aligned}
\{(-)^F, Q_{ai}\} |\psi_B\rangle &= ((-)^F Q_{ai} + Q_{ai}(-)^F) |\psi_B\rangle \\
&= (-)^F Q_{ai} |\psi_B\rangle + Q_{ai}(-)^F |\psi_B\rangle \\
&= -|\psi_F\rangle + |\psi_F\rangle \\
&= (-)^F Q_{ai} |\psi_B\rangle - (-)^F Q_{ai} |\psi_B\rangle \\
&= ((-)^F Q_{ai} - (-)^F Q_{ai}) |\psi_B\rangle \\
\{(-)^F, Q_{ai}\} &= 0.
\end{aligned} \tag{2.100}$$

Multiply equation (2.95) with $(-)^F$, and find an expected value (in the meaning of Tr$\hat{O} = \sum_n \langle n| \hat{O} |n\rangle$ for any operator $\hat{O}$). The equation (2.95) becomes

$$-\frac{1}{2}\sigma_{\mu a\dot{b}}\delta_i^j \,\mathrm{Tr}\big[(-)^F P^\mu\big] = \mathrm{Tr}\Big[(-)^F \{Q_{ai}, \overline{Q}_{\dot{b}}^j\}\Big]. \tag{2.101}$$

By using equation (2.100) and trace cyclic, so that

$$\mathrm{Tr}\Big[(-)^F Q_{ai}\overline{Q}_{\dot{b}}^j + (-)^F \overline{Q}_{\dot{b}}^j Q_{ai}\Big] = \mathrm{Tr}\Big[-Q_{ai}(-)^F \overline{Q}_{\dot{b}}^j\Big] + \mathrm{Tr}\Big[Q_{ai}(-)^F \overline{Q}_{\dot{b}}^j\Big] = 0 \tag{2.102}$$

equation (2.101) becomes

$$\begin{aligned}
\mathrm{Tr}(-)^F &= 0 \\
&= \sum_n \langle n| (-)^F |n\rangle \\
&= \sum_{\text{Boson}} \langle \psi_B| (-)^F |\psi_B\rangle + \sum_{\text{Fermion}} \langle \psi_F| (-)^F |\psi_F\rangle \\
&= \sum_{\text{Boson}} \langle \psi_B|\psi_B\rangle - \sum_{\text{Fermion}} \langle \psi_F|\psi_F\rangle \\
&= n_B - n_F \\
n_B &= n_F
\end{aligned} \tag{2.103}$$

for $P^\mu \neq 0$, which implies that any representation of supersymmetry have the same number of bosonic states and fermionic states .

Massless representation

In this case, the momentum square $P^2 = 0$, which the standard momentum can be chosen as traveling in z-direction denoted by $k^\mu = (E, 0, 0, E)$.

Replace the standard momentum k^μ in the algebra (2.95), so that

$$\begin{aligned}
\{Q_{ai}, \overline{Q}^j_{\dot{b}}\} &= -\frac{1}{2}\delta^i_j(-\delta_{ab}E + \sigma^3_{ab}E) \\
&= \delta^i_j E \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\end{aligned} \tag{2.104}$$

For the massless representation, the central charge vanishes in supersymmetry algebra. The anti-commutators of supercharges are transformed into

$$\{Q_{ai}, Q_{bj}\} = \{\overline{Q}^i_{\dot{a}}, \overline{Q}^j_{\dot{b}}\} = 0, \tag{2.105}$$

and equation (2.104) becomes

$$\{Q_{2i}, \overline{Q}^j_{\dot{2}}\} = \delta^j_i E. \tag{2.106}$$

Equation (2.87) gives the commutator between $J^{\mu\nu}$ and Q_{ai}

$$[\vec{J}, Q_{ai}] = \frac{1}{2}\vec{\sigma}_a^b Q_{bi}, \tag{2.107}$$

where the generator $\vec{J}$, and $\vec{\sigma}$ can be fined by using the relation $J_i = \frac{1}{2}\epsilon_{ijk}J^{jk}$ and $\sigma^i\sigma^j = i\epsilon_{ijk}\sigma^k$, respectively. The conjugation of commutator (2.107) is

$$[\vec{J}, \overline{Q}^i_{\dot{a}}] = -\frac{1}{2}\overline{\sigma}_{\dot{a}}{}^{b}\overline{Q}^i_{\dot{b}}. \tag{2.108}$$

In massless representation, the particle states are defined by the component of spin in the direction of momentum, also known as "helicity" $(h = \frac{\vec{J}.\vec{k}}{|\vec{k}|})$. The states are defined as $|k, h\rangle$, where the helicity h corresponds to eigenvalue of J_3. From equation (2.107) and (2.108), the commutators turn into

$$[J_3, Q_{2i}] = -\frac{1}{2}Q_{2i} \quad \text{and} \quad [J_3, \overline{Q}^i_{\dot{2}}] = \frac{1}{2}\overline{Q}^i_{\dot{2}}, \tag{2.109}$$

so that $\overline{Q}^i_{\dot{2}}$ and Q_{2i} increases and decreases the helicity by $\frac{1}{2}$, respectively.

We can define $\hat{a}_i = \frac{Q_{2i}}{\sqrt{E}}$ and $\hat{a}_i^\dagger = \frac{\overline{Q}^i_{\dot{2}}}{\sqrt{E}}$, resulting in anti-commutator

$$\{\hat{a}_i, \hat{a}_j^\dagger\} = \delta_{ij}. \tag{2.110}$$

The previous equation is in the form of a fermionic harmonic oscillator, and its representation can be found by defining the Clifford vacuum, which is the state that corresponds to the lowest possible helicity $|k, h_{min} = h_0\rangle$ for the system. The conditions necessary to define representation are

$$\hat{a}_i\,|k, h_0\rangle = 0 \quad \text{and} \quad J_3\,|k, h_0\rangle = h_0\,|k, h_0\rangle \tag{2.111}$$

for $i = 1, ..., N$. All states within the representation can be found by applying a specific operator $\hat{a}_i^\dagger$ to a known state $|k, h_0\rangle$, so that all states can be written as

$$\begin{aligned}
&|k, h_0\rangle \\
\left|k, h_0 + \frac{1}{2}; i\right\rangle &= \hat{a}_i^\dagger |k, h_0\rangle \\
|k, h_0 + 1; i, j\rangle &= \hat{a}_j^\dagger \hat{a}_i^\dagger |k, h_0\rangle \\
&\vdots \\
\left|k, h_0 + \frac{n}{2}; i_1 \cdots i_n\right\rangle &= \hat{a}_{i_1}^\dagger \cdots \hat{a}_{i_n}^\dagger |k, h_0\rangle \\
&\vdots \\
\left|k, h_0 + \frac{N}{2}; i_1 \cdots i_N\right\rangle &= \hat{a}_{i_1}^\dagger \cdots \hat{a}_{i_N}^\dagger |k, h_0\rangle . \qquad (2.112)
\end{aligned}$$

The states $\hat{a}_{i_1}^\dagger \cdots \hat{a}_{i_n}^\dagger |k, h_0\rangle$ occupy $h_0 + \frac{n}{2}$ helicity, and the number of all possible states are written as

$$\binom{N}{n} = \frac{N!}{n!(N-n)!}. \qquad (2.113)$$

Because $\{\hat{a}_i^\dagger, \hat{a}_j^\dagger\} = 0$, the state $\hat{a}_{i_1}^\dagger \cdots \hat{a}_{i_n}^\dagger |k, h_0\rangle$ has anti-symmetric indices in $\{i_1 \cdots i_N\}$, and the states created from $N+1$ operators $\hat{a}_i^\dagger$ are equal to zero. The dimension or number of states in massless representation is equal to

$$D(N) = \sum_{n=0}^{N} \binom{N}{n} = (1+1)^N = 2^N. \qquad (2.114)$$

Since the number of boson states and fermion states are equal, the number of boson or fermion states is $\frac{2^N}{2} = 2^{N-1}$. A set of multiple particles in supersymmetric theories known as a supermultiplet serves as a visual representation of the supersymmetry algebra.

Notice that all states resulting from equation (2.112) consist of helicity $(h_0, h_0 + \frac{1}{2}, \cdots, h_0 + \frac{N}{2})$. To correspond to CTP symmetry, which shifts the helicity h to the negative $-h$, the opposite states $(-h_0, -h_0 - \frac{1}{2}, \cdots, -h_0 - \frac{N}{2})$ known as CPT conjugation must be added. In the event that the maximum helicity is equals to the opposite of minimum helicity, we can write the relation as $h_0 + \frac{N}{2} = -h_0$ or $N = -4h_0 = 4|h_0|$. According to quantum field theories that are invariant under

Lorentz symmetry, helicity of particles in physics are $-2 \leq h \leq 2$. As a result, we can impose the maximum number of N as

$$\begin{aligned} h_{max} - h_{min} &\leq 4 \\ h_0 + \frac{N}{2} - h_0 &\leq 4 \\ N &\leq 8. \end{aligned} \tag{2.115}$$

Theoretical physics in four-dimensional space-time has a maximum supersymmetry at $N = 8$ or 32 supercharge.

Consider the most basic case where $N = 1$ supersymmetry.

- Chiral multiplet or scalar multiplet: $h_0 = -\frac{1}{2}$,

$$\left|k, -\frac{1}{2}\right\rangle, \qquad |k, 0\rangle = \hat{a}^\dagger \left|k, -\frac{1}{2}\right\rangle \tag{2.116}$$

The previous state consisted of helicity $(-\frac{1}{2}, 0)$ with no CPT symmetry. When we add CPT symmetry, chiral multiplets become

$$\left\{ \left|-\frac{1}{2}\right\rangle, |0\rangle \right\} \oplus \left\{ |0\rangle, \left|\frac{1}{2}\right\rangle \right\}. \tag{2.117}$$

For its simplicity, momentum k^μ is disregarded. States $|\pm\frac{1}{2}\rangle$ are combined into one Weyl fermion, and $|0\rangle$ provide two scalars.

- Vector (or gauge) multiplet: $h_0 = -1$,

$$\left\{ |-1\rangle, \left|-\frac{1}{2}\right\rangle \right\} \oplus \left\{ \left|\frac{1}{2}\right\rangle, |1\rangle \right\}. \tag{2.118}$$

This multiplet consists of one vector $|\pm 1\rangle$ and one Weyl fermion $|\pm\frac{1}{2}\rangle$. Vector multiplet is the necessary representation to explain gauge fields in a supersymmetric theory.

- Supergravity (or gravity) multiplet: $h_0 = -2$,

$$\left\{ |-2\rangle, \left|-\frac{3}{2}\right\rangle \right\} \oplus \left\{ \left|\frac{3}{2}\right\rangle, |2\rangle \right\}. \tag{2.119}$$

The degrees of freedom are given by a graviton $|\pm 2\rangle$, and the gravitino $|\pm\frac{3}{2}\rangle$, which is the graviton's supersymmetric partner.

- gravitino multiplet: $h_0 = -\frac{3}{2}$,

$$\left\{ \left|-\frac{3}{2}\right\rangle, |-1\rangle \right\} \oplus \left\{ |1\rangle, \left|\frac{3}{2}\right\rangle \right\}. \tag{2.120}$$

The degrees of freedom are those of a gravitino $|\pm\frac{3}{2}\rangle$ and one vector $|\pm 1\rangle$.

All supermultiplets for $1 \leq N \leq 8$ are shown in Table 2.1. It is noteworthy that for $N > 4$, only supermultiplets with $s > \frac{3}{2}$ exist. Furthermore, the supergravity theory of $N = 7$ is physically equivalent to $N = 8$ due to identical field contents.

In the case of $N = 2$, supermultiplets constructed from (2.112) become

$$\left|-\frac{1}{2}\right\rangle, \quad |0;i\rangle = \hat{a}_i^\dagger \left|-\frac{1}{2}\right\rangle, \quad \left|\frac{1}{2};i,j\right\rangle = \hat{a}_i^\dagger \hat{a}_j^\dagger \left|-\frac{1}{2}\right\rangle = \frac{1}{2}\epsilon^{ij}\hat{a}_k^\dagger \hat{a}^{\dagger k}\left|-\frac{1}{2}\right\rangle, \tag{2.121}$$

which consist of states with helicity $(-\frac{1}{2}, 0+0, \frac{1}{2})$ and have CPT symmetry automatically. To let the state with helicity $h = 0$ that provides two scalars correspond to the **2** representation of $SU(2)$ (subgroup of R symmetry $U(2)$ that consists of four real components), the multiplets are combined with two sets of (2.121). Therefore, the new multiplets have four scalars with two spinors and are named hypermultiplets. In addition, notice that the last state in equation (2.121) transforms as a singlet under $SU(2)$, since ϵ_{ij} is invariant tensor and $\hat{a}_k^\dagger \hat{a}^{\dagger k} = \epsilon^{kl}\hat{a}_k^\dagger \hat{a}_l^\dagger$ is a singlet of $SU(2)$.

Supersymmetry plays an important role in constructing supergravity, which we will discuss in the next chapter.

N = 1 supersymmetric field theory

In this section, we only discuss chiral multiplets in $N = 1$ supersymmetry to demonstrate supersymmetry transformation and the action. The field content of the chiral multiplet consists of two complex scalars and a Weyl or Majerana spinor field, see in Table 2.1.

In the theory of quantum fields, vacuum $|0\rangle$ is mapped into particles via a field operator $\Phi(x)$. Particles with momentum p^μ and spin j are created as

$$|p,j\rangle = \Phi(x)\,|0\rangle\,. \tag{2.122}$$

In the case of a vacuum that has supersymmetry, supercharges annihilate the vacuum as

$$Q_{ai} |0\rangle \quad \text{and} \quad \overline{Q}_{\dot{a}i} |0\rangle . \tag{2.123}$$

Take into consideration the scalar field operator $Z(x)$, which produces zero spin particles from vacuum $|0, p\rangle$, denoted as

$$|p, 0\rangle = Z(x) |0\rangle . \tag{2.124}$$

In addition, supercharge $\overline{Q}_{\dot{a}}$ is used to annihilate states in the same way that Q_a annihilated the Clifford vacuum. We can write that

$$\overline{Q}_{\dot{a}} |p, 0\rangle = \overline{Q}_{\dot{a}} Z(x) |0\rangle = 0. \tag{2.125}$$

From equation (2.123), we found that $Z(x)\overline{Q}_{\dot{a}} |0\rangle = 0$, so that $Z(x)$ commute with $\overline{Q}_{\dot{a}}$:

$$[Z(x), \overline{Q}_{\dot{a}}] = 0, \tag{2.126}$$

and $Z(x)$ uncommute with Q_a, since super-Jacobian identity

$$\{[Z(x), Q_a], \overline{Q}_{\dot{a}}\} - \{[\overline{Q}_{\dot{a}}, Z(x)], Q_a\} + [\{Q_a, \overline{Q}_{\dot{a}}\}, Z(x)] = 0, \tag{2.127}$$

provide that

$$[\{Q_a, \overline{Q}_{\dot{a}}\}, Z(x)] = -\frac{1}{2}\sigma^{\mu}_{a\dot{a}} \partial_{\mu} Z(x) = 0. \tag{2.128}$$

In the event that $[Z(X), Q_a] = [Z(x), \overline{Q}_{\dot{a}}] = 0$, $Z(x)$ must be constant in space-time.

The action of Q_a on $|p, 0\rangle$, in the case $[Z(X), Q_a] \neq 0$ provide that

$$Q_a |p, 0\rangle = Q_a Z(x) |0\rangle = \left| p, \frac{1}{2} \right\rangle , \tag{2.129}$$

which increase spin number $\frac{1}{2}$ and momentum p^{μ} from vacuum. Similarly to the scalar, the particle $\left|p, \frac{1}{2}\right\rangle$ can be obtained from the spinor field $\chi(x)_a$ in the form $\left|p, \frac{1}{2}\right\rangle = \chi_a(x) |0\rangle$, so that

$$Q_a Z(x) |0\rangle = [Q_a, Z(x)] |0\rangle = \chi_a(x) |0\rangle . \tag{2.130}$$

From previous equation, we can write the commutator between Q_a and $\chi_a(x)$ as

$$\chi_a(x) = [Q_a, Z(x)]. \tag{2.131}$$

Using the relation $[Z(x), \overline{Q}_{\dot{a}}] = 0$ and super-Jacobian identity

$$[\{\overline{Q}_{\dot{a}}, Q_b\}, Z(x)] + \{[Z(x), \overline{Q}_{\dot{a}}], Q_b\} - \{[Q_b, Z(x)], \overline{Q}_{\dot{a}}\} = 0, \tag{2.132}$$

we can show that

$$\{\overline{Q}_{\dot{a}}, \chi_b(x)\} = -\frac{1}{2}\sigma^{\mu}_{a\dot{a}}\partial_\mu Z(x), \tag{2.133}$$

and consider an action of Q_a on state $|p, \frac{1}{2}\rangle$ written in the term of field operator as

$$\{Q_a, \chi_b(x)\} = \{Q_a, [Q_b, Z(x)]\}. \tag{2.134}$$

When using super-Jacobian identity

$$\{Q_a, [Q_b, Z(x)]\} - \{Q_b, [Z(x), Q_a]\} + [Z(x), \{Q_a, Q_b\}] = 0 \tag{2.135}$$

and the relation $\{Q_a, Q_b\} = 0$ together with (2.131), we can show that

$$\{Q_a, \chi_b(x)\} = \{Q_b, [Z(x), Q_a]\} = -\{Q_b, \chi_a(x)\}. \tag{2.136}$$

This finding implies that $\{Q_a, \chi_b(x)\}$ is anti-symmetric in a and b, so that the result of $\{Q_a, \chi_b(x)\}$ must be scalar $F(x)$, Lorentz tensor that has anti-symmetric spinor indices

$$\{Q_a, \chi_b(x)\} = \epsilon_{ab}F(x). \tag{2.137}$$

Similarly, we can find a commutator between Q_a and $F(x)$ by following:

$$\begin{aligned}\epsilon_{ab}[Q_c, F(x)] &= [Q_c, \{Q_a, \chi_a(x)\}] \\ &= -[Q_a, \{\chi_b(x), Q_c\}] - [\chi_a(x)\{Q_c, Q_a\}] \\ &= -\epsilon_{cb}[Q_a, F(x)].\end{aligned} \tag{2.138}$$

Contracting previous equation with ϵ^{ab}, we obtain

$$[Q_c, F(x)] = 0. \tag{2.139}$$

Finally, we consider the commutator $[\overline{Q}_{\dot{a}}, F(x)]$, which can be shown as

$$\begin{aligned}\epsilon_{ab}[\overline{Q}_{\dot{a}}, F(x)] &= [\overline{Q}_{\dot{a}}, \{Q_a, \chi_b\}] \\ &= -[Q_a, \{\chi_b(x), \overline{Q}_{\dot{a}}\}] - [\chi_b(x), \{\overline{Q}_{\dot{a}}, Q_a\}] \\ &= \frac{1}{2}\sigma^{\mu}_{b\dot{a}}[Q_a, \partial_\mu Z(x)] + \frac{1}{2}\sigma^{\mu}_{a\dot{a}}[\chi_b, P_\mu] \\ &= \frac{1}{2}\sigma^{\mu}_{b\dot{a}}\partial_\mu \chi_a(x) - \frac{1}{2}\sigma^{\mu}_{a\dot{a}}\partial_\mu \chi_b(x). \end{aligned} \tag{2.140}$$

Contracting previous equation with ϵ^{ab}, we obtain

$$[\overline{Q}_{\dot{a}}, F(x)] = -\frac{1}{2}\partial_\mu \chi^a(x)\sigma^{\mu}_{a\dot{a}}. \tag{2.141}$$

According to Noether's theorem, a continuous symmetry results in a corresponding conserved current. In the case of supersymmetry, it possesses a supercurrent J^{μ}_a with both vector and spinor indices. The supercharge Q_a is then defined as

$$Q_a = \int d^3x J^0_a. \tag{2.142}$$

The action of the supercharge transforms the field operator under supersymmetry, resulting in the form

$$\delta\Phi(x) = [\overline{\epsilon}_{\dot{a}}\overline{Q}^{\dot{a}} + \epsilon^a Q_a, \Phi(x)], \tag{2.143}$$

where $\Phi(x)$ represents any field and ϵ_a is a constant spinor (global symmetry) that serves as the transformation parameter. Additionally, ϵ_a possesses the property

$$\{\epsilon_a, Q_b\} = \{\epsilon_a, \Psi_b\} = 0, \tag{2.144}$$

for any spinor fields $\Psi_b(x)$. These relations express that

$$\begin{aligned}[\epsilon^a Q_a, \Psi_b(x)] &= \epsilon^a Q_a \Psi_b(x) - \Psi_b(x)\epsilon^a Q_a \\ &= \epsilon^a (Q_a \Psi_b(x) + \Psi_b(x) Q_a) \\ &= \epsilon^a \{Q_a, \Psi_b(x)\}. \end{aligned} \tag{2.145}$$

Therefore, the transformation rule of the field can be expressed as a commutator in the form (2.143) for both fermions and bosons.

Based on the results derived previously, the transformation of the field content in a chiral multiplet (Z, χ_a, F) under supersymmetry can be written as

$$\delta Z = [\epsilon^a Q_a + \overline{\epsilon}_{\dot{a}}, Z] = \epsilon^a[\overline{Q}_{\dot{a}}, Z] = \epsilon^a \chi_a \tag{2.146}$$

$$\delta\chi_a = \epsilon^c\{Q_c, \chi_a\} - \overline{\epsilon}_{\dot{a}}\{\overline{Q}_{\dot{a}}, \chi_a\} = F\epsilon_a + \frac{1}{2}\sigma^{\mu}_{a\dot{a}}\overline{\epsilon}^{\dot{a}}\partial_\mu Z \tag{2.147}$$

$$\delta F = -\overline{\epsilon}^{\dot{a}}[\overline{Q}_{\dot{a}}, F] = \frac{1}{2}\overline{\epsilon}^{\dot{a}}\partial_\mu\chi^a\sigma^{\mu}_{a\dot{a}}. \tag{2.148}$$

The invariant action for a chiral multiplet can be expressed as:

$$\begin{aligned} S = \int d^4x \Big[& -\partial_\mu\overline{Z}\partial^\mu Z - \overline{\chi}\gamma^\mu\partial_\mu P_L\chi + \overline{F}F \\ & + FW'(Z) - \frac{1}{2}W''(z)\overline{\chi}P_L\chi \\ & + \overline{F}\overline{W}'(\overline{Z}) - \frac{1}{2}\overline{W}''(\overline{Z})\overline{\chi}P_R\chi \Big] \end{aligned} \tag{2.149}$$

where Z, χ, $\overline{Z}$, F, $\overline{F}$ represent the scalar field, the fermionic field, the complex conjugate of the scalar field, the auxiliary field, and the complex conjugate of the auxiliary field, respectively. P_L and P_R are the chirality projection operators, $W(Z)$ is a holomorphic superpotential, and $W'(Z)$ and $W''(Z)$ are its first and second derivatives, respectively.

It should be noted that the kinetic term for the auxiliary field F in the action (2.149) does not have its own dynamics, making it an auxiliary field rather than a physical propagating field. Adding the kinetic term $\partial_\mu F\partial^\mu\overline{F}$ to the action will give this term six dimensions, meaning that it can not renormalization (non-renormalization term is the term that have dimension more than 4).

N	S_{max}	$s=2$	$s=\frac{3}{2}$	$s=1$	$s=\frac{1}{2}$	$s=0$
1	2	1	1			
	$\frac{3}{2}$		1	1		
	1			1	1	
	$\frac{1}{2}$				1	$1+1$
2	2	1	2	1		
	$\frac{3}{2}$		1	2	1	
	1			1	2	$1+1$
	$\frac{1}{2}$				2	$2+2$
3	2	1	3	3	1	
	$\frac{3}{2}$		1	3	3	$1+1$
	1			1	$3+1$	$3+3$
4	2	1	4	6	4	$1+1$
	$\frac{3}{2}$		1	4	$6+1$	$4+4$
	1			1	4	6
5	2	1	5	10	$10+1$	$5+5$
	$\frac{3}{2}$		1	$5+1$	$10+5$	$10+10$
6	2	1	6	$15+1$	$20+6$	$15+15$
	$\frac{3}{2}$		1	6	15	20
7	2	1	$7+1$	$21+7$	$35+21$	$35+35$
8	2	1	8	28	56	70

Table 2.1: Supermultiplets for $1 \leq N \leq 8$ in 4-dimenional space-time

Chapter III

SUPERGRAVITY

Supergravity is a theoretical framework that combines supersymmetry and general relativity principles. It is a kind of quantum field theory that describes the interactions between gravity and matter fields by using both quantum mechanics and general relativity principles. Supergravity is crucial in attempts to combine all of nature's fundamental forces, including the strong, weak, and electromagnetic forces, into a single theory known as a theory of everything.

3.1 Gravitino

In the previous section, the supersymmetry transformation parameter was a constant spinor, ϵ. However, in supergravity, it becomes a spacetime-dependent function, $\epsilon(x)$. The corresponding gauged field in this case is the gravitino $\Psi_\mu(x)$.

In this section, the gravitino field, also known as the Rarita-Schwinger field, will be introduced in a flat spacetime context. The gravitino is a vector-spinor field that has both vector and spinor indices and belongs to the representation

$$(\frac{1}{2},\frac{1}{2}) \otimes \left[(0,\frac{1}{2}) \oplus (\frac{1}{2},0)\right] = (\frac{1}{2},0) \oplus (0,\frac{1}{2}) \oplus (\frac{1}{2},1) \oplus (1,\frac{1}{2}). \tag{3.1}$$

By utilizing the gamma-traceless condition in the form of $\gamma^\mu\Psi_\mu$, the spinor representation $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ is truncated, yielding the representation $(\frac{1}{2},1) \oplus (1,\frac{1}{2})$. This represents a spin $3/2$ field.

The gauge transformation of Ψ_μ can be written as

$$\Psi_\mu(x) \longrightarrow \Psi'_\mu = \Psi_\mu(x) + \partial_\mu\epsilon(x). \tag{3.2}$$

Its gauge-invariant action can be represented as

$$S = -\int d^4x \overline{\Psi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \Psi_\rho. \tag{3.3}$$

It is important to note that although the action is gauge invariant, the Lagrangian density is not. The variation of the Lagrangian density results in a total derivative $\delta\mathcal{L} = -\partial_\mu(\bar{\epsilon}\gamma^{\mu\nu\rho}\partial_\nu\Psi_\rho)$. This is because the fermionic gauge symmetry is a remnant of supersymmetry.

The variation of the previous action with $\overline{\Psi}_\mu$ results in the field equation written as

$$\gamma^{\mu\nu\rho} \partial_\nu \Psi_\rho = 0. \tag{3.4}$$

The off-shell degree of freedom of gravitino can be counted by considering the components of the vector-spinor field and subtracting the number of gauge transformations $(4 \times 2^2 - 2^2 = 12)$. The on-shell degree of freedom of the gravitino can be counted as $\frac{1}{2}(4 \times 2 - 4)$. The $(4 \times 2 - 4)$ comes from the number of degrees of freedom of the vector and spinor components subtracted by the condition $\gamma^i \Psi_i = 0$, and the value of $\frac{1}{2}$ comes from the field equation of Ψ_μ that is first order and has a projection like that of a spinor field.

3.2 N = 1 pure supergravity in four dimenions

The core concept of supergravity is to promote supersymmetry from being global to local. The action is invariant under supersymmetry transformations with the spinor parameter $\epsilon(x)$ being a function of spacetime. As a result, the supersymmetry algebra will include the local translation parameter $\bar{\epsilon}_1\gamma^\mu\epsilon_2$, which must be considered as a diffeomorphism

$$x^\mu \longrightarrow x^{\mu'} + \frac{1}{2}\bar{\epsilon}_2(x)\gamma^\mu\epsilon_1(x), \tag{3.5}$$

which graviton is gauge field. In other words, local supersymmetry necessitates the presence of gravity.

A supergravity theory is a field theory that contains the gravity multiplet as well as, potentially, other matter multiplets of the global supersymmetry algebra. It is nonlinear and involves interactions between its various components. The gravity multiplet in a supergravity theory contains the frame field e_μ^a that describes the graviton, and also a certain number N of gravitino filed ψ_μ^i, where $i = 1, ..., N$. In this case, the number of gravitinos is $N = 1$, and the theory is referred to as $N = 1$ supergravity in $D = 4$ dimensions. The gravity multiplet includes only the graviton e_μ^a and a single Majorana spinor gravitino ψ_μ. The action can be written as

$$\begin{aligned} S &= S_2 + S_{3/2} \\ S_2 &= \frac{1}{2\kappa^2}\int d^4x e e^{a\mu} e^{b\nu} R_{\mu\nu ab}(\omega) \\ S_{3/2} &= -\frac{1}{2\kappa^2}\int d^4x e \overline{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho. \end{aligned} \tag{3.6}$$

The action S_2 describes the dynamics of gravitons in second order formalism, where e represents the determinant of e_μ^a, and $S_{3/2}$ describes the dynamics of the gravitino. The gravitino covariant derivative is defined as:

$$D_\nu \psi_\rho = \partial_\nu \psi_\rho + \frac{1}{4}\omega_{\nu ab}\gamma^{ab}\psi_\rho. \tag{3.7}$$

The supersymmetry transformation corresponding to a graviton and gravitino fields can be written as

$$\delta e_\mu^a = \frac{1}{2}\bar{\epsilon}\gamma^a \psi_\mu \tag{3.8}$$

$$\delta \psi_\mu = D_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\epsilon. \tag{3.9}$$

The transformation of e_μ^a results in the transformation of its inverse and determinant, which can be written as

$$\delta e_a^\mu = -\frac{1}{2}\bar{\epsilon}\gamma^\mu \psi_a \quad \text{and} \quad \delta e = \frac{1}{2}e(\bar{\epsilon}\gamma^\rho \psi_\rho). \tag{3.10}$$

Noted that the action (3.6) is not invariant under supersymmetry transformation. This is because the variation of $S_{3/2}$ with respect to e results in $\epsilon\psi^3$, which cannot make the variation of the action equal to zero. However, the action (3.6) remains invariant at the first order of ψ_μ, which can be verified from the following variation

$$\begin{aligned}\delta S &= \frac{1}{2\kappa^2}\int d^4xe\Big[(2\delta e^{a\mu}e^{b\nu} + e^{-1}\delta e e^{a\mu}e^{b\nu})R_{\mu\nu ab} + e^{a\mu}e^{b\nu}\delta R_{\mu\nu ab} \\ &\quad - \delta\overline{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho - \overline{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\delta\psi_\rho\Big] \\ &= \frac{1}{2\kappa^2}\int d^4xe\left[(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)(-\bar{\epsilon}\gamma^\mu\psi^\nu) - 2\bar{\epsilon}\overleftarrow{\nabla}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho\right]. \end{aligned} \tag{3.11}$$

The second term in the variation of the action (3.11) can be simplified by using integration by parts and the Ricci identity $[D_\mu, D_\nu]\Psi = \frac{1}{4}R_{\mu\nu}{}^{ab}\gamma_{ab}\Psi$, resulting in

$$\begin{aligned}\int d^4xe\bar{\epsilon}\overleftarrow{\nabla}_\mu D_\nu\psi_\rho &= \int d^4xe\bar{\epsilon}\gamma^{\mu\nu\rho}D_\mu D_\nu \\ &= \frac{1}{8}\int d^4xe\bar{\epsilon}\gamma^{\mu\nu\rho}\gamma^{ab}R_{\mu\nu ab}\psi_\rho. \end{aligned} \tag{3.12}$$

Using the relation of gamma matrix

$$\gamma^{\mu\nu\rho}\gamma_{\tau\rho} = \gamma^{\mu\nu\rho}{}_{\tau\sigma} + 6\gamma^{[\mu\nu}{}_{[\tau}\delta^{\rho]}_{\sigma]} + 6\gamma^{[\mu}\delta^{\nu}_{[\tau}\delta^{\rho]}_{\sigma]}, \tag{3.13}$$

so that

$$\begin{aligned}\gamma^{\mu\nu\rho}\gamma^{ab}R_{\mu\nu ab} &= \gamma^{\mu\nu\rho ab}R_{\mu\nu ab} + 6R_{\mu\nu}{}^{[\rho}{}_b\gamma^{\mu\nu]}{}_b + 6\gamma^{[\mu}R_{\mu\nu}{}^{\rho\nu]} \\ &= \gamma^{\mu\nu\rho ab}R_{\mu\nu ab} + 2R_{\mu\nu}{}^{\rho}{}_b\gamma^{\mu\nu b} + 4R_{\mu\nu}{}^{\mu}{}_b\gamma^{\nu\rho b} \\ &\quad + 4\gamma^\mu R_{\mu\nu}{}^{\rho\nu} + 2\gamma^\rho R_{\mu\nu}{}^{\nu\mu}. \end{aligned} \tag{3.14}$$

In four-dimensional spacetime, the first term vanishes due to the property $\gamma^{\mu\nu\rho ab} = 0$. The second term also vanishes because of symmetry of Riemann tensor ($R_{\mu[\nu\rho\sigma]} = 0$), while the third term is equal to zero as $R_{\mu\nu}{}^{\mu}{}_b = R_{\nu b}$ is symmetric in ν and b, but $\gamma^{\nu ab}$ is antisymmetric. The final second term gives rise to the Ricci tensor $R_\mu{}^\rho$ and the Ricci scalar $R_{\mu\nu}{}^{\nu\mu} = -R$, respectively. Equation (2.160) becomes

$$\delta S = \frac{1}{2\kappa^2}\int d^4xe\left[\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)(-\bar{\epsilon}\gamma^\mu\psi^\nu + \bar{\epsilon}\gamma^\mu\psi^\nu)\right] = 0, \tag{3.15}$$

which implies that the action (3.11) is invariant under supersymmetry transformations to first order in ψ_μ.

The construction of an action that is invariant under supersymmetry at all order can be achieved by adding ψ^4 to the action. The simplest method for adding ψ^4 is to use a first-order formalism, in which the gravitino field and the spin connection are varied separately. This action can be rewritten as

$$S = \frac{1}{2\kappa^2}\int d^4xe\Big[R - \overline{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho + \frac{1}{4}(\overline{\psi}_\mu\gamma^\nu\psi_\nu)(\overline{\psi}^\mu\gamma^\rho\psi_\rho) \\ -\frac{1}{16}(\overline{\psi}^\rho\gamma^\mu\psi^\nu)(\overline{\psi}_\rho\gamma_\mu\psi_\nu + 2\overline{\psi}_\rho\gamma_\nu\psi_\mu)\Big]. \qquad (3.16)$$

The action (3.16) is invariant under supersymmetry transformation (3.8) and (3.9).

3.3 N = 2 pure supergravity in four dimenions

Extended supergravity refers to supergravity with $N > 1$ supersymmetry. The first extended supergravity is $N = 2$ supergravity, which field contents shown in Table 2.1. This section focuses on presenting $N = 2$ supergravity without matter multiplets, i.e., fields beyond the supergravity multiplet. The discussion of $N > 2$ supergravity will be addressed in the next chapter.

The $N = 2$ supergravity multiplet consists of a graviton field e^a_μ, two gravitinos $\psi^i_\mu, i = 1, 2$, and a vector field A^μ. The action for this multiplet can be written as

$$S = \int d^4xe\Big[R - \overline{\psi}^i_\mu\gamma^{\mu\nu\rho}D_\nu\psi^i_\rho - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{8}\epsilon_{ij}\overline{\psi}^i_\mu\gamma^{[\mu}\gamma_{\rho\sigma}\gamma^{\nu]}F_{\rho\sigma}\hat{F}^{\rho\sigma}\Big], \qquad (3.17)$$

where $\kappa^2 = \frac{1}{2}$ and field strength tensor $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \qquad (3.18)$$

For tensor $\hat{F}$, it can be written as

$$\hat{F}_{\mu\nu} = F_{\mu\nu} - \epsilon_{ij}\overline{\psi}^i_\mu\psi^j_\nu, \qquad (3.19)$$

which is called supercovariant form of $F_{\mu\nu}$. This supercovariant form means that the transformation of $\hat{F}_{\mu\nu}$ does not contain any derivatives of the supersymmetry parameter ϵ^i, i.e., $\partial_\mu\epsilon^i$.

The action (3.17) invariant under supersymmetry transformation

$$\delta e_\mu^a = \frac{1}{2}\bar{\epsilon}^i \gamma^a \psi_\mu^i \tag{3.20}$$

$$\delta A_\mu = \epsilon_{ij}\bar{\epsilon}^i \psi_\mu^j \tag{3.21}$$

$$\delta \psi_\mu^i = D_\mu \epsilon^i - \frac{1}{8}\epsilon^{ij}\gamma^{\rho\sigma}\gamma_\mu \epsilon^j \hat{F}_{\rho\sigma} \tag{3.22}$$

It should be noted that if we set $\psi_\mu^2 = A_\mu = \epsilon^2 = 0$, the action (3.17) and its transformation rules reduce to $N = 1$ supergravity, which consists of $\psi_\mu = \psi_\mu^1$ and $\epsilon = \epsilon^1$. This observation implies that any N supergravity theory can be constructed by truncating some fields from a higher-supersymmetric $N' > N$ supergravity theory. This method is called truncation. The truncation must satisfy the field equations, meaning that if the truncated fields are set to zero, their field equations provide solutions that are consistent with zero fields.

Chapter IV

GAUGED SUPERGRAVITY

Gauged supergravity is a type of field theory that combines the features of gauge symmetry, gravity, and supersymmetry, and has been the subject of much study in the field of theoretical physics.

In this section, we discuss N supergravity coupled to matter multiplets by dividing the Lagrangian density into two parts, one for bosonic fields and one for fermionic fields. Furthermore, we upgrade the ungauged theory to a gauge theory by using the embedding tensor formalism.

4.1 The action for gauge field and scalar

In gauged supergravity, the vector (gauge) and scalar fields are coupled in a complex form. To understand this coupling, we start by considering the action between the gauge field $A^\mu(x)$ and complex scalar $Z(x)$, which can be written as:

$$S = -\frac{1}{4}\int d^4x \left[\mathrm{Im}\, Z F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\mathrm{Re}\, Z\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\right]. \tag{4.1}$$

When $Z(x)$ is constant, the second term in the action is a total derivative, $\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = 2\epsilon^{\mu\nu\rho\sigma}\partial_\mu(A_\nu F_{\rho\sigma})$, due to $\partial_{[\mu}F_{\rho\sigma]} = 0$. As a result, this term does not affect the field equation of the free vector field.

The field equation of A_μ can be found by taking the variation of the action with respect to A_μ, which is given as

$$\partial_\mu[\mathrm{Im}\, ZF^{\mu\nu} + i\,\mathrm{Re}\, Z\tilde{F}^{\mu\nu}] = 0, \tag{4.2}$$

where $\tilde{F}^{\mu\nu}$ is dual tensor of $F^{\mu\nu}$, see Appendix A.1. If we define tensor

$$G^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}\frac{\delta S}{\delta F^{\rho\sigma}} = -i\,\mathrm{Im}\, Z\tilde{F}^{\mu\nu} + \mathrm{Re}\, ZF^{\mu\nu}, \tag{4.3}$$

then

$$G^{+\mu\nu} = \overline{Z} F^{+\mu\nu} \quad \text{and} \quad G^{-\mu\nu} = Z F^{-\mu\nu}. \tag{4.4}$$

The notation "$+$" and "$-$" denote the self-dual and anti-self-dual parts of the tensor G, respectively. Bianchi identity and field equation can be reexpressed as

$$\partial_\mu \operatorname{Im} F^{-\mu\nu} = \quad \text{and} \quad \partial_\mu \operatorname{Im} G^{-\mu\nu} = 0. \tag{4.5}$$

The Bianchi identity and field equation are invariant under transformation in the form

$$\begin{pmatrix} F'^{-} \\ G'^{-} \end{pmatrix} = S \begin{pmatrix} F^{-} \\ G^{-} \end{pmatrix}, \tag{4.6}$$

where S is a matrix with determinant equal to 1 (element of $SL(2, \mathbb{R})$ group), which can be written as

$$S = \begin{pmatrix} d & c \\ b & a \end{pmatrix}. \tag{4.7}$$

The Bianchi identity and field equation are invariant under this transformation, which ensures that the theory remains self-consistent under duality rotations.

Let the relationship (4.4) invariant under the transformation (4.6), this will result in

$$\begin{aligned} G'^{\mu\nu} &= Z' F'^{\mu\nu} \\ bF^{-\mu\nu} + aG^{-\mu\nu} &= Z'(cG^{-\mu\nu} + dF^{-\mu\nu}). \end{aligned} \tag{4.8}$$

The transformation of the scalar Z under the matrix S can be expressed as

$$Z' = \frac{aZ + b}{cZ + d}, \tag{4.9}$$

using the relationship between $F^{-\mu\nu}$ and $G^{-\mu\nu}$ in equation (4.4).

The action of the scalar, which is invariant under the previous equation, can be expressed as:

$$S = \int d^4x \frac{4\partial_\mu Z \partial^\mu \overline{Z}}{(Z - \overline{Z})^2} = -\int d^4x \frac{\partial_\mu Z \partial^\mu \overline{Z}}{(\operatorname{Im} Z)^2}. \tag{4.10}$$

The momentum-energy tensor of vector field defined from action (4.1) can be written as

$$\Theta^{\mu\nu} = \mathrm{Im}\, Z \left[F^{\mu\rho} F^{\nu}{}_{\rho} - \frac{1}{4}\eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right], \tag{4.11}$$

which constraint that $\mathrm{Im}\, Z > 0$ for positive Θ^{00}.

The extension of the theory into various vector and scalar fields can be performed in a manner similar to previous processes. Let us consider m vector fields A^{a}_{μ}, where a ranges from 1 to m, and n scalar fields ϕ^{i}, where i ranges from 1 to n. The dynamics can be described by the action

$$S = -\frac{1}{4}\int d^4x \left[\mathrm{Re}\, f_{ab} F^{a}_{\mu\nu} F^{b\mu\nu} - i\, \mathrm{Im}\, f_{ab} F^{a}_{\mu\nu} \tilde{F}^{b\mu\nu} \right], \tag{4.12}$$

where $f_{ab}(\phi) = f_{ba}(\phi)$ is symmetric tensor and $\mathrm{Re}\, f_{ab} > 0$.

When expressed in the form of the dual tensor $F^{\pm}_{\mu\nu}$, the action takes the form

$$S = -\frac{1}{4}\int d^4x \left[f_{ab} F^{-a}_{\mu\nu} F^{-b\mu\nu} + f^{*}_{ab} F^{+a}_{\mu\nu} F^{+b\mu\nu} \right], \tag{4.13}$$

and field equations become

$$\partial_\mu\, \mathrm{Im}\, F^{-a\mu\nu} = 0 \quad \text{and} \quad \partial_\mu\, \mathrm{Im}\, G^{-\mu\nu}_{a} = 0, \tag{4.14}$$

where $G^{\mu\nu}_{a}$ and $G^{\pm\mu\nu}_{a}$ are defined as

$$G^{\mu\nu}_{a} = \epsilon^{\mu\nu\rho\sigma} \frac{\delta S}{\delta F^{a\rho\sigma}} = -\,\mathrm{Im}\, f_{ab} F^{b\mu\nu} - i\, \mathrm{Re}\, f_{ab} \tilde{F}^{b\mu\nu} \tag{4.15}$$

$$G^{-\mu\nu}_{a} = i f_{ab} F^{-b\mu\nu}, \qquad G^{+\mu\nu}_{a} = -i f^{*}_{ab} F^{+\mu\nu}. \tag{4.16}$$

A duality transformation, which is a symmetry of the field equations, arises from a $2m \times 2m$ matrix and transforms as

$$\begin{pmatrix} F'^{-} \\ G'^{-} \end{pmatrix} = S \begin{pmatrix} F^{-} \\ G^{-} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F^{-} \\ G^{-} \end{pmatrix}, \tag{4.17}$$

where $A, B, C,$ and D are $m \times m$ real matrix.

In a manner similar to the case of a single vector field, we can impose a transformation on the functions $f_{ab}(\phi)$ in order to make the relation between F and G invariant under the duality transformation. This transformation can be written as

$$if' = (C + iDf)(A + iBf)^{-1}. \tag{4.18}$$

To impose that this transformation provide ${f'}^T = f'$, the condition is written as

$$A^T C = C^T A, \qquad B^T D = D^T B, \qquad A^T D - C^T B = \mathbf{I}. \tag{4.19}$$

This result indicate that S is element of $Sp(2m, \mathbb{R})$, which satisfy the poperty

$$S^T \Omega S = \Omega, \tag{4.20}$$

for $2m \times 2m$ symplectic form

$$\Omega = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}. \tag{4.21}$$

4.2 The action for bosonic field

In extended supergravity with matter multiplets, the bosonic part of the action consists of the graviton $g_{\mu\nu}$, n_s scalar fields ϕ^s, $s = 1, ..., n_s$, and n_v vector fields A^Λ, $\Lambda = 1, ..., n_v$. The interaction between the scalar fields and vectors is expressed through the action (4.12), while the kinetic term for the graviton is given by the Ricci scalar $R(g_{\mu\nu})$. Taking into account all possible configurations, the Lagrangian density that describes the dynamics of the bosonic sector can be written as:

$$e^{-1}\mathscr{L}_B = \frac{1}{2}R - \frac{1}{2}G_{st}\partial_\mu\phi^s\partial^\mu\phi^t + \frac{1}{4}I_{\Lambda\Sigma}(\phi)F^\Lambda_{\mu\nu}F^{\Sigma\mu\nu} + \frac{1}{8}e^{-1}R_{\Lambda\Sigma}(\phi)\epsilon^{\mu\nu\rho\sigma}F^\Lambda_{\mu\nu}F^\Sigma_{\rho\sigma}. \tag{4.22}$$

The matrix $I_{\Lambda\Sigma}$ represents the imaginary part of the complex matrix $\mathscr{N}_{\Lambda\Sigma}$, while the matrix $R_{\Lambda\Sigma}$ represents its real part. The matrix $\mathscr{N}$ is related to the matrix f_{ab} in equation (4.12) through the equation $f = i\mathscr{N}$, which provide $\mathrm{Re}\, f = -\mathrm{Im}\, \mathscr{N}$ and

$\mathrm{Im}\, f = \mathrm{Re}\,\mathscr{N}$. The indices $a, b, ...$ for f_{ab} are used to represent only the vector fields in the gauged multiplet, while the indices $\Lambda, \Sigma, ...$ are used to specify all of the vector fields in the theory.

Given the complex and various field content of supergravity theory, a deep understanding of group theory is necessary to fully comprehend its properties and behaviors. The set of scalar fields in supergravity theory is described as a homogeneous symmetric space, known as the target manifold or scalar manifold, which is often denoted, in the context of group theory, as

$$\mathscr{M} = G/H, \tag{4.23}$$

where H is subgroup of G that is the isometry group of scalar manifold, and scalar ϕ^s is coordinate with n_s dimension, see Appendix. Mathematically, the group G/H is defined as the set of left cosets of H in G:

$$g' = gh \sim g, \tag{4.24}$$

where $g \in G$ and $h \in H$.

The isometry of the scalar manifold $\mathscr{M}$ is given by the elements of the group G, which is generally a non-compact group. Under the transformation, scalar field ϕ^s transform as $\phi^s \to \phi^{s'}(\phi) = g \circ \phi^s$, which the metric G_{st} remains invariant under the isometry, and is written as

$$G_{s't'}(\phi') = \frac{\partial \phi^s}{\partial \phi^{s'}} \frac{\partial \phi^t}{\partial \phi^{t'}} G_{st}(\phi). \tag{4.25}$$

The homogeneous property of the scalar manifold means that all points on $\mathscr{M}$ are connected by an isometry, that is, a transformation that preserves the metric of the scalar manifold.

Group H, which is a subset of G, is called holonomy and it imposes the connection of parallel transport. For example, the coset $SO(3)/SO(2) = S^2$, where $H = SO(2)$, expresses the angle between the original vector and the final vector after parallel transport in a closed loop on S^2. In supergravity, the compact group H is expressed in the form

$$H = H_R \times H_m, \tag{4.26}$$

where H_R is the automorphism group of supersymmetry (R symmetry). For $N < 8$, the R-symmetry group H_R is a unitary group $U(N)$, and for $N = 8$, it is the group $SU(8)$. For $N > 4$, the compact group H_m that acts on the matter fields is absent.

In cases that the isometry of the scalar manifold does not change the point in the manifold, the action forms a subgroup H', called the isotropy group, which satisfies $h'(\phi) \circ \phi^s = \phi^s$. When $\mathscr{M}$ is a symmetric space, the groups H and H' are the same locally. It is important to note that G is the global symmetry while H is the gauge symmetry.

The scalar manifold $\mathscr{M}$ can be represented by the elements of group G in the form of $L(x)$, where the transformation of $L(x)$ under group G construct by multiplying g on the left-hand side. Additionally, in supergravity, the symmetry of group H imposes the transformation of $L(x)$ by multiplication on the right-hand side. The transformation of $L(x)$ under the coset $\mathscr{M} = G/H$ is expressed as

$$L(x) \to L'(x) = gL(x)h(x), \tag{4.27}$$

where $g \in G$, and $h(x) \in H$. We can express the dependence of $L(x)$ on scalar fields in the form $L(\phi(x))$ by using gauge fixing, and this is referred to as the coset representative. $L(\phi)$ depends on $n_s = \text{dim}G - \text{dim}H$, which denote the number of scalar fields. In general, ϕ^s is called the parameter of $L(\phi)$.

Scalars ϕ^s transform under G in the form of

$$gL(\phi) = L(g \circ \phi)h(\phi, g), \tag{4.28}$$

where $g \circ \phi^s$ mean that the scalar field obtain from transformation, and is nonlinear in ϕ^s. The transformation $h(\phi, g)$ is referred to as the compensator because the transformation of g takes $L(\phi)$ out of the original coset representative. The compensator $h(\phi, g)$ process returns $L(g \circ \phi)$ to the original coset representative. It should be noted that the definition of the coset manifold makes it clear that $L(g\circ\phi)$ and $L(g\circ\phi)h(\phi,g)$ are equivalent.

The Lie algebra of group G and H is represented by $\mathfrak{g}$ and $\mathfrak{h}$, respectively. $\mathfrak{g}$ can be written in terms of its complement $\mathfrak{t}$ as

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t} \tag{4.29}$$

where $\mathfrak{t}$ is a vector space that is the complements of $\mathfrak{h}$ in $\mathfrak{g}$. Lie algebra for homogeneous space can be written as

$$[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h},\mathfrak{t}] \subset \mathfrak{t}, \quad [\mathfrak{t},\mathfrak{t}] \subset \mathfrak{h} \oplus \mathfrak{t}. \tag{4.30}$$

The coset representative can be parameterized by using the unitary parametrization, which is a popular method for finding solutions in supergravity. In this parametrization, the coset representative is expressed as

$$L = e^{\phi^s Y_s}, \tag{4.31}$$

where Y_s is basis vector of $\mathfrak{t}$ that also known as coset generator.

The structure of the scalar manifold, $\mathscr{M}$, can be described by the left-invariant 1-form, Ω, defined as

$$\Omega = L^{-1}dL \tag{4.32}$$

which satisfies the Maurer-Cartan equation,

$$d\Omega + \Omega \wedge \Omega = 0. \tag{4.33}$$

Since Ω is in the Lie algebra of the isometry group G of the scalar manifold G/H, $\Omega \in \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{t}$, it can be expressed in terms of the vielbein P and connection Q as

$$\Omega = P + Q. \tag{4.34}$$

Additionally, it can be expressed in a coordinate basis as

$$\Omega_r d\phi^r = P_r d\phi^r + Q_r d\phi^r. \tag{4.35}$$

Consider the transformation of the coset representative under the symmetry of the scalar manifold, equation (4.28):

$$\Omega(g \circ \phi) = h^{-1}L^{-1}g^{-1}d(gL(\phi)h) = h^{-1}L^{-1}(\phi)h + h^{-1}dh. \tag{4.36}$$

Note that under a global symmetry transformation of G, $dg = 0$. Furthermore, Ω can be projected onto the subspace of $\mathfrak{h}$ and $\mathfrak{k}$, giving

$$P(g \circ \phi) = h^{-1}Ph, \tag{4.37}$$

$$Q(g \circ \phi) = h^{-1}dh + h^{-1}Qh. \tag{4.38}$$

It can be seen that the transformation of P is linear, while Q transforms as a non-Abelian gauge field. As a result, Q acts as a connection in the same context as a gauge field. This connection Q is known as a composite connection due to its composition in terms of scalars.

If we impose that $\hat{s}, \hat{t}, ...$ represent the tangent space index of $\mathscr{M}$, we can express P in terms of the basis $\{Y_{\hat{s}}\}$ of $\mathfrak{k}$ as

$$P = P^{\hat{s}}Y_{\hat{s}}. \tag{4.39}$$

Furthermore, we can write $P^{\hat{s}}$ in terms of the coordinate basis as

$$P^{\hat{s}} = P^{\hat{s}}_{s}d\phi^{s}. \tag{4.40}$$

From equation (4.37), vielbein 1-form $P^{\hat{s}}$ transform under group G in the form of

$$P^{\hat{s}}(g \circ \phi) = h_{\hat{t}}{}^{\hat{s}}P^{\hat{t}}. \tag{4.41}$$

If we combine the covariant derivative of L with the connection Q, the result can be expressed as

$$DL = dL - LQ = LP, \quad \text{or equivalently} \quad L^{-1}DL = L^{-1}dL - Q = P. \tag{4.42}$$

The vielbein P satisfies the condition

$$DP = dP + Q \wedge P + P \wedge Q = 0, \tag{4.43}$$

and we can also define the curvature 2-form of the scalar manifold $\mathscr{M}$ in the form

$$R(Q) = dQ + Q \wedge Q = -P \wedge P, \tag{4.44}$$

where its components are given by

$$R(Q) = \frac{1}{2} R_{rs} d\phi^r \wedge d\phi^s \tag{4.45}$$

with $R_{rs} = [P_r, P_s] \in \mathfrak{h}$.

For any field $\Phi(x)$ on the scalar manifold $\mathscr{M}$ that transforms under group H, its covariant derivative can be expressed as

$$D_r\Phi = \partial_r \Phi + Q \circ \Phi, \tag{4.46}$$

where $Q \circ \Phi$ represents the action of Q on the representation of Φ. The derivative D_r satisfies the Ricci identity, which is given by

$$[D_r, D_s]\Phi = R_{rs} \circ \Phi. \tag{4.47}$$

The basis vector $Y_{\hat{s}}$ can be used to define an invariant metric under H transformations in the form

$$\eta_{\hat{s}\hat{t}} = k\,\mathrm{Tr}(Y_{\hat{s}}, Y_{\hat{t}}), \tag{4.48}$$

where k is a positive constant that depends on the representation of $Y_{\hat{s}}$. Consequently, the metric on the scalar manifold $\mathscr{M}$ can be expressed as

$$ds^2 = G_{st} d\phi^s d\phi^t = P_s^{\hat{s}} P_t^{\hat{t}} \eta_{\hat{s}\hat{t}} d\phi^s d\phi^t = k\,\mathrm{Tr}(PP). \tag{4.49}$$

From equation (4.37), the metric is invariant

$$ds^2(g \circ \phi) = ds^2(\phi). \tag{4.50}$$

Furthermore, the Lagraingian density can be reexpressed as

$$\mathscr{L}_{scalar} = \frac{1}{2} e G_{st} \partial_\mu \phi^s \partial^\mu \phi^t = \frac{1}{2} ek\,\mathrm{Tr}\{P_\mu P^\mu\}, \tag{4.51}$$

where $P_\mu = P_s \partial_\mu \phi^s$ that is invariant under G obviously.

The generators of the group G form a Lie algebra with structure constants ${f_{ab}}^c$,

$$[t_a, t_b] = {f_{ab}}^c t_c. \tag{4.52}$$

Under a transformation of G of the form

$$g = \mathbf{I} + \epsilon^a t_a, \tag{4.53}$$

the scalar field transforms as

$$\phi^{s'} = \phi^s + \epsilon^a k_a^s, \tag{4.54}$$

where k_a^s is the killing vector of the symmetry of G.

The transformation under the compensator $h \in H$ can be expressed as

$$h = \mathbf{I} + \epsilon^a {W_a}^{\hat{a}} J_{\hat{a}}, \tag{4.55}$$

where $J_{\hat{a}}$ is the generator of group H. As a result, the transformation of $L(\phi)$ becomes

$$(\mathbf{I} + \epsilon^a t_a) L(\phi) = L(\phi + \epsilon^a k_a)(\mathbf{I} + \epsilon^a {W_a}^{\hat{a}}). \tag{4.56}$$

By expanding $L(\phi + \epsilon^a k_a)$ to the first order of ϵ^a and multiplying the left-hand side by $L^{-1}(\phi)$, we obtain

$$\epsilon^a L^{-1} t_a L(\phi) = \epsilon^a k_a^s L^{-1} \partial_s L + \epsilon^a {W_a}^{\hat{a}} J_{\hat{a}}. \tag{4.57}$$

Using the definition of Ω in (4.32), and the relation (4.34), the previous equation becomes

$$L^{-1} t_a L = k_a^s P_s^{\hat{s}} Y_{\hat{s}} + (k_a^s Q_s^{\hat{a}} + {W_a}^{\hat{a}}) J_{\hat{a}}, \tag{4.58}$$

where

$$Q_s = Q_s^{\hat{a}} J_{\hat{a}}. \tag{4.59}$$

The equation (4.58) can be used to construct killing vector in terms of coset representative and generator of G by projecting $L^{-1}t_aL$ onto subspace $\mathfrak{k}$.

The generator $J_{\hat{a}}$ can be separated into $J_{\tilde{a}}, \tilde{a} = 1, \ldots, \dim(H_R)$ and $J_{\overline{a}}, \overline{a} = 1, \ldots, \dim(H_m)$, which correspond to the groups H_R and H_m, respectively, forming the direct product (4.26). The equation (4.58) is rewritten as

$$L^{-1}t_aL = k_a^s P_s^{\hat{s}} Y_s + \mathscr{P}_a^{\tilde{a}} J_{\tilde{a}} + \mathscr{P}^{\overline{a}} J_{\overline{a}}, \tag{4.60}$$

where

$$\mathscr{P}_a^{\tilde{a}} = k_a^s Q_s^{\tilde{a}} + W_a{}^{\tilde{a}} \quad \text{and} \quad \mathscr{P}_a^{\overline{a}} = k_a^s Q_s^{\overline{a}} + W_a{}^{\overline{a}} \tag{4.61}$$

Generally, the momentum map $\mathscr{P}_a^{\hat{a}} = (\mathscr{P}_a^{\tilde{a}}, \mathscr{P}_a^{\overline{a}})$ is used to express the killing vector as a derivative of $\mathscr{P}_a^{\hat{a}}$. Specifically, we can write

$$k_a^s R_{st}^{\tilde{a}} = D_t \mathscr{P}_a^{\tilde{a}}, \tag{4.62}$$

where $R_{st}^{\hat{a}} = (R_{st}^{\tilde{a}}, R_{st}^{\overline{a}})$ represents the curvature 2-form on the basis of $J_{\hat{a}}$, given by

$$R(Q) = \frac{1}{2} R_{st}^{\hat{a}} J_{\hat{a}} d\phi^s \wedge d\phi^t. \tag{4.63}$$

In order to extend the isometry of the scalar manifold to cover the on-shell symmetry of the action, it is necessary to find a correspondence between isometries of the scalar manifold and electric-magnetic duality transformations of the vector fields.

Given an electric field strength tensor $F_{\mu\nu}^{\Lambda}$ on a spacetime manifold, we can define its magnetic dual tensor

$$G_{\Lambda\mu\nu} = -\epsilon_{\mu\nu\rho\sigma} \frac{\partial \mathscr{L}}{\partial F_{\sigma\rho}^{\Lambda}} = R_{\Lambda\Sigma} F^{\Sigma} - I_{\Lambda\Sigma} * F_{\mu\nu}^{\Sigma} \tag{4.64}$$

via the Hodge star operator, such that

$$*F_{\mu\nu}^{\Lambda} = \frac{1}{2} e \epsilon_{\mu\nu\rho\sigma} F^{\Lambda\rho\sigma}, \tag{4.65}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita symbol. The Bianchi identity, which expresses the conservation of electric and magnetic fluxes, can then be written in terms of F and G, shown as

$$\nabla^{\mu}(*G_{\Lambda\mu\nu}) = 0 \quad \text{and} \quad \nabla^{\mu}(*F^{\Lambda}_{\mu\nu}) = 0. \tag{4.66}$$

We can write the previous equation in differential form as $dG_{\Lambda} = 0$ and $dF^{\Lambda} = 0$, respectively.

To write $*F^{\Lambda}$ in the form of F^{Λ} and G^{Λ}, we can use equation (4.64), and obtain

$$*F^{\Lambda} = I^{\Lambda\Sigma}(R_{\Sigma\Gamma}F^{\Gamma} - G_{\Sigma}), \tag{4.67}$$

wherer $I^{\Lambda\Sigma}$ is inverse of $I_{\Lambda\Sigma}$.

For $*G^{\Lambda}$, taking duality of equation (4.64) and using the previous equation, so that

$$*G_{\Lambda} = (RI^{-1}R + I)_{\Lambda\Sigma}F^{\Sigma} - (RI^{-1})_{\Lambda}{}^{\Sigma}G_{\Sigma}. \tag{4.68}$$

Combine F^{Λ} and G_{Λ} together, we can write the $2n_v$ vector as

$$\mathscr{G}^{M} = \begin{pmatrix} F^{\Lambda} \\ G_{\Lambda}, \end{pmatrix} \tag{4.69}$$

where the index $M = (^{\Lambda}, {}_{\Lambda})$. Field equation and Bianchi identity can be written as

$$d\mathscr{G}^{M} = 0, \tag{4.70}$$

and equation (4.67) together with (4.68) becomes

$$*\mathscr{G} = -\mathbb{C}\mathbb{M}(\phi)\mathscr{G}. \tag{4.71}$$

The matrix $\mathbb{C}$ represents symplectic form expressed as

$$\mathbb{C}^{MN} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_{n_v} \\ -\mathbf{I}_{n_v} & \mathbf{0} \end{pmatrix}, \tag{4.72}$$

where $\mathbf{I}_{n_v}$ is $n_v \times n_v$ identity matrix. The matrix $\mathbb{M}$ is symmetrical matrix that is constructed from scalar fields in the form

$$\mathbb{M}_{MN} = \begin{pmatrix} (RI^{-1}R + I)_{\Lambda\Sigma} & -(RI^{1})_{\Lambda}{}^{\Gamma} \\ -(I^{-1}R)^{\Delta}{}_{\Sigma} & I^{\Delta\Gamma} \end{pmatrix}. \tag{4.73}$$

In addition, the matrix $\mathbb{M}$ also satisfy symlectic property expressed as

$$\mathbb{M}\mathbb{C}\mathbb{M} = \mathbb{C}. \tag{4.74}$$

Similary, we can derive field equation of scalar ϕ^s and Einstein equation from the action (4.1) in the form of duality transformation, which can be written as

$$\mathscr{D}_\mu \partial^\mu \phi^s = \frac{1}{8} G^{st} \mathscr{G}^T_{\mu\nu} \partial_t \mathbb{M} \mathscr{G}^{\mu\nu} \tag{4.75}$$

$$R_{\mu\nu} = G_{rs} \partial_\mu \phi^r \partial_\nu \phi^s + \frac{1}{2} \mathscr{G}^T_{\mu\rho} \mathbb{M} \mathscr{G}_\nu{}^\rho. \tag{4.76}$$

4.3 Global symmetry

In the context of supergravity, the isometry group G that is a symmetry of the scalar field can be extended to become a symmetry of the field equation, known as an on-shell symmetry. This extension is achieved by connecting the nonlinear transformations of the scalar field with the duality transformations of $\mathscr{G} = (F^\Lambda, G_\Lambda$. In other words, any transformation that occurs from an element g of G acting on the scalar field, $\phi \to g \circ \phi$ will have a corresponding $2n_v \times 2n_v$ matrix, $R_v[g]$, that transforms as

$$\mathscr{G}^M = R_v[g]^M{}_N \mathscr{G}^N, \tag{4.77}$$

where R_v is element of $g \in G$ in vector and Hodge duality representation. The symplectic representation of group G is defined by this matrix.

The explicit from of $R_v[g]^M{}_N$ is

$$R_v[g]^M{}_N = \begin{pmatrix} A[g]^\Lambda{}_\Sigma & B[g]^{\Lambda\Sigma} \\ C[g]_{\Lambda\Sigma} & D[g]_\Lambda{}^\Sigma \end{pmatrix}. \tag{4.78}$$

As F^Λ and $*F^\Lambda$ are dependent and related through duality transformations, the transformation matrix $R_v[g]$ that provides symmetry of the field equation must satisfy two properties:

1. The matrix $R_v[g]$ is symplectic matrix

$$R_v[g]^T \mathbb{C} R_v[g] = \mathbb{C} \tag{4.79}$$

2. The matrix $R_v[g]$ provide transformatin of $\mathbb{M}$ in the form

$$\mathbb{M}(g \circ \phi) = (R_v[g]^{-1})^T \mathbb{M}(\phi) R_v[g]^{-1}. \tag{4.80}$$

To ensure invariance of the equation (4.71) under this extended symmetry, the matrix $\mathscr{N}_{\Lambda\Sigma} = R_{\Lambda\Sigma} + iI_{\Lambda\Sigma}$ must transform under $R_v[g]$ as

$$\mathscr{N}(g \circ \phi) = \frac{C[g] + D[g]\mathscr{N}(\phi)}{A[g] + B[g]\mathscr{N}(\phi)}. \tag{4.81}$$

Additionally, a dual representation of R_v, denoted as $R_{v^*} = (R_v^{-1})^T$, satisfy equation:

$$(R_v^{-1})^T = -\mathbb{C} R_v[g] \mathbb{C} \quad \text{or} \quad R_{v^*}[g]_M{}^N = \mathbb{C}_{MP} R_v[g]^P{}_Q \mathbb{C}^{NQ}. \tag{4.82}$$

In supergravity, since supersymmetry connects the vector field and scalar field and imposes transformations on these fields, the conditions of matrix symplecticity (4.79)and the transformation property (4.80) must hold for all extended supergravities.

In supergravity, the actions are constrained within a certian symplectic frame, which includes different actions and their corresponding symmetries. The symplectic frame is determined by the basis of the matrix $\mathbb{M}$, which imposes the embedding of G into $Sp(2n_v, \mathbb{R})$. Any matrices $\mathbb{M}$ in different symplectic frames are connected by a matrix E that belongs to the group $Sp(2n_v, \mathbb{R})$ and has the form

$$\mathbb{M}' = E \mathbb{M} E^T. \tag{4.83}$$

The matrix E that maps one symplectic frame to another belongs to the coset

$$E \in GL(n_v, \mathbb{R}) \setminus Sp(2n_v, \mathbb{R}) / R_{v^*}[G], \tag{4.84}$$

where $GL(n_v,\mathbb{R})$ is the group of invertible $n_v \times n_v$ matrices and $R_{v^*}[G]$ is the element of G in the R_{v^*} representation. This coset means that two matrices E are equivalent if they differ by multiplying an element of $GL(n_v,\mathbb{R})$ on the left and an element of $R_{v^*}[G]$ on the right.

In general, duality symmetry is not a symmetry of the action but rather a symmetry of the field equations and Bianchi identities, on-shell symmetry. In the case of $B[g] \neq 0$, the transformation (4.77) becomes

$$F^{\Lambda'} = A[g]^{\Lambda}{}_{\Sigma} F^{\Sigma} + B[g]^{\Lambda\Sigma} G_{\Sigma}, \tag{4.85}$$

indicating that the Bianchi identity for $F^{\Lambda'}$ is not satisfied unless $dG_{\Sigma} = 0$. Therefore, the symmetry of the action (the off-shell symmetry) must come from the subgroup $G_e \subset G$ of $R_v[g]$ where $B[g]^{\Lambda\Sigma} = 0$. G_e is referred to as the electric subgroup of the global symmetry group G, as the vector fields that appear in the action are known as electric vector fields, while the fields obtained from duality transformations are known as magnetic vector fields.

In the previous discussion, it was established that the matrix transformation $R_v[g]$ for g in the electric subgroup G_e of the global symmetry group G, must be written in the general form:

$$R_v[g]^M{}_N = \begin{pmatrix} A[g]^{\Lambda}{}_{\Sigma} & \mathbf{0} \\ C[g]_{\Lambda\Sigma} & (A[g]^{-1})^T{}_{\Lambda}{}^{\Sigma} \end{pmatrix}, \tag{4.86}$$

where $D = (A^{-1})^T$ due to the symplectic condition. The transformation involving $C[g]_{\Lambda\Sigma}$ is known as the Peccei-Quinn symmetry, which transforms the scalar field called the axion. In this thesis, we only consider the transformations for which $C[g]_{\Lambda\Sigma} = 0$.

For a scalar manifold described by a symmetric space G/H, we can represent it by a set of elements $L(\phi) \in G$ in the R_v representation. This set generates the elements of $Sp(2n_v,\mathbb{R})$ in the same representation as $R_v[L(\phi)] \in Sp(2n_v,\mathbb{R})$. We can also use R_v to map the maximal compact subgroup H of G to $U(n_v) \subset Sp(2n_v,\mathbb{R})$, but $R_v[H]$ is not necessarily unitary. To connect $\tilde{R}_v[h]$ with an orthogonal matrix, we can use $S^N{}_{\overline{N}} \in Sp(2n_v, R)/U(n_v)$ to perform a similarity transformation of the form $\tilde{R}_v[H] = S^{-1}R_vS$. The orthogonal condition for $\tilde{R}_v[h]$ can be expressed as

$$\tilde{R}_v[h]^T\tilde{R}_v[h] = \mathbf{I}, \tag{4.87}$$

for $h \in H$, and $R\tilde{R}_v$ is reducible representation.

Define coset representative in $\tilde{R}_v$ as

$$\tilde{L}^M{}_{\overline{N}} = R_v[L]^M{}_N S^N{}_{\overline{N}}, \tag{4.88}$$

the transformation of $L(\phi)$ in (4.28) becomes

$$R_v[g]\tilde{L}(\phi) = \tilde{L}(g \circ \phi)\tilde{R}_v[h] \tag{4.89}$$

for $g \in G$ and $h \in H$. The indices $M, N, ... = 1, ..., 2n_v$ of $\tilde{L}$ transform under G while the indices $\overline{M}, \overline{N}, ... = 1, ..., 2n_v$ transform under H.

In the form of $\tilde{L}$, the matrix $\mathbb{M}$ can be written as

$$\mathbb{M}_{MN} = \mathbb{C}_{MP}\tilde{L}^P{}_{\overline{L}}\tilde{L}^R{}_{\overline{L}}\mathbb{C}_{RN}. \tag{4.90}$$

By using the symplectic condition of $R_v[g]$ and orthogonal of $\tilde{R}_v[h]$, we can write the transformation of $\mathbb{M}$ as

$$\mathbb{M}(g \circ \phi) = (R_v[g]^{-1})^T\mathbb{M}(\phi)R_v[g]^{-1}, \tag{4.91}$$

which imply that $\mathbb{M}$ is invariant under H, and also write the Lagrangian density of scalar field in the form of $\mathbb{M}$ as

$$\mathscr{L}_{\text{scalar}} = \frac{1}{8}ek\,\mathrm{Tr}\big[(\mathbb{M}^{-1}\partial_\mu\mathbb{M})(\mathbb{M}^{-1}\partial^\mu\mathbb{M})\big]. \tag{4.92}$$

From all result, we can write the transformation of bosonic field under G as

$$\delta\phi^s = \Lambda^a k_a^s \tag{4.93}$$

$$\delta\mathbb{M} = \Lambda^a k_a^s \partial_s\mathbb{M} = \Lambda^a(R_{v*}[t_a]\mathbb{M} + \mathbb{M}R_{v*}[t_a]^T) \tag{4.94}$$

$$\delta\mathscr{G}^M_{\mu\nu} = -\Lambda^a(t_a)_N{}^M\mathscr{G}^N_{\mu\nu}. \tag{4.95}$$

N	$\mathscr{M} = G/H$	$n_s = \text{dim}(G/H)$	n_v	R_v
3	$\frac{SU(3,n)}{SU(3)\times SU(n)\times U(1)}$	$6n$	$3n{+}3$	$(\mathbf{3}+\mathbf{n})+\overline{(\mathbf{3}+\mathbf{n})}$
4	$\frac{SL(2,\mathbb{R})}{SO(2)}\times\frac{SO(6,n)}{SO(6)\times SO(n)}$	$6n+2$	$n+6$	$(\mathbf{2},\mathbf{6}+\mathbf{n})$
5	$\frac{SU(5,1)}{U(5)}$	10	10	$\mathbf{20}$
6	$\frac{SO^*(12)}{U(6)}$	30	16	$\mathbf{32}_c$
8	$\frac{E_{7(7)}}{SU(8)}$	70	28	$\mathbf{56}$

Table 4.1: The scalar manifold, n_s scalar, n_v vector, and R_v representation of supergravity $N > 2$. The number of n in supergravity $N = 3, 4$ represent the number of vector multiplets, and subscription c refer to conjugate spinor

In supergravity, the structure of scalar manifold and vector that have been previously discussed applies to cases with $N > 2$, expressed in the table 4.1. Specifically, in $N > 2$ supergravity, the noncompact group $E_{7(7)}$ represents the group E_7, where the compact subgroup is $SU(8)$. The number 7 indicates the difference between the number of noncompact generators and compact generators. Additionally, the group $SO^*(12)$ represents the special noncompact form of $SO(12)$, where the compact subgroup is $U(6) \sim SU(6) \times U(1)$.

4.4 Fermionic sectors

In the context of $N > 2$ supergravity, it is observed that the spinor field is restricted to only supergravity and vector multiplets. The fermion fields, on the other hand, do not transform under the group G, but rather under the holonomy group H, where the symmetry of H is imposed by the direct product $H_R \times H_m$. The fermion fields are dependent on the number of supersymmetries, which are expressed in the Table 4.2.

The indices $A, B = 1, ..., N$ correspond to the representation of the group

N	Supergravity multiplet	vector multiplet
3	$\psi_{\mu A}, \chi_{ABC}$	$\lambda_{Ai}, \lambda_{ABCi}$
4	$\psi_{\mu A}, \chi_{ABC}$	λ_{Ai}
5	$\psi_{\mu A}, \chi_{ABC}, \chi$	$-$
6	$\psi_{\mu A}, \chi_{ABC}, \chi_A$	$-$
8	$\psi_{\mu A}, \chi_{ABC}$	$-$

Table 4.2: fermion fields in supergravity $N > 2$

$H_R = U(N)$ for $3 \leq N \leq 6$, and $H_R = SU(8)$ for $N = 8$. On the other hand, $i, j = 1, ..., n$ denote the group $H_m = SU(n)$ for $N = 3$ and $H_m = SO(n)$ for $N = 4$. The spinor without indices represents the singlet, whereas the field with anti-symmetry in index $ABC = [ABC]$ belongs to the tensor representation of H_R. Fermion fields $(\psi_{\mu A}, \chi_{ABC}, \lambda_{Ai})$ have positive chirality:

$$\gamma_5 \psi_{\mu A} = \psi_{\mu A}, \quad \gamma_5 \chi_{ABC} = \chi_{ABC}, \quad \gamma_5 \lambda_{Ai} = \lambda_{Ai}, \tag{4.96}$$

whereas the field $(\psi^A_\mu, \chi^{ABC}, \lambda^A_i)$ belong to the conjugate representation of H_R, and have negative chirality:

$$\gamma_5 \psi^A_\mu = -\psi^A_\mu, \quad \gamma_5 \chi^{ABC} = -\chi^{ABC}, \quad \gamma_5 \lambda^A_i = -\lambda^A_i. \tag{4.97}$$

From the Table 2.3, supergravity $N = 3, 5, 6$ consist of special spinor field $\lambda_{ABCi} = \lambda_i \epsilon_{ABC}, \chi, \chi^A$, respectively. These spinor have negative chirality:

$$\gamma_5 \lambda_i = -\lambda_i, \quad \gamma_5 \chi = -\chi, \quad \gamma_5 \chi^A = -\chi^A. \tag{4.98}$$

In addition, we can write all spinors, except $\lambda_{ABCi} = \lambda_i \epsilon_{ABC}, \chi, \chi^A$, in the form

$$\lambda_I = (\chi_{ABC}, \lambda_{Ai}), \quad \gamma \lambda_I = \lambda_I, \tag{4.99}$$

where the index $I = (ABC, Ai)$.

Bosonic fields transform under group G but are invariant under group H, whereas fermionic fields transform under group H and are invariant under group

G. The coupling between fermionic and bosonic fields must use a quantity that has a transformation under both G and H. This quantity is the coset representative $\tilde{L}$, which implies that the scalar field is the connection of interaction between fermions and bosons. Note that the transformations under group G and H have the same form as GCT and LLT, respectively. In other words, bosonic fields (tensors) transform under GCT while fermionic fields (spinors) transform under LLT.

The Lagrangian density that is invariant under group H must be in the form of a covariant derivative with connection Q, because H symmetry is a gauge symmetry. For any fermion field, the covariant derivative can be written as

$$\mathscr{D}_\mu \psi = D_\mu \psi + Q_\mu \circ \psi \tag{4.100}$$

where D_μ represents the spacetime covariant derivative, $Q_\mu = Q_s \partial_\mu \phi^s$ and $Q_\mu \circ \psi$ is the action of connection Q in the representation of ψ. From this definition, the Lagrangian can be written as

$$\mathscr{L}_{f-kinetic} = i\epsilon^{\mu\nu\rho\sigma}(\bar{\psi}^A_\mu \gamma_\nu \mathscr{D}_\rho \psi_{A\sigma} - \bar{\psi}_{A\mu}\gamma_\nu \mathscr{D}_\rho \psi^A_\sigma) - \frac{1}{2}e(\bar{\lambda}^I \gamma^\mu \mathscr{D}_\mu \lambda_I + \bar{\lambda}_I \gamma^\mu \mathscr{D}_\mu \lambda^I). \tag{4.101}$$

The second term of the previous equation can be written explicitly in the form

$$\begin{aligned} -\frac{1}{2}e(\bar{\lambda}^I \gamma^\mu \mathscr{D}_\mu \lambda_I + \bar{\lambda}_I \gamma^\mu \mathscr{D}_\mu \lambda^I) = &-\frac{1}{12}e(\bar{\chi}^{ABC}\gamma^\mu \mathscr{D}_\mu \chi_{ABC} + \bar{\chi}_{ABC}\gamma^\mu \mathscr{D}_\mu \chi^{ABC}) \\ &-\frac{1}{2}e(\bar{\lambda}^{Ai}\gamma^\mu \mathscr{D}_\mu \lambda_{Ai} + \bar{\lambda}_{Ai}\gamma^\mu \mathscr{D}_\mu \lambda^{Ai}). \end{aligned} \tag{4.102}$$

The fermion fields $(\lambda_{ABCi} = \epsilon_{ABC}\lambda_i, \chi, \chi^A)$ for $N = 3, 5, 6$ can be written similarly.

The complete Lagrangian density for $N > 2$ supergravity

The fermion fields belong to the complex representation of group H. Therefore, the interaction between bosonic and fermionic fields can be conveniently explained via the representation $\tilde{R}_v[H] \subset SO(2n_v)$ in complex form. The transformation mentioned previously is constructed using the Cayley matrix

$$A^{\overline{M}}{}_{\overline{N}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{I} & i\mathbf{I} \\ \mathbf{I} & -i\mathbf{I} \end{pmatrix}. \tag{4.103}$$

It's important to note that the Cayley matrix A used to construct the transformation is unitary $A^{\dagger} = A^{-1}$, and it can effectively convert a real symplectic vector $V^{\overline{M}} = (V^{\Lambda}, V_{\Lambda})$ into a complex vector $\mathbf{V}^{\overline{M}} = (\mathbf{V}^{\Lambda}, \mathbf{V}_{\Lambda})$. The transformation can be shown as

$$\mathbf{V}^{\overline{M}} = A^{\overline{M}}{}_{\overline{N}} V^{\overline{N}} = \frac{1}{\sqrt{2}} \begin{pmatrix} V^{\bar{\Lambda}} + iV_{\bar{\Lambda}} \\ V^{\bar{\Lambda}} - iV_{\bar{\Lambda}} \end{pmatrix}. \tag{4.104}$$

Let $R_v[G]$ represent the matrix representation of G in a complex basis obtained from the transformation $\mathbb{R}_v[g] = A\tilde{R}_v A^{\dagger}$. Since $H = H_R \times H_m$, we can separate the index $\bar{\Lambda}, \bar{\Sigma}, ...$ into AB and i. The component in the symplectic form can be written as $V^{\bar{\Lambda}} = (V^{AB}, V^i)$, and its conjugation is $V_{\bar{\Lambda}} = (V^{\bar{\Lambda}})^* = (V_{AB}, V_i)$.

In the reorientation on complex basis, group H is block diagonal because group $U(n_v) \subset Sp(2n_v, \mathbb{R})$ is block diagonal expressed as

$$\begin{pmatrix} \mathbf{u} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{u}} \end{pmatrix}, \tag{4.105}$$

where $\mathbf{U}$ represent matrix in $U(n_v)$. Notice that the fundamental representation of $Sp(2n_v, \mathbb{R})$ can be separated into $\mathbf{n_v^{+1}}$ and $\bar{\mathbf{n}}_\mathbf{v}^{-1}$ that shown as

$$\mathbf{2n_v} \rightarrow \mathbf{n_v^{+1}} + \bar{\mathbf{n}}_\mathbf{v}^{-1}, \tag{4.106}$$

where $\mathbf{n_v^{+1}}$ and $\bar{\mathbf{n}}_\mathbf{v}^{-1}$ are representation in $U(n_v) \sim SU(n_v) \times U(1)$.

The connection in complex representation $\mathbb{Q} = A\tilde{R}_v[Q]A^{\dagger}$ is in block diagonal expressed as

$$\mathbb{Q}^{\overline{M}}{}_{\overline{N}} = \begin{pmatrix} Q^{\bar{\Lambda}}{}_{\bar{\Sigma}} & \mathbf{0} \\ \mathbf{0} & Q_{\bar{\Lambda}}{}^{\bar{\Sigma}} \end{pmatrix}, \tag{4.107}$$

where the $n_v \times n_v$ sub-matrix are in the form

$$Q^{\bar{\Lambda}}{}_{\bar{\Sigma}} = \begin{pmatrix} Q^{AB}{}_{CD} & \mathbf{0} \\ \mathbf{0} & Q^{i}{}_{j} \end{pmatrix} \quad \text{and} \quad Q_{\bar{\Lambda}}{}^{\bar{\Sigma}} = \begin{pmatrix} Q_{AB}{}^{CD} & \mathbf{0} \\ \mathbf{0} & Q_{i}{}^{j} \end{pmatrix}. \tag{4.108}$$

In the previous equation, connection $Q^{AB}{}_{CD}$ and $Q^{i}{}_{j}$ are in the group of H_R and H_m, respectively. In addition, the connection $Q^{AB}{}_{CD}$ can be written in the terms of fundamental representation $Q^{A}{}_{B}$ as $Q^{AB}{}_{CD} = 4\delta^{[A}_{[C} Q^{B]}{}_{D]}$. This form provide the properties with the contraction of two indices:

$$Q^{AC}{}_{BC} = (N-2)Q^{A}{}_{B} + \delta^{A}_{B} Q^{C}{}_{C}, \tag{4.109}$$

where connection $Q^{C}{}_{C}$ is in $U(1) \subset U(N) = H_R$. Notice that $N = 8$ theory gives $Q^{C}{}_{C} = 0$ because of $H_R = SU(8)$.

As all of results, we can construct covariant derivative of $\psi_{A\mu}$, χ_{ABC}, and λ_{Ai}, which are expressed as

$$\mathscr{D}_\mu \psi_{A\nu} = \partial_\mu \psi_{A\nu} - \gamma^{\rho}_{\mu\nu}\psi_{A\rho} + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\psi_{A\nu} + Q_{\mu A}{}^{B}\psi_{B\nu} \tag{4.110}$$

$$\mathscr{D}_\mu \chi_{ABC} = \partial_\mu \chi_{ABC} + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\chi_{ABC} + 3Q_{\mu[A}{}^{D}\chi_{BC]D} \tag{4.111}$$

$$\mathscr{D}_\mu \lambda_{Ai} = \partial_\mu \lambda_{Ai} \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\lambda_{Ai} + Q_{\mu A}{}^{B}\lambda_{Bi} + Q_{\mu i}{}^{j}\lambda_{Aj}. \tag{4.112}$$

In the complex basis, compact generators in coset manifold are off-diagonal, therefore vielbein of scalar manifold $\mathbb{P} = A\tilde{R}_v[P]A^\dagger$ can be written as

$$\mathbb{P}^{\overline{M}}{}_{\overline{N}} = \begin{pmatrix} \mathbf{0} & P^{\bar{\Lambda}\bar{\Sigma}} \\ P_{\bar{\Lambda}\bar{\Sigma}} & \mathbf{0} \end{pmatrix}, \tag{4.113}$$

where $n_v \times n_v$ sub-matrix are in the form

$$P^{\bar{\Lambda}\bar{\Sigma}} = \begin{pmatrix} P^{ABCD} & P^{ABj} \\ P^{iCD} & P^{ij} \end{pmatrix} \quad \text{and} \quad P_{\bar{\Lambda}\bar{\Sigma}} = \begin{pmatrix} P_{ABCD} & P_{ABj} \\ P_{iCD} & P_{ij} \end{pmatrix}. \tag{4.114}$$

The component of vielbein $P^{ABCD} = P^{[ABCD]}$ belong to anti-symmetrical tensor of H_R, which invanish for $N < 4$ supergravity theory.

• $N = 8$ supergravity with $n = 0$: The vielbein is expressed as

$$P^{ABCD} = (P_{ABCD})^* = \frac{1}{24}\epsilon^{ABCDEFGH}P_{EFGH}, \tag{4.115}$$

which provide the Lagrangian density as

$$e^{-1}\mathscr{L}_{scalar} = \frac{1}{48}P_\mu^{ABCD}P^\mu_{ABCD}. \tag{4.116}$$

• $N = 5, 6$ supergravity: $N = 6$ supergravity consists of vielbein P^{ABCD} and P^{AB} that the relation can be written as

$$P^{ABCD} = \frac{1}{2}\epsilon^{ABCDEF}P_{EF}, \tag{4.117}$$

while $N = 5$ only consist of P^{ABCD}. Largrangian density of $N = 5$ and $N = 6$ is the same that can be written as

$$e^{-1}\mathscr{L}_{scalar} = \frac{1}{24}P_\mu^{ABCD}P^\mu_{ABCD}. \tag{4.118}$$

• $N = 4$ supergravity: In this case, the vielbein $P^{ABCD} = \epsilon^{ABCD}P$, which P is vielbein on coset manifold $SL(2,\mathbb{R})/SO(2)$ that describe scalar fields in supergravity multiplet. The coset manifold $SO(6,n)/SO(6) \times SO(n)$ is explained by vielbein P^{iAB} that corresponds to the condition:

$$P^{iAB} = (P_{iAB})^* = \frac{1}{2}\epsilon^{ABCD}P_{iCD}. \tag{4.119}$$

The component P_{ij} depend on P as $P_{ij} = \overline{P}\delta_{ij}$. The Largangian density of $N = 4$ supergravity can be written as

$$e^{-1}\mathscr{L}_{scalar} = \frac{1}{24}P_\mu^{ABCD}P^\mu_{ABCD} + \frac{1}{4}P_\mu^{iAB}P^\mu_{iAB}. \tag{4.120}$$

• $N = 3$ supergravity: In this case, $P^{ABCD} = 0$ because $N = 3$ theory has no scalar field in supergravity multiplet and $P_{ij} = 0$. The Lagrangian density of scalar fields in the form of P_μ^{iAB} can be written as

$$e^{-1}\mathscr{L}_{scalar} = \frac{1}{2}P_\mu^{iAB}P^\mu_{iAB}. \tag{4.121}$$

In the similar way, we can define the coset representative in complex basis by defining $\mathbb{L}(\phi) = \tilde{L}(\phi)A^\dagger$. The component of $\mathbb{L}$ are defined in the form

$$\mathbb{L} = \begin{pmatrix} f^\Lambda{}_{AB} & f^\Lambda{}_i & \bar{f}^{\Lambda AB} & \bar{f}^{\Lambda i} \\ h_{\Lambda AB} & h_{\Lambda i} & \bar{h}^{AB}_\Lambda & \bar{h}^i_\Lambda \end{pmatrix}. \tag{4.122}$$

In this form, the transformation of $\mathbb{L}$ under group G and H is expressed as

$$R_v[g]\mathbb{L}(\phi) = \mathbb{L}(g \circ \phi)\mathbb{R}_v[h], \tag{4.123}$$

and the symmpletic condition is

$$\mathbb{L}^\dagger \mathbb{C} \mathbb{L} = \hat{\mathbb{C}}, \tag{4.124}$$

where $\hat{\mathbb{C}} = A\mathbb{C}A^\dagger$.

The left invariant 1-from can be defined on complex basis in the form $\Omega^c = A\tilde{R}_v[\Omega]A^\dagger$, which explicitly express as

$$\Omega^c = \mathbb{L}^{-1} d\mathbb{L} = \mathscr{P} + \mathscr{Q}, \tag{4.125}$$

the matrix $\mathbb{M}$ can be written as

$$\mathbb{M} = \mathbb{C}\mathbb{L}\mathbb{L}^\dagger\mathbb{C}. \tag{4.126}$$

The matrix $\mathbb{M}$ can be written in the form of matrix $\mathbf{f} = (f^\Lambda{}_{AB}, f^\Lambda{}_i)$ and $\mathbf{h} = (h_{\Lambda AB}, h_{\Lambda i})$ as

$$\mathbb{M} = \begin{pmatrix} -2\mathbf{h}\mathbf{h}^\dagger & 2\mathbf{h}\mathbf{f}^\dagger + i\mathbf{I} \\ 2\mathbf{f}\mathbf{h}^\dagger - i\mathbf{I} & -2\mathbf{f}\mathbf{f}^\dagger \end{pmatrix}. \tag{4.127}$$

From previous equation, we can define the matrix I and R in the from of $\mathbf{f}$ and $\mathbf{h}$ as

$$I = -\frac{1}{2}(\mathbf{I}^{-1})^\dagger \mathbf{f}^{-1} \quad \text{and} \quad \frac{1}{2}(2\mathbf{h} + i(\mathbf{f}^{-1})^\dagger)\mathbf{f}^{-1}. \tag{4.128}$$

The interaction term between vector fields and fermion fields are constructed from anti-symmetric tensor $O^{\overline{M}}_{\mu\nu} = (O^{\bar{\Lambda}}_{\mu\nu}, O_{\bar{\Lambda}\mu\nu})$ that belong to bileinear of fermion. The duality relation (4.71) coupled to fermion field can be reexpressed as

$$*\mathscr{G} = -\mathbb{C}\mathbb{M}(\mathscr{G} + \mathbb{L}O). \tag{4.129}$$

This equation is covariant under G if $O^{\overline{M}}$ transform by compensator of H in the form

$$O'_{\mu\nu} = \mathbb{R}_v[h]O_{\mu\nu}. \tag{4.130}$$

To write supersymetry transformation conveniently, we define composite field strength tensor by multiplying $\mathscr{G}_{\mu\nu}$ with scalar matrix in the form

$$\mathbb{F}_{\mu\nu} = -\mathbb{L}^{\dagger}\mathbb{C}\mathscr{G}, \tag{4.131}$$

which in component is

$$\mathbb{F}^{\overline{M}}_{\mu\nu} = (F^{\bar{\Lambda}}_{\mu\nu}, F_{\bar{\Lambda}\mu\nu}) = -(\mathbb{L}^{*})^{N}{}_{\overline{M}}\mathbb{C}_{NP}\mathscr{G}^{P}_{\mu\nu}. \tag{4.132}$$

$\mathbb{F}^{\overline{M}}_{\mu\nu}$ only transform under group H because of sympletic property of $R_v[g]$.

Self-dual tensor and anti-self-dual tensor are defined in the from

$$F^{\pm}_{\mu\nu} = \frac{1}{2}(F_{\mu\nu} \pm i{*}F_{\mu\nu}), \quad \text{where} \quad i{*}F^{\pm}_{\mu\nu} = \pm F^{\pm}_{\mu\nu}. \tag{4.133}$$

From previous definition, the component of self-dual and anti-self-dual of $\mathbb{F}$ can be written as

$$\mathbb{F}^{\pm}_{\mu\nu} = -\mathbb{L}^{\dagger}\mathbb{C}\mathscr{G}^{\pm}_{\mu\nu}. \tag{4.134}$$

By using symplectic property of $\mathbb{L}$, we can find that

$$O^{-}_{\bar{\Lambda}\mu\nu} = O^{+\bar{\Lambda}}_{\mu\nu} = 0, \tag{4.135}$$

therefore the component of $\mathbb{F}^{\pm}_{\mu\nu}$ becomes

$$\mathbb{F}^{+}_{\mu\nu} = (F^{+AB}_{\mu\nu}, F^{+i}_{\mu\nu}, \frac{i}{2}O^{+}_{AB\mu\nu}, \frac{i}{2}O^{+}_{i\mu\nu}), \quad \text{and} \quad \mathbb{F}^{-}_{\mu\nu} = (-\frac{i}{2}O^{-AB}_{\mu\nu}, -\frac{i}{2}O^{-i}_{\mu\nu}, F^{-}_{AB\mu\nu}, F^{-}_{i\mu\nu}). \tag{4.136}$$

In addition, we also write the component of $G^{\pm}_{\Lambda}$ as

$$G^{+}_{\Lambda} = \mathscr{N}_{\Lambda\Sigma}F^{+\Sigma} + iI_{\Lambda\Sigma}\bar{f}^{\Sigma\bar{\Gamma}}O_{\bar{\Gamma}} \tag{4.137}$$

$$G^{-}_{\Lambda} = \bar{\mathscr{N}}_{\Lambda\Sigma}F^{-\Sigma} - iI_{\Lambda\Sigma}f^{\Sigma}{}_{\bar{\Gamma}}O^{\bar{\Gamma}}. \tag{4.138}$$

When we use the definition of tensor $G^{\pm}_{\Lambda\mu\nu}$ in the form

$$G^{\pm}_{\Lambda\mu\nu} = \pm\frac{2i}{e}\frac{\partial\mathscr{L}}{\partial F^{\pm\Lambda\mu\nu}} \tag{4.139}$$

we found that the Lagrangian density of vector field becomes

$$\begin{aligned} e^{-1}\mathscr{L}_{\text{vector}} =& \frac{i}{4}(\overline{\mathscr{N}}_{\Lambda\Sigma}F^{-\Lambda}_{\mu\nu}F^{-\Sigma\mu\nu} - \mathscr{N}_{\Lambda\Sigma}F^{+\Lambda}_{\mu\nu}F^{+\Sigma\mu\nu}) \\ &+ \frac{1}{2}(F^{+\Lambda\mu\nu}I_{\Lambda\Sigma}\bar{f}^{\Sigma\bar{\Gamma}}O_{\bar{\Gamma}\mu\nu} + F^{-\Lambda\mu\nu}I_{\Lambda\Sigma}f^{\Sigma}{}_{\bar{\Gamma}}O^{\bar{\Gamma}}_{\mu\nu}). \end{aligned} \tag{4.140}$$

The second term of previous equation also known as Pauli term, which describe interaction between fermion fields and tensor $F^{\pm\Lambda}_{\mu\nu}$.

In the form of fermion field λ_I, $O_{\bar{\Lambda}}$ can be written as

$$O_{AB\mu\nu} =2\bar{\psi}_{A\rho}\gamma^{[\rho}\gamma_{\mu\nu}\gamma^{\sigma]}\psi_{B\sigma} + C_{AB,C}{}^{I}\bar{\psi}^{C}_{\rho}\gamma_{\mu\nu}\gamma^{\rho}\lambda_I + C_{AB,IJ}\bar{\lambda}^{I}\gamma_{\mu\nu}\lambda^{J} \tag{4.141}$$

$$O_{i\mu\nu} =C_{i,C}{}^{I}\bar{\psi}^{C}_{\rho}\gamma_{\mu\nu}\gamma^{\rho}\lambda_I + C_{i,IJ}\bar{\lambda}^{I}\gamma_{\mu\nu}\lambda^{J}, \tag{4.142}$$

where $C_{AB,C}{}^{I}$, $C_{AB,IJ}$, and $C_{i,IJ}$ are coefficient tensor that depend on the number of supersymmetry.

The complete $N > 2$ Lagrangian density can be constructed by using all results discussed, as (without the forth power term of fermion fields)

$$\begin{aligned} e^{-1}\mathscr{L} =& \frac{1}{2}R - \frac{1}{2}ek\,\mathrm{Tr}(P_\mu P^\mu) + \frac{i}{4}(\overline{\mathscr{N}}_{\Lambda\Sigma}F^{-\Lambda}_{\mu\nu}F^{-\Sigma\mu\nu} - \mathscr{N}_{\Lambda\Sigma}F^{+\Lambda}_{\mu\nu}F^{+\Sigma\mu\nu}) \\ &+ ie^{-1}\epsilon^{\mu\nu\rho\sigma}(\bar{\psi}^{A}_{\mu}\gamma_\nu\mathscr{D}_\rho\psi_{A\sigma} - \bar{\psi}_{A\mu}\gamma_\nu\mathscr{D}_\rho\psi^{A}_{\sigma}) - \frac{1}{2}(\bar{\lambda}^{I}\gamma^{\mu}\mathscr{D}_\mu\lambda_I + \bar{\lambda}\gamma^{\mu}\mathscr{D}_\mu\lambda^{I}) \\ &+ \frac{1}{2}(F^{+\Lambda\mu\nu}I_{\Lambda\Sigma}\bar{f}^{\Sigma\bar{\Gamma}}O_{\bar{\Gamma}\mu\nu} + F^{-\Lambda\mu\nu}I_{\Lambda\Sigma}f^{\Sigma}{}_{\bar{\Gamma}}O^{\bar{\Gamma}}_{\mu\nu}) \\ &+ \bar{\lambda}^{I}\gamma^{\mu}\gamma^{\nu}\psi^{B}_{\nu}\partial_\nu\phi^{s}P_{sIB} + \bar{\lambda}_{I}\gamma^{\mu}\gamma^{\nu}\psi_{B\mu}\partial_\nu\phi^{s}P^{IB}_{s}. \end{aligned} \tag{4.143}$$

The last term is the interaction term between scalar field and fermion fields. For $N = 3, 5, 6$ supergravity, terms of fermion fields λ_i, χ, and χ^A will be added as we discussed previously.

The action obtained from the equation (4.143) is invariant under supersymmetry transformations:

$$\delta e^a_\mu = \bar{\epsilon}^A \gamma^a \psi_{A\mu} + \bar{\epsilon}_A \gamma^a \psi^A_\mu \tag{4.144}$$

$$\delta A^\Lambda_\mu = \mathbb{L}^\Lambda{}_{\overline{M}} O^{\overline{M}}_\mu = \frac{1}{2} f^\Lambda{}_{AB} O^{AB}_\mu + f^\Lambda{}_i O^i_\mu + \text{h.c.} \tag{4.145}$$

$$P^{ABCD}_s \delta\phi^s = \Sigma^{ABCD}, \quad P^{iAB}_s \delta\phi^s = \Sigma^{iAB} \tag{4.146}$$

$$\delta\psi_{A\mu} = \mathscr{D}_\mu \epsilon_A + \frac{i}{8} F^-_{\rho\sigma AB} \gamma^{\rho\sigma} \gamma_\mu \epsilon^B \tag{4.147}$$

$$\delta\chi_{ABC} = P_{sABCD} \partial_\mu \phi^s \gamma^\mu \epsilon^D + \frac{3}{4} i F^-_{\mu[AB} \gamma^{\mu\nu} \epsilon_{C]} \tag{4.148}$$

$$\delta\lambda_{Ai} = P_{siAB} \partial_\mu \phi^s \gamma^\mu \epsilon^D + \frac{3}{4} i F^-_{\mu\nu i} \gamma^{\mu\nu} \epsilon_A. \tag{4.149}$$

For extra fermion fields in $N = 3, 5, 6$ supergravity, the supersymmetry transformations can be written as

$$N = 3: \quad \delta\lambda_i = \frac{1}{2} P_{siAB} \partial_\mu \phi^s \gamma^\mu \epsilon_C \epsilon^{ABC} \tag{4.150}$$

$$N = 5: \quad \delta\chi = \frac{1}{24} \epsilon^{ABCDE} P_{sABCD} \partial_\mu \phi^s \gamma^\mu \epsilon_E \tag{4.151}$$

$$N = 6 = \quad \delta\chi_F = \frac{1}{24} \epsilon_{FABCDE} P^{ABCD}_s \partial_\mu \phi^s \gamma^\mu \epsilon^E + \frac{i}{4} \tilde{F}^-_{\mu\nu} \gamma^{\mu\nu} \epsilon_F. \tag{4.152}$$

Tensor Σ_{ABCD} and Σ_{iAB} are the component of coset generator $\Sigma \in \mathfrak{t}$ on complex basis $\mathbb{R}_v[\Sigma]$, where Σ are defined in the form of transformation

$$\Sigma = (L^{-1}\partial_s L)|_{\mathfrak{t}} \delta\phi^s = P_s \delta\phi^s \tag{4.153}$$

or

$$\delta L = \partial_s L \delta\phi^s = L\Sigma. \tag{4.154}$$

The variation of vielbein $P = (L^{-1}dL)|_{\mathfrak{t}}$ can be written as

$$\delta P = d\Sigma + Q\Sigma = D\Sigma, \tag{4.155}$$

by using the relation $\delta L^{-1} = -L^{-1}\delta L L^{-1}$, and $[\mathfrak{h}, \mathfrak{t}] \subset \mathfrak{t}$.

By using these relations, we can write the variation of scalar term as

$$\delta\mathscr{L}_{\text{scalar}} = ek\,\text{Tr}(P_\mu \delta P^\mu) = ek\,\text{Tr}(P_\mu D^\mu \Sigma). \tag{4.156}$$

Tensor Σ_{ABCD} and Σ_{iAB} for all supergravity are expressed as

$$N = 3: \quad \Sigma^{iAB} = \epsilon^{ABC}\bar{\epsilon}_C\lambda_i - 2\bar{\epsilon}^{[A}\lambda^{B]i} \tag{4.157}$$

$$N = 4: \quad \Sigma^{ABCD} = -4\bar{\epsilon}^{[A}\chi^{BCD]}, \tag{4.158}$$

$$\Sigma^{iAB} = -2\bar{\epsilon}^{[A}\lambda^{B]i} - \epsilon^{ABCD}\bar{\epsilon}_{[C}\lambda_{D]i} \tag{4.159}$$

$$N = 5: \quad \Sigma^{ABCD} = -4\bar{\epsilon}^{[A}\chi^{BCD]} + \epsilon^{ABCDE}\bar{\epsilon}_E\chi \tag{4.160}$$

$$N = 6: \quad \Sigma^{ABCD} = -4\bar{\epsilon}^{[A}\chi^{BCD]} - \epsilon^{ABCDEF}\bar{\epsilon}_E\chi_F \tag{4.161}$$

$$N = 8: \quad \Sigma^{ABCD} = -4\bar{\epsilon}^{[A}\chi^{BCD]} - \frac{1}{6}\epsilon^{ABCDEFGH}\bar{\epsilon}_E\chi_{FGH}. \tag{4.162}$$

Tensor O^{AB}_μ and O^i_μ in equation (2.316) are defined as

$$O^{AB}_\mu = -\bar{\epsilon}_C\gamma_\mu\chi^{ABC} - 4\bar{\epsilon}^{[A}\psi^{B]}_\mu \quad \text{and} \quad O^i_\mu = -\bar{\epsilon}_A\gamma_\mu\lambda^{Ai}. \tag{4.163}$$

In $N = 6$ supergravity, 16 vector fields consist of A^{AB}_μ and $\tilde{A}_\mu$, which belong in representation **15** and **1** of group $U(6)$, respectively. The tensor O^{AB}_μ still obtain from equation (2.334). while tensor $\tilde{O}$ can be obtainde from

$$\tilde{O}_\mu = -\bar{\epsilon}_A\gamma_\mu\chi^A. \tag{4.164}$$

4.5 Gauged supergravity

Based on previous considerations, it has been determined that the symmetry of $N > 2$ supergravity corresponds to the isometry of a scalar manifold, which results in the occurrence of duality transformations of vector fields. This symmetry is global and independent of spacetime. In this section, we shall discuss the gauging of supergravity, namely, the process of elevating a subgroup G_0 of the global symmetry G to a gauge symmetry.

Gauging and minimal coupling

The symmetry group G is an on-shell symmetry, while G_e is an off-shell symmetry that is only a subgroup of G. As such, each action has a different G_e, which results in a different gauging of subgroup $G_0 \in G_e$. G_e is considered to be the electric subgroup of G, and the vector fields that appear in ungauged supergravity action are referred to as electric vectors. In the context of supergravity, the gauging process that is both general and universal is the embedding tensor formalism. Furthermore, the embedding tensor formalism can accommodate the use of a gauge symmetry G_0 that is a subset of G, even if G_0 is not a subset of G_e.

The first condition for gauging any G_0 is that the number of dimensions of G_0 must be less than or equal to the number of vector fields that will become the gauge field, that is,

$$\text{dim}G_0 \leq n_v. \tag{4.165}$$

Here, $A^{\hat{\Lambda}}$ represents the electric gauge fields that become the gauge field of G_0 and belong to the adjoint representation of G_0. By letting Ω_g represent the gauge connection, we obtain

$$\Omega_{g\mu} = gA^{\hat{\Lambda}}_{\mu}X_{\hat{\Lambda}}, \tag{4.166}$$

where g is the coupling constant and $X_{\hat{\Lambda}}$ represents the generator of group G_0 that corresponds to the Lie algebra

$$[X_{\hat{\Lambda}}, X_{\hat{\Sigma}}] = f_{\hat{\Lambda}\hat{\Sigma}}{}^{\hat{\Gamma}}X_{\hat{\Gamma}}. \tag{4.167}$$

In order for G_0 to be a subgroup with a closure property, the structure constant $f_{\hat{\Lambda}\hat{\Sigma}}{}^{\hat{\Gamma}}$ must satisfy the Jacobi identity

$$f_{[\hat{\Lambda}\hat{\Sigma}}{}^{\hat{\Gamma}}f_{\hat{\Delta}]\hat{\Gamma}}{}^{\hat{\Pi}} = 0. \tag{4.168}$$

The generator $X_{\hat{\Lambda}}$ can be written in symplectic form of R_v representation,

$$(X_{\hat{\Lambda}})^{\hat{M}}{}_{\hat{N}} = R_v[X_{\hat{\Lambda}}]^{\hat{M}}{}_{\hat{N}} = \begin{pmatrix} X_{\hat{\Lambda}}{}^{\hat{\Sigma}}{}_{\hat{\Gamma}} & \mathbf{0} \\ -X_{\hat{\Lambda}\hat{\Sigma}\hat{\Gamma}} & X_{\hat{\Lambda}\hat{\Sigma}}{}^{\hat{\Gamma}} \end{pmatrix}. \tag{4.169}$$

$X_{\hat{\Lambda}\hat{\Sigma}\hat{\Gamma}}$ correspond to $C[g]_{\Lambda\Sigma}$ in the transformation (4.86) and we set $X_{\hat{\Lambda}\hat{\Sigma}\hat{\Gamma}} = 0$ throughout this thesis.

The symplectic form of $(X_{\hat{\Lambda}})^{\hat{M}}{}_{\hat{N}}$, $X_{\hat{\Lambda}\hat{M}}{}^{\hat{P}}\mathbb{C}_{\hat{P}\hat{N}} = X_{\hat{\Lambda}\hat{N}}{}^{\hat{P}}\mathbb{C}_{\hat{M}\hat{P}}$, provides the relation between the components of $(X_{\hat{\Lambda}})^{\hat{M}}{}_{\hat{N}}$ as

$$X_{\hat{\Lambda}}{}^{\hat{\Sigma}}{}_{\hat{\Gamma}} = -X_{\hat{\Lambda}\hat{\Gamma}}{}^{\hat{\Sigma}}. \tag{4.170}$$

Upon comparing the gauge transformation of $F^{\hat{\Lambda}}$ in the form $\delta F^{\hat{\Lambda}} = \xi^{\hat{\Gamma}} f_{\hat{\Gamma}\hat{\Sigma}}{}^{\hat{\Lambda}} F^{\hat{\Sigma}}$ ($\xi^{\hat{\Lambda}}$ represent transformation parameter) to the transformation obtained from $\delta\mathscr{G}^{\hat{M}} = \xi^{\hat{\Lambda}}(X_{\hat{\Lambda}})^{\hat{M}}{}_{\hat{N}}\mathscr{G}^{\hat{N}}$, we can obtain $\delta F^{\hat{\Lambda}} = \xi^{\hat{\Gamma}} X_{\hat{\Lambda}}{}^{\hat{\Sigma}}{}_{\hat{\Gamma}} F^{\hat{\Sigma}}$, which leads to the relation

$$f_{\hat{\Gamma}\hat{\Sigma}}{}^{\hat{\Lambda}} = -X_{\hat{\Gamma}\hat{\Sigma}}{}^{\hat{\Lambda}}. \tag{4.171}$$

According to this result, the algebra (2.338) becomes

$$[X_{\hat{\Lambda}}, X_{\hat{\Sigma}}] = -X_{\hat{\Lambda}\hat{\Sigma}}{}^{\hat{\Gamma}} X_{\hat{\Gamma}}. \tag{4.172}$$

This equation shown that the generator $X_{\hat{\Lambda}}$ is invariant under gauge transformation $\delta_{\hat{\Lambda}} X_{\hat{\Sigma}}$, and the equation (4.172) is called the quadratic constraint. In addition, the relation (4.171) also impose that

$$X_{(\hat{\Gamma}\hat{\Sigma})}{}^{\hat{\Lambda}} = 0, \tag{4.173}$$

because of $f_{\hat{\Gamma}\hat{\Sigma}}{}^{\hat{\Lambda}} = -f_{\hat{\Sigma}\hat{\Gamma}}{}^{\hat{\Lambda}}$.

In the case of $X_{\hat{\Gamma}\hat{\Sigma}\hat{\Lambda}} \neq 0$, the condition of symmetry under Peccei–Quinn transformation requires that

$$X_{(\hat{\Lambda}\hat{\Sigma}\hat{\Gamma})} = 0. \tag{4.174}$$

The condition (4.173) and (4.174) are called linear constraint.

Under gauge transformation $g(x) \in G_0 \subset G$, the connection transforms in the form

$$\Omega_g' = g(x)\Omega_g g^{-1}(x) + dg(x)g^{-1}(x). \tag{4.175}$$

Upon writing $\Omega_g' = gA^{\hat{\Lambda}'}X_{\hat{\Lambda}}$, and consideration of infinitesimal transformation $g(x) = \mathbf{I} + g\zeta^{\hat{\Lambda}}(x)X_{\hat{\Lambda}}$, we obtain

$$\delta A_\mu^{\hat{\Lambda}} = A_\mu^{\hat{\Lambda}'} - A_\mu^{\hat{\Lambda}} = \nabla_\mu \zeta^{\hat{\Lambda}}, \tag{4.176}$$

where the covariant derivative ∇_μ is defined as

$$\nabla_\mu \zeta^{\hat{\Lambda}} = \partial_\mu \zeta^{\hat{\Lambda}} + gX_{\hat{\Sigma}\hat{\Gamma}}{}^{\hat{\Lambda}} A_\mu^{\hat{\Sigma}} \zeta^{\hat{\Gamma}}. \tag{4.177}$$

In the following, the covariant derivative ∇_μ will serve as the covariant derivative under GCT and LLT, as well as the symmetry group H and the gauge group G_0.

Field strength tensor can be found from 2-form curvature $R(\Omega_g) = F^{\hat{\Lambda}}X_{\hat{\Lambda}}$. From definition

$$R(\Omega_g) = \frac{1}{g}(d\Omega_g - \Omega_g \wedge \Omega_g), \tag{4.178}$$

we obtain the component of $F_{\mu\nu}^{\hat{\Lambda}}$ as

$$F_{\mu\nu}^{\hat{\Lambda}} = \partial_\mu A_\mu^{\hat{\Lambda}} - \partial_\nu A_\mu^{\hat{\Lambda}} + gX_{\hat{\Gamma}\hat{\Sigma}}{}^{\hat{\Lambda}} A_\mu^{\hat{\Gamma}} A_\nu^{\hat{\Sigma}}. \tag{4.179}$$

It should be notice that $R(\Omega_g)$ transform in covariant form $R(\Omega_g)' = g(x)R(\Omega_g)g^{-1}(x)$, and $F_{\mu\nu}^{\hat{\Lambda}}$ satisfies Bianchi identity

$$\nabla F^{\hat{\Lambda}} = dF^{\hat{\Lambda}} + gX_{\hat{\Sigma}\hat{\Gamma}}{}^{\hat{\Lambda}} A^{\hat{\Sigma}} \wedge A^{\hat{\Gamma}} = 0. \tag{4.180}$$

It can be summarized that when we gauge the symmetry G_0 as a subgroup of G, the abelian field strength tensor $F^{\hat{\Lambda}} = dA^{\hat{\Lambda}}$ is replaced by a non-abelian tensor, and the derivative $\mathscr{D}_\mu$ is replaced by the covariant derivative $\nabla_\mu = \mathscr{D}_\mu - gA^{\hat{\Lambda}}X_{\hat{\Lambda}}$. The derivative ∇_μ lead to identity

$$[\nabla_\mu, \nabla_\mu] = -gF_{\mu\nu}^{\hat{\Lambda}}X_{\hat{\Lambda}} + ..., \tag{4.181}$$

where ... is the term of curvature tensor of space-time and curvature tensor on scalar manifold.

Since fermion fields do not transform under the group G, they do not transform directly under the group G_0. However, the derivative of the coset representative $L(\phi)$, which is used to define the connection Q_μ, must be changed to the covariant derivative

$$\nabla_\mu \phi^s = \partial_\mu \phi^s - g A_\mu^{\hat{\Lambda}} k_{\hat{\Lambda}}^s(\phi). \tag{4.182}$$

This results in the connection Q_μ, where the term involving gauge fields is added from the definition $\hat{\Omega}_\mu = \hat{P}_\mu + \hat{Q}_\mu$ in the form

$$\hat{\Omega}_\mu = L^{-1}\nabla_\mu L = L^{-1}(\partial_\mu - g A_\mu^{\hat{\Lambda}} X_{\hat{\Lambda}})L. \tag{4.183}$$

Thus, the inclusion of gauge fields in the covariant derivative of fermion fields related to the connection $\hat{Q}_\mu$ corresponds to the idea that these fermion fields transform under the group H, which serves as a compensator for transformations from $G_0 \subset G$, and note that $\partial_\mu L = \partial_s L \partial_\mu \phi^s$.

Using the relation $L^{-1}dL = Q + P$ and equation (4.182), we obtain

$$\hat{P}_\mu = P_\mu - g A_\mu^{\hat{\Lambda}} P_{\hat{\Lambda}} \quad \text{and} \quad \hat{Q}_\mu = Q_\mu - g A_\mu^{\hat{\Lambda}} Q_{\hat{\Lambda}}, \tag{4.184}$$

where $P_{\hat{\Lambda}}$ and $Q_{\hat{\Lambda}}$ represent projection of $L^{-1}X_{\hat{\Lambda}}L$ on subspace $\mathfrak{k}$ and $\mathfrak{h}$, respectively:

$$P_{\hat{\Lambda}} = L^{-1}X_{\hat{\Lambda}}L|_{\mathfrak{k}} \quad \text{and} \quad Q_{\hat{\Lambda}} = L^{-1}X_{\hat{\Lambda}}L|_{\mathfrak{h}}. \tag{4.185}$$

From definition of $\hat{\Omega}$, we can find gauge transformation of $\hat{P}$ and $\hat{Q}$ as

$$\hat{P}(g(x)\circ\phi) = h^{-1}\hat{P}h \quad \text{and} \quad \hat{Q}(g(x)\circ\phi) = h^{-1}\hat{Q}h + h^{-1}dh. \tag{4.186}$$

In addition, we also obtain Maurer-Cartan equation in gauge form expressed as

$$d\hat{\Omega} + \hat{\Omega}\wedge\hat{\Omega} = -gL^{-1}R(\Omega_g)L. \tag{4.187}$$

The projection on subspace $\mathfrak{k}$ and $\mathfrak{h}$ provides

$$\mathscr{D}\hat{P} = d\hat{P} + \hat{Q} \wedge \hat{P} + \hat{P} \wedge \hat{Q} = -gF^{\hat{\Lambda}}P_{\hat{\Lambda}} \tag{4.188}$$

$$\hat{R}(\hat{Q}) = d\hat{Q} + \hat{Q} \wedge \hat{Q} = -\hat{P} \wedge \hat{P} - gF^{\hat{\Lambda}}Q_{\hat{\Lambda}}. \tag{4.189}$$

Based on the results discussed, we can write covaraint derivative of any fermion fields as

$$\nabla_\mu \Psi = D_\mu \Psi + \hat{Q} \circ \Psi, \tag{4.190}$$

which requires replacing P and Q with $\hat{P}$ and $\hat{Q}$ when gauge symmetry G_0 is considered.

Gauge symmetry and embedding tensor

This section discusses the significance and properties of the embedding tensor in order to understand the relationship between gauging in the symplectic frame of action and general gauging that is covariant under the symmetry of group G and independent of the symplectic frame.

In symplectic frame of action, embedding tensor is represented by $\Theta_{\hat{\Lambda}}{}^{\sigma}$, which projects Lie algebra $\mathfrak{g}_e$ of group G_e with generator t_σ onto Lie algebra $\mathfrak{g}_0$ of gauge symmetry G_0. Therefore, we can write the projection in the form

$$X_{\hat{\Lambda}} = \Theta_{\hat{\Lambda}}{}^{\sigma} t_\sigma, \tag{4.191}$$

where tensor $\Theta_{\hat{\Lambda}}{}^{\sigma}$ belong in representation $\mathbf{n_v} \otimes \mathrm{adj}(G_e)$ with $\hat{\Lambda} = 1, 2, ..., n_v$ and $\sigma = 1, 2, ..., \mathrm{dim} G_e$.

Consider generator in covariant form

$$X_M = (X_\Lambda, X^\Lambda), \tag{4.192}$$

the relation between generator in symplectic frame of action $X_{\hat{\Lambda}}$ and matrix X_M is expressed as

$$\begin{pmatrix} X_{\hat{\Lambda}} \\ 0 \end{pmatrix} = E \begin{pmatrix} X_{\Lambda} \\ X^{\Lambda} \end{pmatrix}. \tag{4.193}$$

For any vector $A^M = (A^{\Lambda}, A_{\Lambda})$, the contraction can be shown as

$$A^{\hat{\Lambda}} X_{\hat{\Lambda}} = A^{\Lambda} X_{\Lambda} + A_{\Lambda} X^{\Lambda} = A^M X_M. \tag{4.194}$$

Notice that A^{Λ} and A_{Λ} are not independence, since there can be obtained from

$$A^{\Lambda} = E_{\hat{\Lambda}}{}^{\Lambda} A^{\hat{\Lambda}} \quad \text{and} \quad A_{\Lambda} = E_{\hat{\Lambda}\Lambda} A^{\hat{\Lambda}}. \tag{4.195}$$

In any symplectic frame, we can write X_M in the form of embedding tensor as

$$X_M = \Theta_M{}^a t_a, \tag{4.196}$$

where t_a represent the generator of group G. Tensor $\Theta_M{}^a$ belong in representation $R_{v^*} \otimes \text{adj}(G)$ and consist of component $\Theta_M{}^a = (\Theta_{\Lambda}{}^a, \Theta^{\Lambda a})$.

Similar to vector A_{μ}^M, the components of $\Theta_M{}^a$ are related by

$$\Theta_{\hat{\Lambda}}{}^a = E_{\hat{\Lambda}}{}^M \Theta_M{}^a \quad \text{and} \quad \Theta^{\hat{\Lambda} a} = E^{\hat{\Lambda} M} \Theta_M{}^a = 0. \tag{4.197}$$

The symplectic condition of $E_{\hat{M}}{}^M$ shown that the tensor $\Theta_M{}^a$ satisfies

$$\mathbb{C}^{MN} \Theta_M{}^a \Theta_N{}^b = 0, \tag{4.198}$$

that is called locality constraint. In addition, it is found that the dimension of gauge symmetry must satisfy

$$\text{dim} G_0 = \text{rank} \Theta_{\hat{\Lambda}}{}^a = \text{rank} \Theta_M{}^a \leq n_v. \tag{4.199}$$

The generator X_M in representation R_{v*} can be written as

$$X_{MN}{}^P = R_{v^*}[X_M]_N{}^P = \Theta_M{}^a t_{aN}{}^P, \tag{4.200}$$

which relate to $X_{\hat{M}\hat{N}}{}^{\hat{P}}$ in symplectic frame of action as

$$X_{MN}{}^{P} = (E^{-1})_{M}{}^{\hat{N}}(E^{-1})_{N}X_{\hat{M}\hat{N}}{}^{\hat{P}}E_{\hat{P}}{}^{P}. \tag{4.201}$$

In the form of X_{MNP}, The linear condition (4.173) and (4.174) takes the form

$$X_{(MNP)} = 0. \tag{4.202}$$

By replacing the generator X_M from (4.196) into quadratic equation (4.172) and using Lie algebra $\mathfrak{g}$, the condition becomes

$$\Theta_{M}{}^{a}\Theta_{N}{}^{b}f_{ab}{}^{c} + \Theta_{M}{}^{a}t_{aN}{}^{P}\Theta_{P}{}^{c} = 0. \tag{4.203}$$

The previous equation mean that the embedding tensor is invariant under gauge transformation, $\delta_M\Theta^a_N = 0$.

The condition (4.202) corresponds to projection of $R_{v^*} \otimes \text{Adj}(G)$ on specific representation (R_Θ) of G. From decomposition of $R_{v^*} \otimes \text{Adj}(G)$ under group G, we can write it as

$$R_{v^*} \otimes \text{Adj}(G) \to R_\Theta \oplus ..., \tag{4.204}$$

Therefore, we can rewrite linear condition in (4.202) as

$$\mathbb{P}_\Theta\Theta = \Theta, \tag{4.205}$$

where $\mathbb{P}_\Theta$ represent projection operator from $R_{v^*} \otimes \text{Adj}(G)$ onto R_Θ. The condition (4.203) and (4.205) are used to determine whether a given subgroup G_0 of the global symmetry group G can be gauged in supergravity by using group theory.

Lagrangian density of gauged supergravity

The minimal coupling procedure for gauging a subgroup of the original symmetry group in a supergravity theory involves introducing a gauge field associated with the subgroup, and replacing ordinary derivatives with covariant derivatives that include the gauge field. However, this modification to the action can break supersymmetry. In order to restore supersymmetry, additional terms must be added to the Lagrangian that cancel the unwanted contributions from the gauge fields. This can be done using the embedding tensor formalism, which allows for a systematic and gauge-invariant construction of the complete supersymmetric action for gauged supergravity theories.

Consider kinetic term of gravitino

$$\mathscr{L}_{\psi_{A\mu}} = -e\bar{\psi}^A_\mu \gamma^{\mu\nu\rho}\nabla_\nu \psi_{A\rho} + \text{h.c.}, \tag{4.206}$$

and supersymmetry transformation of the form

$$\delta\psi_{A\mu} = \nabla_\mu \epsilon_A + ..., \tag{4.207}$$

the variation becomes

$$\delta\mathscr{L}_{\psi_{A\mu}} = -2e\bar{\psi}^A_\mu \gamma^{\mu\nu\rho}\nabla_\nu\nabla_\rho \epsilon_A + ... = ge\bar{\psi}^A_\mu \gamma^{\mu\nu\rho} F^{\hat{\Lambda}}_{\nu\rho} Q_{\hat{\Lambda}A}{}^B \epsilon_B + ... \tag{4.208}$$

The kinetic term of λ^I is in the form

$$\mathscr{L}_{\lambda^I} = -\frac{1}{2}e\bar{\Lambda}_I \gamma^\mu \lambda^I + ..., \tag{4.209}$$

and use supersymmetry transformation

$$\delta\lambda^I = \hat{P}^{AI}_\mu \gamma^\mu \epsilon_A + ... \tag{4.210}$$

The variation of (4.209) becomes

$$\delta\mathscr{L}_{\lambda_I} = -e\bar{\lambda}_I \gamma^\mu \gamma^\nu \nabla_\mu \hat{P}^{AI}_\nu \epsilon_A + ... = \frac{1}{2}ge\bar{\lambda}_I \gamma^{\mu\nu} F^{\hat{\Lambda}}_{\mu\nu} P_{\hat{\Lambda}}{}^{AI} \epsilon_A + ... \tag{4.211}$$

It's found that the variation of fermion Lagrangian density is modified as

$$\delta\mathscr{L}_{\psi_{A\mu}} \sim g\bar{\psi}^A_\mu \gamma^{\mu\nu\rho} F^{\hat{\Lambda}}_{\nu\rho} (L^{-1}X_{\hat{\Lambda}}L|_{\mathfrak{h}})_A{}^B \epsilon_B + ... \tag{4.212}$$

$$\delta\mathscr{L}_{\lambda_I} \sim g\bar{\lambda}_I \gamma^{\mu\nu} F^{\hat{\Lambda}}_{\mu\nu} (L^{-1}X_{\hat{\Lambda}}L|_{\mathfrak{t}})^{IA} \epsilon_A + ... \tag{4.213}$$

By writing $F^{\hat{\Lambda}}L^{-1}X_{\hat{\Lambda}}L$ in the covariant form of group G

$$F^{\hat{\Lambda}}L^{-1}X_{\hat{\Lambda}}L = F^{\hat{\Lambda}}E^M_{\hat{\Lambda}}L^{-1}X_M L = \mathscr{G}^M L^{-1}X_M L, \tag{4.214}$$

the previous transformation relate to tensor of group H in the form

$$T_{\overline{M}} = \mathbb{L}_{\overline{M}}{}^{N} L^{-1} X_N L, \tag{4.215}$$

where $\mathbb{L}_{\overline{M}}{}^{N} = (\mathbb{L}^T)^N{}_{\overline{M}}$, and the tensor $T_{\overline{M}}$ is called T-tensor. The component of tensor $T_{\overline{M}}$ on complex basis can be written as

$$T_{\overline{MN}}{}^{\overline{N}} = \mathbb{L}_{\overline{M}}{}^{M} \mathbb{L}_{\overline{N}}{}^{N} X_{MN}{}^{P} (\mathbb{L}^{-1})_P{}^{\overline{P}}. \tag{4.216}$$

The definition (4.185) including the relation $Q_{\hat{\Lambda}} = E_{\hat{\Lambda}}{}^{M} Q_M$ and $P_{\hat{\Lambda}} = E_{\hat{\Lambda}}{}^{M} P_M$ are used to construct T-tensor in the form of Q_M and P_M, expressed as

$$T_{\overline{M}} = \mathbb{L}_{\overline{M}}{}^{M} (P_M + Q_M), \tag{4.217}$$

and in the form of subgroup H generator with coset generator as

$$T_{\overline{M}} = \mathbb{L}_{\overline{M}}{}^{M} \Theta_N{}^{b} L_b{}^{a} t_a = \mathbb{L}_{\overline{M}}{}^{a} t_a. \tag{4.218}$$

The tensor $\mathbb{L}_{\overline{M}}{}^{a}$ is defined as $\mathbb{L}_{\overline{M}}{}^{a} = \mathbb{L}_{\overline{M}}{}^{N} \Theta_N{}^{b} L_b{}^{a}$, in which $L_b{}^{a}$ is coset representative in adjoint representation of group G. From definition of T-tensor, we can explicitly verify that $T_{\overline{MN}}{}^{\overline{N}}$ transform under group H only.

For T-tensor, we can write locality, linear and quadratic constraints as

$$\mathbb{C}^{\overline{MN}} T_{\overline{M}}{}^{a} T_{\overline{N}}{}^{b} = 0 \tag{4.219}$$

$$T_{(\overline{MNP})} = 0 \tag{4.220}$$

$$[T_{\overline{M}}, T_{\overline{N}}] + T_{\overline{MN}}{}^{\overline{P}} T_{\overline{P}} = 0. \tag{4.221}$$

The Yukawa term is added to the Lagrangian density in order to preserve supersymmetry. It is a term that couples the scalar fields and the fermion fields, and it is of order g. The form of the Yukawa term depends on the particular supergravity theory being considered, and it can be determined using the embedding tensor formalism. The Yukawa term ensures that the gauging of the symmetry group G_0 is consistent with supersymmetry and that the resulting theory is still a supergravity theory. The Yukawa term is of form

$$e^{-1}\mathscr{L}_{\text{Yukawa}} = g(-2\bar{\psi}^A_\mu \gamma^{\mu\nu} \psi^B_\nu S_{AB} + \bar{\lambda}^I \gamma^\mu \psi_{A\mu} N^A_I + \bar{\lambda}^I \lambda^J M_{IJ}) + \text{h.c.} \tag{4.222}$$

In gauged supergravity, the Yukawa term is often referred to as the fermion mass-like term. The tensors S_{AB}, $N_I{}^A$, and M_{IJ} can be written in terms of the T-tensor. The complex conjugate of these tensor are expressed as

$$S^{AB} = (S_{AB})^*, \quad N^I{}_A = (N_I{}^A)^*, \quad M^{IJ} = (M_{IJ})^*. \tag{4.223}$$

When we gauge a supergravity theory, we need to modify the supersymmetry transformation rules of the fermion fields to include the coupling to the gauge fields by adding the term in first order in g. The modified transformation can be written as

$$\delta\psi_{A\mu} = \nabla_\mu \epsilon_A - g S_{AB} \gamma_\mu \epsilon^B + ... \tag{4.224}$$

$$\delta\lambda_I = \hat{P}^A_{A\mu} \gamma^\mu \epsilon_A + g N_I{}^A \epsilon_A + ... \tag{4.225}$$

Sometime, these tensor S_{AB}, $N_I{}^A$, and M_{IJ} are called fermion-shift matrix.

The Yukawa term (4.222) gives the variations that consist of second order in g, which obtain from variation of $\psi_{A\mu}$ and λ_I. To cancel this second order term in g, we need to add the term that consist of second order of S_{AB} and $N_I{}^A$ into the action. The term added into the action is nonlinear in scalar fields, and it's called scalar potential. In gauged supergravity, the terms that are added are carefully chosen such that they preserve the supersymmetry of the ungauged theory.

The representation of embedding tensor R_Θ can be separated into irreducible representations of group H as

$$R_\Theta \rightarrow R_S \oplus R_N \oplus R_M \oplus ..., \tag{4.226}$$

where R_S, R_N, and R_M are representation of tensor S_{AB}, $N_I{}^A$, and M_{IJ} respectively. The term ... refers to any representation that are invisible in supersymmetry transformation of the action.

To verify the supersymmetry of the Lagrangian density of gauged supergravity, only the terms related to the coupling constant are considered. This is because any term in ungauged supergravity is cancelled by supersymmetry without minimal coupling, and complex conjugate terms vanish automatically. To illustrate this, we can start by considering the kinetic term with the Yukawa term for gravitinos in the form of

$$\mathscr{L}_{\psi_A,\mathrm{KY}} = -e\bar{\psi}^A_\mu \gamma^{\mu\nu\rho}\nabla_\nu \psi_{A\rho} - 2g\bar{\psi}^A_\mu \gamma^{\mu\nu}\psi^B_\nu S_{AB}. \tag{4.227}$$

When we vary the previous equation respect to $\psi_{A\mu}$, we see that

$$\begin{aligned}\delta\mathscr{L}_{\psi_A,\mathrm{KY}} =& -2e\bar{\psi}^A_\mu \gamma^{\mu\nu\rho}\nabla_\nu(\nabla_\rho \epsilon_A - gS_{AB}\gamma_\rho \epsilon^B)\\ & - 4g\bar{\psi}^A_\mu \gamma^{\mu\nu}(\nabla_\nu \epsilon^B - gS^{BC}\gamma_\nu \epsilon_C)S_{AB}.\end{aligned} \tag{4.228}$$

By using the identity $\gamma^{\mu\nu\rho}\gamma_\rho = 2\gamma^{\mu\nu}$, the previous becomes

$$\begin{aligned}\delta\mathscr{L}_{\psi_A,\mathrm{KY}} =& -2e\bar{\psi}^A_\mu \gamma^{\mu\nu\rho}\nabla_\nu\nabla_\rho \epsilon_A + 4eg\bar{\psi}^A_\mu \gamma^{\mu\nu}\nabla_\nu \epsilon^B S_{AB}\\ & - 4eg\bar{\psi}^A_\mu \gamma^{\mu\nu}\nabla_\nu \epsilon^B S_{AB} + 4g^2\bar{\psi}^A_\mu \gamma^{\mu\nu}\gamma_\nu S^{BC}S_{AB}\epsilon_C\\ =& -2e\bar{\psi}^A_\mu \gamma^{\mu\nu\rho}\nabla_\nu\nabla_\rho \epsilon_A + 4g^2\bar{\psi}^A_\mu \gamma^{\mu\nu}\gamma_\nu S^{BC}S_{AB}\epsilon_C\end{aligned} \tag{4.229}$$

Therefore, in order to preserve supersymmetry in the presence of gauge fields, we need to add terms of order g in the supersymmetry transformations.

The term $\nabla_\nu\nabla_\rho\epsilon_A$ is in the same form as (4.212) and (4.213), and it's canceled by allowing tensor S_{AB} and $N_I{}^A$ depend on T-tensor in the form

$$S_{AB} = T[R_S]_{AB}, \quad N_I{}^A = T[R_N]_I{}^A, \tag{4.230}$$

which can be written cearly (the index I split into Ai and ABC)

$$T_C{}^{DAB} = \mathbb{L}^{MAB}Q_M{}^D{}_C = -\frac{1}{2}N^{DAB}{}_C - 2S^{D[A}\delta^{B]}_C \tag{4.231}$$

$$(T_{AB})^{CDEF} = -\mathbb{L}^M{}_{AB}\Theta_M{}^a k^s_a P^{CDEF}_s = -4\delta^{[C}_{[A}N^{DEF]}{}_{B]} \tag{4.232}$$

$$(T_{AB})^{CDi} = -\mathbb{L}^M{}_{AB}\Theta_M{}^a k^s_a P^{CDi}_s = -2\delta^{[C}_{[A}N^{D]i}{}_{B]} \tag{4.233}$$

$$(T_i)_A{}^B = \mathbb{L}^M{}_i\Theta_M{}^a P^{\hat{a}}_a (J_{\hat{a}})_A{}^B = N_{iA}{}^B, \tag{4.234}$$

where $T_A{}^{BCD}$ relate to $(T^{AB})^{CD}{}_{EF}$ of T-tensor in the form

$$(T^{AB})^{CD}{}_{EF} = 4\delta^{[C}_{[E}T_{F]}{}^{D]AB}, \quad \text{and} \quad (T_{AB})^{CD}{}_{EF} = -4\delta^{[C}_{[E}T^{D]}{}_{F]AB}. \tag{4.235}$$

The term that relate to $\nabla_\nu S_{AB}$ from the last term of (4.229) is in the from

$$4eg\bar{\psi}^A_\mu\gamma^{\mu\nu}\epsilon^B\nabla_\nu S_{AB}. \tag{4.236}$$

The term that is in the same form of the previous equation obtains from the transformation of

$$e\bar{\lambda}^I\gamma^\mu\gamma^\nu\psi^B_\mu\hat{P}_{\nu BI} + eg\bar{\lambda}\gamma^\mu\psi_{A\mu}N_I{}^A \tag{4.237}$$

varying with respect to λ^I by using identity $\bar{\lambda}^I\psi^B_\mu$ and $\bar{\lambda}^I\gamma^{\mu\nu}\psi^B_\mu = -\bar{\psi}^B_\mu\gamma^{\mu\nu}\lambda^I$, we can show that

$$\bar{\lambda}^I\gamma^\mu\gamma^\nu\psi^B_\mu = \bar{\lambda}^I(\eta^{\mu\nu}+\gamma^{\mu\nu})\psi^B_\mu = \bar{\psi}^B_\mu(\eta^{\mu\nu}-\gamma^{\mu\nu})\lambda^I = \bar{\psi}^B_\mu\gamma^\mu\gamma^\nu\lambda^I. \tag{4.238}$$

As a results of the previous equation, the variation of (4.237) with respect to λ^I becomes

$$egN^I{}_A\bar{\psi}^B_\mu\gamma^\nu\gamma^\nu\epsilon^A\hat{P}_{\nu BI} - egN^I{}_B\bar{\psi}^B_\mu\gamma^\mu\gamma^\nu\epsilon^A\hat{P}_{\nu AI} - eg^2N_I{}^AN^I{}_B\bar{\psi}_{A\mu}\gamma^\mu\epsilon^B. \tag{4.239}$$

By using the identity $\gamma^\mu\gamma^\nu = \gamma^{\mu\nu}+\eta^{\mu\nu}$, the variation term at order g becomes

$$-2eg\bar{\psi}^A_\mu\gamma^{\mu\nu}\epsilon^BN^I{}_{(B}\hat{P}_{\nu A)I} - 2egN^I{}_B\bar{\psi}^{[A}_\mu\epsilon^{B]}\hat{P}^\mu_{AI}. \tag{4.240}$$

The first term is canceled by the term in equation (4.236), if $2\nabla_\mu S_{AB} = N^I{}_{(B}P_{\mu A)I}$ or

$$\mathscr{D}_s S_{AB} = \frac{1}{2}N^I{}_{(B}P_{sA)I}, \tag{4.241}$$

where $\mathscr{D}_s$ mean that covariant derivative for connection H.

Consider kinetic term of scalar fields $\frac{1}{2}eG_{rs}\nabla_\mu\phi^r\nabla^\mu\phi^s$, the variation with respect to vector field becomes

$$\begin{aligned} eG_{rs}\nabla^\mu\phi^r\delta A^M_\mu k^s_M = &-2egG_{rs}\nabla^\mu\phi^r\mathbb{L}^M{}_{AB}k^s_M\bar{\psi}^A_\mu\epsilon^B + egG_{rs}\nabla^\mu\phi^r\mathbb{L}^M{}_ik^s_M\bar{\lambda}^{Ai}\gamma_\mu\epsilon_A \\ &+\frac{1}{2}egG_{rs}\nabla^\mu\phi^r\mathbb{L}^M{}_{AB}k^s_M\bar{\chi}^{ABC}\gamma_\mu\epsilon_C. \end{aligned} \tag{4.242}$$

It's found that the first term of previous equation is cancelled by the second term of equation (4.240), if we impose the condition

$$\mathbb{L}^M{}_{AB}k^s_M = -G^{sr}N^I{}_{[B}P_{rA]I}. \tag{4.243}$$

It should be noted that if we define

$$\mathbb{L}^M{}_I{}^A = (\mathbb{L}^M{}_{iB}{}^A, \mathbb{L}^M{}_{BCD}{}^A) = (\mathbb{L}^M{}_i \delta^A_B, 3\mathbb{L}^M{}_{[BC}\delta^A_{D]}), \tag{4.244}$$

the equation (4.242) becomes (without the first term)

$$egG_{rs}\nabla^\mu \phi^r \mathbb{L}^M{}_I{}^A k^s_M \bar{\lambda}^I \gamma_\mu \epsilon_A. \tag{4.245}$$

The variation of kinetic term of λ^I and coupling term between scalar fields and fermion fields together with Yukawa term is written in the form (variation respect to $\psi_{A\mu}$)

$$\begin{aligned} \delta\mathscr{L}_{\psi_{A\mu},\lambda} =& - e\bar{\lambda}^I\gamma^\mu\nabla_\mu\delta\lambda_I + eg\bar{\lambda}^I\gamma^\mu N_I{}^A\delta\psi_{A\mu} + e\bar{\lambda}^I\gamma^\mu\gamma^\nu\hat{P}_{\nu BI}\delta\psi^B_\mu \\ & + e\bar{\psi}^B_\mu\gamma^\nu\gamma^\mu\delta\lambda^I\hat{P}_{\nu BI} + 2eg\bar{\lambda}^J M_{IJ}\delta\lambda^I. \end{aligned} \tag{4.246}$$

At order of g and g^2

$$\begin{aligned} \delta\mathscr{L}^{(g,g^2)}_{\psi_{A\mu},\lambda} =& - eg\bar{\lambda}^I\gamma^\mu(N_I{}^A\nabla_\mu\epsilon_A + \epsilon_A\nabla_\mu N_I{}^A) + eg\bar{\lambda}^I\gamma^\mu N_I{}^A(\nabla_\mu\epsilon_A - gS_{AB}\gamma_\mu\epsilon^B) \\ & - eg\bar{\lambda}^I\gamma^\mu\gamma^\nu\hat{P}_{\nu BI}S^{BC}\gamma_\nu S^{BC}\gamma_\mu\epsilon_C + 2eg\bar{\lambda}^J M_{IJ}(\gamma^\mu\hat{P}^{AI}_\mu\epsilon_A g N^I{}_A\epsilon^A), \end{aligned} \tag{4.247}$$

we found that the term related to $\nabla_\mu\epsilon_A$ in the first line is cancelled in the same way as gravitino field.

The term in order g and the term about $\bar{\lambda}^I\gamma^\mu\epsilon_A$ are cancelled by equation (4.245), if we impose that

$$\mathscr{D}_r N_I{}^A = G_{rs}\mathbb{L}^M{}_I{}^A k^s_M + 2P_{rBI}S^{BA} + 2M_{IJ}P^{AJ}_r. \tag{4.248}$$

When we consider term at order g^2 from equation (4.228), (4.239) and (4.247) by using identity $\gamma^{\mu\nu}\gamma_\nu = 3\gamma^\nu$, we can see that

$$eg^2\bar{\psi}^A_\mu\gamma^\mu\epsilon_C(12S_{AB}S^{BC} - N^I{}_A N_I{}^C) + eg^2\bar{\lambda}^I\epsilon^A(2M_{IJ}N^J{}_A - 4N_I{}^B S_{BA}). \tag{4.249}$$

This tern can be cancelled by adding scalar potential in the form

$$V(\phi) = \frac{1}{N}g^2(N_I{}^A N^I{}_A - 12S^{AB}S_{AB}). \tag{4.250}$$

In Lagrangian density, this term is added in the form $-eV(\phi)$, which provide the variation that respect to e^A_μ as

$$-\delta eV = -ee^\mu_a\delta e^a_\mu V = e\bar{\psi}^A_\mu\gamma^\mu\epsilon_A V. \tag{4.251}$$

This term is cancelled by order g^2 term that relate to $\bar{\psi}^A_\mu\gamma^\mu\epsilon_C$ in (4.249), if we impose

$$\delta^A_B V = g^2(N^I{}_A N_I{}^B - 12 S_{AC}S^{CB}), \tag{4.252}$$

which obviously reproduces the definition of scalar potential in equation (4.250) when we contract the indices between A and B.

The term of order g^2 that relates to $\bar{\lambda}^I\epsilon^A$ in equation (4.249) is cancelled by variation of scalar potential with respect to ϕ^s, if we use the condition

$$\frac{\partial V}{\partial\phi^s}P^s_{IA} = 2g^2(M_{IJ}N^J{}_A - 2N_I{}^B S_{AB}), \tag{4.253}$$

by using equation (4.252) and varying with respect to ϕ^s, relation (4.241), and (4.248) together with

$$\Theta_M{}^a\mathbb{L}^M{}_I{}^A N^I{}_A + \Theta_M{}^a\mathbb{L}^{MI}{}_A N_I{}^A = 0. \tag{4.254}$$

This condition can be found from the relation between fermion-shift tensor and the component of T-tensor in equation (4.231) – (4.234).

The formula that is important for finding vacuum of theory is written in the form of derivative of scalar potential with respect to scalar field expressed as

$$\frac{\partial V(\phi)}{\partial\phi^s} = \frac{g^2}{N}(2M_{IJ}N^J{}_A P^{IA}_s - 4S_{AB}N_I{}^B P^{IA}_s) + \text{c.c.}, \tag{4.255}$$

where c.c represent complex conjugate term.

The modification of an ungauged theory into a gauged theory with non-Abelian symmetry can be achieved by adding the minimal coupling term to the ungauged theory and modifying the field strength tensor of the gauge field into a non-Abelian form. To preserve supersymmetry, we must also add the Yukawa term and scalar potential into the supergravity action, at the first and second order of coupling constant, respectively. Additionally, the supersymmetry transformation of fermions is modified by adding a first-order term of the coupling constant.

Base on all results discussed, the full lagrangian density that is invariant under gauge non-abelian symmetry is

$$
\begin{aligned}
e^{-1}\mathscr{L}_{\text{gauge}} &= e^{-1}\mathscr{L}_{\text{ungauge}}(\partial \to \nabla, dA + A \wedge A) \\
&\quad + g(-2\bar{\psi}^A_\mu \gamma^{\mu\nu}\psi^B_\nu S_{AB} + \bar{\lambda}^I \gamma^\mu \psi_{A\mu} N_I{}^A + \bar{\lambda}^I \lambda^J M_{IJ}) + \text{h.c.} - V(\phi), \qquad (4.256)
\end{aligned}
$$

where scalar potential V is defined in (4.250).

The action obtained form (4.256) is invariant under supersymmetry transformation

$$\delta e^a_\mu = \bar{\epsilon}^A \gamma^a \psi_{A\mu} + \bar{\epsilon}_A \gamma^a \psi^A_\mu \qquad (4.257)$$

$$\delta A^\Lambda_\mu = \mathbb{L}^\Lambda{}_{\bar{M}} O^{\bar{M}}_\mu \qquad (4.258)$$

$$P^{ABCD}_s \delta\phi^s = \Sigma^{ABCD} \qquad (4.259)$$

$$P^{iAB}_s \delta\phi^s = \Sigma^{iAB} \qquad (4.260)$$

$$\delta\psi_{A\mu} = \nabla_\mu \epsilon_A - g S_{AB}\gamma_\mu \epsilon^B + \frac{i}{8} F^-_{\rho\sigma AB}\gamma^{\rho\sigma}\gamma_\mu \epsilon^B \qquad (4.261)$$

$$\delta\chi_{ABC} = P_{sABCD}\partial_\mu \psi^s \gamma^\mu \epsilon^D + \frac{3}{4} i F^-_{\mu\nu[AB}\gamma^{\mu\nu}\epsilon_{C]} + g N_{ABC}{}^D \epsilon_D \qquad (4.262)$$

$$\delta\lambda_{Ai} = P_{siAB}\partial_\mu \phi^s \gamma^\mu \epsilon^B + \frac{1}{4} i F^-_{\mu\nu i}\gamma^{\mu\nu}\epsilon_A + N_{IA}{}^B \epsilon_B \qquad (4.263)$$

Chapter V

THE ADS/CFT CORRESPONDENCE

The AdS/CFT correspondence, first suggested by Maldacena , is a duality between gravity theory in anti de Sitter space (AdS) background and conformal field theory (CFT). The gravity theory on AdS is a combination of Einstein's general theory of relativity and supersymmetry, called supergravity. On the other hand, a quantum field theory that is invariant under conformal transformations is known as a conformal field theory (CFT). The term "duality" refers to the relationship between AdS_{d+1} and CFT_d, which can explain quantum systems in d dimension by calculating in $d+1$ dimensional supergravity on AdS, shown as

$$AdS_{d+1} \times M^{D-d-1} \longleftrightarrow CFT_d, \tag{5.1}$$

where M^{D-d-1} is compact manifold, and $D = 10, 11$. In other words, the supergravity approximation can be used to calculate correlation functions of strongly coupled conformal field theories by computing the on-shell action of supergravity. This is known as the AdS/CFT correspondence, or the holographic principle.

5.1 Conformal field theory

Conformal field theory is a type of quantum field theory that possesses conformal symmetry, which means that it is invariant under conformal transformations. Conformal symmetry is a powerful tool in understanding the behavior of quantum field theories, and CFT have been used to describe a wide range of physical systems, including critical phenomena, statistical physics, and string theory.

The conformal transformation consist of scale transformation or dilatation and special conformal transformation, which can be expressed as

$$\text{dilatation}: \quad x^{\mu'} = \lambda x^{\mu} \tag{5.2}$$

$$\text{special conformal transformation}: \quad x^{\mu'} = \frac{x^{\mu} + b^{\mu}x^2}{1 + 2b.x + b^2x^2}, \tag{5.3}$$

where λ and b^{μ} are parameter of dilatation and special conformal transformation, respectively. Generally, the special conformal transformation is commonly expressed in terms of translations $x^{\mu'} = x^{\mu} + a^{\mu}$ and inversions

$$I: \quad x^{\mu} \to x^{\mu'} = \frac{x^{\mu}}{x^2}. \tag{5.4}$$

In other words, the special conformal transformation can be obtained from translations and inversions as shown in the following process:

$$x^{\mu} \quad I:\longrightarrow \quad \frac{x^{\mu}}{x^2} \quad T:\longrightarrow \quad \frac{x^{\mu} + a^{\mu}}{x^2 + a^2 + 2a.x} \quad I:\longrightarrow \quad \frac{x^{\mu} + a^{\mu}x^2}{1 + 2a.x + a^2x^2} \tag{5.5}$$

In addition, the inversion results in a change in the metric tensor, given by

$$g'_{\mu\nu}(x') = e^{2\omega(x)} g_{\mu\nu}(x), \tag{5.6}$$

where $\omega(x)$ is an arbitrary function of the coordinates. Clearly, scale transformations are a particular case of conformal transformations with constant $e^{2\omega}$. In the case of an infinitesimal transformation $x^{\mu'} = x^{\mu} - \epsilon^{\mu}(x)$ and $e^{2\omega(x)} = 1 + 2\omega(x)$ on $g_{\mu\nu} = \eta_{\mu\nu}$, we can express the variation of the metric tensor as

$$\delta g_{\mu\nu} = \partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = 2\omega(x)\eta_{\mu\nu}. \tag{5.7}$$

When we contract the previous equation with $\eta^{\mu\nu}$, it becomes

$$\partial_{\mu}\epsilon^{\mu} = \omega(x)d. \tag{5.8}$$

Next, we substitute the parameter $\omega = \frac{1}{d}\partial_{\mu}\epsilon^{\mu}$ into the equation (5.7) and take the second-order derivative in the form $\partial_{\mu}\partial_{\nu}$ together with using the relation (5.8) again. This results in the condition on the parameter that imposes a conformal transformation, which is given by

$$\left[\eta_{\mu\nu}\partial_{\rho}\partial^{\rho} + (d-2)\partial_{\mu}\partial_{\nu}\right]\partial_{\lambda}\epsilon^{\lambda} = 0. \tag{5.9}$$

It is well known that in two dimensions, the above condition yields the Cauchy-Riemann equations, commonly employed in the analysis of complex functions. Consequently, there exists an infinite set of solutions for the parameter ϵ. This implies that conformal field theories in $d = 2$ possess an infinite symmetry.

For $d > 2$, we can expand ϵ^μ in the form of a power series to find a solution to the condition (5.9), which is given by:

$$\epsilon^\mu = a^\mu + \omega^\mu{}_\nu x^\nu \lambda x^\mu + b^\mu x^2 - 2(b \cdot x)x^\mu, \tag{5.10}$$

where $x^2 = x^\mu x_\mu$ and $\omega_{\mu\nu} = -\omega_{\nu\mu}$. Note that the first and second terms correspond to the Poincaré transformation, which means that the Poincaré group is a subgroup of the conformal group.

The operator for a conformal transformation can be expressed as:

$$U(a, \omega, \lambda, b) = \mathbf{I} + a_\mu P^\mu + \frac{1}{2}\omega_{\mu\nu} J^{\mu\nu} + \lambda D + b_\mu K^\mu, \tag{5.11}$$

where P^μ and $J^{\mu\nu}$ represent the Poincaré generators, and D and K^μ represent the generators of dilation and special conformal transformations, respectively.

The condition imposed by the direct product between the elements of $U(a, \omega, \lambda, b)$ expresses that the generators $(P^\mu, J^{\mu\nu}, D, K^\mu)$ correspond to the conformal algebra given by

$$\begin{aligned}
&[J_{\mu\nu}, J_{\rho\sigma}] = 4\eta_{[\mu[\rho} J_{\sigma]\nu]}, \quad [P_\mu, J_{\nu\rho}] = 2\eta_{\mu[\nu} P_{\rho]} \\
&[K_\mu, J_{\nu\rho}] = 2\eta_{\mu[\nu} K_{\rho]}, \quad [P_\mu, K_\nu] = 2(\eta_{\mu\nu} D + J_{\mu\nu}) \\
&[D, P_\mu] = P_\mu, \quad [D, K_\mu] = -K_\mu
\end{aligned} \tag{5.12}$$

The first line of Lie algebra corresponds to the Poincaré subalgebra, while the second shows that K_μ is a vector under the Lorentz group. It should be noted that the Lie algebra between P_μ and K_μ provides $J_{\mu\nu}$ with D that commute with each other. In the last line of equation (5.12), we find that the generators P_μ and K_μ have unit charges under the generator D but opposite in sign.

From the Lie algebra $[D, J_{\mu\nu}] = 0$, it is found that the conformal group consists of subgroups that commute, namely $SO(1,1) \times SO(1,d-1)$, which have generators D and $J_{\mu\nu}$, respectively. However, other Lie algebra expresses that the conformal group forms a non-compact group $SO(2,d)$, which can be shown as follows. Let the indices $a = 0, 1, ..., d+1, d$, and $\tilde{\eta} =$ dia$(-1, 1, ..., 1, -1)$, which the component $0, 1, 2, .., d+1 = \mu$ represent $SO(1, d+1)$ and component d represent $SO(1,1)$. We can write generator $\tilde{J}_{ab} = -\tilde{J}_{ba}$ of $SO(2,d)$ in the form of $(P^{\mu}, J^{\mu\nu}, D, K^{\mu})$ as

$$\begin{aligned} &\tilde{J}_{\mu\nu} = J_{\mu\nu}, \quad \tilde{J}_{d,d+1} = D \\ &\tilde{J}_{\mu d} = \frac{1}{2}(K_{\mu} - P_{\mu}), \quad \tilde{J}_{\mu,d+1} = \frac{1}{2}(P_{\mu} + K_{\mu}). \end{aligned} \tag{5.13}$$

When we use Lie algebra of $SO(2,d)$ in the form

$$[\tilde{J}_{ab}, \tilde{J}_{cd}] = 4\tilde{\eta}_{[a[c}\tilde{J}_{d]b]}, \tag{5.14}$$

we obtain the conformal algebra in equation (5.12). This leads us to conclude that the conformal group in d dimensions is isomorphic to the group $SO(2,d)$. It should be noted that the conformal group $SO(2,d)$ is a non-compact simple group, whereas the Poincaré group is a non-semisimple group.

To find the representation of the conformal group, we can consider the transformation of the field at the position $x^{\mu} = 0$ and use the generators P_{μ} to translate it to a different position x^{μ}. We consider $\Phi(0)$ representing the fields that transform under translation of Lorentz group in the form

$$[J_{\mu\nu}, \Phi(0)] = M_{\mu\nu}\Phi(0). \tag{5.15}$$

The transformation of field $\Phi(0)$ under dilatation is in the form

$$[D, \Phi(0)] = \Delta\Phi(0) \tag{5.16}$$

where Δ is called scaling dimension or only dimension of $\Phi(0)$. Under dilation of $x^{\mu'} = \lambda x^{\mu}$, the field $\Phi(0)$ transform as

$$\Phi'(x') = \lambda^{-\Delta}\Phi(x). \tag{5.17}$$

The irreducible representation can be specified by the spin s under the Lorentz group $SO(1, d-1)$ and Δ under dilatation $SO(1,1)$. Since $[J_{\mu\nu}, D] = 0$, we can use the eigenvalues of $J_{\mu\nu}$ and D to specify a state in the representation. We start from the state $\Phi(0)$ with dimension Δ, which correspond to the condition

$$[K_\mu, \Phi(0)] = 0. \tag{5.18}$$

From the algebra (5.12), we found that the generator K_μ decreases the dimension Δ of a field, while the generator P_μ increases it. The fields that correspond to the lowest possible dimension are called conformal primary fields. Any other state can be obtained by applying the operator P_μ to a primary field, and the resulting state is called a conformal descendant.

The unitarity condition (which requires that all states in the Hilbert space have a positive norm) imposes a lower bound on the value of Δ, which is called the unitarity bound. This bound can be found by considering the norm of the state $|\Delta\rangle$:

$$\langle\Delta| K_\mu K_\nu P_\rho P_\sigma |\Delta\rangle \geq 0, \tag{5.19}$$

where $|\Delta\rangle$ represent the primary state that satisfies the condition $K_\mu |\Delta\rangle = 0$ and $D|\Delta\rangle = \Delta|\Delta\rangle$. For example, if $|\Delta\rangle$ represent scalar field ($J_{\mu\nu}|\Delta\rangle = 0$), we can use algebra (5.12) to express the bound as follows:

$$\Delta \geq \frac{d-2}{2}. \tag{5.20}$$

In the similar way, we can express that

$$\Delta \geq \frac{d-1}{2}, \quad \text{and} \quad \Delta \geq d+s-2 \tag{5.21}$$

for state $|\Delta\rangle$ that contain spin $\frac{1}{2}$ and $s > \frac{1}{2}$, respectively.

The Noether theorem can be used to express that the symmetry generated by D and K_μ provides conserved currents:

$$J^{(D)}_\mu = x^\nu T_{\mu\nu}, \quad \text{and} \quad J^{(K)}_{\mu\nu} = x^2 T_{\mu\nu} - 2x_\nu x^\rho T_{\mu\rho}, \tag{5.22}$$

which can be verified directly by the following conditions:

$$\partial^{\mu} J_{\mu}^{(D)} = T^{\mu}{}_{\mu} = 0 \tag{5.23}$$

$$\partial^{\mu} J_{\mu\nu}^{(K)} = -2x_{\nu} T^{\mu}{}_{\mu} = 0. \tag{5.24}$$

It indicates that the conservation law of conformal symmetry implies that the momentum-energy tensor is traceless $T^{\mu}{}_{\mu} = 0$.

5.2 Anti-de Sitter space time

The Anti-de Sitter (AdS) space is a maximally symmetrical space with negative curvature. In this context, we will discuss the geometrical structure of the space AdS_{d+1} in $d+1$ dimensions.

The curvature tensor of AdS_{d+1} can be expressed in terms of the metric tensor as follows:

$$R_{\mu\nu\rho\sigma} = -\frac{1}{L^2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}), \tag{5.25}$$

where L represent curvature radius of the AdS_{d+1} space. From the tensor (2.459), we can calculate directly to show that

$$R_{\mu\nu} = -\frac{1}{L^2} d g_{\mu\nu}, \quad \text{and} \quad R = -\frac{1}{L^2}(d+1)d. \tag{5.26}$$

The space AdS_{d+1} can be defined as a surface embedded in $\mathbb{R}^{2,d}$, where the signature of $\mathbb{R}^{2,d}$ is $(-,-,+,...,+)$. This means that AdS_{d+1} has one negative and d positive dimensions, preserving the signature of the ambient space. Let $Y^A, A = 0, 1, ..., d, d+1$ represent coordinates of $\mathbb{R}^{2,d}$, the space AdS_{d+1} is defined as the surface that corresponds to

$$Y^A Y^B \eta_{AB} = -(Y^0)^2 - (Y^{d+1})^2 + \sum_{i=1}^{d} (Y^i)^2 = -L^2. \tag{5.27}$$

The metric on AdS_{d+1} can be obtained from the expression:

$$ds^2 = \eta_{AB} dY^A dY^B, \tag{5.28}$$

where Y^A corresponds to the coordinates given in equation (2.461). In this form, it is easy to see that the AdS_{d+1} space has an isometry group of $SO(2,d)$, which means it preserves the structure and symmetries under this group.

The coordinates of AdS_{d+1} commonly used in the AdS/CFT context can be defined by transforming the coordinates $Y^A \to (x^0, x^i, u)$, where $i = 1, 2, ..., d-1$. The transformation is given by the following expressions:

$$\begin{aligned} &Y^0 = Lux^0, \quad Y^i = Lux^i \\ &Y^d = \frac{1}{2u}[u^2(L^2 - x^2) - 1], \quad Y^{d+1} = \frac{1}{2u}[u^2(L^2 + x^2) + 1], \end{aligned} \tag{5.29}$$

where $x^2 = -(x^0)^2 + \sum_{i=1}^{d-1}(x^i)^2$ by using the Minkowski metric in d dimensions given by $\eta_{\alpha\beta} dx^\alpha dx^\beta$. These coordinate transformations relate the AdS_{d+1} coordinates Y^A to the new coordinates (x^0, x^i, u), and L represents the curvature radius of AdS_{d+1}.

When we substitute Y^A in the equation (5.28), we obtain the metric

$$ds^2 = L^2 \left[\frac{du^2}{u^2} + u^2 \eta_{\alpha\beta} dx^\alpha dy^\beta \right]. \tag{5.30}$$

If we change the coordinate again by using $u = \frac{1}{z}$, we obtain

$$ds^2 = \frac{L^2}{z^2}(\eta_{\alpha\beta} dx^\alpha dx^\beta + dz^2), \tag{5.31}$$

where the coordinates (x^α, z) is called Poincare patch coordinates.

Another popular set of coordinates used in AdS_{d+1} is given by (x^α, r), where $\alpha = 0, 1, ..., d-1$, and r is the radial coordinate. The transformation between these coordinates and the original AdS_{d+1} coordinates can be expressed as:

$$e^{\frac{r}{L}} = \frac{L}{z}, \tag{5.32}$$

where z is a parameter related to the radial coordinate. In these new coordinates, the metric takes the form:

$$ds^2 = e^{\frac{2r}{L}} \eta_{\alpha\beta} dx^\alpha dx^\beta + dr^2. \tag{5.33}$$

5.3 Holographic renormalization

Feynman diagrams are a powerful tool in quantum field theory for calculating and visualizing particle interactions. However, when these diagrams are evaluated in theories with higher order corrections, they often lead to infinite values. To make sense of these calculations, a process called renormalization is employed.

Renormalization involves introducing counterterms to cancel out the infinities arising in the calculations. These counterterms are chosen such that the physical observables remain finite and well-defined. Renormalization also involves adjusting the values of coupling constants and scale parameters in the theory.

The renormalization group is a concept closely related to renormalization. After renormalization, the coupling constant will depend on the energy scale or distance, exhibiting group properties. For example, the transformation from scale μ_1 to μ_2 and then to μ_3 is equivalent to a direct transformation from μ_1 to μ_3. In certain cases, the coupling constants in a theory exhibit scale invariance, meaning they remain unchanged under scale transformations. This is expressed by the condition

$$\beta = \mu \frac{\partial g}{\partial \mu} = 0 \tag{5.34}$$

where β is the beta function and μ is the scale parameter. The presence of a scale-invariant coupling constant is a key feature of theories exhibiting conformal symmetry.

In a conformal field theory, certain coupling constants, denoted as g, may satisfy the condition $\beta(g^*) = 0$, where $\beta(g)$ is the beta function that describes how the coupling constant changes under scale transformations. This implies the existence of a conformal fixed point, where the coupling constant does not flow under scale transformations. At the conformal fixed point, the theory exhibits enhanced symmetry and is said to be scale invariant.

However, in most cases, quantum field theories do not possess exact conformal symmetry, and the coupling constants do not satisfy $\beta(g^*) = 0$ for any value of g. Nevertheless, it is still possible to have approximate conformal symmetry in the vicinity of a conformal fixed point. This is where the concept of the renormalization group flow becomes relevant.

The renormalization group flow describes how coupling constants change as the energy scale or distance scale is varied. When a conformal field theory is perturbed away from the conformal fixed point at high energy scales (UV), the theory undergoes a renormalization group flow and flows towards a different theory at low energy scales (IR). This flow breaks the conformal symmetry, and the resulting theory may exhibit new phenomena and different physical properties.

In the context of AdS/CFT, the scalar potential of gauged supergravity on AdS plays a significant role in understanding conformal field theory (CFT) via the AdS/CFT correspondence. The scalar potential contains important information about the conformal fixed points and the renormalization group (RG) flows in the corresponding quantum field theory.

By analyzing the scalar potential in gauged supergravity on AdS, one can gain insights into the properties and dynamics of the dual field theory. This approach is known as holographic RG flow, as it allows us to study the RG flows of the field theory by examining the gravitational solutions in the AdS background.

In the case of RG flows between different conformal phases or non-conformal fixed points, the corresponding gravitational solution takes the form of a domain wall solution, often referred to as an ”AAdS” (asymptotically Anti-de Sitter) spacetime. The metric of this domain wall solution is given by:

$$ds^2 = e^{2A(r)} dx^\mu dx^\nu \eta_{\mu\nu} + dr^2, \tag{5.35}$$

where $A(r)$ is a function that characterizes the RG flow. For RG flows from one conformal fixed point to another conformal fixed point, the domain wall solution approaches an AdS_4 space, and in this case, $A(r)$ takes the form $A(r) = \frac{r}{L}$, where L is the curvature radius of the AdS space.

The relationship between the coordinate r and the coordinate z can be expressed as $z = Le^{-\frac{r}{L}}$, where z represents the radial direction in the AdS space. This relation allows us to map the behavior of the RG flow in the field theory to the geometric properties of the AdS space.

By studying the domain wall solutions in gauged supergravity, one can extract valuable information about the RG flows, phase transitions, and other properties of the dual field theory. This provides a powerful tool for exploring the dynamics of quantum field theories using the AdS/CFT correspondence.

Chapter VI

REVIEW OF LITERATURE

The gauge/gravity duality, also known as AdS/CFT correspondence, is a powerful tool for understanding the dynamics of certain quantum field theories. The duality states that there is an equivalence between a gravitational theory in Anti-de Sitter space (AdS) and a conformal field theory (CFT) living on the boundary of that space.

The initial suggestion of this duality was made by Juan Maldacena (Maldacena, 1999). It proposed a connection between supergravity in AdS and a specific conformal field theory. In order to apply the AdS/CFT correspondence to describe $N = 2$ models, we start with the truncation of $N = 8$ supergravity. This truncation involves removing certain fields and retaining only those relevant to the $N = 2$ sector.

By focusing on the structure of $N = 6$ supergravity, which shares the same bosonic sector as our model, we can find solutions that describe RG flows. These solutions correspond to transitions between different phases of the dual $N = 2$ SCFT.

In the next subsection, we will provide a brief overview of various approaches to constructing $N = 2$ supergravity theories and the process of finding RG flows solution.

6.1 The Large N Limit of Superconformal field theories and supergravity

The paper (Maldacena, 1999) proposes that the large N limit of certain conformal field theories in various dimensions includes a sector describing supergravity on the product of Anti-deSitter spacetimes with spheres and other compact manifolds. This is shown by taking some branes in the full M/string theory and then taking a low energy limit where the field theory on the brane decouples from the bulk. The enhanced supersymmetries of the near horizon geometry correspond to the extra supersymmetry generators present in the superconformal group. The t Hooft limit of $3+1$ $N = 4$ super-Yang-Mills at the conformal point is shown to be dual to IIB strings on $AdS_5 \times S^5$. The paper also conjectures that compactifications of M/string theory on various Anti-de Sitter spacetimes are dual to various conformal field theories.

6.2 Exceptional N = 6 and N = 2 4D gauged Supergravity

In the domain of supergravity, the theory encompasses different levels of supersymmetry, reaching a maximum at $N = 8$, indicating the existence of eight gravitinos. However, for lower levels of supersymmetry, such as $N = 2, 3, ..., 6$, it's possible to derive these theories by truncating certain field components from the $N = 8$ supergravity theory. The paper (Andrianopoli et al., 2009) outlines the methodology for deriving $N = 2$ and $N = 6$ supergravity theories using concepts from group theory. Through this truncation process, the outcome includes ungauged $N = 6$ supergravity and $N = 2$ supergravity coupled to 15 vector multiplets.

Furthermore, the paper highlights an interesting finding: $N = 6$ supergravity shares the same bosonic fields with $N = 2$ supergravity coupled to 15 vector multiplets. This essentially means that the $N = 2$ supergravity model can be parameterized by the same scalar manifold that characterizes the $N = 6$ theory. Specifically, this scalar manifold is described as $SO^*(12)/U(6)$.

6.3 Supersymmetric solutions from N=6 gauged supergravity

The focus of our interest lies in the $N = 2$ gauged supergravity model coupled to 15 vector multiplets. Importantly, the bosonic sector of this $N = 2$ model is identical to that of the $N = 6$ gauged supergravity model.

Consequently, we can leverage the insights gained from studying the $N = 6$ gauged supergravity model to construct the $N = 2$ supergravity model with the same gauged symmetry, which is $SO(2) \times SO(6)$. The approach we are following to achieve this construction is outlined in the paper (Karndumri and Seeyangnok, 2021).

In addition to the gauge construction process, we are also exploring an understanding of the Renormalization Group (RG) flow. This involves comprehending how physical quantities change as energy scales are modified. The techniques we are employing encompass explanations of these dynamic changes in the context of these supergravity models.

By combining the insights from $N = 6$ gauged supergravity, the gauge construction procedure outlined in the paper, and an exploration of RG flow, we are working towards a comprehensive understanding of the $N = 2$ gauged supergravity model coupled to 15 vector multiplets within the $SO(2) \times SO(6)$ gauge symmetry framework. This endeavor involves linking different theoretical aspects and mathematical techniques to unveil the intricate relationships between these theories and their underlying principles.

Chapter VII

HOLOGRAPHIC RG FLOWS FROM 4-DIMENSIONAL N=2 GAUGED SUPERGRAVITIES

In this chapter, we investigate the holographic RG flow of four-dimensional $N = 2$ gauged supergravity with the $SO(2) \times SO(6)$ gauged group. The structure of $N = 2$ supergravity can be obtained from the truncation of the maximal $N = 8$ supergravity. This truncation provides the $N = 2$ supergravity coupled to 15 vector multiplets, and also provides the general $N = 6$ gauged supergravity.

7.1 Twin N = 6 and N = 2 gauged supergravity

The maximal supersymmetry that corresponds to the theory of gravity is the amount of $N = 8$ supersymmetry (that is 32 supercharges). The four-dimensional maximal supergravity, first proposed in (de Wit et al., 2007), is a mathematical model that describes a single massless graviton with maximal supersymmetry. The graviton $g_{\mu\nu}$ ($\mu, \nu, ... = 0, 1, 2, 3$ that is space-time indices), 8 spin-3/2 gravitini ψ^i_μ ($i = 1, ..., 8$) transforming in the fundamental representation of the R-symmetry group $SU(8)$, 28 vector field A^{ij}_μ, 56 spin-1/2 dilatini χ_{ijk} in the **56** of $SU(8)$, and 70 real scalar field ϕ^{ijkl} are the field content of $N = 8$ theory. Group theory allows for the decomposition of the essential fermionic $SU(8)$ representations with respect to $SU(6) \times SU(2) \times U(1)$ in order to produce the $N = 6$ gauged supergravity, the process shown in (Andrianopoli et al., 2009).

The theory would be reduced to the $N = 2$ theory if we truncated the multiplets of the six gravitini fields ψ^A_μ in the $N = 8$ theory instead. The $N = 2$ theory that resulted from truncating out of ψ_μ^A has the same bosonic content as the $N = 6$ theory. This result describes the $N = 2$ supergravity coupling to 15 vector multiplets. The field content of the $N = 6$ theory that resulted from the $N = 8$ truncation consists of six gravitini ψ_μ^A that belong in $(\mathbf{6}, \mathbf{1})_{+\frac{1}{2}}$, sixteen vectors A_μ^{AB} and $A_\mu^{\alpha\beta}$ in $(\mathbf{15}, \mathbf{1})$, twenty-six spin–12 fields χ^{ABC} and χ^A in $(\mathbf{20}, \mathbf{1})_{+\frac{3}{2}} + (\mathbf{6}, \mathbf{1})_{-\frac{5}{2}}$, along with complex scalars $\phi^{AB\alpha\beta}$ in $(\mathbf{15}, \mathbf{1})_{-2} + (\overline{\mathbf{15}}, \mathbf{1})_{+2}$, while $\phi^{ABC\alpha}$, $A_\mu^{A\alpha}$, ψ_μ^α and $\chi^{AB\alpha}$ are truncated. Similarly, the bosonic part of the $N = 2$ theory is the same as the $N = 6$ theory ($\phi^{AB\alpha\beta}$,A_μ^{AB} and $A_\mu^{\alpha\beta}$), and the different in fermionic content consists of two gravitini ψ_μ^α in $(\mathbf{1}, \mathbf{2})_{-\frac{3}{2}}$ representation, together with thirty spin-1/2 fields $\chi^{AB\alpha}$ in $(\mathbf{15}, \mathbf{2})_{-\frac{1}{2}}$. The fields that were truncated in order to obtain the $N = 2$ theory are $\phi^{ABC\alpha}$, $A_\mu^{A\alpha}$, ψ_μ^A, χ^{ABC} and χ^A . Therefore, the scalar manifolds of both theories can be shown as

$$\mathcal{M} = \frac{SO^*(12)}{U(6)}. \tag{7.1}$$

The gauging of these $N = 6$ and $N = 2$ theories will now be discussed. As we'll see, all of these gauged theories can be built as truncations of the $N = 8$ theory using the necessary gauging. When the $N = 8$ theory is gauged, the terms of fermion shifts that define the fermion mass terms and the scalar potential are added to the Lagrangian. All terms in the fermions' Lagrangian bilinear theory for $N = 8$ theory are shown in (de Wit and Nicolai, 1982), where we change the notation of the fermions shift as $S^{ij} = S^{ji} = -\frac{1}{\sqrt{2}} A_1^{ij}$, $N_l{}^{ijk} = -\sqrt{2} A_{2l}^{ijk}$. So that the terms in the Lagrangian bilinear (first order of coupling constant g) in the fermions can be written as

$$g\left(4S_{ij}\bar{\psi}_\mu^i \gamma^{\mu\nu}\psi_\nu^j + \frac{1}{6}N^l{}_{ijk}\bar{\chi}^{ijk}\gamma^\mu\psi_{l\mu}\right) + h.c.\ . \tag{7.2}$$

The supersymmetry variations of the fermion fields are expressed as follows:

$$\delta\psi_\mu^i = ... + igS^{ij}\gamma_\mu\epsilon_j, \tag{7.3}$$

$$\delta\chi^{ijk} = ... + gN_l{}^{ijk}\epsilon^l. \tag{7.4}$$

These fermion shifts correspond to the scalar potential of the $N = 8$ theory, shown as

$$V^{(N=8)}(\phi) = g^2 \left(\frac{1}{48} N_i{}^{jkl} N^i{}_{jkl} - \frac{3}{2} S_{ij} S^{ij} \right). \tag{7.5}$$

As follows, supersymmetry transformations of order g can be decomposed as

$$\delta\psi^A_\mu = ... + ig\left(S^{AB}\gamma_\mu\epsilon_B + S^{A\beta}\gamma_\mu\epsilon_\beta\right), \tag{7.6}$$

$$\delta\psi^\alpha_\mu = ... + ig\left(S^{\alpha B}\gamma_\mu\epsilon_B + S^{\alpha\beta}\gamma_\mu\epsilon_\beta\right), \tag{7.7}$$

$$\delta\chi^{ABC} = ... + g\left(N_D{}^{ABC}\epsilon^D + N_\beta{}^{ABC}\epsilon^\beta\right), \tag{7.8}$$

$$\delta\chi^{AB\alpha} = ... + g\left(N_D{}^{AB\alpha}\epsilon^D + N_\beta{}^{AB\alpha}\epsilon^\beta\right), \tag{7.9}$$

$$\delta\chi^A = ... + g\left(N_B{}^A\epsilon^B + N_\beta{}^A\epsilon^\beta\right). \tag{7.10}$$

The equation (7.2) can be written in the form of all truncated fermion fields as

$$\begin{aligned} g\bigg(& 4S_{A\alpha}\bar{\psi}^A_\mu\gamma^{\mu\nu}\psi^\alpha_\nu + \frac{1}{6}N^\alpha{}_{BCD}\bar{\chi}^{BCD}\gamma^\mu\psi_{\alpha\mu} \\ & + \frac{1}{2}N^A{}_{BC\alpha}\bar{\chi}^{CD\alpha}\gamma^\mu\psi_{A\mu} + N^\alpha{}_A\bar{\chi}^A\gamma^\mu\psi_{\alpha\mu}\bigg) + h.c.\,. \end{aligned} \tag{7.11}$$

In the case of the $N = 6$ and $N = 2$ theories, the fermion fields that are truncated consist of ψ^α_μ (truncated in $N = 6$ case), $\chi^{AB\alpha}$, ψ^A_μ(truncated in $N = 2$ case) , χ^{ABC} and χ^A. To ensure that the $N = 6$ and $N = 2$ theories are a consistent truncation, the fermion shift $S_{A\alpha}$,N^α_{BCD}, $N^A_{BC\alpha}$ and N^α_A must vanish. The following transformation rules are then included in the $N = 6$ and $N = 2$ theories:

$$\delta\psi^A_\mu = ... + igS^{AB}\gamma_\mu\epsilon_B, \tag{7.12}$$

$$\delta\chi^{ABC} = ... + gN_D^{ABC}\epsilon^D, \tag{7.13}$$

$$\delta\chi^A = ... + gN_B{}^A\epsilon^B, \tag{7.14}$$

for $N = 6$ theory, while for the $N = 2$ theory we have:

$$\delta\psi^\alpha_\mu = ... + igS^{\alpha\beta}\gamma\epsilon^\beta, \tag{7.15}$$

$$\delta\chi^{AB\alpha} = ... + gN_\beta{}^{\alpha AB}\epsilon^\beta. \tag{7.16}$$

It indicates that $S^{\alpha\beta}$ and $N_\beta{}^{\alpha AB}$ vanish for the $N = 6$ theory, and $S^{AB}\gamma_\mu$, $N_D{}^{ABC}$ and $N_B{}^A$ vanish for the $N = 2$ theory. These fermion shifts produce a scalar potential for both theories, which can be formulated from the $N = 8$ Ward identity:

$$\delta^i_j V^{(N=8)} = g^2 \left(-12 S^{ik} S_{kj} + \frac{1}{6} N_j{}^{klm} N^i{}_{klm} \right). \tag{7.17}$$

By decomposing and tracing the above identities, we obtain the scalar potential written in terms of $N = 6$ and $N = 2$ quantities respectively:

$$V^{(N=8)} \approx V^{(N=6)} = g^2 \left(-2 S^{AB} S_{AB} + \frac{1}{36} N_A{}^{BCD} N^A_{CDE} + N_B{}^C N^A_C \right), \tag{7.18}$$

$$V^{(N=8)} \approx V^{(N=2)} = g^2 \left(-6 S^{\alpha\beta} S_{\alpha\beta} + \frac{1}{4} N_\alpha{}^{\beta AB} N^\alpha_{\beta AB} \right). \tag{7.19}$$

The scalar potential contains important properties about conformal fixed points, which are indispensable for describing holographic RG flow. Finding scalar potential requires using embedding tensor formalism to elevate the ungauged theory into the gauged theory (de Wit et al., 2003, 2005; de Wit and Nicolai, 1982). We consider the gauging of an extended supergravity with n_v vector fields $A^\Lambda_\mu, \Lambda = 1, ..., n_v$ and a scalar manifold of the form G/H, where G represents the on-shell (classical) global symmetry group and H is maximal compact subgroup. The gauging procedure includes promoting a suitable subgroup G_0 of the Lagrangian's global symmetry group to local symmetry, which is gauged by the theory's electric potentials. The choice of the gauge algebra inside the Lie algebra of G can be mapped into a subset of the electric group by using an embedding tensor $\theta_M{}^m$, where index M labels the symplectic representation **R** of G ($V^M = V^\Lambda, V_\Lambda$) and index t_n belongs to **Adj**(**G**). The embedding tensor expresses the gauge generators X_M as a linear combination of the generators t_m of G : $X_M = \theta_M{}^m t_m$, where $\theta_M{}^m$ belong to the product $\mathbf{R} \times$ **Adj**(**G**) and we can also define the tensor $X_{MN}{}^P = \theta_M^n (t_n)_N{}^P$ in the same representation as θ_M^n. Consistency of the construction of a gauged extended supergravity requires θ_M^n to satisfy some G-covariant constraints consisting of a linear condition on $X_{MN}{}^P$:

$$X_{(MN}{}^L \Omega_{P)L} = 0, \tag{7.20}$$

and the following quadratic conditions

$$\theta_M{}^m \theta_N{}^n f_{mn}{}^p + X_{MN}{}^P \theta_P{}^p = 0, \tag{7.21}$$

$$\theta_M{}^m \theta_N{}^n \Omega^{MN} = 0, \tag{7.22}$$

where $f_{mn}{}^p$ are the structure constants of G and Ω^{MN} is the symplectic invariant matrix. In the case of the $N = 6$ and $N = 2$ theories, the coset manifold is $SO^*(12)/U(6)$, in which the global symmetry can be identified with the maximal subgroup $SO^*(12) \times SU(2)$ of $E_{7(7)}$. So that, $\mathbf{R} = (\mathbf{32}, \mathbf{1})$, $\mathbf{Adj}(\mathbf{G}) = (\mathbf{66}, \mathbf{1}) + (\mathbf{1}, \mathbf{3})$ and the decomposition of $\mathbf{R} \times \mathbf{Adj}(\mathbf{G})$ reads

$$(\mathbf{32}, \mathbf{1}) \times [(\mathbf{66}, \mathbf{1}) + (\mathbf{1}, \mathbf{3})] \longrightarrow (\mathbf{32}, \mathbf{1}) + (\mathbf{1728}, \mathbf{1}) + (\mathbf{352}, \mathbf{1}) + (\mathbf{32}, \mathbf{3}). \tag{7.23}$$

Note that constraint (4.20) states that the representations from the previous decomposition have a symmetric product of $(\mathbf{32}, \mathbf{1})$, which means that

$$[(\mathbf{32}, \mathbf{1}) \times (\mathbf{32}, \mathbf{1}) \times (\mathbf{32}, \mathbf{1})]_{sym.} \longrightarrow (\mathbf{32}, \mathbf{1}) + (\mathbf{4224}, \mathbf{1}) + (\mathbf{1728}, \mathbf{1}), \tag{7.24}$$

should vanish. Therefore, we conclude that the generic gaugings in both theories are defined by an embedding tensor in the following representations:

$$\theta_M{}^n \in (\mathbf{352}, \mathbf{1}) + (\mathbf{32}, \mathbf{3}). \tag{7.25}$$

7.2 N = 6 gauged supergravity with SO(2)xSO(6) gauge group

The only supermultiplet in $N = 6$ supersymmetry is the gravity multiplet with the field content

$$(e_\mu^{\hat{\mu}}, \psi_\mu^A, A_\mu^{AB}, A_\mu^{\alpha\beta}, \chi^{ABC}, \chi^A, \phi^{AB\alpha\beta}). \tag{7.26}$$

The components obtained from truncation are described in section 1. The coset representative in representation **32** of $SO^*(12)$ of the form

$$\mathcal{V}_M{}^{\underline{M}} = \mathcal{A}^\dagger e^Y \tag{7.27}$$

can characterize the 30 real scalars in $\phi_{AB\alpha\beta}$ as coordinates of the scalar manifold $SO^*(12)/U(6)$ with the Cayley matrix

$$\mathcal{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{I}_{16} & i\mathbb{I}_{16} \\ \mathbb{I}_{16} & -i\mathbb{I}_{16} \end{pmatrix}, \tag{7.28}$$

and

$$Y = \begin{pmatrix} 0 & 0_{1\times 15} & 0 & \phi_{CD} \\ 0_{15\times 1} & 0_{15\times 15} & \phi_{AB} & \frac{1}{2}\epsilon_{ABCDEF}\bar{\phi}^{EF}\phi_{CD} \\ 0 & \bar{\phi}^{CD} & 0 & 0_{1\times 15} \\ \bar{\phi}^{AB} & \frac{1}{2}\epsilon^{ABCDEF}\phi_{EF} & 0_{15\times 1} & 0_{15\times 15} \end{pmatrix}. \tag{7.29}$$

We also note that $\phi_{AB} = \phi_{AB0}$ is a singlet under $SU(2)$ and $\bar{\phi}^{AB} = (\phi_{AB})^*$. In the analysis that follows, it is helpful to identify the 16×16 submatrices of the $\mathcal{V}_M{}^{\underline{M}}$ by the identification

$$\mathcal{V}_M{}^{\underline{M}} = \begin{pmatrix} \bar{h}_\Lambda{}^{\underline{\Lambda}} & h_{\Lambda\underline{\Lambda}} \\ \bar{f}^{\Lambda\underline{\Lambda}} & f^\Lambda{}_{\underline{\Lambda}} \end{pmatrix}, \tag{7.30}$$

where $\mathbf{f}$, $\mathbf{h}$, $\bar{\mathbf{f}}$, and $\bar{\mathbf{h}}$ satisfy the relations

$$(\mathbf{f}\mathbf{f}^\dagger)^T = \mathbf{f}\mathbf{f}^\dagger, \quad (\mathbf{h}\mathbf{h}^\dagger)^T = \mathbf{h}\mathbf{h}^\dagger, \quad \mathbf{f}\mathbf{h}^\dagger - \bar{\mathbf{f}}\mathbf{h}^T = i\mathbb{I}_{16}, \tag{7.31}$$
$$\mathbf{f}^\dagger\mathbf{h} - \mathbf{h}^\dagger\mathbf{f} = -i\mathbb{I}_{16}, \quad \mathbf{f}^T\mathbf{h} - \mathbf{h}^T\mathbf{f} = 0.$$

We can write the general expression of $(\mathcal{V}_M{}^{\underline{M}})^{-1} = \mathcal{V}_{\underline{M}}{}^M$ by using the above properties as

$$\mathcal{V}_{\underline{M}}{}^M = \begin{pmatrix} -if^\Lambda{}_{\underline{\Lambda}} & ih_{\Lambda\underline{\Lambda}} \\ i\bar{f}^{\Lambda\underline{\Lambda}} & -i\bar{h}_\Lambda{}^{\underline{\Lambda}} \end{pmatrix}. \tag{7.32}$$

The result of the combination of the sixteen electric gauge fields A^0 and A^{AB} is $A^\Lambda = (A^0, A^{AB})$. The magnetic dual A_Λ and the gauge fields combine with the gauge fields into the 32 representation of $SO^*(12)$ as

$$A^M = (A^\Lambda, A_\Lambda). \tag{7.33}$$

The relationship between the fermion shift and the embedding tensor is defined via the T-tensor identity

$$T_{\underline{M},\underline{N}}{}^{\underline{P}} = [\mathcal{V}^{-1} \circ X]_{\underline{M},\underline{N}}{}^{\underline{P}} = \mathcal{V}^{-1}{}_{\underline{M}}{}^M \mathcal{V}^{-1}{}_{\underline{N}}{}^N \mathcal{V}_P{}^{\underline{P}} X_{MN}{}^P, \tag{7.34}$$

and we can use the corresponding $N = 8$ relations and reduce them to the $N = 6$ theory. The relation between T-tensor and fermion shifts in maximal gauged supergravity is written as

$$T_{ij,kl}{}^{pq} = -\frac{1}{2\sqrt{2}}\delta^{[p}_{[k}N^{q]}_{l]ij} - \sqrt{2}\delta^{[p}_{[k}S_{l][i}\delta^{q]}_{j]}. \tag{7.35}$$

Indices $i, j, ...$ are further subdivided into $A, B, ...$ and finally :

$$N^A{}_B = -2\sqrt{2}T_{\alpha\beta,BC}{}^{AC},\ N_{AB} = -\frac{8}{3}\sqrt{2}T_{C[A,B]E}{}^{CE}, \tag{7.36}$$
$$N^A{}_{BCD} = -2\sqrt{2}T_{[CD,B]E}{}^{AE} - \frac{1}{4}\delta^A_{[B}N_{CD]},\ S_{AB} = \frac{\sqrt{2}}{5}T_{C(A,B)E}{}^{CE}.$$

Let us now consider the gauging of $\mathcal{G} = SO(6) \times SO(2)$. By definition of gauged generator X_M, the embedding tensor functions as a mapping of the global symmetry group into gauged group $\mathcal{G}$. From equation (7.21), we can imply that the gauged generator forms the algebra:

$$[X_M, X_N] = -X_{MN}{}^P X_P. \tag{7.37}$$

For gauge group $\mathcal{G} = SO(6) \times SO(2)$ that is embedded electrically in $U(6) \subset SO^*(12)$, the gauged generator is $SO(6)$ generators. So that the non vanishing components of tensor $X_{MN}{}^P$ read:

$$X_{I_1J_1,I_2J_2}{}^{I_3J_3} = 4g\delta^{[I_3}_{[I_1}\delta_{J_1][I_2}\delta^{J_3]}_{J_2]},\quad X_{I_1I_2}{}^{I_3J_3}{}_{I_2J_2} = -X_{I_1J_1,I_2J_2}{}^{I_3J_3}. \tag{7.38}$$

There are no $X^\Lambda{}_M{}^N$ components that couple to magnetic gauge fields, and the indices $\Lambda, \Sigma, ...$ are split into $(0, [IJ])$.

7.3 N = 2 gauged supergravity

In the context of $N = 2$ supergravity with 15 vector multiplets and gauge group $SO(2) \times SO(6)$, the scalar potential plays a crucial role in understanding the dynamics of the theory. The scalar potential determines the vacuum structure of the theory and governs the interactions among the scalar fields.

The field content in $N=2$ gravity multiplet can be shown as

$$\left(1\times e_{\mu}^{\hat{\mu}},\ 2\times\psi_{\mu}^{\alpha},\ 1\times A_{\mu}^{\alpha\beta}\right) \tag{7.39}$$

with 15 vector multiplets

$$\left(15\times A_{\mu}^{AB},\ \ 30\times\chi^{AB\alpha},\ 15\times\phi^{AB}\right). \tag{7.40}$$

In $N=2$ supergravity with 15 vector multiplets, the scalar fields arise from the complex scalars in the vector multiplets. These scalar fields parameterize the scalar manifold $\mathscr{M}$. In this case, the scalar manifold $\mathscr{M}$ has the same form as in $N=6$ supergravity and is given by

$$\mathscr{M}=\frac{SO^{*}(12)}{U(6)}. \tag{7.41}$$

The specific form of the scalar manifold reflects the symmetry structure and the transformation properties of the scalar fields in the theory.

The component of the symplectic section are denoted by

$$\mathcal{V}\equiv\begin{pmatrix}\mathcal{L}^{\Lambda}\\ \mathcal{M}_{\Lambda}\end{pmatrix},\quad \mathcal{U}_i\equiv\begin{pmatrix}f^{\Lambda}{}_{i}\\ h_{\Lambda i}\end{pmatrix}, \tag{7.42}$$

where $i=1,2,...,15$ denote the number of scalar field. In the term of equation (4.30), we define 16×16 matrix as

$$f^{\Lambda}{}_{\underline{\Lambda}}=\begin{pmatrix}f_i^{\Lambda}\\ \overline{\mathcal{L}}^{\Lambda}\end{pmatrix},\quad h_{\Lambda\underline{\Lambda}}=\begin{pmatrix}h_{\Lambda i}\\ \overline{\mathcal{M}}_{\Lambda}\end{pmatrix}, \tag{7.43}$$

the period matrix is now introduced via the relations:

$$\overline{\mathcal{N}}_{\Lambda\Sigma}=h_{\Lambda\underline{\Lambda}}\circ(f^{-1})^{\underline{\Lambda}}{}_{\Sigma}. \tag{7.44}$$

The bosonic Lagrangian of $N=2$ gauged supergravity, which shares the same form as $N=6$ supergravity, can be expressed as follows:

$$e^{-1}\mathcal{L}=+\frac{1}{2}R-\nabla^{\mu}\phi^{i}\nabla_{\mu}\bar{\phi}_{i}-\frac{i}{4}(\mathcal{N}_{\Lambda\Sigma}F_{\mu\nu}^{+\Lambda}F^{+\Sigma\mu\nu}-\overline{\mathcal{N}}_{\Lambda\Sigma}F_{\mu\nu}^{-\Lambda}F_{\mu\nu}^{-\Sigma})-V, \tag{7.45}$$

where we define $D_\mu = \nabla_\mu - gA_\mu^M X_M$ as the covariant derivative implementing the minimal coupling of various fields, and ∇_μ is the usual space-time covariant derivative including the local $U(6)$ composite connection.

For gauging of $N = 2$ supergravity with gauge group $SO(2) \times SO(6)$, we can write the fermion shift matrix in the form of momentum map $\mathcal{P}_\Lambda$ and killing vector k_Λ^i by given

$$S_{\alpha\beta} = \frac{i}{2} (\sigma^x)_\alpha{}^\gamma \epsilon_{\beta\gamma} \mathcal{P}_\Lambda{}^x \mathcal{L}^\Lambda \tag{7.46}$$

$$W^{i\alpha\beta} = i (\sigma^x)_\gamma{}^\beta \epsilon^{\gamma\alpha} \mathcal{P}^x{}_\Lambda \bar{f}^{\Lambda i} \tag{7.47}$$

$$W^i = \frac{1}{2} \bar{\mathcal{L}}^\Lambda k_\Lambda^i = -\frac{i}{2} \bar{f}^{\Lambda i} \mathcal{P}_\Lambda. \tag{7.48}$$

Here, $\alpha, \beta = 1, 2$ represents the index of $SO(2)$ group. The complex conjugate of the fermion shift matrix is denoted by the asterisk $*$. The scalar potential $V(\phi)$, as given in equation (7.19), can be expressed in terms of these fermion shift matrices:

$$V^{(N=2)} = -6S^{\alpha\beta} S_{\alpha\beta} + \frac{1}{2} W^{i\alpha\beta} W^{*i}{}_{\alpha\beta} + W^i W^{*i}. \tag{7.49}$$

In this expression, the index A, B is mapped to i. The term $W^i W^{*i}$ represents the contraction of the complex conjugate of W^i with W^i, and similarly for the other terms.

In the gauging procedure, we need to find the form of embedding tensor in equation (7.25), which decompose under compact symmetry group $SU(6) \times SU(2) \times U(1)$, given by

$$\begin{aligned}(\mathbf{352}, \mathbf{1}) + (\mathbf{32}, \mathbf{3}) \longrightarrow\ & (\mathbf{35}, \mathbf{1})_{+\mathbf{3}} + (\mathbf{21} + \mathbf{15} + \mathbf{105}, \mathbf{1})_{+\mathbf{1}} + (\overline{\mathbf{21}} + \overline{\mathbf{15}} + \overline{\mathbf{105}}, \mathbf{1})_{-\mathbf{1}} \\ & + (\overline{\mathbf{35}}, \mathbf{1})_{-\mathbf{3}} + (\mathbf{1}, \mathbf{3})_{+\mathbf{3}} + (\mathbf{15}, \mathbf{3})_{+\mathbf{1}} + (\mathbf{1}, \mathbf{3})_{-\mathbf{3}} + (\mathbf{15}, \mathbf{3})_{-\mathbf{1}}.\end{aligned} \tag{7.50}$$

The correspondence of the above representations with the fermion shifts introduced in (7.12) and $((7.16)$ is:

$$N = 6 : (\mathbf{35}, \mathbf{1})_{+\mathbf{3}} \equiv N_B{}^A, \quad (\overline{\mathbf{21}} + \overline{\mathbf{105}} + \overline{\mathbf{15}}, \mathbf{1})_{-\mathbf{1}} \equiv (S^{AB}, N_D{}^{ABC}) \tag{7.51}$$

$$N = 2 : (\mathbf{1}, \mathbf{3})_{+\mathbf{3}} \equiv S^{\alpha\beta}, \quad (\overline{\mathbf{15}}, \mathbf{1})_{-\mathbf{1}} + (\overline{\mathbf{15}}, \mathbf{3})_{-\mathbf{1}} \equiv N_\beta{}^{\alpha AB}. \tag{7.52}$$

From equation (7.52), we write $N_{\beta}{}^{\alpha AB}$ in the term of W^{AB} and $W^{AB\alpha\beta}$ by given

$$N_{\beta}{}^{\alpha AB} = -c_1 \delta^{\alpha}_{\beta} W^{AB} + c_2 i (\sigma^x)^{\alpha}{}_{\beta} W^{ABx}, \tag{7.53}$$

where $(\sigma^x)^{\alpha}{}_{\rho} = (\sigma^x)_{\gamma}{}^{\beta} \epsilon^{\gamma\alpha} \epsilon_{\beta\rho}$. When we substitute equation (7.53) into the scalar potential (7.19) and map the index $[A, B]$ to i, we obtain the scalar potential (7.49). Therefore, by using the expressions for the embedding tensor and the corresponding fermion shifts, the scalar potential (7.19) is indeed consistent with the scalar potential (7.49) when mapping the index $[A, B]$ to i.

The explicit form of momentum map $\mathcal{P}_{\Lambda}$ and killing vector k^i_{Λ} are written as

$$\mathcal{P}_{\Lambda} = f_{\Lambda\Sigma}{}^{\Delta} (\mathcal{M}_{\Delta} \overline{\mathcal{L}} + \overline{\mathcal{M}}_{\Delta} \mathcal{L}^{\Sigma}) \tag{7.54}$$

$$k_{\Lambda}{}^{i} = i f_{\Lambda\Sigma}{}^{\Delta} (\overline{f}^{\Sigma i} \mathcal{M}_{\Delta} + \mathcal{L}^{\Sigma} \overline{h}_{\Delta}{}^{i}), \tag{7.55}$$

where $f_{\Lambda\Sigma}{}^{\Delta}$ is structure constant of $SO(6)$ gauge group.The momentum map $\mathcal{P}_{\Lambda}{}^{x}$ is a constant vector that is related to the embedding tensor $\mathcal{P}_{M}{}^{x}$, which map global symmetry group $SU(2)$ into gauge group. The quadratic conditions impose that

$$\mathcal{P}_{M}{}^{x} \mathcal{P}_{N}{}^{y} \epsilon_{xy}{}^{z} = 0. \tag{7.56}$$

Since the gauge generator of $SO(2)$ is abelian, $X_{MN}{}^{P} = 0$, the only non-zero momentum map is $\mathcal{P}_{0}{}^{x}$, which is given by

$$\mathcal{P}_{0}{}^{x} = \delta^{x}_{1}, \quad \mathcal{P}_{\Lambda}{}^{x} = 0 \quad \text{for} \quad \Lambda \neq 0. \tag{7.57}$$

The gauging of the $SU(2)$-symmetry by Abelian gauge group ($X_{MN}{}^{P} = 0$) are known as Fayet-Iliopoulos (FI) gauging.

In the case of a non-Abelian gauge group $SO(6)$, we calculate fermion-shift matrix using equation (7.54) or (7.55), where the index $\Lambda, \Sigma, \ldots$ are mapped into $SO(6)$ indices $[I, J]$ with $I, J = 1, \ldots, 6$. The structure constants of $SO(6)$ can be expressed in terms of the gauged generators (7.38) $(f_{\Lambda\Sigma}{}^{\Delta} \to X_{I_1 J_1, I_2 J_2}{}^{I_3 J_3})$, giving us

$$\mathcal{P}_{I_1 J_1} = 2g \left[\delta_{J_1 I} (f^{IJ} \bar{h}_{I_1 J} + h_{I_1 J} \bar{f}^{IJ}) - \delta_{I_1 I} (f^{IJ} \bar{h}_{J_1 J} + h_{J_1 J} \bar{f}^{IJ}) \right] \tag{7.58}$$

$$k_{I_1 J_1}{}^{i} = -2gi \left[\delta_{I_1 I} (f^{IJ} \bar{h}^{i}_{J_1 J} + h_{J_1 J} \bar{f}^{IJi}) - \delta_{J_1 I} (f^{IJ} \bar{h}^{i}_{I_1 J} + h_{I_1 J} \bar{f}^{IJi}) \right], \tag{7.59}$$

where $f^{IJ} = f^{IJ}{}_0 = \overline{\mathcal{L}}^{IJ}$, and $h_{IJ} = h_{IJ0} = \overline{\mathcal{M}}_{IJ}$.

Supergravity transformation rules of the Fermi fields can be written as

$$\delta\psi_{\mu\alpha} = D_\mu \epsilon_\alpha + \left[H^-{}_{\mu\nu}\epsilon_{\alpha\beta} + iS_{\alpha\beta}\eta_{\mu\nu}\right]\gamma^\nu \epsilon^\beta \tag{7.60}$$

$$\delta\chi^{\alpha i} = iD_\mu \phi^i \gamma^\mu \epsilon^\alpha + \left[(H^{-i}{}_{\mu\nu}\gamma^{\mu\nu} + W^i)\epsilon^{\alpha\beta} + W^{i\alpha\beta}\right]\epsilon_\beta, \tag{7.61}$$

where $H^-{}_{\mu\nu}$ and $H^{-i}{}_{\mu\nu}$ are field strengths associated with the gauge fields, which will vanish in the BPS equations. The exact expressions for $H^-{}_{\mu\nu}$ and $H^{-i}{}_{\mu\nu}$ depend on the specific model that can be found in (Trigiante, 2017).

7.4 Holographic RG flows

If conformal fixed points are deformed by the operator $\mathcal{O}_\Delta$, RG flows can occur, leading to either another conformal fixed point or a field theory that no longer possesses conformal symmetry. In the context of gauged supergravity, the solutions that correspond to the conformal field theory (CFT) can be found in the form of domain walls, as given by equation (5.35). In this framework, the beta function can be expressed as

$$\beta^s = \frac{d\phi^s}{dr}, \tag{7.62}$$

where ϕ^s are functions of r determined by the conditions that preserve supersymmetry. These functions describe the RG flow of the scalar fields along the domain wall solution and play a crucial role in understanding the deformation of conformal field theories.

To find the domain wall solution in gauged supergravity, we start by considering the BPS equation. In a vacuum that preserves the Lorentz symmetry of the supergravity theory, the background fields that do not vanish must be scalas which are singlets under the Lorentz group. Let $\phi_0^s(x) = \langle \phi^s(x) \rangle$ represent the vacuum expectation value of the scalar field $\phi^s(x)$. The vacuum that preserves Poincare symmetry under translations of ϕ_0^s is independent of spacetime coordinates. In this case, the field equation for $\phi^s(x) = \phi_0^s$ becomes

$$\frac{\partial V}{\partial \phi^s}|_{\phi^s=\phi_0^s} = 0, \tag{7.63}$$

which imposes the condition that the scalar potential $V(\phi)$ has a critical point, corresponding to a conformal fixed point by the conjecture of AdS/CFT. The scalar potential at vacuum $V_0 = V(\phi_0)$ is also called as cosmological constant.

In a vacuum state $|0\rangle$ that preserves supersymmetry ($\bar{\epsilon}^A Q_A |0\rangle = 0$), the supersymmetry transformation of any field $\Phi(x)$ is given by

$$\delta\Phi(x) = \langle 0| [\bar{\epsilon}^A Q_A, \hat{\Phi}(x)] |0\rangle = 0, \tag{7.64}$$

where $\hat{\Phi}(x)$ represents the field operator associated with $\Phi(x)$. This implies that vacuum preserves supersymmetry if supersymmetry transformations of all field vanish.

Indeed, in a supersymmetric vacuum, the fermion fields must vanish, resulting in zero supersymmetry transformations for the Bosonic fields. The remaining condition can be expressed as

$$\delta\Phi_F(x) = 0. \tag{7.65}$$

To verify the preservation of supersymmetry in a vacuum, one can examine the scalar potential at that vacuum point. For supersymmetry to be preserved, the scalar potential $V(\phi_0)$ evaluated at the vacuum expectation values ϕ_0 of the scalar potential should be negative, i.e.,

$$V_0 = V(\phi_0) < 0. \tag{7.66}$$

This condition ensures that there is a balance between the bosonic and fermionic degrees of freedom, indicating the presence of unbroken supersymmetry.

Therefore, at the vacuum, the only non-vanishing bosonic field is the scalar field. The supersymmetry transformations of the fermionic fields, $\psi_{A\mu}$ and λ_I, can be written as

$$\delta\psi_{A\mu} = \nabla_\mu \epsilon_A - g S_{AB} \gamma_\mu \epsilon^B = 0 \tag{7.67}$$

$$\delta\lambda_I = g {N_I}^A \epsilon_A = 0, \tag{7.68}$$

where S_{AB} and ${N_I}^A$ are the matrices that depend on the scalar field $\phi^s(x)$ evaluated at the vacuum expectation values ϕ_0^s. These equations represent the supersymmetry transformations set to zero, and they are known as the BPS equations.

The domain wall metric given by equation (5.35) can be written in terms of the vielbein as follows:

$$e^{\hat{\mu}} = e^{A(r)} dx^\mu, \quad e^{\hat{r}} = dr, \tag{7.69}$$

where $\hat{\mu} = 0, 1, 2$ represent the space time indices associated with the three-dimensional coordinates.

The non-vanishing components of the spin connection can be determined using the relation

$$\omega^{\mu r} = e^\nu \left(\partial_\nu e^r_\mu - \Gamma^\rho_{\nu\rho} e^r_\mu \right), \tag{7.70}$$

where $\Gamma^\sigma_{\nu\rho}$ are the Christoffel symbols associated with the metric. For the domain wall metric, we find that the non-vanishing component of the spin connection is given by

$$\omega^{\hat{\mu}\hat{r}} = A' e^{\hat{\mu}}, \tag{7.71}$$

where A' is the derivative of the warp factor $A(r)$ with respect to r.

In the from of Majorana spinor $\tilde{\epsilon}$, we can write parameter ϵ^α and ϵ_α as

$$\epsilon_\alpha = \frac{1}{2}(1 + \gamma_5)\tilde{\epsilon}^A, \quad \epsilon^\alpha = \frac{1}{2}(1 - \gamma_5)\tilde{\epsilon}^A, \tag{7.72}$$

by using Majorna representation that provide all gamma matrices γ^a are real, and γ_5 is imaginary. Therefore, the relation between ϵ^α and ϵ_α is $\epsilon_\alpha = (\epsilon^\alpha)^*$.

To satisfy the condition $\delta\lambda^{\alpha i} = 0$, we introduce the projection condition:

$$\gamma^{\hat{r}}\epsilon_\alpha = e^{i\Lambda}\epsilon^\alpha, \tag{7.73}$$

where Λ is a real function. This condition relates the spinor components along the radial projection. Taking the conjugate of this condition, we have:

$$\gamma^{\hat{r}}\epsilon^\alpha = e^{-i\Lambda}\epsilon_\alpha \tag{7.74}$$

These conditions determine the behavior of the spinors along the radial direction and are used to find the BPS equation for the domain wall.

For convenience, we define superpotential $\mathcal{W}$ as eigenvalue of matrix $S_{\alpha\beta}$, given by

$$S_{\alpha\beta} = -\frac{i}{2}\mathcal{W}\delta_{\alpha\beta}. \tag{7.75}$$

The superportential depend on scalar field only, which we will show the result in next chapter.

By using the condition $\delta\psi_{\mu\alpha} = 0$ for $\mu = 0, 1, 2$, and employing the expression for the covariant derivative D_μ provided in the Appendix, we obtain the following equation:

$$\frac{1}{2}A'\gamma_{\hat{\mu}}\gamma_{\hat{r}}\epsilon_\alpha - \frac{1}{2}\mathcal{W}\gamma_{\hat{\mu}}\epsilon^\alpha = 0. \tag{7.76}$$

Next, we multiply this equation by $\gamma^{\hat{\mu}}$ and make use of the projection condition $\gamma^{\hat{r}}\epsilon_\alpha = e^{i\Lambda}\epsilon^\alpha$. This yields:

$$A'e^{i\Lambda} - \mathcal{W} = 0. \tag{7.77}$$

This equation provides a constraint on the warp factor $A(r)$ and the superpotential $\mathcal{W}$, relating them through the phase factor $e^{i\Lambda}$.

By writing the superpotential $\mathcal{W}$ in the form of its absolute value and complex phase as $\mathcal{W} = |\mathcal{W}|e^{i\omega} = We^{i\omega}$, we can separate the equation (7.77) into its imaginary and real parts:

$$e^{i\Lambda} = \pm e^{i\omega} = \pm\frac{|\mathcal{W}|}{W}, \quad \text{and} \quad A' = \pm W. \tag{7.78}$$

The previous relation gives the relation between the warp factor and scalar field. The projection condition $\gamma^{\hat{r}}\epsilon_{\alpha} = e^{i\Lambda}\epsilon^{\alpha}$ used in $\delta\lambda^{\alpha i}$ condition determines the dependence of the scalar fields ϕ^i on the coordinate r. The explicit forms of the warp factor $A(\phi)$ and the scalar fields $\phi^i(r)$ will be determined in the subsequent chapters. These solutions will provide a complete description of the domain wall configuration in the theory.

Chapter VIII

DOMAIN WALL SOLUTIONS

In this chapter, we will explore the solutions for the domain wall configurations using the relations derived in the previous chapter. These domain walls serve as connections between the supersymmetric AdS_4 vacuum and either another AdS_4 vacuum (if such a vacuum exists) or a singular geometry. These solutions represent RG flows in the dual UV $N = 2$ SCFT, indicating transitions either towards another conformal fixed point or into a non-conformal phase in the IR region.

8.1 Truncation of scalar fields

Truncation of scalar fields in gauged supergravity is a common technique used to simplify the calculations and focus on specific aspects of the theory. By selecting a subset of scalar fields to work with, the complexity of the equations can be significantly reduced, allowing for more manageable calculations and insights into the system's behavior.

The process of truncation involves setting some of the scalar fields to zero or neglecting their dynamics, effectively reducing the dimensionality of the scalar field space. This simplification is motivated by the fact that certain scalar fields may be less relevant to the specific phenomena under investigation or may have negligible contributions to the observables of interest.

The truncation method based on symmetry considerations allows us to focus on a specific subset of scalar fields that are singlets under a chosen subgroup of the gauge symmetry. By selecting these singlet fields and setting the remaining fields to zero, we simplify the analysis while preserving the self-consistency of the truncation.

In this approach, we start by considering a critical point of the scalar potential, denoted by $(\phi_0^i, \chi_0^a = 0)$, where ϕ_0^i represents the scalar fields that are singlets under the chosen subgroup $H_0 \subset SO(2) \times SO(6)$ and χ_0^a are the remaining scalar fields in arbitrary representations. We then expand the scalar potential $V(\phi, \chi)$ around this critical point as

$$V(\phi, \chi) = V_0 + \frac{\partial V}{\partial \phi^i}\bigg|_{\phi=\phi_0^i, \chi^a=0} (\phi^i - \phi_0^i) + \frac{\partial V}{\partial \chi^a}\bigg|_{\phi^i=\phi_0^i, \chi^a=0} \chi^a + ..., \tag{8.1}$$

where $V_0 = V(\phi_0, \chi_0 = 0)$ is the value of scalar potential at critical point.

The scalar potential for H_0 singlet scalars admits the aforementioned critical point if the following condition is satisfied:

$$\frac{\partial V}{\partial \phi^i}\bigg|_{\phi=\phi_0^i, \chi^a=0} = 0. \tag{8.2}$$

This condition ensures that the potential is stationary at the critical point $(\phi_0^i, \chi^a = 0)$ with respect to variations in the singlet scalar fields ϕ^i. Additionally, since χ^a are non-singlets of H_0, in order to obtain the scalar potential that is H_0 invariant, we must have $\frac{\partial V}{\partial \chi^a}\big|_{\phi=\phi_0, \chi=0} = 0$. Therefore, the critical points found within the H_0 singlet scalars are essentially the critical points of scalar potential on the full scalar manifold.

In the upcoming sections, we will explore solutions that describe RG flows by considering the singlet scalar ϕ^i under $SO(2) \times SO(4)$ and $U(3)$, which are subgroup of $SO(2) \times SO(4)$. By focusing on these fields, we can study the transitions between different phases in the dual UV $N = 2$ SCFT.

8.2 SO(2)xSO(4) singlet scalars

For $SO(2) \times SO(4) \subset SO(6)$ symmetry, we can express the 15 complex scalars ϕ^i as ϕ_{AB}, of the form

$$\phi_{AB} = \phi(\delta^1_A \delta^2_B - \delta^1_B \delta^2_A). \tag{8.3}$$

To further analyze the scalar fields, we write the complex scalars ϕ as:

$$\phi = \varphi e^{i\zeta}, \tag{8.4}$$

where φ and ζ are real scalars. The domain wall also imposes that the scalar fields depend only on the radial coordinate r, such that we can write $\varphi(r)$ and $\zeta(r)$.

From equation (7.27) – (7.32), we can define the component of matrices in equation (7.43) for $SO(2) \times SO(4)$ singlet. In this result, the scalar potential can be written as

$$V = -\frac{1}{2}\xi^2[\cosh(2\varphi) + 2]. \tag{8.5}$$

It is noteworthy that the above scalar potential arises solely from the term associated with the coupling constant ξ corresponding to the $SO(2)$ gauge group. The term involving the coupling constant g associated with the $SO(6)$ gauge group vanishes. This vanishing result is obtained from the calculation of equation (7.58), which is equal to zero, and from $\overline{\mathcal{L}}^{\Lambda} k^i_{\Lambda} = 0$. Hence, the gauging of $SO(6)$ has no effect on the scalar potential (8.5) for the $SO(2) \times SO(4)$ singlet.

The scalar potential (8.5) has a critical point at $\varphi = 0$, which corresponds to a vanishing value for the scalar field. At this critical point, the potential takes the value $V_0(\varphi = 0) = -\frac{3}{2}\xi^2$. This implies that the model, in accordance with the AdS/CFT correspondence, suggests a dual description in terms of an $N = 2$ superconformal field theory (SCFT) in three dimensions. The negative cosmological constant $V_0(\varphi = 0)$ is a characteristic feature of the AdS geometry.

The fermions shift matrix $S_{\alpha\beta}$ can be written as,

$$S_{\alpha\beta} = -\frac{i\xi \cosh(\varphi)}{2\sqrt{2}}\delta_{\alpha\beta}, \tag{8.6}$$

so that the superpotential defined in (7.75) is expressed as

$$\mathcal{W} = \frac{\xi \cosh(\varphi)}{2\sqrt{2}}. \tag{8.7}$$

Therefore, we can further simplify equation (7.78) to:

$$A' = \frac{\xi \cosh(\varphi)}{\sqrt{2}}, \quad \text{and} \quad e^{i\Lambda} = 1. \tag{8.8}$$

We choose the positive value for A' to ensure that the supersymmetric AdS_4 critical point corresponds to $r \to \infty$.

The BPS equation corresponding to equation $\delta\lambda^{\alpha i}$, after imposing the projector (7.73) given by:

$$\varphi' = -\frac{\xi \sinh(\varphi)}{\sqrt{2}}, \quad \text{and} \quad \zeta' = 0. \tag{8.9}$$

These equations (8.8) and (8.9) represent the BPS equations that satisfy all the supersymmetry conditions. Furthermore, it can be confirmed that these equations also imply the second-order field equations.

The BPS equations (8.8) and (8.9) can be analytically solved, yielding the following solutions:

$$\xi r = \sqrt{2}[\ln(1+e^{\varphi}) - \ln(1-e^{\varphi})] \tag{8.10}$$

$$A = \varphi - \ln\left(1 - e^{2\varphi}\right) \tag{8.11}$$

In these equations, we have neglected the constants as they can be absorbed by shifting the radial coordinate and rescaling the $x^{0,1,2}$ coordinates, respectively. In the limit $r \to \infty$, we find that

$$\varphi \sim e^{-\frac{\xi}{\sqrt{2}}r} \sim e^{-\frac{r}{L}} \quad \text{and} \quad A \sim \frac{\xi}{\sqrt{2}}r \sim \frac{r}{L} \tag{8.12}$$

where L is the AdS_4 radius associated with the cosmological constant, given by

$$L = \sqrt{-\frac{3}{V_0}} = \frac{\sqrt{2}}{\xi} \tag{8.13}$$

for $\xi > 0$.

In the supersymmetric AdS_4 vacuum, all scalars have masses $m^2L^2 = -2$, which corresponds to operators of dimensions $\Delta = 1, 2$ in the dual $N = 2$ SCFT. These operators are given by scalar and fermion bilinears (mass terms), respectively. The mass term for scalars corresponds to an operator with dimension $\Delta = 1$, while the mass term for fermions corresponds to an operator with dimension $\Delta = 2$. These mass terms play an important role in the dynamics of the dual SCFT and contribute to the spectrum and correlation functions of the theory.

In the limit $r \to 0$, the solutions in equations (8.10) and (8.11) become:

$$\varphi \sim \pm \ln\left(\frac{\xi}{\sqrt{2}}r\right) \quad \text{and} \quad A \sim \ln\left(\frac{\xi}{\sqrt{2}}r\right), \tag{8.14}$$

This implies that φ diverges as $\varphi \sim \pm\infty$ near the singularity. The solution then describes an RG flow from the UV $N = 2$ SCFT to a non-conformal phase in the IR. At $r \to \infty$, the solution approaches AdS_4 space, while at $r \to 0$, the solution is singular. The flow is driven by an operator of dimensions $\Delta = 1, 2$, corresponding to scalar or fermion mass terms in three dimensions. This non-conformal phase in the IR reflects the spontaneous breaking of the conformal symmetry and the emergence of new dynamics in the low-energy regime of the theory.

To ensure that the singularity in the IR of the supergravity solution correctly corresponds to a sensible RG flow in the quantum field theory (QFT), we can examine the scalar potential and use the criterion based on (Gubser, 2000). By rewriting the scalar potential as:

$$V = -\frac{\xi^2}{4}(e^{-2\varphi} + e^{2\varphi} + 4), \tag{8.15}$$

we observe that at the singularity $\varphi \to \pm\infty$, the scalar potential diverges to negative infinity ($V \to -\infty$). This indicates that the singularity is physically meaningful in the context of the dual QFT.

8.3 U(3) singlet scalars

In this section, we consider scalar fields that are invariant under the $U(3) \sim SU(3) \times U(1) \subset SO(6)$. These scalar can be written in the form:

$$\phi_{AB} = \begin{pmatrix} 0_{3\times 3} & \phi \mathbb{I}_{3\times 3} \\ -\phi \mathbb{I}_{3\times 3} & 0_{3\times 3} \end{pmatrix} = \phi J_{AB}, \tag{8.16}$$

where $\mathbb{I}_{3\times 3}$ is the 3×3 identity matrix, and J_{AB} is the Kähler form of CP^3.

By using the form of scalar (8.4), we obtain the scalar potential as

$$V = -\frac{3}{2}\xi^2 \cosh(2\varphi). \tag{8.17}$$

This potential has a critical point at $\varphi = 0$ and preserves supersymmetry, with the vacuum energy $V_0 = -\frac{3}{2}\xi^2$. This critical point is dual to an $N = 2$ SCFT in three dimensions as in the previous case.

The superpotential, defined as in equation (7.75), can be written as:

$$\mathcal{W} = \frac{\xi}{4\sqrt{2}} e^{-3\varphi} \left[\left(e^{2\varphi} + 1\right)^3 + e^{-3i\zeta} \left(e^{2\varphi} - 1\right)^3 \right]. \tag{8.18}$$

It should be noted that in this case, the superpotential is complex. The condition $\delta\chi^{\alpha i} = 0$ provide

$$e^{-i\Lambda}(\varphi' \pm \varphi\zeta') = -\frac{\xi}{8\sqrt{2}} e^{-3\varphi - 2i\zeta} \left(e^{4\varphi} - 1\right) \left[\left(1 + e^{3i\zeta}\right) e^{2\varphi} - e^{3i\zeta} + 1 \right], \tag{8.19}$$

which imply that $\zeta' = 0$ for $\varphi(r) \neq 0$.

The BPS equations for the scalar singlet are given by

$$\zeta' = 0 \tag{8.20}$$

$$A' = \frac{\xi e^{-\varphi}\left(e^{4\varphi} + 3\right)}{2\sqrt{2}} \tag{8.21}$$

$$\varphi' = -\frac{\xi e^{-\varphi}\left(e^{4\varphi} - 1\right)}{2\sqrt{2}}. \tag{8.22}$$

The solution to these equations is:

$$\frac{\xi}{\sqrt{2}}r = \tan^{-1} e^{\varphi} - \ln(1-e^{\varphi}) + \ln(1+e^{\varphi}) \tag{8.23}$$

$$A = \frac{3\varphi}{2} - \frac{1}{2}\ln\big(1-e^{4\varphi}\big). \tag{8.24}$$

Indeed, the behavior of the solution confirms the presence of singularities at $r = 0$. As r approaches 0:

$$\varphi \sim \ln(\xi r) \quad \longrightarrow \quad A \sim 3\ln(\xi r) \tag{8.25}$$

$$\varphi \sim -\ln(\xi r) \quad \longrightarrow \quad A \sim \ln(\xi r), \tag{8.26}$$

which leads to the scalar potential

$$V = \frac{-3}{4}\xi^2 e^{-2\varphi}\left(e^{4\varphi}+1\right) \to -\infty. \tag{8.27}$$

These singularities are physically meaningful, indicating a breakdown of the confor-mmal symmetry and suggesting the presence of non-conformal behavior or a phase transition in the dual field theory. The holographic dual solution describes RG flows from the UV $N = 2$ (SCFT) to non-conformal phases in the IR. As in the previous case, the RG flows are driven by an operator of dimensions $\Delta = 1, 2$.

Chapter IX

DISCUSSION

9.1 Conclusion

In this work, we have considered a model of $N = 2$ gauged supergravity with the gauge group $SO(2) \times SO(6)$, which is obtained by truncating $N = 8$ supergravity by removing the sixth gravitinos. This model has a holographic dual description to an $N = 2$ superconformal field theory (SCFT) in three dimensions. By studying the scalar fields invariant under $SO(2) \times SO(4)$ and $U(3)$, we have found domain wall solutions that capture the RG flows in the corresponding CFT.

The domain wall solutions exhibit interesting behavior, with the scalar fields providing the dynamics associated with the $SO(2)$ gauge group. The solutions reveal RG flows from the UV $N = 2$ SCFT to non-conformal phases in the IR. The scalar potentials of both singlet scalars have been shown to be physically sensible based on criteria established in (Gubser, 2000).

Overall, our work provides insights into the holographic description of RG flows in $N = 2$ SCFTs and sheds light on the strongly coupled dynamics of these theories in the infrared regime.

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Appendix I

ADDITIONAL THEORIES AND NOTATION

A.1 Electromagnetic duality

In a four-dimensional spacetime, the 2-form Hodge duality provides a 2-form. This allows us to define the dual tensor of the tensor $H_{\mu\nu}$ as

$$\tilde{H}_{\mu\nu} = -i(*H)_{\mu\nu} = -\frac{i}{2}\epsilon_{\mu\nu\rho\sigma}H^{\rho\sigma}, \tag{A.1}$$

where $*$ denotes the Hodge dual $*(*H) = *^2 H = -H$ and μ, ν are indices for the spacetime coordinates.

From the definition of $\tilde{H}_{\mu\nu}$, we can construct linear combinations

$$H^{\pm}_{\mu\nu} = \frac{1}{2}(H_{\mu\nu} \pm \tilde{H}_{\mu\nu}), \tag{A.2}$$

which satisfy $(H^{\pm})^*_{\mu\nu} = H^{\mp}_{\mu\nu}$, and

$$\tilde{H}^{\pm}_{\mu\nu} = -\frac{i}{2}\epsilon_{\mu\nu\rho\sigma}H^{\pm\rho\sigma} = \pm H^{\pm}_{\mu\nu}. \tag{A.3}$$

Tensor $H^{+}_{\mu\nu}$, and $H^{-}_{\mu\nu}$ are called self-dual tensor and anti-self-dual tensor, respectively.

Let $G_{\mu\nu}$ be another anti-symmetric tensor with $G^{\pm}_{\mu\nu}$ defined as in (A.2). We have the following relations:

$$G^{+}_{\mu\nu}H^{-\mu\nu} = 0 \quad \text{and} \quad \tilde{G}^{\mu\nu}\tilde{H}_{\mu\nu} = -G^{\mu\nu}H_{\mu\nu}. \tag{A.4}$$

Duality for one free electromagnetic field

In the simplest case, namely a single free gauge field, the Maxwell and Bianchi equations become

$$\partial_\mu F^{\mu\nu} = 0, \quad \text{and} \quad \partial_\mu \tilde{F}^{\mu\nu} = 0, \tag{A.5}$$

note that Bianchi equations is in the form $\partial_\mu \tilde{F}^{\mu\nu} = -\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\partial_{[\mu}F_{\rho\sigma]} = 0$.

In this form, the Maxwell and Bianchi equations take the same form. Furthermore, these equations also transform under

$$F^{\mu\nu} \longrightarrow F'^{\mu\nu} = i\tilde{F}^{\mu\nu}. \tag{A.6}$$

In the form of electric and magnetic fields, the previous transformation provides

$$E_i^{'} = -B_i \quad \text{and} \quad B_i^{'} = E_i, \tag{A.7}$$

where $E^i = F^{0i}$ and $B_i = \frac{1}{2}\epsilon_{ijk}F^{jk}$.

Furthermore, this symmetry cannot be elevated to the level of the vector potential, which suggests that the duality symmetry is only an on-shell symmetry.

A.2 Nonlinear sigma model

The nonlinear sigma model is a theory that describes the nonlinear interaction of n scalar fields $\phi^i(x)$,$i = 1, ...n$, expressed in terms of the action on Minkowski spacetime

$$S = -\frac{1}{2}\int d^4x g_{ij}(\phi)\partial_\mu\phi^i\partial_\nu\phi^j\eta^{\mu\nu}, \tag{A.8}$$

where g_{ij} is $n \times n$. In the event that $g_{ij}(\phi) = \delta_{ij}$, the action provides the theory that explains free scalar fields. The nonlinear sigma model can be used to describe the behavior of the pions, which are the lightest mesons composed of quarks and antiquarks. Pions are not fundamental particles, but are instead composite particles made up of quarks and gluons, and the nonlinear sigma model can be used to describe their interactions.

In mathematics, the field $\phi^i(x)$ is a mapping from the spacetime into a target manifold, which has a metric tensor $g_{ij}(\phi)$ and coordinates represented by ϕ^i. The variation of the action leads to the field equation, which can be expressed as

$$\partial_\mu \partial^\mu \phi^i + \Gamma^i_{jk} \partial_\mu \phi^j \partial^\mu \phi^k = 0, \tag{A.9}$$

where Γ^i_{jk} are the Christoffel symbols, defied as

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} (\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}), \tag{A.10}$$

where g^{il} is the inverse metric tensor, and ∂_j denotes the partial derivative with respect to the coordinate ϕ^j on the target manifold.

The field equation A.9 is covariant under the transformation

$$\phi^i \to \phi^{i'}(\phi) \tag{A.11}$$

if g_{ij} transforms as

$$g^{'}_{ij}(\phi^{'}) = \frac{\partial \phi^k}{\partial \phi^{i'}} \frac{\partial \phi^l}{\partial \phi^{j'}} g_{kl}(\phi) \tag{A.12}$$

This transformation is called a diffeomorphism on the target manifold, which is a smooth invertible map between two manifolds that preserves differentiable structure.

If metric tensor g_{ij} is invariant under transformation A.11, wriiten as

$$g^{'}_{ij}(\phi^{'}) = g_{ij}(\phi), \tag{A.13}$$

this transformation will be isometry of target manifold and symmetry of action. This symmetry can be written as

$$\phi^{i'} = \phi^i + \theta^a k^i_a(\phi). \tag{A.14}$$

Using the transformation law of the metric tensor A.12 and expanding A.13 to first order, we can express the variation of the metric tensor as:

$$\delta g_{ij}(\phi) = g^{'}_{ij}(\phi) - g_{ij}(\phi) = -\theta^a \mathscr{L}_{k_a} g_{ij}(\phi) = 0, \tag{A.15}$$

where $\mathscr{L}_{k_a} g_{ij}$ is Lie derivative in the direction of k^i_a defined as

$$\mathscr{L}_{k_a} g_{ij} = k^l_a \partial_l g_{ij} + \partial_i k^l_a g_{ij} + \partial_j k^l_a g_{li}. \tag{A.16}$$

Using the relation $\nabla_i g_{ik} = 0$, the condition of isometry transformation can be expreesed as

$$\nabla_i k_{aj} + \nabla_j k_{ai} = 0, \tag{A.17}$$

where

$$k_{ai} = k^j_a g_{ij} \quad \text{and} \quad \nabla_i k_{aj} = \partial_i k_{aj} - \Gamma^l_{ij} k_{al}. \tag{A.18}$$

The equation A.17 is called killing equation, and the generator k^i_a is killing vector.

Notice that equation A.14 is internal symmetry and transformation of ϕ becomes

$$\delta\phi^i = \theta^a k^i_a. \tag{A.19}$$

The action A.8 is invariant under this transformation, which can be verified by using relation $\delta g_{ij} = \partial_k g_{ij} \delta\phi^k$ and $\delta\partial_\mu\phi^i = \theta^a \partial_\mu k^i_a$. The variation of A.8 becomes

$$\begin{aligned}
\delta S &= -\frac{1}{2}\int d^4x\theta^a \left[\partial_k g_{ij} k^k_a \partial_\mu\phi^i\partial^\mu\phi^j + g_{ij}\partial_\mu\phi^k(\partial_k k^i_a \partial^\mu\phi^j + \partial_k k^j_a \partial^\mu\phi^i)\right] \\
&= -\frac{1}{2}\int d^4x\theta^a \mathscr{L}_{k_a} g_{ij}\partial_\mu\phi^i\partial^\mu\phi^j \\
&= -\frac{1}{2}\int d^4x\theta^a (\nabla_i k_{aj} + \nabla_j k_{ai})\partial_\mu\phi^i\partial^\mu\phi^j \\
&= 0
\end{aligned} \tag{A.20}$$

To find Neother current, using relation $\delta\phi^i = \theta^a\nabla_a\phi^i = \theta^a k^i_a$ shown that

$$J^\mu_a = \frac{\mathscr{L}}{\partial\partial_\mu\phi^i}\nabla_a\phi^i = g_{ij}k^j_a\partial^\mu\phi^i = k_{ai}\partial^\mu\phi^i. \tag{A.21}$$

A.3 Notations in the coset manifolds

SU(2) and Sp(2k) metrics

The flat SU(2) and Sp(2k) metrics satisfy:

$$\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = -\delta^{\alpha}_{\gamma}, \quad \epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}, \quad \epsilon^{12} = \epsilon_{12} = +1 \tag{A.22}$$

$$\mathbb{C}^{\alpha\beta}\mathbb{C}_{\beta\gamma} = -\delta^{\alpha}_{\gamma}, \quad \mathbb{C}^{\alpha\beta} = \mathbb{C} = -\mathbb{C}^{\beta\alpha}, \quad \mathbb{C}^{12} = \mathbb{C}_{21} = +1 \tag{A.23}$$

For any $SU(2)$ vector V_α we have:

$$\epsilon_{\alpha\beta}V^{\beta} = V_{\alpha}, \tag{A.24}$$

$$\epsilon^{\alpha\beta}V_{\beta} = -V^{\alpha} \tag{A.25}$$

Pauli matrices

The standard Pauli matrices $(\sigma^x)_\alpha{}^\beta$, which $x = 1, 2, 3$ can be written as

$$(\sigma^1)_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (\sigma^2)_\alpha{}^\beta = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (\sigma^3)_\alpha{}^\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.26}$$

The matrices with two indices at the same level are defined as follows:

$$(\sigma^x)_{\alpha\beta} = (\sigma^x)_\alpha{}^\gamma \epsilon_{\beta\gamma}, \tag{A.27}$$

$$(\sigma^x)^{\alpha\beta} = (\sigma^x)_\gamma{}^\beta \epsilon^{\alpha\gamma}, \tag{A.28}$$

which are related by the relation:

$$(\sigma^x_{\alpha\beta})^* = -(\sigma^x)^{\alpha\beta}. \tag{A.29}$$

Finally, we define $(\sigma^x)^\alpha{}_\beta$ as

$$(\sigma^x)^\alpha{}_\beta = -\epsilon^{\alpha\gamma}(\sigma^x)_{\gamma\beta}. \tag{A.30}$$

N = 6 supergravity coset manifold

In the previous review, it was established that the bosonic sector of $N = 2$ supergravity is equivalent to $N = 6$ supergravity, which consists of 30 scalar fields that parametize the coordinates of the coset manifolds $SO^*(12)/U(6)$. In this section, we will examine additional relations, which were not presented in the previous chapter.

The complex self-dual and anti-self-dual gauge field strengths are defined by

$$F^{\pm\Lambda}_{\mu\nu} = \frac{1}{2}\left(F^{\Lambda}_{\mu\nu} \pm \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}F^{\Lambda\rho\sigma}\right) \tag{A.31}$$

with $F^{\Lambda}_{\mu\nu}$ given by

$$F^{\Lambda}_{\mu\nu} = \partial_\mu A^{\Lambda}_{\nu} - \partial_\nu A^{\Lambda}_{\mu} + X_{\Gamma\Sigma}{}^{\Lambda}A^{\Gamma}_{\mu}A^{\Sigma}_{\nu}. \tag{A.32}$$

The chiralities of the fermionic fields

$$\gamma_5\psi_{\mu\alpha} = -\psi_{\mu\alpha}, \quad \gamma_5\chi_{\alpha i} = -\chi_{\alpha i}. \tag{A.33}$$

The tensors $\hat{F}^{+}_{\mu\nu AB} = \left(\hat{F}^{-AB}_{\mu\nu}\right)^*$ can be obtained from

$$\hat{F}^{-AB}_{\mu\nu} = \mathcal{V}_M{}^{AB}G^{-M}_{\mu\nu} \tag{A.34}$$

with

$$G^{M}_{\mu\nu} = \begin{pmatrix} F^{\Lambda}_{\mu\nu} \\ G_{\Lambda\mu\nu} \end{pmatrix} \tag{A.35}$$

and $G_{\Lambda\mu\nu} = i\epsilon_{\mu\nu\rho\sigma}\frac{\partial\mathcal{L}}{\partial F_{\rho\sigma}}$. Similarly, we have $\hat{F}^{+}_{\mu\nu} = \left(\mathcal{V}^{0}_{M}G^{-M}_{\mu\nu}\right)^*$.

The covariant derivative of ϵ_A is defined by

$$D_\mu\epsilon_A = \partial_\mu\epsilon_A + \frac{1}{4}\omega^{ab}_{\mu}\gamma_{ab}\epsilon_A + \frac{1}{2}Q^{B}_{\mu A}\epsilon_B. \tag{A.36}$$

The connection $Q_{\mu A}{}^{B}$ is given by

$$Q_{\mu A}{}^{B} = \frac{2i}{3}\left(h_{\Lambda AC}\partial_\mu\bar{f}^{\Lambda AB} - f^{\Lambda}{}_{AC}\partial_\mu\bar{h}_{\Lambda}{}^{BC}\right) - gA^{M}_{\mu}Q_{MA}{}^{B} \tag{A.37}$$

with $Q_{MA}{}^{B}$ obtained from

$$Q_{MAB}{}^{CD} = \mathcal{V}_{AB}{}^{P} X_{MP}{}^{N} \mathcal{V}_{N}{}^{CD} \tag{A.38}$$

by the relation $Q_{MAB}{}^{CD} = 4\delta^{[C}_{[A} Q_{MB]}{}^{D]}$.

A.4 N = 2 Momentum Map and Killing Vector

From gauge generator:

$$X_{I_1J_1,I_2J_2}{}^{I_3J_3} = 4g\delta^{[I_3}_{[I_1}\delta_{J_1][I_2}\delta^{J_3]}_{J_2]}, \quad X_{I_1I_2}{}^{I_3J_3}{}_{I_2J_2} = -X_{I_1J_1,I_2J_2}{}^{I_3J_3}. \tag{A.39}$$

we can show that

$$\mathcal{P}_{I_1J_1} = 2g\left[\delta_{J_1I}(f^{IJ}\bar{h}_{I_1J} + h_{I_1J}\bar{f}^{IJ}) - \delta_{I_1I}(f^{IJ}\bar{h}_{J_1J} + h_{J_1J}\bar{f}^{IJ})\right] \tag{A.40}$$

$$k_{I_1J_1}{}^{i} = -2gi\left[\delta_{I_1I}(f^{IJ}\bar{h}^{i}_{J_1J} + h_{J_1J}\bar{f}^{IJi}) - \delta_{J_1I}(f^{IJ}\bar{h}^{i}_{I_1J} + h_{I_1J}\bar{f}^{IJi})\right] \tag{A.41}$$

by the following. First, we start with the expanding of (A.39) written as

$$\begin{aligned} X_{I_1J_1,I_2J_2}{}^{I_3J_3} =& \frac{g}{2}\Big[\delta_{J_1I_2}(\delta^{I_3}_{I_1}\delta^{J_3}_{J_2} - \delta^{J_3}_{I_1}\delta^{I_3}_{J_2}) + \delta_{J_1J_2}(\delta^{J_3}_{I_1}\delta^{I_3}_{I_2} - \delta^{I_3}_{I_1}\delta^{J_3}_{I_2}) \\ &+ \delta_{I_1I_2}(\delta^{J_3}_{J_1}\delta^{I_3}_{J_2} - \delta^{I_3}_{J_1}\delta^{J_3}_{J_2}) + \delta_{I_1J_2}(\delta^{I_3}_{J_1}\delta^{J_3}_{I_2} - \delta^{J_3}_{J_1}\delta^{I_3}_{I_2})\Big], \end{aligned} \tag{A.42}$$

and substitute (A.42) into (4.54):

$$\begin{aligned} \mathcal{P}_{I_1J_1} =& X_{I_1J_1,I_2J_2}{}^{I_3J_3}(\overline{h}_{I_3J_3}f^{I_2J_2} + h_{I_3J_3}\overline{f}^{I_2J_2}) \\ \mathcal{P}_{I_1J_1} =& \frac{g}{2}\Big[\delta_{J_1I_2}(\delta^{I_3}_{I_1}\delta^{J_3}_{J_2} - \delta^{J_3}_{I_1}\delta^{I_3}_{J_2}) + \delta_{J_1J_2}(\delta^{J_3}_{I_1}\delta^{I_3}_{I_2} - \delta^{I_3}_{I_1}\delta^{J_3}_{I_2}) \\ &+ \delta_{I_1I_2}(\delta^{J_3}_{J_1}\delta^{I_3}_{J_2} - \delta^{I_3}_{J_1}\delta^{J_3}_{J_2}) + \delta_{I_1J_2}(\delta^{I_3}_{J_1}\delta^{J_3}_{I_2} - \delta^{J_3}_{J_1}\delta^{I_3}_{I_2})\Big]\overline{h}_{I_3J_3}f^{I_2J_2} \\ &+ \frac{g}{2}\Big[\delta_{J_1I_2}(\delta^{I_3}_{I_1}\delta^{J_3}_{J_2} - \delta^{J_3}_{I_1}\delta^{I_3}_{J_2}) + \delta_{J_1J_2}(\delta^{J_3}_{I_1}\delta^{I_3}_{I_2} - \delta^{I_3}_{I_1}\delta^{J_3}_{I_2}) \\ &+ \delta_{I_1I_2}(\delta^{J_3}_{J_1}\delta^{I_3}_{J_2} - \delta^{I_3}_{J_1}\delta^{J_3}_{J_2}) + \delta_{I_1J_2}(\delta^{I_3}_{J_1}\delta^{J_3}_{I_2} - \delta^{J_3}_{J_1}\delta^{I_3}_{I_2})\Big]h_{I_3J_3}\overline{f}^{I_2J_2} \\ =& 2g\left[\delta_{J_1I}(f^{IJ}\bar{h}_{I_1J} + h_{I_1J}\bar{f}^{IJ}) - \delta_{I_1I}(f^{IJ}\bar{h}_{J_1J} + h_{J_1J}\bar{f}^{IJ})\right]. \end{aligned} \tag{A.43}$$

In the last step we use the property of ant-symmetric tensor $f^{IJ} = -f^{JI}$. For killing vector, we obtain from (4.55) shown as

$$k_{I_1J_1}{}^{i} = iX_{I_1J_1,I_2J_2}{}^{I_3J_3}(\overline{f}^{I_2J_2i}\overline{h}_{I_3J_3} + \overline{f}^{I_2J_2}\overline{h}_{I_3J_3}{}^{i}), \tag{A.44}$$

and follow the calculation as the same as (A.43) to get (A.40)

Appendix II

LIST OF PUBLICATIONS

B.1 International Conference Proceeding

1. Nutthaphat Lunrasri (2023). Holographic RG flows from four-dimensional N = 2 gauged supergravities. In 2023 18th Siam Physics Congress.

Biography

Nutthaphat Lunrasri was born on September 30, 1996, in Banphai, Khon Kean, Thailand, his fascination with the intricacies of the universe led him to pursue a career in physics. His goal is to unravel the mysteries behind the origin and functioning of the universe itself.

After completing his education at Banphai School, Nutthaphat Lunrsri enrolled at Khon Kean University, where he obtained a Bachelor of Science degree in physics. During this phase of his education, he delved into the realm of cosmology, seeking to elucidate one of the universe's enigmatic phenomena—Dark Matter. His undergraduate thesis focused on the fields of modified gravity and cosmology, with the aim of constructing models that provide an explanation for Dark Matter. The title of his thesis was "Dark Matter from Kaluza-Klein theory."

In 2019, Nutthaphat Lunrsri pursued further academic pursuits by enrolling in a Master's degree program in theoretical physics at Chulalongkorn University. Here, his academic journey converged with his passion for string theory, supergravity, and cosmology—disciplines renowned for their profound contributions to our understanding of the fundamental principles governing the universe's existence and evolution.