Chapter III

Results about the logic

Many results in traditional predicate logic are preserved in our logic but not the Completeness theorem. In this chapter, we will state and prove some of them and show why Completeness theorem is not true.

3.1 Basic results on formal proofs.

We start with a proposition listing a few basic results about proofs.

Proposition 3.1.1 *Let Σ and Γ be sets of formulas, and let τ be a formula.*

(i) $\Sigma \vdash \sigma$ *for all* $\sigma \in \Sigma$.

(ii) If $\Sigma \vdash \tau$ *and* $\Sigma \subseteq \Gamma$, *then* $\Gamma \vdash \tau$ *also.*

(iii) If $\Sigma \vdash \gamma$ *for all* $\gamma \in \Gamma$ *and* $\Gamma \vdash \tau$, *then* $\Sigma \vdash \tau$ *also.*

Proof:

(i) Note that σ is a proof of σ from Σ.

(ii) Note that a proof of τ from Σ is also a proof from Γ, since $\Sigma \subseteq \Gamma$.

(iii) Since $\Gamma \vdash \tau$, there exists a proof of τ from Γ. Let $\tau_1, ..., \tau_k \equiv \tau$ be a proof of τ from Γ. Let $i_1, ..., i_\ell \in \{1, ..., k\}$ be such that $\tau_{i_j} \in \Gamma$ for $j = 1, ..., \ell$. By assumption, for each $j \in \{1, ..., \ell\}$ there is a proof $\sigma_{j1}, ..., \sigma_{jm_j} \equiv \tau_{i_j}$ of τ_{i_j} from Σ. Then

$$\sigma_{11}, ..., \sigma_{1m_1}, ... \sigma_{\ell 1}, ..., \sigma_{\ell m_\ell}, \tau_1, ..., \tau_k$$

is a proof of τ from Σ.

3.2 Meta-Theorems

Many important meta-theorems are preserved. They are stated and proved here.

Theorem 3.2.1 (Generalization Theorem) *If* $\Sigma \vdash \varphi$, *then* $\Sigma \vdash \forall x^T \varphi$.

Proof: This is obvious, because of the generalization rules of inference.

Definition 3.2.2 *Let $\varphi_1, ..., \varphi_n, \varphi$ be formulas. We say* $\{\varphi_1, ..., \varphi_n\}$ **tautologically implies** φ *iff* $((\cdots(\varphi_1 \wedge \varphi_2) \wedge \cdots) \wedge \varphi_n) \Rightarrow \varphi$ *is a propositional axiom.*

Theorem 3.2.3 (Tautology Theorem) *If* $\Sigma \vdash \varphi_1, ..., \Sigma \vdash \varphi_n$ *and* $\{\varphi_1, ..., \varphi_n\}$ *tautologically implies* φ, *then* $\Sigma \vdash \varphi$.

Proof: Assume that $\Sigma \vdash \varphi_1, ..., \Sigma \vdash \varphi_n$ and $\{\varphi_1, ..., \varphi_n\}$ tautologically implies φ. Then $\psi_1 \equiv ((\cdots(\varphi_1 \wedge \varphi_2) \wedge \cdots) \wedge \varphi_n) \Rightarrow \varphi$ is a logical axiom. Note that $\psi_2 \equiv (((\cdots(\varphi_1 \wedge \varphi_2) \wedge \cdots) \wedge \varphi_n) \Rightarrow \varphi) \Rightarrow (\varphi_1 \Rightarrow (\varphi_2 \Rightarrow \cdots(\varphi_n \Rightarrow \varphi)\cdots))$ is also a logical axiom and $\psi_3 \equiv \varphi_1 \Rightarrow (\varphi_2 \Rightarrow \cdots(\varphi_n \Rightarrow \varphi)\cdots)$ can be deduced from ψ_1 and ψ_2 by modus ponens.

For each $i \in \{1, ..., n\}$, let $\psi_{i1}, ..., \psi_{im_i}$ be a proof of φ_i from Σ. Now the following is a proof of φ from Σ :

$\psi_1, \psi_2, \psi_3,$

$\psi_{11}, \psi_{12}, ..., \psi_{1m_1}, \tau_1 \equiv \varphi_2 \Rightarrow (\varphi_3 \Rightarrow \cdots(\varphi_n \Rightarrow \varphi)\cdots))$ $(\psi_3, \psi_{1m_1},$ and MP),

$\psi_{21}, \psi_{22}, ..., \psi_{2m_2}, \tau_2 \equiv \varphi_3 \Rightarrow (\varphi_4 \Rightarrow \cdots(\varphi_n \Rightarrow \varphi)\cdots))$ $(\tau_1, \psi_{2m_2},$ and MP),

$\vdots$

$\psi_{n-1,1}, \psi_{n-1,2}, ..., \psi_{n-1,m_{n-1}}, \tau_{n-1} \equiv \varphi_n \Rightarrow \varphi$ $(\tau_{n-2}, \psi_{n-1,m_{n-1}},$ and MP),

$\psi_{n,1}, \psi_{n,2}, ..., \psi_{n,m_n}, \varphi$ $(\tau_{n-1}, \psi_{n1,m_n},$ and MP).

Theorem 3.2.4 (Deduction Theorem) *Let* Σ *be a set of sentences,* σ *a sentence, and* φ *a formula. If* $\Sigma \cup \{\sigma\} \vdash \varphi$, *then* $\Sigma \vdash (\sigma \rightarrow \varphi)$.

Proof: We will prove this theorem by induction on the length of a shortest proof of φ from $\Sigma \cup \{\sigma\}$. Let $\varphi_1, ..., \varphi_n$ be a shortest proof of φ from $\Sigma \cup \{\sigma\}$.

If $n = 1$, then $\varphi = \varphi_1$, so φ is a logical axiom or $\varphi \in \Sigma \cup \{\sigma\}$. If $\varphi \equiv \sigma$, then $\sigma \Rightarrow \varphi \equiv \sigma \Rightarrow \sigma$ is a logical axiom, so it is a proof of $\sigma \Rightarrow \varphi$ from Σ, and hence $\Sigma \vdash \sigma \Rightarrow \varphi$. If $\varphi \in \Sigma$ or φ is a logical axiom, then since $\varphi \Rightarrow (\sigma \Rightarrow \varphi)$ is a propositional axiom, φ, $\varphi \Rightarrow (\sigma \Rightarrow \varphi)$, $\sigma \Rightarrow \varphi$ is a proof of $\sigma \Rightarrow \varphi$ from Σ.

Assume that $n > 1$ and that for all formulas ψ in $\mathcal{L}$ such that there is a proof of ψ from $\Sigma \cup \{\sigma\}$ of length less than n, we have $\Sigma \vdash \sigma \Rightarrow \psi$. Since $n > 1$, $\varphi \equiv \varphi_n$ must follow from some formula(s) in $\{\varphi_1, ..., \varphi_{n-1}\}$ by a rule of inference.

Case (i): There exist $k, \ell \in \{1, ..., n-1\}$ such that φ_n follows from φ_k and φ_ℓ by modus ponens. Without loss of generality we may assume $\varphi_\ell \equiv \varphi_k \Rightarrow \varphi_n$. Then $\varphi_1, ..., \varphi_k$ and $\varphi_1, ..., \varphi_\ell$ are proofs of φ_k and φ_ℓ from $\Sigma \cup \{\sigma\}$ of length less than n. By induction $\Sigma \vdash \sigma \Rightarrow \varphi_k$ and $\Sigma \vdash \sigma \Rightarrow \varphi_\ell$. Let $\psi_1, ..., \psi_p$ be a proof of $\sigma \Rightarrow \varphi_k$ from Σ and $\chi_1, ..., \chi_q$ be a proof of $\sigma \Rightarrow \varphi_\ell$ from Σ. Since $(P \Rightarrow Q) \Rightarrow ((P \Rightarrow (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R))$ is a tautology, $\tau \equiv (\sigma \Rightarrow \varphi_k) \Rightarrow ((\sigma \Rightarrow (\varphi_k \Rightarrow \varphi_n)) \Rightarrow (\sigma \Rightarrow \varphi_n))$ is a propositional axiom. Hence, $\psi_1, ..., \psi_p, \tau, (\sigma \Rightarrow (\varphi_k \Rightarrow \varphi_n)) \Rightarrow (\sigma \Rightarrow \varphi_n), \chi_1, ..., \chi_q, \sigma \Rightarrow \varphi_n$ is a proof of $\sigma \Rightarrow \varphi$ from Σ.

Case (ii): There exists $k \in \{1, ..., n-1\}$ such that φ_n follows from φ_k by generalization. So $\varphi_n \equiv \forall x^T \varphi_k$ for some $x^T \in \mathbb{V}$. As in case (i), by induction $\Sigma \vdash \sigma \Rightarrow \varphi_k$. Since σ is a sentence, $x^T \notin \mathbb{FV}(\sigma)$. Thus $\tau \equiv$

$(\forall x^T(\sigma \Rightarrow \varphi_k) \Rightarrow (\sigma \Rightarrow \forall x^T\varphi_k)) \equiv (\forall x^T(\sigma \Rightarrow \varphi_k)) \Rightarrow (\sigma \Rightarrow \varphi_n)$ is a quantifier axiom. Let $\psi_1, ..., \psi_p$ be a proof of $\sigma \Rightarrow \varphi_k$ from Σ. Then $\psi_1, ..., \psi_p, \forall x^T(\sigma \Rightarrow \varphi_k), \tau, \sigma \Rightarrow \varphi_n$ is a proof of $\sigma \Rightarrow \varphi$ from Σ.

Case (iii): There exists $k \in \{1, ..., n-1\}$ such that φ_n follows from φ_k by specialization. So, there exists $\chi \in \mathbb{F}$, $x^T \in \mathbf{V}$, and $t^T \in \mathbb{T}$ such that $\varphi_k \equiv \forall x^T\chi$ and $\varphi_n \equiv [t^T/x^T]\chi$. As in case (i), $\Sigma \vdash \sigma \Rightarrow \forall x^T\chi$. Let $\psi_1, ..., \psi_p$ be a proof of $\sigma \Rightarrow \forall x^T\chi$ from Σ. Since $(P \Rightarrow Q) \Rightarrow ((R \Rightarrow P) \Rightarrow (R \Rightarrow Q))$ is a tautology, $\tau \equiv (\forall x^T\chi \Rightarrow \chi) \Rightarrow ((\sigma \Rightarrow \forall x^T\chi) \Rightarrow (\sigma \Rightarrow \chi))$ is a logical axiom. Hence,

$\psi_1, ..., \psi_p \equiv \sigma \Rightarrow \forall x^T\chi$
$\psi_{p+1} \equiv \tau$ (PA)
$\psi_{p+2} \equiv \forall x^T\chi \Rightarrow \chi$ (QA)
$\psi_{p+3} \equiv (\sigma \Rightarrow \forall x^T\chi) \Rightarrow (\sigma \Rightarrow \chi)$ ($\psi_{p+1}, \psi_{p+2},$ and MP)
$\psi_{p+4} \equiv \sigma \Rightarrow \chi$ ($\psi_p, \psi_{p+3},$ and MP)
$\psi_{p+5} \equiv \forall x^T(\sigma \Rightarrow \chi)$ ($\psi_{p+4},$ and GN)
$\psi_{p+6} \equiv [t^T/x^T](\sigma \Rightarrow \chi)$ ($\psi_{p+5},$ and SP).

By the definition of substitution, $[t^T/x^T](\sigma \Rightarrow \chi) \equiv ([t^T/x^T]\sigma \Rightarrow [t^T/x^T]\chi)$. Since $x^T \notin \mathbf{FV}(\sigma)$, $([t^T/x^T]\sigma \Rightarrow [t^T/x^T]\chi) \equiv \sigma \Rightarrow [t^T/x^T]\chi$. Then $\psi_{p+6} \equiv \sigma \Rightarrow [t^T/x^T]\chi$. Hence, we have a proof of $\sigma \Rightarrow \varphi$ from Σ.

Theorem 3.2.5 (Contrapositive Theorem) *Let Σ be a set of sentences and let φ and ψ be sentences. Then $\Sigma \cup \{\varphi\} \vdash \neg\psi$ iff $\Sigma \cup \{\psi\} \vdash \neg\varphi$.*

Proof: Because of the symmetry between φ and ψ, it suffices to prove that if $\Sigma \cup \{\varphi\} \vdash \neg\psi$, then $\Sigma \cup \{\psi\} \vdash \neg\varphi$. Assume that $\Sigma \cup \{\varphi\} \vdash \neg\psi$. By the Deduction theorem, $\Sigma \vdash \varphi \Rightarrow \neg\psi$. Let $\chi_1, ..., \chi_n$ be a proof of $\varphi \Rightarrow \neg\psi$ from Σ. Note that $(\varphi \Rightarrow \neg\psi) \Rightarrow (\psi \Rightarrow \neg\varphi)$ is a propositional axiom. Then

$$\chi_1, ..., \chi_n \equiv (\varphi \Rightarrow \neg\psi), (\varphi \Rightarrow \neg\psi) \Rightarrow (\psi \Rightarrow \neg\varphi), \psi \Rightarrow \neg\varphi, \psi, \neg\varphi$$

is a proof of $\neg\varphi$ from $\Sigma \cup \{\psi\}$.

3.3 The Soundness theorem

The Soundness theorem is still true in our logic. This theorem must be true; otherwise our logic has some problems.

Lemma 3.3.1 *If χ is a logical axiom, then for all structures $\mathfrak{A}$ and all variable assignments α, $I^{\mathfrak{A}}(\chi, \alpha) = T$.*

Proof: Let χ be a logical axiom. There are three case that we need to consider.

Case i: χ is a propositional axiom. This case follows directly from Proposition 2.4.4

Case ii: χ is a quantifier axiom. Let φ and ψ be formulas and x^T a variable not in $\mathbb{FV}(\varphi)$.

Subcase: $\chi \equiv (\forall x^T(\varphi \Rightarrow \psi)) \Rightarrow (\varphi \Rightarrow \forall x^T\psi)$. Let $\mathfrak{A}$ be a structure and α a variable assignment. By Lemma 2.3.9, it suffices to show that we cannot have $I^{\mathfrak{A}}(\forall x^T(\varphi \Rightarrow \psi), \alpha) = T$ and $I^{\mathfrak{A}}(\varphi \Rightarrow \forall x^T\psi, \alpha) = F$. Assume for a contradiction that $I^{\mathfrak{A}}(\forall x^T(\varphi \Rightarrow \psi), \alpha) = T$ and $I^{\mathfrak{A}}(\varphi \Rightarrow \forall x^T\psi, \alpha) = F$. Then $I^{\mathfrak{A}}(\varphi, \alpha) = T$ and $I^{\mathfrak{A}}(\forall x^T\psi, \alpha) = F$. Thus, there is a variable assignment β with $\beta(v^S) = \alpha(v^S)$ for all $v^S \not\equiv x^T$ such that $I^{\mathfrak{A}}(\psi, \beta) = F$. Since $x^T \notin \mathbb{FV}(\varphi)$, $I^{\mathfrak{A}}(\varphi, \beta) = T$, so $I^{\mathfrak{A}}(\varphi \Rightarrow \psi, \beta) = F$. It follows that $I^{\mathfrak{A}}(\forall x^T(\varphi \Rightarrow \psi), \alpha) = F$, a contradiction.

Subcase: $\chi \equiv \forall x^T\psi \Rightarrow \psi$: Let $\mathfrak{A}$ be a structure and α a variable assignment. Assume that $I^{\mathfrak{A}}(\forall x^T\psi, \alpha) = T$. Then for all variable assignments β s.t. $\beta(v^S) = \alpha(v^S)$ for all $v^S \not\equiv x^T$, $I^{\mathfrak{A}}(\psi, \beta) = T$. Since α is such variable assignment, $I^{\mathfrak{A}}(\psi, \alpha) = T$. So we have $I^{\mathfrak{A}}(\chi, \alpha) = T$.

Subcase: $\chi \equiv (\forall x^T\neg\psi) \Rightarrow (\neg\exists x^T\psi)$: Let $\mathfrak{A}$ be a structure and α a variable assignment. As with the first subcase, it suffices to show that $I^{\mathfrak{A}}(\forall x^T\neg\psi, \alpha) = T$ and $I^{\mathfrak{A}}(\neg\exists x^T\psi, \alpha) = F$ leads to a contradiction. Suppose $I^{\mathfrak{A}}(\forall x^T\neg\psi, \alpha) = T$ and $I^{\mathfrak{A}}(\neg\exists x^T\psi, \alpha) = F$. Then $I^{\mathfrak{A}}(\exists x^T\psi, \alpha) = T$, so there is a variable assignment β such that $\beta(v^S) = \alpha(v^S)$ for all $v^S \not\equiv x^T$ and $I^{\mathfrak{A}}(\psi, \beta) = T$. Hence $I^{\mathfrak{A}}(\neg\psi, \beta) = F$, so that $I^{\mathfrak{A}}(\forall x^T\neg\psi, \alpha) = F$, a contradiction.

Subcase: $\chi \equiv (\neg\exists x^T\psi) \Rightarrow (\forall x^T\neg\psi)$: This is similar to the previous subcase.

Case iii: φ is an equality axiom. Let $x^S, y^S, v^S \in \mathbb{V}, t^T \in \mathbb{T}$, and φ be an atomic formula.

Subcase: $\chi \equiv (x^S = x^S)$: Trivial.

Subcase: $\chi \equiv (x^S = y^S) \Rightarrow ([x^S/v^S]t^T = [y^S/v^S]t^T)$: Let $\mathfrak{A}$ be a structure and α a variable assignment. Assume that $I^{\mathfrak{A}}((x^S = y^S), \alpha) = T$. Let β be the variable assignment defined by $\beta(z^U) = \alpha(z^U)$ for all variables $z^U \not\equiv v^S$ and $\beta(v^S) = \alpha(x^S) = I^{\mathfrak{A}}(x^S, \alpha)$. Since $I^{\mathfrak{A}}((x^S = y^S), \alpha) = T$, $\beta(v^S) = \alpha(x^S) = \alpha(y^S) = I^{\mathfrak{A}}(y^S, \alpha)$. By Proposition 2.3.11, $I^{\mathfrak{A}}([x^S/v^S]t^T, \alpha) = I^{\mathfrak{A}}(t^T, \beta) = I^{\mathfrak{A}}([y^S/v^S]t^T, \alpha)$. Thus, $I^{\mathfrak{A}}([x^S/v^S]t^T = [y^S/v^S]t^T, \alpha) = T$.

Subcase: $\chi \equiv (x^S = y^S) \Rightarrow ([x^S/v^S]\varphi \Rightarrow [y^S/v^S]\varphi)$: This subcase is similar to the previous subcase.

Theorem 3.3.2 *Let Σ be a set of sentences, φ a sentence. If $\Sigma \vdash \varphi$, then $\Sigma \models \varphi$.*

Proof: We will prove this theorem by induction on the length of a shortest proof of φ from Σ. However, since some steps in the proof of φ may not be sentences, we will need to prove the following stronger result: Let Σ be a set of sentences and φ a formula. If $\Sigma \vdash \varphi$, then for all structures $\mathfrak{A}$, if $\mathfrak{A} \models \Sigma$, then for all variable assignments α we have that $I^{\mathfrak{A}}(\varphi, \alpha) = T$.

Assume that $\Sigma \vdash \varphi$. Let $\mathfrak{A}$ be a structure such that $\mathfrak{A} \models \Sigma$, α a variable assignment, and let $\varphi_1, ..., \varphi_n$ be a shortest proof of φ from Σ. If $n = 1$, then $\varphi \in \Sigma$ or φ is a logical axiom. If $\varphi \in \Sigma$, then φ is a sentence and $\mathfrak{A} \models \varphi$. So we have $I^{\mathfrak{A}}(\varphi, \alpha) = T$. The case that φ is a logical axiom is handled by the previous lemma.

Assume that $n > 1$ and that for all formulas ψ such that there is a proof of ψ from Σ of length less than n, we have $I^{\mathfrak{B}}(\psi, \beta) = T$ for all variable assignments β and structures $\mathfrak{B}$ which are models of Σ. Since $n > 1$, $\varphi \equiv \varphi_n$ must follow from some formula(s) in $\{\varphi_1, ..., \varphi_{n-1}\}$ by a rule of inference.

Case(i): There exist $k, \ell \in \{1, ..., n-1\}$ such that φ_n follows from φ_k and φ_ℓ by modus ponens. Without lose of generality, we may assume $\varphi_\ell \equiv \varphi_k \Rightarrow \varphi_n$. Then $\varphi_1, ..., \varphi_k$ and $\varphi_1, ..., \varphi_\ell$ are proofs of φ_k and φ_ℓ from Σ of length less than n. By induction, $I^{\mathfrak{A}}(\varphi_k, \alpha) = T$ and $I^{\mathfrak{A}}(\varphi_\ell, \alpha) = T$. Then $I^{\mathfrak{A}}(\varphi_n, \alpha) = T$.

Case(ii): There exists $k \in \{1, ..., n-1\}$ such that φ_n follows from φ_k by generalization. Then $\varphi_n \equiv \forall x^T \varphi_k$ for some variable x^T. As in Case(i), we can use induction to conclude that $I^{\mathfrak{A}}(\varphi_k, \beta) = T$ for all variable assignments β. In particular, $I^{\mathfrak{A}}(\varphi_k, \alpha) = T$ for all variable assignments β such that $\beta(v^S) = \alpha(v^S)$ for all variables $v^S \not\equiv x^T$. This implies that $I^{\mathfrak{A}}(\forall x^T \varphi_k, \alpha) = T$.

Case(iii): There exists $k \in \{1, ..., n-1\}$ such that φ_n follows from φ_k by specialization. So, there exists $\psi \in \mathbb{F}$, $x^T \in \mathbb{V}$, and $t^T \in \mathbb{T}$ such that $\varphi_k \equiv \forall x^T \psi$ and $\varphi_n \equiv [t^T/x^T]\psi$. Define a variable assignment β by $\beta(v^S) = \alpha(v^S)$ for all variables $v^S \not\equiv x^T$ and $\beta(x^T) = I^{\mathfrak{A}}(t^T, \alpha)$. As in Case(ii) $I^{\mathfrak{A}}(\forall x^T \psi, \beta) = T$. This implies that $I^{\mathfrak{A}}(\psi, \beta) = T$. By Proposition 2.3.11, $I^{\mathfrak{A}}([t^T/x^T]\psi, \alpha) = I^{\mathfrak{A}}(\psi, \beta) = T$.

3.4 Failure of the Compactness and Completeness theorems

3.4.1 A counterexample to the Compactness theorem.

Recall that the Compactness theorem states that if Σ is a set of sentences such that every finite subset of Σ has a model, then Σ itself has a model. We will show that the Compactness theorem fails in our logic, by giving a counterexample.

Let $\mathcal{L}$ be a language over a type system $\mathbb{T}$. Fix P a primitive type in $\mathcal{P}$. Let σ be a sentence which says that "Every one-to-one mapping from P to P must also be onto", and for each $n \in \mathbb{N}$, let σ_n be a sentence which says that "There are at least n distinct elements of type P". These can all be written as sentences in our logic, as follows:

$$\sigma \equiv \forall f^{P\to P}((\forall x_1^P \forall x_2^P (f^{P\to P}(x_1^P) =^P f^{P\to P}(x_2^P) \Rightarrow x_1^P =^P x_2^P)) \Rightarrow (\forall x_1^P \exists x_2^P (x_1^P =^P f^{P\to P}(x_2^P)))).$$

and for each $n \in \mathbb{N}$,

$$\sigma_n \equiv \exists x_1^P \exists x_2^P \cdots \exists x_n^P (x_1^P \neq x_2^P \wedge x_1^P \neq x_3^P \wedge \cdots \wedge x_1^P \neq x_n^P \wedge x_2^P \neq x_3^P \wedge \cdots \wedge x_2^P \neq x_n^P \wedge \cdots \wedge x_{n-1}^P \neq x_n^P).$$

Let $\Sigma = \{\sigma, \sigma_1, \sigma_2, ...\}$.

Lemma 3.4.1 *Every finite subset of Σ has a model.*

Proof: Let Σ' be a finite subset of Σ. If $\Sigma' = \{\sigma\}$, then any structure $\mathfrak{A}$ with $\mathcal{P}^{\mathfrak{A}}(P)$ finite will be a model of Σ'. Now suppose $\Sigma' \neq \{\sigma\}$. Then $I = \{n \in \mathbb{N} \mid \sigma_n \in \Sigma'\}$ is finite. Let $m = max\ I$. Choose a structure $\mathfrak{A}$ with $\mathcal{P}^{\mathfrak{A}}(P)$ finite and $|\ \mathcal{P}^{\mathfrak{A}}(P)\ | \geq m$. Then $\mathfrak{A} \models \sigma$ and $\mathfrak{A} \models \sigma_n$ for all $n \leq m$. Thus $\mathfrak{A} \models \Sigma'$.

Lemma 3.4.2 *The set Σ has no models.*

Proof: Let $\mathfrak{A}$ be a structure for $\mathcal{L}$ over $\mathbb{T}$.

Case $\mathcal{P}^{\mathfrak{A}}(P)$ is finite: Say $m = |\ \mathcal{P}^{\mathfrak{A}}(P)\ |$. Then $\mathfrak{A} \not\models \sigma_{m+1}$.

Case $\mathcal{P}^{\mathfrak{A}}(P)$ is infinite: Define $g : \mathbb{N} \to \mathcal{P}^{\mathfrak{A}}(P)$ by induction as follows. Since $\mathcal{P}^{\mathfrak{A}}(P) \neq \emptyset$, let $a_1 \in \mathcal{P}^{\mathfrak{A}}(P)$, and define $g(1) = a_1$. Assume that $n > 1$ and that for all $m < n$, $g(m)$ is defined. Since $\mathcal{P}^{\mathfrak{A}}(P)$ is infinite, $\mathcal{P}^{\mathfrak{A}}(P) \setminus \{g(1), ..., g(n-1)\} \neq \emptyset$. Let $a_n \in \mathcal{P}^{\mathfrak{A}}(P) \setminus \{g(1), ..., g(n-1)\}$, and define $g(n) = a_n$. Clearly g is one-to-one.

Let $R_g = \{a_1, a_2, ...\}$. Define $h : R_g \to R_g$ by $h(a_n) = a_{n+1}$ for each $n \in \mathbb{N}$. Clearly h is one-to-one but not onto.

Finally, define $f : \mathcal{P}^{\mathfrak{A}}(P) \to \mathcal{P}^{\mathfrak{A}}(P)$ by

$$f(x) = \begin{cases} h(x) & \text{if } x \in R_g \\ x & \text{otherwise.} \end{cases}$$

Then f is one-to-one but not onto. Thus $\mathfrak{A} \not\models \sigma$.

The two previous lemmas show that every finits subset of Σ has a model but Σ itself has no model. This shows that the Compactness theorem is not true in our logic.

3.4.2 A counterexample to the Completeness theorem.

Recall that the Completeness theorem states that for any set of sentences Σ and any sentence φ, if $\Sigma \models \varphi$, then $\Sigma \vdash \varphi$. In this section we will show that the Completeness theorem fails in our logic also, by giving a counterexample based on the example in the previous section. First, we observe that the "Compactness theorem for proofs" is true.

Theorem 3.4.3 *Let Σ be a set of formulas and φ a formula. If $\Sigma \vdash \varphi$, then there exist a finite subset Σ' of Σ such that $\Sigma' \vdash \varphi$.*

Proof: Let φ be a formula, Σ a set of formulas such that $\Sigma \vdash \varphi$. Say $\varphi_1, ..., \varphi_n$ is a proof of φ from Σ. Let $\Sigma' = \Sigma \cap \{\varphi_1, ..., \varphi_n\}$. Then Σ' is a finite subset of Σ, and it is clear from the definition that $\Sigma' \vdash \varphi$.

Now we can use the example of the previous section to construct a counterexample to the Completeness theorem. Let φ be the formula $\exists x^P \neg(x^P = x^P)$ and let Σ be as in the previous section. Since Σ has no models, $\Sigma \models \varphi$. Suppose that $\Sigma \vdash \varphi$. By the above theorem there would exists a finite subset Σ' of Σ such that $\Sigma' \vdash \varphi$. By the Soundness theorem, we would have $\Sigma' \models \varphi$. But Lemma 3.4.1 tells us that Σ' has a model, say $\mathfrak{A}$, so we would have $\mathfrak{A} \models \varphi$, which is clearly impossible. Thus, we must have that $\Sigma \nvdash \varphi$.